

# A note on the connection between product-form Jackson networks and counting lattice walks in the quarter plane

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## 1 Introduction

The ultimate goal is to find explicit solutions for counting generating functions (CGF) of some random walks in the quarter plane, starting from the wellknown product-form for the stationary distribution of customers in Jackson's stochastic networks. The first step is to solve a *relatively simple* boundary value problem (BVP) for the queueing system. Then, by using a variational principle based on the continuity with respect to adequate parameters, it is possible to transform some smooth curves in a continuous fashion and to get explicitly formulas for the conformal mappings of Riemann's theorem. This might be a new way of computing the gluing functions appearing in [1, 3], which are a key ingredient in the explicit formulas for CGFs.

## 2 An illustrative example of walks with steps East, South, North-West: tandem queues

We start by considering the related system consisting of two M/M/1 service stations in tandem, under Poisson (intensity  $\lambda$ ) external arrival process and exponential service time distributions with equal service rates  $\mu$  (this does not change the nature of the results).

Setting  $\rho \stackrel{\text{def}}{=} \frac{\lambda}{\mu}$ , the stationary distribution for the joint distribution of the number of units in the system satisfies the functional equation

$$Q(x, y)F(x, y) = \left(1 - \frac{y}{x}\right)F(0, y) + \left(1 - \frac{1}{y}\right)F(x, 0), \quad (2.1)$$

with  $Q(x, y) = \rho(1 - x) + \left(1 - \frac{y}{x}\right) + \left(1 - \frac{1}{y}\right)$ .

For  $\rho < 1$ , the solution of (2.1) is well-known to have the simple product-form

$$F(x, y) = \frac{(1 - \rho)^2}{(1 - \rho x)(1 - \rho y)}. \quad (2.2)$$

Now, assume we do not know the simple expression (2.2). Then, according the method invented in [1], it is always possible to set a boundary value problem (BVP) of Riemann-Dirichlet-Carleman type for the function  $F(0, y)$ , which here has a simple formulation. We shall sketch the mains steps.

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As usual, we first restrict the equation (2.1) to the algebraic curve

$$\{(x, y) \in \mathbb{C}^2 : xyQ(x, y) = 0\}.$$

Solving  $xyQ(x, y) = 0$  with respect to  $y$  yields

$$y^2 - x(\rho + 2 - \rho x)y + x = 0, \quad (2.3)$$

which has two roots  $Y_1(x), Y_2(x)$  (the two branches of the global algebraic function  $Y(x)$ ), such that

$$|Y_1(x)| \leq 1 \leq |Y_2(x)|, \forall |x| = 1, \quad \text{and} \quad |Y_1(x)| \leq |Y_2(x)|, \forall x \in \mathbb{C}.$$

It is also worth noting

$$Y_1(x) \cdot Y_2(x) = x, \quad \forall x \in \mathbb{C}. \quad (2.4)$$

The discriminant of equation (2.3)

$$D(x) = x^2(\rho + 2 - \rho x)^2 - 4x = x(x - 1)(\rho^2 x^2 - \rho(\rho + 4)x + 4)$$

has four real zeros:  $x_1 = 0 < x_2 = 1 < x_3 < x_4$ , which are the branch points of the algebraic curve  $Y(x)$  in the complex plane  $\mathbb{C}_x$ .

For  $x \in [0, 1]$ , we have  $Y_1(x) = \overline{Y_2(x)}$ , since the two roots of  $Q(x, y)$  considered as a polynomial in  $y$  are complex conjugate, and  $Y(x)$  describes a simple closed contour  $\mathcal{L}$ . Indeed  $\mathcal{L}$  is the *smallest* component of a quartic curve, which in the  $\mathbb{C}_y$ -plane has the following equation, with  $y = u + iv$ ,

$$\rho(u^2 + v^2)^2 - (\rho + 2)(u^2 + v^2) + 2u = 0. \quad (2.5)$$

The contour  $\mathcal{L}$  is situated entirely in the right half- $\mathbb{C}_y$ -plane, inside the unit circle, and crosses the real axis at  $u = 0$  and  $u = 1$ , where it has two vertical tangents. The interior domain bounded by  $\mathcal{L}$  will be denoted by  $\mathcal{L}^+$ .

Letting  $x$  tend successively to the upper and lower edge of the slit (cut)  $[0, 1]$ , we can eliminate  $F(x, 0)$  which is continuous on  $[0, 1]$ . So, after a quick algebra, setting for short

$$\Phi(y) \stackrel{\text{def}}{=} F(0, y), \quad A(y) \stackrel{\text{def}}{=} \left( \frac{y}{1 - y} \right)^2,$$

we can state the following BVP for  $F(0, y)$ .

*Find a function  $\Phi$  holomorphic in  $\mathcal{L}^+$ , the limiting values of which on the contour  $\mathcal{L}$  are continuous and satisfy the relation*

$$\boxed{A(t)\Phi(t) - A(\bar{t})\Phi(\bar{t}) = 0, \quad t \in \mathcal{L}}. \quad (2.6)$$

**Remark 2.1.** The problem (2.6) is a priori of generalized Dirichlet-Carleman type, *since the function  $A(\cdot)$  has a zero at  $t = 0$  and a pole at  $t = 1$  on the contour*, but is otherwise analytic in  $\mathcal{L}^+$ . Therefore slightly sharper technicalities appear.

Setting

$$B(t) \stackrel{\text{def}}{=} \frac{A(\bar{t})}{A(t)} = \left(1 - \frac{1}{t}\right)^2 / \left(1 - \frac{1}{\bar{t}}\right)^2$$

the BVP (2.6) can be rewritten in the form

$$\boxed{\Phi(t) = B(t)\Phi(\bar{t}), \quad t \in \mathcal{L}}. \quad (2.7)$$

Checking that  $B(t)$  is continuous on the contour  $\mathcal{L}$ , where it satisfies a Hölder condition, we get a classical Riemann-Hilbert problem, see [2].

In order to solve it, we introduce the conformal mapping  $T(z)$  of the closed unit disc  $\mathcal{D} = \{z : |z| \leq 1\}$  onto the region  $\mathcal{L} \cup \mathcal{L}^+$ . By symmetry with respect to the real axis in the respective planes, we can take

$$y = T(z), \quad \bar{y} = \overline{T(z)} = T(\bar{z}) = T(1/z), \quad \forall |z| \leq 1,$$

and we shall also choose  $T(-1) = 0$ , which implies necessarily  $T(1) = 1$ .

Introduce the new functions  $\Psi(z) \stackrel{\text{def}}{=} \Phi(T(z))$ ,  $C(z) \stackrel{\text{def}}{=} B(T(z))$ , and denote by  $\mathcal{U}$  the unit circle  $|z| = 1$ .

Then condition (2.7) can be put into Hilbert's standard form

$$\boxed{\Psi^+(t) = C(t)\Psi^-(t), \quad t \in \mathcal{U}}, \quad (2.8)$$

where  $\Psi(\cdot)$  is sought to be sectionally analytic, of finite degree at infinity, and the solution in  $\mathcal{L}^+$  is given by (see e.g. [2])

$$\Psi^+(z) = e^{\Gamma(z)} \quad \text{with} \quad \Gamma(z) = \frac{1}{2i\pi} \int_{\mathcal{U}} \frac{\log[t^{-\chi}C(t)]dt}{t-z}, \quad \forall |z| < 1. \quad (2.9)$$

where  $\chi$  is the *index*, defined by

$$\chi = \mathcal{I}nd[C]_{\mathcal{U}} \stackrel{\text{def}}{=} \frac{1}{2\pi} [\arg C]_{\mathcal{U}} = \frac{1}{2i\pi} [\log C]_{\mathcal{U}}, \quad (2.10)$$

which represents the variation of the argument of  $C(u)$ , as  $u$  moves along the unit circle  $\mathcal{U}$  in the positive direction. Hence, the function  $\log[u^{-\chi}C(u)]$  is one-valued on  $\mathcal{U}$ .

From the expression of  $B(t)$  given in (2.7), it appears that  $|C(u)| = 1$  for  $|u| = 1$ , and  $C(u)$  is also continuous at the points  $u = 0, 1$ . From these remarks, it is not difficult to prove that, for the special model under consideration,  $\chi = 0$  [details are omitted].

Starting from the expression

$$C(t) = \left(1 - \frac{1}{T(t)}\right)^2 / \left(1 - \frac{1}{T(1/t)}\right)^2,$$

we can write  $\log[C(t)]$  in the following form, which takes into account the singularities of the logarithms at  $t = 1$  and  $t = -1$  (respectively  $y = 1$  and  $y = -1$  in the original  $\mathbb{C}_y$  plane).

$$\log[C(t)] = 2 \left[ \log \frac{1-T(t)}{1-t} - \log \frac{T(t)}{t+1} - \log \frac{1-T(1/t)}{1-t} + \log \frac{T(1/t)}{t+1} \right]. \quad (2.11)$$

Now the computation of  $\Gamma(z)$  defined in (2.9) becomes straightforward by using the Cauchy residue theorem for the integrand (2.11). This yields

$$\Psi^+(z) = \left[ \frac{(1+z)(1-T(z))}{(1-z)T(z)} \right]^2, \quad \forall |z| \leq 1. \quad (2.12)$$

On the other hand, we have by (2.2)

$$F(0, T(z)) \equiv \Psi^+(z) = \frac{(1-\rho)^2}{1-\rho T(z)} \quad \forall |z| \leq 1,$$

so that, comparing with (2.12), we see that  $T(z)$  satisfies a polynomial equation of third degree. It is in fact simpler to write a closed form formula for the inverse mapping  $S(y)$ , which transforms conformally  $\mathcal{L}_+$  onto the unit disc  $|z| \leq 1$ . We obtain

$$S(y) = \frac{(1-\rho)y - (1-\rho\sqrt{1-\rho y})}{(1-\rho)y + (1-\rho\sqrt{1-\rho y})}.$$

### 3 More general two-queue Jackson networks

In Fig. 3.1, we have drawn three generators representing the evolution of two-queue Jackson networks, which means that the invariant measure (assuming ergodicity conditions to be satisfied) has a product-form solution. From left to right, we refer to them as models  $M1, M2, M3$ . Then, in the context of Section 4 and according to the definition given in [1], the corresponding walks have respective *groups of order* 6, 6, 8.

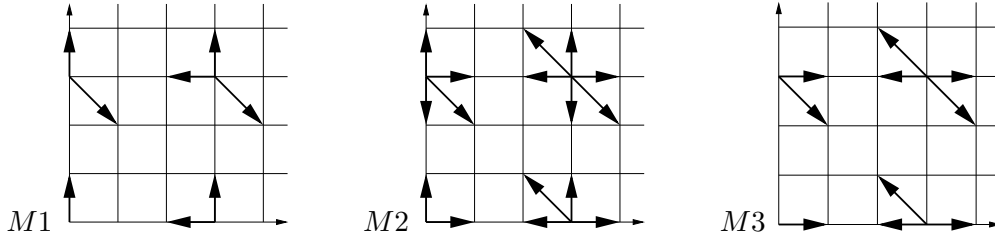


Fig. 3.1: Three examples of coupled queues representing a product-form Jackson network.

#### 3.1 A BVP for the models of figure 3.1

Again we shall view  $M2$  as a simple stochastic network of two coupled  $M/M/1$  queues. The following choice for the rates of the generator of the underlying Markov process will be made, with obvious notations for the cardinal points, and  $0 \leq p_i \leq 1$ ,  $p_i + q_i = 1, i = 1, 2$ .

$$\lambda_1 \text{ (E)}, \quad \lambda_2 \text{ (N)}, \quad p_1\mu \text{ (W)}, \quad p_2\mu \text{ (S)}, \quad q_1\mu \text{ (N-W)}, \quad q_2\mu \text{ (S-E)}.$$

Setting  $\rho_i = \frac{\lambda_i}{\mu}, i = 1, 2$ , it is not difficult to check that the invariant measure satisfies the following functional equation.

$$\begin{cases} R(x, y)G(x, y) = y[x - (p_1 + q_1y)]G(0, y) + x[y - (p_2 + q_2x)]G(x, 0), \\ R(x, y) = xy[\rho_1(1-x) + \rho_2(1-y)] + y[x - (p_1 + q_1y)] + x[y - (p_2 + q_2x)]. \end{cases} \quad (3.1)$$

Then the solution of (3.1) turns out to have the nice product-form

$$G(x, y) = \frac{(1-r_1)(1-r_2)}{(1-r_1x)(1-r_2y)}, \quad (3.2)$$

where

$$r_1 = \frac{\lambda_1 + q_2\lambda_2}{(1-q_1q_2)\mu}, \quad r_2 = \frac{\lambda_2 + q_1\lambda_1}{(1-q_1q_2)\mu},$$

assuming the ergodicity conditions  $r_i < 1, i = 1, 2$  are satisfied.

Then, after setting for short  $H(y) \stackrel{\text{def}}{=} G(0, y)$ , we return to (3.1) to obtain the following BVP in the  $\mathbb{C}_y$ -plane (the contour  $\mathcal{L}$  being, as before, the convenient component of a quartic curve), which is of the same type as (2.6) and obtained by eliminating  $xG(x, 0)$  in (3.1).

Find  $H(\cdot)$  meromorphic in the domain  $\mathcal{L}^+$  and satisfying

$$K(t)H(t) - K(\bar{t})H(\bar{t}) = 0, \quad t \in \mathcal{L}, \quad (3.3)$$

where

$$K(t) = \frac{t[X_1(t) - (p_1 + q_1 t)]}{[t - (p_2 + q_2 X_1(t))]},$$

noting that  $H(\cdot)$  might have a pole inside  $\mathcal{L}^+ \setminus \mathcal{D}$ , and the function  $X_1(\cdot)$  is the analogous of  $Y_1(\cdot)$  defined in (2.4) for the kernel  $R(x, y)$ .

The next step is to use the above product form. Indeed, by (3.2), the right-hand side of (3.1) yields the following algebraic system

$$\begin{cases} \frac{y[x - (p_1 + q_1 y)]}{1 - r_2 y} + \frac{x[y - (p_2 + q_2 x)]}{1 - r_1 x} = 0, \\ \{(x, y) \in \mathbb{C}^2 : R(x, y) = 0\}, \quad |x|, |y| \leq 1. \end{cases} \quad (3.4)$$

We note first that the BVP (3.3) is equivalent to the BVP

$$(1 - r_2 t)H(t) - (1 - r_2 \bar{t})H(\bar{t}) = 0, \quad t \in \mathcal{L}, \quad (3.5)$$

which in the present situation admits (as expected!) precisely the unique solution  $H(y) = \frac{H(0)}{1 - r_2 y}$ .

To proceed as in Section 2 to find the conformal mapping  $T(z)$  for the model  $M_2$ , it is in fact necessary to solve *directly* the BVP (3.3). This is the matter of an ongoing work.

## 4 Application to counting walks in the quarter-plane

Let us recall below the basic assumptions.

- Walks start at the origin.
- The set  $\mathcal{S}$  of admissible steps is included in the set of the 8 nearest neighbors, i.e.,  $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ . By using an extended Kronecker's delta, we shall write

$$\delta_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in \mathcal{S}, \\ 0 & \text{if } (i, j) \notin \mathcal{S}. \end{cases}$$

Let  $f(i, j, k) \stackrel{\text{def}}{=}$  the number of paths in  $\mathbb{Z}_+^2$  starting from  $(0, 0)$  and ending at  $(i, j)$  after  $k$  steps. Then

$$F(x, y, s) \stackrel{\text{def}}{=} \sum_{i,j,k \geq 0} f(i, j, k) x^i y^j s^k \quad (4.1)$$

satisfies the functional equation

$$\boxed{K(x, y, s)F(x, y, s) = c(x)F(x, 0, s) + \tilde{c}(y)F(0, y, s) + c_0(x, y, s)}, \quad (4.2)$$

a priori valid in the domain  $|x| \leq 1, |y| \leq 1, |s| < 1/|\mathcal{S}|$ , where

$$\begin{cases} K(x, y; s) = xy \left[ \sum_{(i,j) \in \mathcal{S}} x^i y^j - 1/s \right], \\ c(x) = \sum_{i \leq 1} \delta_{i,-1} x^{i+1}, \quad \tilde{c}(y) = \sum_{j \leq 1} \delta_{-1,j} y^{j+1}, \\ c_0(x, y, s) = -\delta_{-1,-1} F(0, 0, s) - xy/s, \end{cases}$$

As for the special model of Section 2, we have

$$K(x, y, s) = y^2 - x(1/s - x)y + x.$$

which we compare with the kernel given by (2.3), namely

$$xyQ(x, y) = y^2 - x(\rho + 2 - \rho x)y + x.$$

#### 4.1 A varational principle for the curve $\mathcal{L}$ in the east, South and North-West model

The idea is to find a linear homeomorphism  $\mathcal{H} : (x, y) \rightarrow (X, Y)$  as simple as possible, transforming  $xyQ(x, y)$  into  $K(X, Y, z) = 0$ , for convenient values of the parameter  $\rho$ , so that  $z$  remains unchanged (note that we consider  $z$  as a parameter).

The change of variables  $x = aX, y = bY$ , with  $ab \neq 0$ , in (2.3) gives (after having divided by  $b^2$ )

$$Y^2 - \frac{aX(\rho + 2 - a\rho X)}{b}Y + \frac{aX}{b^2} = 0.$$

Hence, taking  $a = b^2$ , the identification of  $\frac{a(\rho + 2 - a\rho X)}{b}$  with  $\frac{1}{s} - X$  yields the two following necessary conditions:

$$b(\rho + 2) = \frac{1}{s}, \quad \text{and} \quad b^3\rho = 1.$$

Finally, we can completely specify the sought homeomorphism by

$$x = b^2X, \quad y = bX, \quad \rho = b^{-3}, \quad s = \frac{b^2}{1 + 2b^3}. \quad (4.3)$$

Two facts should be quoted.

- (i) Here  $b$  is real positive and, since  $\rho < 1$ , we have  $b > 1$ . Moreover,  $s$  given by (4.3) is smaller than  $1/3$  and thus belongs to the initial domain of definition of (4.1).
- (ii) In the sequel,  $s$  will not be necessarily real.

## References

- [1] Guy Fayolle, Roudolph Iasnogorodski, and Vadim A. Malyshev. *Random Walks in the Quarter Plane: Algebraic Methods, Boundary Value Problems, Applications to Queueing Systems and Analytic Combinatorics*, volume 40 of *Probability Theory and Stochastic Modelling*. Springer International Publishing, February 2017. *Second edition*, <https://hal.inria.fr/hal-01651919>.
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