HAL
open science

A short-list of pairing-friendly curves resistant to Special TNFS at the 128-bit security level

Aurore Guillevic

To cite this version:

Aurore Guillevic. A short-list of pairing-friendly curves resistant to Special TNFS at the 128-bit security level. 2019. hal-02396352v1

HAL Id: hal-02396352

https://inria.hal.science/hal-02396352v1

Preprint submitted on 5 Dec 2019 (v1), last revised 5 Feb 2020 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A short-list of pairing-friendly curves resistant to Special TNFS at the 128-bit security level

Aurore Guillevic

Université de Lorraine, CNRS, Inria, LORIA, Nancy, France
aurore.guillevic@inria.fr

Abstract. There have been notable improvements in discrete logarithm computations in finite fields since 2015 and the introduction of the Tower Number Field Sieve (TNFS) algorithm for extension fields. The *Special* TNFS is very efficient in finite fields that are target groups of pairings on elliptic curves, where the characteristic is special (e.g. sparse). The key sizes for pairings should be increased, and alternative pairing-friendly curves can be considered. We revisit the Special variant of TNFS for pairing-friendly curves. In this case the characteristic is given by a polynomial of moderate degree (between 4 and 38) and tiny coefficients, evaluated at an integer (a seed). We present a polynomial selection with a new practical trade-off between degree and coefficient size. As a consequence, the security of curves computed by Barbulescu, El Mrabet and Ghammam should be revised: we obtain a smaller estimated cost of STNFS for all curves except BLS12 and BN. To obtain TNFS-secure curves, we reconsider the Brezing–Weng generic construction of families of pairing-friendly curves and estimate the cost of our new Special TNFS algorithm for these curves. This improves on the work of Fotiadis and Konstantinou, Fotiadis and Martindale, and Barbulescu, El Mrabet and Ghammam. We obtain a short-list of interesting families of curves that are resistant to the Special TNFS algorithm, of embedding degrees 10 to 16 for the 128-bit security level. We conclude that at the 128-bit security level, a BLS-12 curve over a 440 to 448-bit prime seems to be the best choice for pairing efficiency. We also give a brief overview of the 192-bit security level.

1 Introduction

A cryptographic pairing is a bilinear non-degenerate map from two groups $\mathbb{G}_1$ and $\mathbb{G}_2$ to a target group $\mathbb{G}_T$, where the three groups share a common prime order r. The first two groups are distinct subgroups of the group of points $E(\mathbb{F}_{p^k})$ of an elliptic curve E defined over a prime field $\mathbb{F}_p$, and the third group is a multiplicative subgroup of order r of a finite field $\mathbb{F}_{p^k}$, where k is the minimal integer such that $r \mid p^k - 1$, and is called the embedding degree. Pairing-friendly curves such that k is small (between 1 and 20 for example) should be designed on purpose, as the embedding degree is usually very large, of the magnitude of r.

Freeman, Scott and Teske presented a taxonomy of pairing-friendly curves in [21]. At that time, the size of the target finite field $\mathbb{F}_{p^k}$ was chosen to be

the same as a prime field $\mathbb{F}_q$ offering the desired security, that is, a 3072-bit (or 3200-bit) finite field for a 128-bit security level. The size of $\mathbb{F}_q$ is deduced from the asymptotic complexity of the Number Field Sieve $L_p(1/3, c) = \exp((c + o(1))(\ln p)^{1/3}(\ln \ln p)^{2/3})$, where $c = (64/9)^{1/3} \approx 1.923$ for general prime fields and $c = (32/9)^{1/3} \approx 1.526$ for special primes having a very sparse representation. Barreto–Naehrig (BN) curves became very popular. A BN curve defined over a prime field has prime order and embedding degree 12, hence choosing p and r of 256 bits gives 128 bits of security one the curve, then p^k is about 3072-bit long, as desired to match the 128-bit security level in $\mathbb{F}_{p^k}$. But it turned out that prime fields and extension fields of the same total size q and p^k do not offer the same security. The state of affairs for extension fields is complicated, with many different cases.

In 2015 and 2016, Barbulescu, Gaudry and Kleinjung, followed by Kim and Barbulescu and Kim and Jeong [5,28,29] revisited Schirokauer's Tower Number Field Sieve algorithm (TNFS) and applied this new setting to finite fields of composite extension degrees. The asymptotic complexity of this new algorithm decreased significantly, from $L_Q(1/3, 2.201)$ to $L_Q(1/3, 1.526)$ and in particular, below the complexity of a generic DL computation in a prime field, in $L_Q(1/3, 1.923)$. This makes mandatory to revisit the sizes and choices of pairing-friendly curves.

Fotiadis and Konstantinou [18] revisited the Brezing–Weng method to generate families of pairing-friendly curves and identified a list of interesting choices of moderate embedding degrees to match the 128-bit security level. However, they considered the asymptotic complexity of STNFS to deduce the security offered by the curves. It gives a first hint on the sizes of finite fields to choose but is not precise enough. Later Menezes, Sarkar and Singh [31], then Barbulescu and Duquesne [3] and more recently Guillevic and Singh [24] refined the analysis of STNFS to obtain more precise sizes of finite fields to match a given security level. Fotiadis and Martindale [19] focus on composite embedding degrees ($k \in \{8, 9, 10, 12\}$ for the 128-bit security level), Guillevic, Masson and Thomé [23] consider a modification of the Cocks–Pinch method for $k \in \{5, 6, 7, 8\}$, and Barbulescu, El Mrabet and Ghammam span embedding degrees from 9 to 53.

This is an active topic: the standardisation of pairings is under discussions at IETF [34] and at ISO for updating the standard on pairing-friendly curves [25]. Particular pairing-friendly curves (e.g. cycles of curves [13]) are also needed in zero-knowledge proofs and blockchains (ZCash uses a BLS12-381 curve [8,37], Ethereum a BN-256 curve [16], and Zexe a BLS12-377 curve and a Cocks–Pinch curve of embedding degree 6 [9, Table 16]).

Our Contributions

We introduce a practical variant of special polynomial selection for STNFS that applies to target finite fields of pairing-friendly curves. It does not change the asymptotic complexity of STNFS but it changes the estimated cost of STNFS as computed by Barbulescu and Duquesne.

We extend the work of Fotiadis and Konstantinou [18], and identify another criterion to be resistant to STNFS: the polynomial $p(x)$ defining the field characteristic should have no automorphism. Then we build on the work of Guillevic and Singh [24] to estimate finely the cost of a discrete logarithm computation with STNFS. We write a SageMath script to automatically and systematically compare many polynomial selections, and in particular, change of variables on $p(x)$. We consider embedding degrees from 9 to 17 at the 128-bit security level. This is a complement to the work of Fotiadis and Martindale [19], where embedding degrees 8, 9, 10 and 12 are considered at this security level. We also identify non-optimal parameter choices in the recent preprint of Barbulescu, El Mrabet and Ghammam [4], resulting in over-estimated cost of STNFS and under-estimated finite field size. We conclude with a short-list of STNFS-secure pairing-friendly curves of embedding degrees from 10 to 16.

The work in [23] showed that a pairing-friendly curve over a non-special prime, and with a prime embedding degree $k = 5, 7$ gives a slow pairing computation, about three times slower than the best candidate: a BLS12-curve over a 446-bit prime field. Here we foresee that a curve of prime embedding degree k with a special prime will not provide a competitive pairing computation, despite a smaller prime p, of 333 bits for $k = 11$ and 310 bits for $k = 13$, compared to a 446-bit prime p for BLS12 curves, but may provide a faster arithmetic in $\mathbb{G}_1$ (elliptic curve scalar multiplication over $\mathbb{F}_p$) thanks to a smaller finite field.

Organisation of the paper. In Section 2 we recall briefly the special tower number field sieve algorithm and the approximation of running-time made in [24]. We present our variant of special polynomial selection for pairing-friendly curves. In Section 3 we recall the Brezing–Weng construction for pairing-friendly curves, then we list the possible curves for the 128-bit security level, and we present the results of simulation of STNFS for each curve. We select a short-list of nine secure curves. In Section 4 we roughly estimate the cost of the Miller loop for an optimal ate pairing computation on the curves of the short-list that do not appear in previous works. In Section 5 we estimate the cost of STNFS for curves in [4] at the 192-bit security level. This is more complex than the 128-bit security level. We identify underestimated sizes for many embedding degrees from 9 to 22, and in particular, for $k = 14, 15$. We conclude in Section 6.

2 The Special Tower Number Field Sieve

In this section, we sketch the TNFS algorithm. We refer to [5,28,29,24] for an extended description of TNFS. The TNFS algorithm falls in the broader Number Field Sieve algorithms. To compute a discrete logarithm in a finite field, one first computes a large amount of precomputed data. A first important ingredient is the *factor basis*. A finite field $\mathbb{F}_{p^k}$ has no factorisation of elements into irreducible elements or prime elements. However a number field has a ring of integers, and factorisation of ideals in prime ideals. Equipped with a map from a (sub)ring of integers of a number field to a finite field, one can factor ideals in prime

ideals, then map each prime factor to the finite field to obtain a factorisation in $\mathbb{F}_{p^k}$. (There are now well-defined procedures to handle non-principal ideals and non-torsion units). The factor basis is made of the prime ideals (usually of degree one) of small norm, bounded by the *smoothness bound* B. The first step of the algorithm is to define two non-isomorphic number fields with two irreducible polynomials f and g, which share a common irreducible factor of degree k modulo p (a common root if one targets a prime field $\mathbb{F}_p$), so that one has two maps from the ring of integers of number fields defined by f and g, to the same finite field $\mathbb{F}_{p^k}$.

The next step is to collect a large number of relations involving the primes of the factor basis. We will say that an algebraic integer is B-smooth if it factors in prime ideals of degree one and norm bounded by B (B is an integer). Once enough relations are collected, taking the logarithm of the multiplicative relations, one obtains a large set of linear equations whose unknown are the discrete logarithms of the prime ideals of the factor basis. Solving the system, one obtains the discrete logarithms of the factor basis elements. Finally, to compute the discrete logarithm of a given target in the finite field, one lifts the target in the number field, and try to find a smooth decomposition of this target over the prime ideals whose logarithms are known.

In the Number Field Sieve setting, two distinct number fields are needed, so that their ring of integers can be mapped to the finite field $\mathbb{F}_{p^k}$. In the Tower NFS setting, one consider two extensions of a same number field. Let k be the extension degree, and $k = \eta\kappa$ where η, κ are integers ($\eta = k$ and $\kappa = 1$ if k is prime). One chooses an irreducible monic polynomial $h(Y) \in \mathbb{Z}[Y]$ of degree η and small coefficients. Define the number field $K_h = \mathbb{Q}[Y]/(h(Y))$, and let y denotes a root of h in K_h. Let $\mathcal{O}_h$ denotes the ring of integers of K_h, and let $\mathbb{Z}_y$ be a subring of $\mathcal{O}_h$ (we take the same notations as [24]). Let $\mathfrak{p} = (p, h(Y))$ be a prime ideal of $\mathcal{O}_h$ above p. One selects a pair of polynomials $f_y(X), g_y(X)$ so that reduced modulo $(p, h(Y))$, they share a common irreducible factor $\psi_y(X)$ of degree κ. Let $K_{y,f}$ and $K_{y,g}$ the number fields defined above K_h by $f_y(X)$ and $g_y(X)$ respectively, and $\mathcal{O}_{y,f}$, $\mathcal{O}_{y,g}$ their ring of algebraic integers. Let $\alpha_{y,f}$ a root of $f_y(X)$ in $K_{y,f}$ and $\alpha_{y,g}$ a root of $g_y(X)$ in $K_{y,g}$. We have the following setting (Figure 1) and commutative diagram (Figure 2).

In the relation collection step, one enumerates all $\boldsymbol{a} = a_0 + a_1 Y + \ldots + a_{\eta-1} Y^{\eta-1}$, $\boldsymbol{b} = b_0 + b_1 Y + \ldots + b_{\eta-1} Y^{\eta-1}$ such that $|a_i|, |b_i|$ are bounded by the *relation collection bound* A. The aim is to compute the norms of $\boldsymbol{a} + \boldsymbol{b}X$ in $K_{y,f}$ and $K_{y,g}$ and keep the $(\boldsymbol{a}, \boldsymbol{b})$ whose norms are B-smooth. Assuming $h(Y), f_Y(X)$ are monic, the norm is

$$N_f = \mathrm{Norm}_{K_{y,f}/\mathbb{Q}}(\boldsymbol{a} + \boldsymbol{b}X) = \mathrm{Res}_Y(\mathrm{Res}_X(\boldsymbol{a} + \boldsymbol{b}X, f_Y(X)), h(Y)) \ . \quad (1)$$

and for a non-monic $g_y(X)$ of leading coefficient $\mathrm{lc}(g_y)$,

$$N_g = \mathrm{Norm}_{K_{y,g}/\mathbb{Q}}(\boldsymbol{a} + \boldsymbol{b}X) |\,\mathrm{lc}(g_y)|^n = \mathrm{Res}_Y(\mathrm{Res}_X(\boldsymbol{a} + \boldsymbol{b}X, g_Y(X)), h(Y)) \ .$$

The schedule of TNFS can be summarised in four important steps.

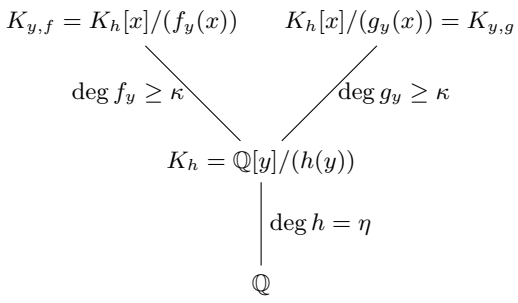

Fig. 1. Extensions of number field for TNFS

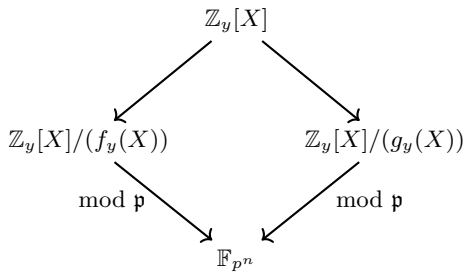

Fig. 2. Commutative diagram for TNFS.

1. Polynomial selection: choosing $h(Y), f_y(X), g_y(X)$ so as to minimise the norms N_f and N_g;
2. Relation collection: obtaining many $a(y) + b(y)X$ whose absolute norms in $K_{y,f}$ and $K_{y,g}$ w.r.t. $\mathbb{Q}$ are B-smooth. The coefficients a_i, b_i have absolute value bounded by A, where $a(y) = a_0 + a_1 y + \ldots + a_i y^{\eta-1}$, $b(y) = b_0 + b_1 y + \ldots + b_i y^{\eta-1}$;
3. Linear algebra: each relation encodes a row of a large sparse matrix. After a filtering step (preprocessing of the matrix to remove the singletons and small cliques) the right kernel is computed with the Block–Wiedemann algorithm;
4. Individual discrete logarithm computation: obtain the database of discrete logarithms of the prime ideals of factor basis. Then given a target in $\mathbb{F}_{p^n}$, lift in one of the number fields, $K_{y,f}$ or $K_{y,g}$, and obtain a smooth decomposition. Sum the discrete logarithms of the factor basis involved in the smooth decomposition to obtain the logarithm of the target.

2.1 Estimation of TNFS cost

This is an important concern to know the finite field size needed to match a security level such as 128 bits. Lenstra and Verheul designed an approach to extrapolate prime field sizes from the asymptotic complexity of NFS [30].

Unfortunately this cannot be applied straightforward for extension fields [31] as the asymptotic complexity assumes a ratio of the extension degree $n = \eta\kappa$ as $p^n = Q$ tends to infinity (see [28, Table 4] for the Kim–Barbulescu variants of TNFS: with Conjugation, $\kappa = (\ln Q/(12 \ln \ln Q))^{1/3}$ is required, and for STNFS, one needs p to be d-SNFS (i.e. $p = P(u)$ and P is a polynomial of degree d and very small coefficients) with $d = ((2/3)^{1/3} + o(1))(\ln Q/(\ln \ln Q))^{1/3}/\kappa)$, but in practice, n is fixed to a small integer an p ranges (roughly) from 256 to 512 bits.

To circumvent this theoretical limitation, Menezes, Sarkar and Singh bounded the size of norms for a given input. later Barbulescu and Duquesne averaged the size of norms over a sample of about 26000 random inputs. Then Guillevic and Singh computed the smoothness bias of the resultants with respect to integers of the same size (α value of polynomials), simulated the relation collection of TNFS, and averaged the smoothness probability over random samples, as a TNFS variant of the Murphy E function. This estimate should be done for each set of parameters $(p(x), u)$. We build on these two previous works [3,24]. In particular, we model the relation collection cost as [24, Eq. 6.3] and the linear algebra cost as [24, Eq. 6.5].

$$\text{Cost of relation collection } = \frac{(2A+1)^{2\cdot\eta} \cdot \log(\log(B))}{2 \cdot (\#\operatorname{aut}(h) \gcd(\deg(f), \deg(g)))} \tag{2}$$

where A is the bound on the coefficients a_i, b_i in the relation collection. The $\boldsymbol{a} = a_0 + a_1 y + \ldots + a_{\eta-1} y^{\eta-1}$ and $\boldsymbol{b}$ have η coefficients each in $[-A, A]$, there are $(2A+1)^{2\eta}$ such pairs $(\boldsymbol{a}, \boldsymbol{b})$. For each pair, one computes the norms N_f, N_g and test for B-smoothness, this is estimated as costing $\log \log B$. The process can be speeded-up for specific choices of h, f_y, g_y where automorphisms are available, hence the denominator.

$$\text{Cost of Linear Algebra} = \texttt{cnst} \cdot \texttt{wt} \cdot (\#\mathcal{B} \div \texttt{flt})^2 \tag{3}$$

where `cnst` is a constant representing the cost of a multiplication modulo ℓ, `wt` is the weight per row (number of non-zero entries), $\#\mathcal{B}$ is the total size of the factor basis (f-side and g-side), and `flt` is the reducing factor of the filtering step. Following [24], $\texttt{cnst} = \lfloor \ell/64 \rfloor$ is the machine-word size of ℓ, $\texttt{wt} = 200$ and $\texttt{flt} = 20$.

For each pairing-friendly curve parameters $(p(x), u)$ we run Algorithm 2.1 to estimate the number of relations obtained for given inputs A, B. The Dickman-ρ function is denoted by D_ρ. We write a SageMath code to automatically adjust the parameters A, B so that enough relations are obtained and the cost of linear algebra and relation collection are finely balanced, in order to minimise the total estimated cost of TNFS.

2.2 Special Polynomial Selection

We refine the special polynomial selection introduced in [5] and present a variant particularly suited for certain families of pairing-friendly curves that appear in the recent preprint [4].

Algorithm 2.1: Monte-Carlo approximation of Murphy's E for TNFS [24, Alg. 6.1] (computes an estimation of the number of relations)

Input: Valid polynomials f_y, g_y, h, α_f, α_g, parameter $A \in \mathbb{N}$, smoothness bound B, $N \approx 10^5$

Output: Yield estimate (number of relations)

$P_f \leftarrow 0$; $P_g \leftarrow 0$
for $n := 1$ *to* N **do**
  $\boldsymbol{a} \leftarrow$ random vector in $\{-A, A\}^{2\deg h}$
  $\boldsymbol{b} \leftarrow$ random vector in $\{-A, A\}^{2\deg h-1} \times \{0, A\}$
  if $\gcd(\boldsymbol{a}, \boldsymbol{b}) \neq 1$ **then** gcd of an array of integers
    continue
  $\mathfrak{a} \leftarrow \boldsymbol{a}\mathcal{O}_h$, $\mathfrak{b} \leftarrow \boldsymbol{b}\mathcal{O}_h$
  if *the ideals* $\mathfrak{a}, \mathfrak{b}$ *are not coprime* ($\mathfrak{a} + \mathfrak{b} \neq 1$) **then**
    continue
  $N_f \leftarrow |\operatorname{Res}(h, \operatorname{Res}(f_y, \boldsymbol{a} - \boldsymbol{b}x))|$
  $N_g \leftarrow |\operatorname{Res}(h, \operatorname{Res}(g_y, \boldsymbol{a} - \boldsymbol{b}x))|$
  $u_f \leftarrow (\ln N_f + \alpha_f)/\ln B$; $p_f \leftarrow D_\rho(u_f) + (1-\gamma)D_\rho(u-1)/\ln N_f$
  $u_g \leftarrow (\ln N_g + \alpha_g)/\ln B$; $p_g \leftarrow D_\rho(u_g) + (1-\gamma)D_\rho(u-1)/\ln N_g$
  $P_{fg} \leftarrow P_{fg} + p_f p_g$
$P_{fg} \leftarrow P_{fg}/N$
$w \leftarrow$ index of group of torsion units of $\mathcal{O}_h$
$V \leftarrow (2A+1)^{2\deg h}/(2w\zeta_{K_h}(2))$
return $V \times P_{fg}$

Pairing-friendly curves have a special characteristic p, given by a polynomial $p(x)$ of small degree evaluated at an integer u. For BLS12 curves, we have $p(x) = (x^6 - 2x^5 + 2x^3 + x + 1)/3$, and for a 381-bit prime p, $u = -(2^{63} + 2^{62} + 2^{60} + 2^{57} + 2^{48} + 2^{16})$ [8]. Joux and Pierrot introduced a dedicated polynomial selection that takes advantage of the polynomial form $p = p(u)$ [26]. The adaptation to the Tower setting is the following.

Joux–Pierrot polynomial selection for TNFS. Assume there exists an integer $u \approx p^{1/d}$ and a polynomial $P(U)$ of degree d and small coefficients ($\|P(U)\|_\infty = O(1)$) such that $P(u) = 0 \bmod p$. Select a monic polynomial $S_y(X)$ of degree κ and small coefficients ($\|S_y(X)\|_\infty = O(1)$) such that $g_y(X) = S_y(X) - u$ and $f_y(X) = P(S_y(X))$ are irreducible. Finally select a monic irreducible $h(Y)$. Then $(h(Y), f_y(X), g_y(X))$ are STNFS polynomials.

Joux–Pierrot polynomial selection for TNFS with automorphism. We recall a variant of the Joux–Pierrot method to obtain a pair of polynomials (f_y, g_y) admitting an automorphism, when k is not prime. First select an auxiliary polynomial with automorphism, for example from the list in [17].

- $\kappa = 2$: $c_t(X) = X^2 - tX + 1$, $\sigma : X \mapsto 1/X$; $c_t(X) = X^2 + t$, $\sigma : X \mapsto -X$;
- $\kappa = 3$: $c_t(X) = X^3 - tX^2 - (t+3)X - 1$, $\sigma : X \mapsto -(X+1)/X$;

- $\kappa = 4$: $c_t(X) = X^4 - tX^3 - 6X^2 + tX + 1$, $\sigma : X \mapsto -(X+1)/(X-1)$;
- $\kappa = 6$: $c_t(X) = X^6 - 2tX^5 - (5t+15)X^4 - 20X^3 + 5tX^2 + (2t+6)X + 1$, $\sigma : X \mapsto -(2X+1)/(X-1)$.

If $\gcd(\kappa, \eta) = 1$, define $f_y(X) = \mathrm{Res}_U(c_U(X), P(U))$ and $g_y(X) = c_u(X)$. If $\gcd(\kappa, \eta) > 1$, define $f_y(X) = \mathrm{Res}_U(c_{Uy}(X), P(U))$ and $g_y(X) = c_{uy}(X)$, or alternatively, $f_y(X) = \mathrm{Res}_U(c_{U+y}(X), P(U))$ and $g_y(X) = c_{u+y}(X)$. If f_y, g_y are irreducible, select a monic irreducible $h(Y)$. Then $(h(Y), f_y(X), g_y(X))$ are STNFS polynomials.

Example 1 ([24, Table 7]). To minimise the size of norms and the total estimated cost of STNFS for BLS12-381 curves, one chooses h of degree 6, and f_y, g_y share a common irreducible factor of degree 2 modulo $(p, h(Y))$. The prime p of BLS12 curves satisfies $p = P(u)/3$, where $P(x) = x^6 - 2x^5 + 2x^3 + x + 1$. The polynomials selected in [24, Table 7] are $h = Y^6 - Y^2 + 1$, $f_y = \mathrm{Res}_U(P(U), X^2 - UY) \bmod h(Y) = X^{12} - 2yX^{10} + 2y^3X^6 + y^5X^2 + y^2 - 1$ and $g_y = X^2 - uy = X^2 + 15132376222941642752y$.

Improvements on the Joux–Pierrot method. The pairing-friendly curves of Section 3 all have a characteristic of a polynomial form $p = p(u)$ for a seed u, where $p(x)$ has very small coefficients and degree from 4 (BN curves) to 46 (Construction 6.7 for $k = 9$, Table 2). We observed that when the degree of $p(x)$ is larger than 12, the average size of norms obtained with Algorithm 2.1 is not satisfying. In other words, for a same size of finite field $\mathbb{F}_{p^k}$ but different families of curves with $p(x)$ of very different degrees, one obtain very different estimated costs of STNFS. We explain in the following our method to obtain a lower estimated cost of STNFS when the degree of $p(x)$ is too large and the Joux–Pierrot method does not give good enough results.

In [5, §5.2] and in the SageMath script provided with [4], one observes that when it is possible, the degree of the polynomial P is divided by two without increasing the size of the coefficients. We name it Variant 1.

Variant 1 (Even polynomial $p(x)$) *When $p(x)$ is an even polynomial (that is, with only even degree monomials, and one has $p(x) = p(-x)$), then one defines $P(x)$ such that $P(x^2) = p(x)$, and P has degree $\deg(p(x))/2$. The pair of polynomials (for TNFS) $(P(x), x - u^2)$ satisfies $\mathrm{Res}_x(P(x), x - u^2) = P(u^2) = p(u) = p$ as desired.*

We adapt this technique to palindrome polynomials (also mentioned in [5, §5.2]).

Variant 2 (Palindrome polynomial $p(x)$) *When $p(x) = p(1/x)x^{\deg p(x)}$, then we define $P(x)$ to be the minimal polynomial of $\alpha + 1/\alpha$ in the number field defined by $p(x)$, $K = \mathbb{Q}[x]/(p(x)) = \mathbb{Q}(\alpha)$. Then $P(x)$ has degree $\deg(p(x))/2$ and small coefficients (as long as $p(x)$ has small coefficients). The pair of polynomials (for TNFS) is $(P(x), ux - (u^2+1) = u(x - (u + 1/u)))$, and $\mathrm{Res}_x(P(x), x - (u + 1/u)) = P(u + 1/u) \equiv 0 \bmod p(u)$ as desired.*

Variant 3 (Polynomial $p(x)$ with automorphism) *More generally when there is an automorphism available for $p(x)$, say σ, of order two i.e. $\sigma^2(a) = a$, then we define $P(x)$ to be the minimal polynomial of $a + \sigma(a)$ (the trace of the automorphism is invariant). Then $P(x)$ has degree $\deg(p(x))/2$ and small coefficients (as long as $p(x)$ has small coefficients). The second polynomial for TNFS is $x - (u + \sigma(u))$. If $a + \sigma(a)$ does not have a good expression (a fraction of linear polynomials in a), then one computes a half-extended GCD of $p(x)$ and $x + \sigma(x)$ to obtain $x + \sigma(x) = s_1(x)/s_2(x)$. If the degrees of s_1 and s_2 are small, one can define $s_2(u)x - s_1(x)$ as the second polynomial for NFS. We have $\mathrm{Res}_x(P(x), x - s_1(u)/s_2(u)) = P(u + \sigma(u)) \equiv 0 \bmod p(u)$.*

These three variants already allow more possibility of trade-off between f and g in terms of degrees and coefficient size: one divides the degree of f by two and increases the coefficient size of g by a factor two ($\|g_y\|_\infty \approx u^2$ instead of u).

Variant 4 *When $p(x)$ has tiny coefficients and a high degree, it might be worth doing the following transformation, knowing the seed u. Write $p(x) = \sum_{i=0}^{d} p_i x^i$ where $d = \deg p(x)$ and p_i are tiny integer coefficients. Then for an integer l in the range $2 \le l \le d/2$, define*

$$P(x) = \sum_{i=0}^{d} p_i u^{i \bmod l} x^{\lfloor i/l \rfloor} .$$

Then $P(x)$ has degree $\lfloor d/l \rfloor$ (the floor integer of the number d/l) and coefficients at most u^{l-1}, and $P(u^l) = p(u)$. The pair of polynomials (for TNFS) is $(P(x), x - u^l)$, and $\mathrm{Res}_x(P(x), x - u^l) = P(u^l) \equiv 0 \bmod p(u)$ as desired.

This is possible to combine Variant 4 with one of Variants 1, 2 or 3. With these alternative pairs of polynomials, we can have more balanced size of norms, hence a higher smoothness probability, and a lower DL cost estimation. Our results are given in the right-most column of Table 3. It has direct impact on many curves of embedding degrees 9, 10, 11, 13, 14, 17, in particular, the curves whose polynomial $p(x)$ has a high degree.

Example 2. Let us consider a curve of embedding degree $k = 13$, discriminant $D = 3$, following Construction 6.6. The polynomial defining the characteristic is $p(x) = (x^{28} + x^{27} + x^{26} + x^{15} - 2x^{14} + x^{13} + x^2 - 2x + 1)/3$. It has no automorphism. We define $P(x) = (u + 1)x^9 + u^2x^8 + x^5 + u(1 - 2u)x^4 + u^2 - 2u + 1$ such that $P(u^3) = 3p(u)$, and u is a seed for a particular curve. A degree 13 irreducible polynomial $h(Y)$ and the pair $(f, g) = (P(x), x - u^3)$ can be used for polynomial selection with STNFS.

Example 3. Consider a curve of embedding degree 17, named Construction 6.6 in Section 3. It has $p(x) = (x^{36} + x^{35} + x^{34} + x^{19} + 4x^{18} + x^{17} + x^2 + x + 1)/3$ and automorphism $\sigma : x \mapsto 1/x$. Variant 2 gives $P(x) = x^{18} + x^{17} - 17x^{16} - 17x^{15} + 119x^{14} + 119x^{13} - 442x^{12} - 442x^{11} + 935x^{10} + 935x^9 - 1122x^8 - 1122x^7 + 714x^6 + 714x^5 - 204x^4 - 204x^3 + 17x^2 + 18x + 4$ such that $P(x + 1/x)x^{18} = 3p(x)$.

Applying Variant 4, we obtain $P(x) = u(x^9 + (v-17)x^8 - (17v-119)x^7 + (119v-442)x^6 - (442v-935)x^5 + (935v-1122)x^4 - (1122v-714)x^3 + (714v-204)x^2 - (204v-17)x + 18v + 4)$, where $v = (u + 1/u) = (u^2+1)/u$ (we multiply by u to get integer coefficients). The pair $(P(x), u^2x - (u^2+1)^2)$ can be used for STNFS. Since $\deg p(x) = 36$, the seed u will be very small, and the coefficients of P in u^2 are small.

3 Complete Families of Pairing-Friendly Curves

We will apply our new special polynomial selection to paring-friendly curves whose parameters are given by polynomials, such as BN and BLS12 curves. We recall the generic Brezing–Weng construction of families of pairing-friendly curves. A family will be encoded by three parameters: the embedding degree k, the discriminant D, and a choice e_0 to compute the trace. It allows to capture all cyclotomic constructions of pairing-friendly curves with three parameters. The BN curves, KSS curves and Fotiadis–Konstantinou and Fotiadis–Martindale curves [7,27,18,19] do not fall in the cyclotomic framework because $r(x)$ is not a cyclotomic polynomial.

3.1 Brezing–Weng constructions of pairing-friendly curves

A set of the complete families presented in the Freeman, Scott and Teske paper [21] are special instances of the generic Brezing–Weng construction [10] that we recall in Algorithm 3.1. In this framework, $r(x)$ is chosen to be a cyclotomic polynomial, and we name it a cyclotomic construction. For BN curves, $r(x)$ is one factor of an Aurifeuillean factorisation of a cyclotomic polynomial. For KSS curves, $r(x)$ is a minimal polynomial of an algebraic element of a cyclotomic field. Freeman,

Algorithm 3.1: $\textsc{Cyclo}(k, D, e_0)$ – Cyclotomic construction of pairing-friendly curves

if $D = 1$ **then** $m \leftarrow 4/\gcd(4, k)$
else if $D = 2$ **then** $m \leftarrow 8/\gcd(8, k)$
else if $D = 3$ **then** $m \leftarrow 3/\gcd(3, k)$
else $m \leftarrow 1$
$r_x \leftarrow \Phi_{km}(x)$
if *$-D$ is not a square mod r_x* **then return** $\perp$
if $\gcd(e_0, k) \neq 1$ **then return** $\perp$
$t_x \leftarrow x^{e_0 m} + 1 \bmod r_x$
$y_x \leftarrow \textsc{Polynomial}((t_x - 2)\sqrt{-D}/D) \bmod r_x$
$p_x = (t_x^2 + Dy_x^2)/4$
if *p_x is not irreducible* **then return** $\perp$
if *p_x does not represent primes* **then return** $\perp$
return (p_x, r_x, t_x, y_x, D)

Scott and Teske [21] obtain complete families that correspond to specific choices of trace in Algorithm 3.1. We recall the BLS construction [6], with $D = 3$. The

$$\begin{aligned}
k &= 3^i \\
r(x) &= \Phi_{3^i}(x)/3 = (x^{2\cdot 3^{i-1}} + x^{3^{i-1}} + 1)/3 \\
t(x) &= x + 1 \\
c(x) &= (x-1)^2 \\
y(x) &= (x-1)(2x^{3^{i-1}} + 1)/3 \\
p(x) &= (t^2(x) + 3y^2(x))/4 = (x^2 + x + 1 + (x-1)^2 x^{3^{i-1}}(x^{3^{i-1}} + 1))/3
\end{aligned}$$

$$\begin{aligned}
k &= 2^i \cdot 3 \\
r(x) &= \Phi_{2^i\cdot 3}(x) = (x^{2^i} - x^{2^{i-1}} + 1) \\
t(x) &= x + 1 \\
c(x) &= (x-1)^2/3 \\
y(x) &= (x-1)(2x^{2^{i-1}} - 1)/3 \\
p(x) &= (t^2(x) + 3y^2(x))/4 = (x^2 + x + 1 + (x-1)^2 x^{2^{i-1}}(x^{2^{i-1}} - 1))/3
\end{aligned}$$

Table 1. Polynomials of the BLS families for $k = 3^i$ and $k = 2^i \cdot 3$ (for example $k \in \{6, 9, 12, 24, 27, 48\}$). In practice, it is very popular for $k = 12$.

construction is generalised in [21] as Construction 6.6, and gives polynomial families for any k such that $18 \nmid k$, and $D = 3$. Constructions 6.6 and BLS give the same polynomials for $k = 24$, for other embedding degrees, only the ρ-value is the same. The BLS construction gives a very simple Miller loop of ate pairing, of length $x = t - 1$ (without extra Frobenius and line computation), which is optimal. Constructions 6.2, 6.3 and 6.4 in [21] are polynomial families with $D = 1$ and $k = 1 \bmod 2$, $k = 2 \bmod 4$ and $k = 4 \bmod 8$ respectively. We report the construction number from [21] in Table 2.

Unfortunately, [4] does not consider cyclotomic methods with small discriminants other than 1, 2 and 3. In [10, p.137], Brezing and Weng give alternatives such as $D = 5$ for $k = 10$. Recently, Fotiadis and Konstantinou used the Brezing–Weng method with small discriminants D to generate other pairing-friendly curves whose ρ-value is slightly larger but that are more resistant to TNFS [18]. For $k = 10$, Fotiadis and Konstantinou list alternatives with $D = 5$ and $D = 15$, for $k = 11$, with $D = 11$, for $k = 13$, with $D = 13$. For smaller embedding degrees, between 5 and 8, the ρ value is larger than 2. We refer to [23] for TNFS-resistant curves in this case with a modification of the Cocks–Pinch method.

3.2 Reducing the possibilities

For BLS12 and BN curves, the finite field size identified as secure for 128 bits of security is about $12 \times 448 = 5376$. The arithmetic on these curves is already very well optimised. Hence we decided to reduce the investigation of other families of curves to those where p^k is smaller than 5376 bits.

The minimum size of r is 256 bits to ensure the security on the curve, and the size of p is given by the ρ-value defined as the ratio between the degree of $p(x)$ and $r(x)$. We choose the sharp constraint (at the 128-bit security level)

$$3072 \leq 256\rho k \leq 5376 \tag{4}$$

to reduce the number of families to consider. If $\rho = 1$ we obtain the upper bound $k \leq 21$, and if $\rho = 2$ then we obtain the lower bound $k \geq 6$. We obtain candidates with $9 \leq k \leq 17$, in Table 2.

Small embedding degrees up to 8. Embedding degree 1 is considered in [11]. Embedding degrees 2 and 3 are obtained with supersingular curves [22, § IX.13 p.204]. Embedding degrees 3, 4, and 6 are obtained with MNT curves. Embedding degrees 5 to 8 were compared in [23]. We focus on embedding degrees 9 to 17 for the 128-bit security level.

Embedding degree 9. There are three families of pairing-friendly curves of embedding degree $k = 9$, discriminant $D = 3$ and $\rho = 4/3$. We focus on $D = 3$ to have a twist of order three since $3 \mid k$. Alternatives are $D = 1$ and $\rho = 11/6$, $D = 2$ and $\rho = 23/12$. Another family with $D = 3$ is given in [35, §4.4] from the Aurifeuillean factorisation of $\Phi_9(-3x^2)$.

Embedding degree 10. We will consider three additional families for $k = 10$: with $D = 1$ and trace $t = x^{18} + 1 \bmod r(x)$ (in [10, p.137] and [21, Construction 6.5]), with $D = 5$, $r(x) = \Phi_{20}(x)$ and $t = x^{18}+1 \bmod r(x)$ ([18, Table 2 and Example 5]), and with $D = 15$, $r(x) = \Phi_{30}(x)$ and $t(x) = x^3 + 1$ [18, Table 2]. With $D = 3$, no cyclotomic construction is valid, we consider the $\rho = 2$ option in [4]. With $D = 2$, the construction is not interesting: the polynomial $p(x)$ has degree 30 and the choices of seeds u are very limited. There were no choice of u to get a pair of primes (p, r) such that r is 256-bit long or more, and p is at most 512-bit long.

Embedding degree 11. With $D = 1$, $r(x) = \Phi_{44}(x)$ and $t(x) = x^{24} + 1 \bmod r(x)$, the family has $\rho = 1.3$, this is Construction 6.2 in [4]. The other possibilities of $t(x) = x^{4e_0} + 1 \bmod r(x)$ are $e_0 \in \{1, 2, 7\}$. We discard $e_0 = 2$ since no seed u was found so that $p^k \leq 5376$. With $D = 3$, $e_0 = 4$ is Construction 6.6, and $e_0 \in \{8, 1\}$ gives two other valid families of curves. With $D = 11$, we obtain two families of curves with $e_0 \in \{4, 8\}$ ($e_0 = 8$ appears in [18, Table 4]).

Embedding degree 12. For embedding degree 12, we concentrate on $D = 3$ to maximise the twist. The BLS12 and BN curves are the most popular curves of embedding degree 12, and recently Fotiadis and Martindale suggested a third interesting family in [19]. Curves of discriminant $D = 1$ have a twist of degree 4. Construction 6.4 from [21] produces a family with $\rho = 2$, the size of p is not suited. Applying the Brezing–Weng method, we do not obtain other families ($p(x)$ does not produce primes). With $D = 2$ there is one family of curves and $\rho = 7/4$. Note that in this case, only a quadratic twist is available, the pairing computation will be slower compared to BLS12 curves with $D = 3$ and sextic twists.

Embedding degree 13. Since -13 is not a square in $\mathbb{Q}(\zeta_{13})$, we concentrate on $D = 1$ with $r(x) = \Phi_{4\times 13}(x)$ and $D = 3$ with $r(x) = \Phi_{3\times 13}(x)$. For $D = 1$, the trace is $x^{4e_0} + 1$ where $e_0 \in \{1, 7\}$ give valid families of curves, and $e_0 = 7$ corresponds to Construction 6.2. For $D = 3$, the trace is $t(x) = x^{3e_0} + 1$ and $e_0 = 9$ corresponds to Construction 6.6. We also consider $e_0 \in \{1, 2, 10\}$.

Embedding degree 14. We concentrate on Construction 6.3 and 6.6. The other choices of e_0 in the Brezing–Weng construction do not produce families of curves satisfying the bounds on the size of p^k. In particular, $D = -7$ produces an alternative family whose ρ-value is too large.

Embedding degree 17. In addition to Construction 6.2 and 6.6, we consider $D = 3$ and trace $t(x) = x^{3\times 12} + 1 \bmod r(x)$ where $r(x) = \Phi_{3\times 17}$. Actually because of the very large degree of $p(x)$ (36 and 38), it was not possible to find a seed u so that p^k is smaller than 5376 bits. However for a comparison to [4], we include the three families of curves in our security estimate.

Other embedding degrees. Embedding degrees 15 and above 17 do not satisfy the conditions (4).

For $9 \leq k \leq 17$, we list in Table 2 the available families satisfying Equation (4). Moreover we will later restrict to $D = 3$ when $3 \mid k$ and $D = 1$ when $4 \mid k$ to ensure the higher degree of twist.

3.3 Security estimate for the finite field

The next step is to determine the size of the finite field $\mathbb{F}_{p^k}$ to ensure the required security w.r.t. a DL computation with any variant of the NFS algorithm.

Barbulescu–El Mrabet–Ghammam results. In the preprint [4], Barbulescu, El Mrabet and Ghammam presented a consequent list of pairing-friendly curves of embedding degrees 6 to 53 for the three common security levels of 128, 192 and 256 bits. There were about 150 distinct curves. We compare the curves of [4] that are listed in Table 2.

We obtain lower DL cost estimates in the embedding field of these curves, except for $k = 9$ construction LZZW (that we set in the BLS framework). Sometimes the cost for STNFS is not given in [4], we give our estimate. We investigated these differences by running the scripts provided with [4] and developing a second implementation based on the SageMath code available with [24,23]. We develop the following improvements.

1. Given p^k as input, for each possible decomposition $k = \eta\kappa$ with $\eta > 1$, we generate many irreducible polynomials h of degree η and pairs of polynomials (f_y, g_y).
2. For each set of polynomials (h, f_y, g_y), the code iterates and adjusts automatically the parameters A, B (sieving bound, smoothness bound) in order to find the best combination that balances the costs of relation collection

k	Construction	D	m	e_0	ρ	deg $p(x)$	$\sigma_p(x)$	$\lceil 256\rho \rceil$	$\lceil 256\rho k \rceil$
9	Cyclo (BLS)	3	1	1	$1.33 = 4/3$	8	$\frac{x^4-x^3-1}{x^2+x}$	342	3072
9	Cyclo	3	1	4	$1.33 = 4/3$	8	$1/x$	342	3072
9	Cyclo (6.6)	3	1	7	$1.33 = 4/3$	8	$\frac{x^4+x^3+x^2+x-1}{1-x^2}$	342	3072
9	Cyclo (6.2)	1	4	5	$1.83 = 11/6$	22	$-x$	470	4224
9	Cyclo (6.7)	2	8	1	$1.92 = 23/12$	46	$-x$	491	4416
9	Cyclo (FM10)	3	1	5	$2.00 = 2$	12	Id	512	4608
10	Cyclo (6.5)	1	2	9	$1.50 = 3/2$	12	$-x$	384	3840
10	Cyclo (6.3) (FM13)	1	2	1	$1.75 = 7/4$	14	$-x$	448	4480
10	Cyclo (FM16)	2	4	9	$1.88 = 15/8$	30	$-x$	480	4800
10	6.6	3	3	1	$2.00 = 2$	16	Id	512	5120
10	Cyclo (FM14)	5	2	9	$1.75 = 7/4$	14	$-x$	448	4480
10	Cyclo (FM15)	15	3	1	$1.75 = 7/4$	14	Id	448	4480
11	Cyclo (6.2)	1	4	6	$1.30 = 13/10$	26	$-x$	333	3661
11	Cyclo	1	4	1	$1.50 = 3/2$	30	$-x$	384	4224
11	Cyclo	1	4	7	$1.70 = 17/10$	34	$-x$	436	4788
11	Cyclo (6.6)	3	3	4	$1.20 = 6/5$	24	$1/x$	308	3380
11	Cyclo	3	3	8	$1.30 = 13/10$	26	Id	333	3661
11	Cyclo	3	3	1	$1.40 = 7/5$	28	Id	359	3943
11	Cyclo	11	1	4	$1.60 = 8/5$	16	Id	410	4506
11	Cyclo	11	1	8	$1.60 = 8/5$	16	$1/x$	410	4506
12	BN (6.8)	3	1	1	$1.00 = 1$	4	$1/(6x)$	256	3072
12	Cyclo (BLS)	3	1	1	$1.50 = 3/2$	6	Id	384	4608
12	FM17	3	1	–	$1.50 = 3/2$	6	Id	384	4608
12	FM19	3	1	–	$1.50 = 3/2$	6	Id	384	4608
12	FM20	3	1	–	$1.50 = 3/2$	6	Id	384	4608
12	Cyclo (6.7) (FM18)	2	2	1	$1.75 = 7/4$	14	$-x$	448	5376
12	6.4	1	1	1	$2.00 = 2$	8	$-1/x$	512	*6144*
13	Cyclo (6.2)	1	4	7	$1.25 = 5/4$	30	$-x$	320	4160
13	Cyclo	1	4	1	$1.42 = 17/12$	34	$-x$	363	4715
13	Cyclo (6.6)	3	3	9	$1.17 = 7/6$	28	Id	299	3883
13	Cyclo	3	3	1	$1.33 = 4/3$	32	Id	342	4438
13	Cyclo	3	3	10	$1.42 = 17/12$	34	Id	363	4715
13	Cyclo	3	3	2	$1.58 = 19/12$	38	Id	406	5270
14	Cyclo (6.3)	1	2	1	$1.50 = 3/2$	18	$-x$	384	5376
14	Cyclo (6.6)	3	3	5	$1.33 = 4/3$	16	Id	342	4779
15	Cyclo (BLS)	3	1	1	$1.50 = 3/2$	12	Id	384	*5760*
15	Cyclo (6.6)	3	1	11	$1.50 = 3/2$	12	Id	384	*5760*
16	KSS16 (6.11)	1	–	–	$1.25 = 5/4$	10	Id	320	5120
17	Cyclo (6.2)	1	4	9	$1.18 = 19/16$	38	$-x$	304	5168
17	Cyclo (6.6)	3	3	6	$1.12 = 9/8$	36	$1/x$	288	4896
17	Cyclo	3	3	12	$1.19 = 19/16$	38	Id	304	5168

Table 2. Pairing-friendly Constructions for $9 \leq k \leq 17$ such that $3072 \leq 256\rho k \leq 5376$. The parameters m and e_0 correspond to the parameters m and e_0 in the Cyclo construction of Algorithm 3.1. The value 256ρ is an approximation of the minimal bit-size of p required to ensure r to be of 256 bits, so that the curve $E(\mathbb{F}_p)$ offers 128 bits of security. We include $k = 12$ Construction 6.4, and $k = 15$ although $256\rho k$ is too large, for they are refereed in Tables 3 and 4.

and linear algebra, so that the total cost is minimised. When plugging these values into the former scripts and adding a tiny offset if needed, one obtains the new results[1].

3. We implement the improvements of the Special setting described in the variants 1-4: automorphisms and changes of variables on $p(x)$ to minimise the average size of norms.
4. We compute the joint average size of norms and smoothness probabilities for $K_{y,f}$ and $K_{y,g}$ simultaneously. This allows to compute the ratio of non-coprime ideals $\boldsymbol{a}\mathcal{O}, \boldsymbol{b}\mathcal{O}$ and validates the formula $1/\zeta_{K_h}(2)$.

We obtain the results of Tables 3, 4 and 5. In Table 3, we reproduce the results of Barbulescu, El Mrabet and Ghammam [4, §3.4]. We hereafter make the following remarks.

Remark 1.

- We do not consider even embedding degrees k with Construction 6.2. As explained in [21], 6.2 is valid for odd embedding degrees, 6.3 is for $k = 2 \bmod 4$, and 6.4 for $k = 4 \bmod 8$. Hence we do not report even k with 6.2 in Table 3.
- For $k = 10$ and construction 6.3, we obtained a lower DL cost with $\eta = 10$ instead of $\eta = 5$. We obtained 2^{122} instead of 2^{134}.
- For all curves but BN and BLS12, we obtain a lower estimated cost with optimised parameters A, B.
- When the degree of $p(x)$ is large, we apply one of the variants using automorphisms 1, 2 or 3 if applicable, so that $\deg P = {}^1\!/\!_2 \deg p(x)$. We compared without the polynomial variants and observed a lower DL cost estimate with variants 1 or 2 when the degree of $p(x)$ is more than 12. Note that the variant 2 is commented in the Python script of [4] for $k = 17(6.6)$.
- We observed that when the degree of P is more than 12 (after applying variants 1, 2 or 3 if applicable), applying our improvement 4 reduces further the estimated complexity of STNFS. We obtained the smallest cost with P of degree between 4 and 12. This case is reported in the right-most column of Tables 3 and 4. The curves involved with this improvement are $k = 9(6.7)$, $k = 10(6.6)$, $k = 11(6.2)$, $k = 13(6.2)$ and $k = 13(6.6)$, $k = 14(6.6)$ and $k = 17(6.2)$, $k = 17(6.6)$.

Moreover, we applied our work to the parameter seeds of [4]. The previous remarks apply: we do not consider the seeds of even k with Construction 6.2 ([4, Table 10]). We identified five seeds that produce insecure curves because the STNFS estimated cost in $\mathbb{F}_{p^k}$ is below 2^{128}: these are $k = 9$ BLS (denoted LZZW in [4, Table 23]), $k = 9$ (6.2), $k = 10$ (6.3), $k = 11$ (6.2) and $k = 11$ (6.6). Our DL security estimate is given in bold coloured font in Table 4.

[1] This improvement was suggested in [4, page 9]: *"Instead of a blind program to guess the polynomials automatically, we made all the choices manually using our experience on computation records of discrete logarithms. It is a good research project to write a program which reproduces our choices."*

For $k = 10$ (6.3), the size of u is smaller than the minimum size recommended in [4, §3.4] ($p(u)$ is 433-bit long instead of 446, and $r(u)$ is 249-bit long, smaller than 256 bits). For $k13$(6.2), the minimum size recommended in [4, §3.4] is $p(u)$ of 329 bits, but the seed produces a 599-bit prime p. The security is much larger than 2^{128}. These two cases are reported in italic coloured font in Table 4.

In Table 5 we present our estimations of STNFS security. For each curve family in Table 2, we first generate seeds and parameters so that r is a 256-bit prime. Then we run our estimation of STNFS, trying many combinations of degrees of $h(Y)$ and of $P(x)$. When the cost is smaller than 2^{128}, we increase the size of the seed u and generate larger parameters $r(u)$ and $p(u)$. We report the minimum size of p so that r is at least 256-bit long, and the security in $\mathbb{F}_{p^k}$ is at least 2^{128}.

For each embedding degree k, we highlight in coloured background the family that has no automorphism available in $p(x)$ so that the variants 1, 2 and 3 do not apply, and so that $p(u)$ has minimal possible size. We eliminate the embedding degree $k = 17$. Since $p(x)$ has large degree of 36 or 38, it was not possible to find a seed u so that $p(u)$ and $r(u)$ are prime, and $p^k(u)$ is less than 5376 bits (constrain of Eq. (4)). We eliminate embedding degree $k = 9$: the curves whose $p(x)$ has no automorphism do not satisfy $p^k(u) \leq 5376$.

There are eleven highlighted families in Table 5. The families of Fotiadis and Martindale [19] with $k = 12$ and $D = 3$ (denoted FM17, FM19 and FM20) have very similar properties and like in [19], we only include FM17 in our final short list (for the same bitsize of $p(u)$, FM17 produces $r(u)$ one bit larger than FM19 and four bits larger than FM20).

We are left with a final short-list of nine STNFS-secure pairing-friendly curves that we summarise in Table 6. We give the polynomials $p(x), r(x), t(x)$ as Curves 1, 2, 3, 4, 5. We add the modified Cocks–Pinch curve with $k = 8$ from [23] as it looks quite promising in terms of pairing efficiency [1].

Remark 2. The curves listed below all admit a fast endomorphism from the complex multiplication, because their discriminant $-D$ is small. For curves with $-D = -4$ and j-invariant 1728, the endomorphism is $(x, y) \mapsto (-x, iy)$, where $i^2 = -1$ (in short Weierstrass representation). The curves are ordinary, $p \equiv 1 \bmod 4$, and there exists $i \in \mathbb{F}_p$ such that $i^2 = -1 \bmod p$. More precisely, we can easily precompute i. The characteristic p has form $p = (t^2 + y^2)/4$ where t is the trace, and $t^2 - 4p = -y^2$. Then $\sqrt{-1} \equiv t/y \bmod p$. The endomorphism has characteristic polynomial $x^2 + 1$, and eigenvalue $\sqrt{-1} \bmod r$, where $r \mid p + 1 - t$ and r is prime. Writing $p+1-t = ((t-2)^2+y^2)/4$, one has $\sqrt{-1} \equiv (t-2)/y \bmod r$. This is explained in details in [36]. When the cofactor c of the elliptic curve such that $r \cdot c = p + 1 - t$ is larger (not just 1 or 2 for example), and the curve has parameters of polynomial form, one can reduce the lattice spanned by the rows $(r(x), 0)$ and $(y(x), t(x) - 2)$ to obtain a short basis. The Magma language for example allows lattice reduction over polynomials.

For curves with $-D = -3$, the endomorphism is $(x, y) \mapsto (\omega x, y)$, where $\omega \in \mathbb{F}_p$ is a third root of unity, such that $\omega^2 + \omega + 1 = 0$. The endomorphism has characteristic polynomial $x^2 + x + 1$ and eigenvalue $\lambda \bmod r$ such that

k	Construction, D, m, e_0	deg $p(x)$	p bits	p^k bits	η, STNFS variant [4]	deg $P(x)$	DL cost $\mathbb{F}_{p^k}$ [4]	DL cost new A,B	DL cost new P	deg $P(x)$	our variant
9	Cyclo (BLS) 3,1,1	8	591	5314	9, $P(x) = 3p(x)$	8	128	128			
9	Cyclo (6.6) 3,1,7	8	535	*4810*	9, $P(x) = 3p(x)$	8	129	**123**			
9	Cyclo (6.2) 1,4,5	22	484	*4356*	9, $P(x^2) = 4p(x)$	11	134	**116**			
9	Cyclo (6.7) 2,8,1	46	520	4672	9, $P(x^2) = 8p(x)$	23	266	220	140	11	$P(u^4) = 8p(u)$
10	Cyclo (6.3) 1,2,1	14	446	*4460*	*5*, $P(x^2) = 4p(x)$	7	134	**122**			$\eta = 10$
10	6.6 3,3,1	16	511	5104	10, $P(x) = 3p(x)$	16	166	152	145	8	$P(u^2) = 3p(u)$
11	Cyclo (6.2) 1,4,6	26	337	*3697*	11, $P(x^2) = 4p(x)$	13	173	**123**	**121**	6	$P(u^4) = 4p(u)$
11	Cyclo (6.6) 3,3,4	24	311	*3421*	11, $P(x) = 3p(x)$	24	$\emptyset$	232	**114**	12	$x^{12}P(x+{}^1/_x)=3p(x)$
12	BN (6.8) 3,–,–	4	462	5534	6, $P(x) = p(x)$	4	128	135			
12	Cyclo (BLS) 3,1,1	6	461	5525	6, $P(x) = 3p(x)$	6	128	135			
12	6.7 2,2,1	14	448	5340	12, $P(x^2) = 8p(x)$	7	148	134			
12	6.4 1,1,1	6	510	6120	12, $P(x) = 4p(x)$	6	$\emptyset$	138			
13	Cyclo (6.2) 1,4,7	30	329	4265	13, $P(x^2) = 4p(x)$	15	325	143	140	7	$P(u^4) = 4p(u)$
13	Cyclo (6.6) 3,3,9	28	309	4008	13, $P(x) = 3p(x)$	28	$\emptyset$	288	140	9	$P(u^3) = 3p(u)$
14	Cyclo (6.3) 1,2,1	18	394	5516	14, $P(x^2) = 4p(x)$	9	148	130			
14	Cyclo (6.6) 3,3,5	16	351	4906	14, $P(x) = 3p(x)$	16	175	151	151	8	$P(u^2) = 3p(u)$
15	Cyclo (6.6) 3,1,11	12	383	5736	15, $P(x) = 3p(x)$	12	175	137			
15	Cyclo (BLS) 3,1,1	12	383	5745	15, $P(x) = 3p(x)$	12	286	137			
16	KSS16 (6.11) 1,–,–	10	331	5281	16, $P(x)=980p(x-1)$	10	154	140			
17	Cyclo (6.2) 1,4,9	38	304	5152	17, $P(x^2) = 4p(x)$	19	254	189	155	9	$P(u^4) = 4p(u)$
17	Cyclo (6.6) 3,3,6	36	348	5914	17, $P(x+{}^1/_x)x^{18}=3p(x)$	18	$\emptyset$	186	168	9	$u^{36}P((u+{}^1/_u)^2)=3p(u)$

Table 3. Pairing-friendly Constructions for $9 \leq k \leq 17$ from Table 2 and their security estimate in [4]. The parameter η is the degree of the base extension in TNFS, and η divides k. In several cases the data in [4] was missing or unexpected (it seems that the parameters A, B were not optimised).

k	family	min p bits [4]	seed u, Table number in [4]	$r(u)$ bits	$p(u)$ bits	$p^k(u)$ bits	DL cost estim.	η, STNFS variant
9	BLS	535	[4, Tab.23] $2^{74}+2^{35}-2^{22}+2$	443	591	5314	128	9 $P(x)=3p(x)$
9	BLS	535	[20, §8.1] $2^{70}+2^{59}+2^{46}+2^{41}+1$	419	559	5026	**126**	9 $P(x)=3p(x)$
9	6.2	483	[4, Tab.6] $-1+2^3+2^4+2^5+2^9+2^{10}+2^{22}$	265	483	4339	**116**	9 $P(x^2)=4p(x)$
9	6.7	520	[4, Tab.19] $-1-2^4+2^6+2^9+2^{11}=$ `0xa2f`	273	520	4672	140	9 $P(u^4)=8p(u)$
10	6.3	446	[4, Tab.7] $1+2^3-2^5+2^{10}+2^{13}+2^{31}$	*249*	*433*	*4321*	**120**	10 $P(x^2)=4p(x)$
11	6.2	337	[4, Tab.6] $-1+2^8+2^{14}$	281	363	3993	**124**	11 $P(u^4)=4p(u)$
11	6.6	311	[4, Tab.16] $2^4+2^6+2^7+2^9+2^{10}+2^{14}$	283	338	3718	**118**	11 $x^{12}P(x+1/x)=3p(x)$
12	BLS	461	[3] $-2^{77}+2^{50}+2^{33}$	308	461	5525	135	6 $P(x)=p(x)$
12	BN	462	[3] $2^{114}+2^{101}-2^{14}-1$	462	462	5535	135	6 $P(x)=3p(x)$
12	6.7	446	[4, Tab.18] $1+2^{14}+2^{17}+2^{32}$	257	446	5341	134	12 $P(x^2)=8p(x)$
12	6.4	510	[4, Tab.8] $1+2+2^3+2^8+2^9+2^{11}+2^{64}$	257	511	6121	138	12 $P(x)=4p(x)$
13	6.2	329	[4, Tab.6] $1+2+2^3+2^4+2^8+2^{10}+2^{14}+2^{20}$	**481**	**599**	**7784**	163	13 $P(x^2)=4p(x)$
13	6.6	309	[4, Tab.16] $2^4+2^7+2^{10}+2^{11}+2^{13}=$ `0x2c90`	324	376	4886	153	13 $P(u^3)=3p(u)$
14	6.3	394	[4, Tab.7] $1-2^2+2^6+2^9-2^{12}-2^{15}-2^{19}+2^{22}$	262	391	5464	130	14 $P(x^2)=4p(x)$
14	6.6	351	[4, Tab.15] $-1+2^6+2^7+2^9+2^{10}+2^{13}+2^{17}+2^{22}$	265	352	4917	151	14 $P(u^2)=3p(u)$
15	BLS	383	[20, §8.1] $2^2+2^5+2^{19}+2^{31}$	*249*	371	5557	135	15 $P(x)=3p(x)$
15	6.6	383	[4, Tab.14] $1+2^2+2^{12}+2^{16}+2^{32}$	257	383	5737	137	15 $P(x)=3p(x)$
15	BLS	383	[4, Tab.23] $2+2^{10}+2^{16}+2^{19}+2^{32}$	257	383	5737	137	15 $P(x)=3p(x)$
16	KSS	331	[3] $-2^{34}+2^{27}-2^{23}+2^{20}-2^{11}+1$	257	330	5280	140	16 $P(x)=980p(x-1)$

Table 4. Seeds provided in [4]. No seed is given for $k=9$, $k=10$ with 6.6, $k=17$. The seeds for $k=12,16$ are from [3].

$\lambda^2+\lambda+1=0 \bmod r$. We can easily precompute ω and λ. Since $p=(t^2+3y^2)/4$, then $\sqrt{-3}\equiv t/y \bmod p$, and $\omega\equiv(-1+\sqrt{-3})/2\equiv(-y+t)/(2y) \bmod p$. We also have $\sqrt{-3}\equiv(t-2)/y \bmod r$. The eigenvalue is $\lambda\equiv(-1+\sqrt{-3})/2\equiv(-y+t-2)/(2y) \bmod r$. Since the square roots are given up to sign, in practice one obtains equality up to sign ($[\pm\lambda](x_p,y_P)=(\omega x_P,y_P)$ or $[\pm\lambda](x_P,y_P)=(\omega^2x_P,y_P)$), that is, a practival adjustment is required.

We give a polynomial form of low degree for $\beta=\sqrt{-D} \bmod p$ and $\lambda=\sqrt{-D} \bmod r$ for the curves below.

Curve 1 *A pairing-friendly curve $y^2=x^3+ax+b$ with the Brezing–Weng method, $k=10$, $D=15$, $m=3$, $e_0=1$, $\rho=7/4=1.75$ ([18, Table 2]). The shortest Miller loop for optimal ate pairing is $x-p^2+x^2p^3$.*

$r=\Phi_{30}(x)=x^8+x^7-x^5-x^4-x^3+x+1$

$p=(4x^{14}+4x^{13}+x^{12}-12x^{11}-12x^{10}-7x^9+11x^8$
$\quad+17x^7+15x^6-3x^5-11x^4+x^3-2x^2+3x+6)/15$

$t=x^3+1$

$y=(x-1)(4x^6+6x^5+6x^4-3x^2-5x-3)/15$

$c=(x-1)(2x^2+x+2)(2x^2+3x+3)/15$

$u=1,3,6,13 \bmod 15$

The Hilbert class polynomial is $H(-15)=x^2+191025x-121287375$ of discriminant $5(3^3\cdot5\cdot7^2\cdot13)^2$. For a root $j_0=135(-1415\pm637\sqrt{5})/2$ of $H(-15)$ modulo p, one has $a=-3j_0/(j_0-1728)$, $b=2j_0/(j_0-1728)$. A simplified pair is

k	Construction,	D, m, e_0	deg $p(x)$	p bits	p^k bits	r bits	η	STNFS variant	deg P,	$O(\|P\|_\infty)$,	P	DL cost in $\mathbb{F}_{p^k}$
9	Cyclo (BLS)	3,1,1	8	608	*5472*	456	9		8	1	$P(x) = 3p(x)$	130
9	Cyclo (6.2)	1,4,5	22	640	*5752*	350	9	(1)	11	1	$P(x^2) = 4p(x)$	130
9	Cyclo (6.7)	2,8,1	46	520	4672	273^+	9	(1+4)	11	u^2	$P(u^4) = 8p(u)$	140
9	Cyclo (FM10)	3,1,5	12	608	*5472*	304	9		12	1	$P(x) = 3p(x)$	133
10	Cyclo (6.5)	1,2,9	12	480	4800	322	5	(1)	6	1	$P(x^2) = 3p(x)$	128
10	Cyclo (6.3 FM13)	1,2,1	14	512	5120	294	10	(1)	7	1	$P(x^2) = 4p(x)$	129
10	Cyclo (FM16)	2,4,9	30	488	4871	262^+	10	(1)	15	1	$P(x^2) = 8p(x)$	141
10	6.6	3,3,1	16	511	5104	256	10	(4)	8	u	$P(u^2) = 3p(u)$	145
10	Cyclo (FM14)	5,2,9	14	480	4800	276	10	(1)	7	1	$P(x^2) = 20p(x)$	128
10	Cyclo (FM15)	15,3,1	14	446	4460	256	10		14	1	$P(x) = 15p(x)$	133
11	Cyclo (6.2)	1,4,6	26	414	4554	320	11	(1)	13	1	$P(x^2) = 4p(x)$	130
11	Cyclo	1,4,1	30	391	4297	262^+	11	(1+4)	7	u^2	$P(u^4) = 4p(u)$	136
11	Cyclo	1,4,7	34	444	4876	262^+	11	(1+4)	8	u^2	$P(u^4) = 4p(u)$	146
11	Cyclo (6.6)	3,3,4	24	446	4899	373	11	(2)	12	1	$x^{12}P(x+{}^1\!/_x)=3p(x)$	128
11	Cyclo	3,3,8	26	333	3663	258^+	11	(4)	8	u^2	$P(u^3) = 3p(u)$	131
11	Cyclo	3,3,1	28	355	3901	255^+	11	(4)	9	u^2	$P(u^3) = 3p(u)$	135
11	Cyclo	3,3,1	28	373	4101	268^+	11	(4)	9	u^2	$P(u^3) = 3p(u)$	139
11	Cyclo	11,1,4	16	411	4521	256	11	(4)	8	u	$P(u^2) = 11p(u)$	145
11	Cyclo	11,1,8	16	480	4280	298	11	(2)	8	1	$x^8P(x+{}^1\!/_x)=11p(x)$	130
12	BN (6.8)	3,–,–	4	446	5376	446	6		4	1	$P(x) = p(x)$	132
12	Cyclo (BLS)	3,1,1	6	446	5376	299	6		6	1	$P(x) = 3p(x)$	132
12	FM17	3,–,–	6	446	5352	296	6		6	1	$P(6x + 2) = 108p(x)$	136
12	FM19	3,–,–	6	446	5352	295	6		6	1	$P(x) = 225p(x)$	135
12	FM20	3,–,–	6	446	5352	292	6		6	1	$P(x + 3) = 1425p(x)$	137
12	Cyclo (6.7,FM18)	2,2,1	14	445	5329	256	12	(1)	7	1	$P(x^2) = 8p(x)$	134
13	Cyclo (6.2)	1,4,7	30	339	4396	256^*	13	(1+4)	7	u^2	$P(u^4) = 4p(u)$	142
13	Cyclo	1,4,1	34	380	4931	270^+	13	(1+4)	8	u^2	$P(u^4) = 4p(u)$	141
13	Cyclo (6.6)	3,3,9	28	310	4027	267^+	13	(4)	9	u^2	$P(u^3) = 3p(u)$	140
13	Cyclo	3,3,1	32	348	4512	262^+	13	(4)	10	u^2	$P(u^3) = 3p(u)$	139
13	Cyclo	3,3,10	34	388	5037	275^+	13	(4)	8	u^2	$P(u^4) = 3p(u)$	144
13	Cyclo	3,3,2	38	403	5233	256	13	(4)	6	u^2	$P(u^6) = 3p(u)$	150
14	Cyclo (6.3)	1,2,1	18	382	5376	256	14	(1)	9	1	$P(x^2) = 4p(x)$	130
14	Cyclo (6.6)	3,3,5	16	340	4755	256	14	(4)	8	u	$P(u^2) = 3p(u)$	148
16	KSS16 (6.11)	1,–,–	10	330	5280	256	16		10	1	$P(x) = 980p(x-1)$	140
17	Cyclo (6.2)	1,4,9	38	382	*6494*	262^*	17	(1+4)	9	u^2	$P(u^4) = 4p(u)$	167
17	Cyclo (6.2)	1,4,9	38	359	*6087*	254^*	17	(1+4)	9	u^2	$P(u^4) = 4p(u)$	164
17	Cyclo (6.6)	3,3,6	36	374	*6358*	281^*	17	(2+4)	9	u^2	$P((u+{}^1\!/_u)^2)u^{36}=3p(u)$	172
17	Cyclo	3,3,12	38	337	*5718*	255^*	17	(4)	9	u^3	$P(u^4) = 3p(u)$	165

Table 5. Pairing-friendly Constructions for $9 \leq k \leq 17$ from Table 2 and our new security estimate. For $k = 17$, r is a prime divisor of $r(u)$ but $r(u)$ itself is not prime, there is a cofactor (symbol *). For many families with $k = 11$ and $k = 13$, it was not possible to find a seed u such that r is 256-bit long (symbol $^+$) because $r(x)$ has a high degree.

k	Construction,	D, m, e_0	deg $p(x)$	seed u	p bits	p^k bits	r bits	DL cost in $\mathbb{F}_{p^k}$
8	Cocks–Pinch	1,–,–	8	$2^{64} - 2^{54} + 2^{37} + 2^{32} - 4$[23]	544	4352	256	131[23]
10	Cyclo (FM15)	15,3,1	14	$2^{32} - 2^{26} - 2^{17} + 2^{10} - 1$, $a = -3$	446	4460	256	133
11	Cyclo	3,3,8	26	$-2^{13} + 2^{10} - 2^8 - 2^5 - 2^3 - 2 = -$`0x1d2a`, $b = 13$	333	3663	258^+	131
11	Cyclo	11,1,4	16	$-2^{26} + 2^{21} + 2^{19} - 2^{11} - 2^9 - 1$, $a = 2$	412	4522	256	145
12	BN (6.8)	3,–,–	4	$2^{110} + 2^{36} + 1$, $b = 257$ [33]	446	5376	446	132[24]
12	Cyclo (BLS)	3,1,1	6	$-(2^{74} + 2^{73} + 2^{63} + 2^{57} + 2^{50} + 2^{17} + 1)$, $b = 1$ [23,24]	446	5376	299	132[24]
12	FM17	3,–,–	6	$-2^{72} - 2^{71} - 2^{36}$, $b = -2$ [19, §4(b)]	446	5352	296	136
13	Cyclo (6.6)	3,3,9	28	$2^{11} + 2^8 - 2^6 - 2^4 =$ `0x8b0`, $b = -17$	310	4027	267^+	140
14	Cyclo (6.6)	3,3,5	16	$2^{21} + 2^{19} + 2^{10} - 2^6$, $b = -4$	340	4755	256	148
16	KSS16 (6.11)	1,–,–	10	$-2^{34} + 2^{27} - 2^{23} + 2^{20} - 2^{11} + 1$, $a = 1$ [3]	330	5280	257	140[24]
16	KSS16 (6.11)	1,–,–	10	$2^{34} - 2^{30} + 2^{26} + 2^{23} + 2^{14} - 2^5 + 1$, $a = 1$	330	5268	256	140

Table 6. Our short-list of STNFS-secure pairing-friendly curves at the 128-bit security level.

$(a, b) = (-3(245 \pm 416\sqrt{5}), 154(\pm 416 + 49\sqrt{5}))$. *Moreover if* $\omega = j_0/(j_0 - 1728) = 5^2/11^2 \pm 2^5 \cdot 5 \cdot 13\sqrt{5}/(7^2 11^2)$ *is a square modulo* p*, one can have* $a' = -3$*,* $b' = b/\omega^{3/2}$*. If the curve* $y^2 = x^3 + ax + b$ *is the quadratic twist (of order* $p+1+t$ *instead of* $p+1-t$*), then* $y^2 = x^3 + a\nu^2 x + b\nu^3$ *is the curve we want, where* ν *is a non-square modulo* p*.*

Since the discriminant is small, the curve has a fast endomorphism for efficient GLV scalar multiplication. The short eigenvalue of the endomorphism (see [36]) is $\lambda = \sqrt{-15} \equiv 2x^7 - 2x^5 - 4x^4 - 2x^3 - 2x^2 + 4x + 3 \equiv (2x^4 + x^3 - 4x^2 + x + 2)/(x^3 - x) \bmod r(x)$*. Note that the square root is defined up to sign. We also have* $\sqrt{-15} \equiv (-64x^{13} - 24x^{12} + 8x^{11} + 250x^{10} + 92x^9 + 32x^8 - 448x^7 - 226x^6 - 146x^5 + 398x^4 + 222x^3 + 32x^2 - 42x - 159)/45 \bmod p(x)$*. The endomorphism can be obtained from a 3-isogeny and a 5-isogeny. There are two 3-isogenies and two 5-isogenies, one combination gives an endomorphism (we were able to check it on a numerical example in Magma, and obtained the eigenvalue* $-\lambda(u)$*).*

Curve 2 *A pairing-friendly curve* $y^2 = x^3 + b$ *with the Brezing–Weng method,* $k = 11$*,* $D = 3$*,* $m = 3$*,* $e_0 = 8$*,* $\rho = 13/10 = 1.30$*. Since* $D = 3$*,* $a = 0$*. The shortest Miller loop for optimal ate pairing is* $x + x^2 p^5 + p^6$*.*

$$
\begin{aligned}
r &= \Phi_{33}(x) = x^{20} - x^{19} + x^{17} - x^{16} + x^{14} - x^{13} + x^{11} - x^{10} + x^9 - x^7 + x^6 \\
&\quad - x^4 + x^3 - x + 1 \\
p &= (x^{26} + x^{24} + x^{22} + x^{15} - 2x^{13} + x^{11} + x^4 - 2x^2 + 1)/3 \\
t &= x^{3\times 8} + 1 \bmod r = -x^{13} - x^2 + 1 \\
y &= (x^{13} + 2x^{11} - x^2 + 1)/3 \\
c &= (x^2 - x + 1)(x^2 + x + 1)^2/3 \\
u &= 1, 2 \bmod 3
\end{aligned}
$$

The eigenvalue of the endomorphism $(x, y) \mapsto (\omega x, y)$ *is* $\lambda \equiv (-1 + \sqrt{-3})/2 \equiv x^{11} \equiv (x^{10} - x^9 + x^7 - x^6 + x^4 - x^3 + x - 1)/(x^9 - x^8 + x^6 - x^5 + x^3 - x^2 + 1) \bmod r(x)$*, and* $\omega \equiv (-1 + \sqrt{-3})/2 \equiv (2x^{25} - x^{24} + 5x^{23} + 7x^{21} - x^{20} + 8x^{19} + x^{18} + 7x^{17} - x^{16} + 8x^{15} + 3x^{14} + 6x^{13} - 2x^{12} + 6x^{11} - x^{10} - 2x^9 + x^8 + 2x^7 - x^6 - 2x^5 + x^4 + 4x^3 - 2x^2 - 3x - 1)/5 \equiv (x^{11} - x^2 + 1)/(x^{13} + x^2 - 1) \bmod p(x)$*.*

Curve 3 *A pairing-friendly curve* $y^2 = x^3 + ax + b$ *with the Brezing–Weng method,* $k = 11$, $D = 11$, $m = 1$, $e_0 = 4$, $\rho = 8/5 = 1.6$. *The shortest Miller loop for optimal ate pairing is* $x - p^3$.

$r = \Phi_{11}(x) = x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
$p = (x^{16} + 2x^{15} + x^{14} - x^{12} - 3x^{11} - x^5 + 9x^4 - x^3 + x + 3)/11$
$t = x^4 + 1$
$y = (2x^8 + 2x^7 - x^4 - 2x^3 + 2x^2 - 2x - 1)/11$
$c = (x - 1)^2(x^4 + 3x^3 + 4x^2 + 4x + 3)$
$u = 1 \mod 11$

The j*-invariant of a curve of discriminant* -11 *is* $-32768 = -2^{15}$, *and* $(a, b) = (-264, 1694)$. *Moreover if* 22 *is a square modulo* p, *one can define* $(a', b') = (-3, 7\sqrt{22}/2^4)$.

The endomorphism can be obtained from a 11-isogeny. The eigenvalue is $\lambda \equiv \sqrt{-11} \equiv 2x^9 + 2x^5 + 2x^4 + 2x^3 + 2x + 1 \equiv (2x^5 + x^4 - 2x^3 + 2x^2 - x - 2)/(x^4 + x) \bmod r(x)$.

Curve 4 *A pairing-friendly curve* $y^2 = x^3 + b$ *with the Brezing–Weng method,* $k = 13$, $D = 3$, $m = 3$, $e_0 = 9$ *(this is (6.6)),* $\rho = 7/6 = 1.17$. *Since* $D = 3$, $a = 0$. *The shortest Miller loop for optimal ate pairing is* $x^2 + xp + p^2$.

$$r = \Phi_{39}(x) = x^{24} - x^{23} + x^{21} - x^{20} + x^{18} - x^{17} + x^{15} - x^{14} + x^{12} - x^{10} + x^9 - x^7 + x^6 - x^4 + x^3 - x + 1$$
$p = (x^{28} + x^{27} + x^{26} + x^{15} - 2x^{14} + x^{13} + x^2 - 2x + 1)/3$
$t = (x^{3\times 9} + 1) \bmod r = -x^{14} - x + 1$
$y = (x^{14} + 2x^{13} - x + 1)/3$
$c = (x^2 + x + 1)^2/3$
$u = 1 \mod 3$

The endomorphism is $(x, y) \mapsto (\omega x, y)$ *where* $\omega \equiv (-1 + \sqrt{-3})/2 \equiv x^{26} + x^{25} + 2x^{24} + x^{23} + 2x^{22} + x^{21} + 2x^{20} + x^{19} + 2x^{18} + x^{17} + 2x^{16} + x^{15} + 2x^{14} + x^{13} - x^{12} + x^{11} - x^{10} + x^9 - x^8 + x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1 \bmod p(x)$. *The eigenvalue is* $\lambda \equiv (-1 + \sqrt{-3})/2 \equiv x^{13} \equiv (x^{11} - x^{10} + x^8 - x^7 + x^5 - x^4 + x^2 - x)/(x^{12} - x^{11} + x^9 - x^8 + x^6 - x^5 + x^3 - x^2 + 1) \bmod r(x)$.

Curve 5 *A pairing-friendly curve* $y^2 = x^3 + b$ *with the Brezing–Weng method,* $k = 14$, $D = 3$, $m = 3$, $e_0 = 5$ *(this is (6.6)),* $\rho = 4/3 = 1.33$. *Since* $D = 3$, $a = 0$. *The shortest Miller loop for optimal ate pairing is* $x^2 + xp + p^2$.

$r = \Phi_{42}(x) = x^{12} + x^{11} - x^9 - x^8 + x^6 - x^4 - x^3 + x + 1$
$p = (x^{16} + x^{15} + x^{14} - x^9 + 2x^8 - x^7 + x^2 - 2x + 1)/3$
$t = (x^{3\times 5} + 1) \bmod r = x^8 - x + 1$
$y = (x^8 + 2x^7 + x - 1)/3$
$c = (x^2 - x + 1)(x^2 + x + 1)/3$
$u = 1 \mod 3$

The endomorphism is $(x, y) \mapsto (\omega x, y)$ *where* $\omega \equiv (-1 + \sqrt{-3})/2 \equiv (2x^{15} + 3x^{14} + 5x^{13} + 4x^{12} + 5x^{11} + 4x^{10} + 5x^9 + 2x^8 + 5x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + 3x - 4)/3 \bmod p(x)$. *The eigenvalue is* $\lambda \equiv (-1 + \sqrt{-3})/2 \equiv x^{13} \equiv (x^5 + x^4 - x^2 - x)/(x^6 - x^4 - x^3 + x + 1) \bmod r(x)$.

k	curve	DoubleLine and AddLine	Vertical Line	Update1 and Update2	ref
		Weierstrass model			
any k	$y^2 = x^3 + ax + b$	$5\mathbf{m}_k + 6\mathbf{s}_k + 2k\mathbf{m}$ $10\mathbf{m}_k + 3\mathbf{s}_k$	$k\mathbf{m}$	$4\mathbf{m}_k + 2\mathbf{s}_k$ $4\mathbf{m}_k$	[23, Alg. 3,4,5]
any k	$y^2 = x^3 - 3x + b$	$6\mathbf{m}_k + 4\mathbf{s}_k + 2k\mathbf{m}$ $10\mathbf{m}_k + 3\mathbf{s}_k$	$k\mathbf{m}$	$4\mathbf{m}_k + 2\mathbf{s}_k$ $4\mathbf{m}_k$	[23, Alg. 3,4,5]
any k	$y^2 = x^3 + b$	$5\mathbf{m}_k + 5\mathbf{s}_k + 2k\mathbf{m}$ $10\mathbf{m}_k + 3\mathbf{s}_k$	$k\mathbf{m}$	$4\mathbf{m}_k + 2\mathbf{s}_k$ $4\mathbf{m}_k$	[23, Alg. 3,4,5]
$2 \mid k$	$y^2 = x^3 + b$ quadratic twist	$2\mathbf{m}_{k/2} + 7\mathbf{s}_{k/2} + k\mathbf{m}$ $14\mathbf{m}_{k/2} + 2\mathbf{s}_{k/2} + k\mathbf{m}$	0	$\mathbf{m}_k + \mathbf{s}_k$ $\mathbf{m}_k$	[14, §5,Tab.3]
$2 \mid k$	$y^2 = x^3 + ax + b$ quadratic twist	$5\mathbf{m}_{k/2} + 6\mathbf{s}_{k/2} + k\mathbf{m}$ $10\mathbf{m}_{k/2} + 3\mathbf{s}_{k/2} + k\mathbf{m}$	0	$\mathbf{m}_k + \mathbf{s}_k$ $\mathbf{m}_k$	[12]
$2 \mid k$	$y^2 = x^3 - 3x + b$ quadratic twist	$6\mathbf{m}_{k/2} + 4\mathbf{s}_{k/2} + k\mathbf{m}$ $10\mathbf{m}_{k/2} + 3\mathbf{s}_{k/2} + k\mathbf{m}$	0	$\mathbf{m}_k + \mathbf{s}_k$ $\mathbf{m}_k$	[12]

Table 7. Miller loop cost from [12,14,2,23].

4 Optimal Ate Pairing Computation

We left to future work to include a theoretical estimate of the number of multiplications in $\mathbb{F}_p$ required to compute an optimal ate pairing on the curves of Table 6, to give a ranking of the curves in terms of pairing efficiency. We predict that BLS12 curves over a 446-bit prime field will be the best option. We only sketch a rough estimate below. In [23, Table 10], the Miller loop of optimal ate pairing on a BLS12 curve over a 446-bit prime field (7 machine-words of 64 bits) costs $7805\mathbf{m}$ (multiplications in $\mathbb{F}_p$), and for a KSS16 curve over a 339-bit prime field (6 machine-words of 64 bits), the Miller loop costs $7691\mathbf{m}$.

For curves $y^2 = x^3 + ax + b$ with j-invariant 0 ($a = 0$) and 1728 ($b = 0$), we reproduce the counts from [14]. For prime embedding degrees ($k = 11, 13$), we apply the formulas from [23, Table 7] for prime embedding degree $k = 5, 7$. We obtain Table 7.

The cost of optimal ate pairing computation is given by Eq. (5), where nbits is the bitlength and $\mathrm{HW}_{\text{2-NAF}}$ is the Hamming weight in 2-non-adjacent form, and $\mathbf{i}_k$ an inversion in $\mathbb{F}_{p^k}$.

$$\begin{aligned}\mathrm{Cost}_{\text{MillerLoop}} =&(\mathrm{nbits}(u) - 1)\,(\mathrm{Cost}_{\text{DoubleLine}} + \mathrm{Cost}_{\text{VerticalLine}})\\ &+ (\mathrm{nbits}(u) - 2)\mathrm{Cost}_{\text{Update1}}\\ &+ (\mathrm{HW}_{\text{2-NAF}}(u) - 1)(\mathrm{Cost}_{\text{AddLine}} + \mathrm{Cost}_{\text{VerticalLine}} + \mathrm{Cost}_{\text{Update2}})\\ &+ (\text{only if } k \in \{11, 13\})\mathbf{i}_k. \qquad (5)\end{aligned}$$

4.1 Prime Embedding Degrees 11 and 13.

We give a first estimate of a pairing computation on the curves of prime embedding degrees. Let $\mathbf{m}_k$ denotes a multiplication in $\mathbb{F}_{p^k}$, $\mathbf{m}$ a multiplication in $\mathbb{F}_p$, $\mathbf{s}_k$ a

square in $\mathbb{F}_{p^k}$ and $\mathbf{s}$ a square in $\mathbb{F}_p$. We follow [23] where an estimate for $k = 5, 7$ is given with Eq. (5).

For Curve 2 ($k = 11$, $D = 3$, p of 333 bits, $u =$ `-0x1d2a`), the optimal ate Miller loop has length $x + x^2p^5 + p^6$. The main part has length x^2. For u of 13 bits, u^2 is 26-bit long and has $\mathrm{HW}_{\text{2-NAF}}(u^2) = 11$. Moreover, since $D = 3$, we have $a = 0$. No twist is available. We obtain from Eq. (5) $25(5\mathbf{m}_k + 5\mathbf{s}_k + 2k\mathbf{m} + k\mathbf{m}) + 24(4\mathbf{m}_k + 2\mathbf{s}_k) + 10(10\mathbf{m}_k + 3\mathbf{s}_k + k\mathbf{m} + 4\mathbf{m}_k) + \mathbf{i}_k = 361\mathbf{m}_k + 203\mathbf{s}_k + 85k\mathbf{m} + \mathbf{i}_k$. A schoolbook implementation of multiplication and squaring would give $\mathbf{m}_k = k^2\mathbf{m} = 121\mathbf{m}$ and $\mathbf{s}_k = k(k-1)\mathbf{m} = 110\mathbf{m}$. We obtain the upper bound $66946\mathbf{m} + \mathbf{i}_k$. An optimised Karatsuba multiplication in $\mathbb{F}_{p^k}$ would require at least $k^{\log_2 3}\mathbf{m}$, that is, $45\mathbf{m}$. Assuming that $\mathbf{s}_k \geq 45\mathbf{m}$, we obtain the lower bound $26315\mathbf{m} + \mathbf{i}$. Note that in [23, Table 10], the Miller loop on a KSS16 curve over a 339-bit prime p costs $7691\mathbf{m}$: this is more than three times smaller.

For Curve 3 ($k = 11$, $D = 11$, p of 412 bits, $u =$ `-0x3d80a01`), the optimal ate Miller loop has length $x - p^3$. We have u of 26 bits, $\mathrm{HW}_{\text{2-NAF}}(u) = 6$, and $a = 2$. No twist is available. We obtain from Eq. (5) $25(5\mathbf{m}_k + 6\mathbf{s}_k + 2k\mathbf{m} + k\mathbf{m}) + 24(4\mathbf{m}_k + 2\mathbf{s}_k) + 5(10\mathbf{m}_k + 3\mathbf{s}_k + k\mathbf{m} + 4\mathbf{m}_k) + \mathbf{i}_k = 291\mathbf{m}_k + 213\mathbf{s}_k + 80k\mathbf{m} + \mathbf{i}_k$. With the upper bound $\mathbf{m}_k = k^2\mathbf{m}$ and $\mathbf{s}_k = k(k-1)\mathbf{m}$, the count is $59521\mathbf{m} + \mathbf{i}_k$. With the lower bound $\mathbf{m}_{11} = \mathbf{s}_{11} = 45\mathbf{m}$, the count is $23560\mathbf{m} + \mathbf{i}_k$. This is again three times more than the count of $7805\mathbf{m}$ for the Miller loop of a BLS12 curve over a 446-bit prime field reported in [23, Table 10].

For Curve 4 ($k = 13$, $D = 3$, (6.6), p of 310 bits, $u =$ `0x8b0`), the optimal ate Miller loop has length $x^2 + xp + p^2$. We have u of 12 bits, u^2 of 23 bits, $\mathrm{HW}_{\text{2-NAF}}(u^2) = 6$, and $a = 0$, but no twist is available. We obtain from Eq. (5) $22(5\mathbf{m}_k + 5\mathbf{s}_k + 2k\mathbf{m} + k\mathbf{m}) + 21(4\mathbf{m}_k + 2\mathbf{s}_k) + 5(10\mathbf{m}_k + 3\mathbf{s}_k + k\mathbf{m} + 4\mathbf{m}_k) + \mathbf{i}_k = 264\mathbf{m}_k + 167\mathbf{s}_k + 71k\mathbf{m} + \mathbf{i}_k$. With the upper bound $\mathbf{m}_{13} = k^2\mathbf{m} = 169\mathbf{m}$ and $\mathbf{s}_{13} = k(k-1)\mathbf{m} = 156\mathbf{m}$, the count is $71591\mathbf{m} + \mathbf{i}_k$. With the lower bound $\mathbf{m}_{13} = \mathbf{s}_{13} = 13^{\log_2 3}\mathbf{m} = 59\mathbf{m}$, the count is $26352\mathbf{m} + \mathbf{i}_k$. Again this is not competitive compared to KSS16 and BLS12 curves for a pairing computation.

4.2 Even embedding degrees 10 and 14.

For Curve 5 ($k = 14$, $D = 3$, (6.6), p of 340 bits, $u =$ `0x2803c0`), the optimal ate Miller loop has length $x^2 + xp + p^2$. We have u of 22 bits, u^2 of 43 bits, $\mathrm{HW}_{\text{2-NAF}}(u^2) = 10$, $a = 0$, and a quadratic twist is available. We obtain from Eq. (5) $42(2\mathbf{m}_{k/2} + 7\mathbf{s}_{k/2} + k\mathbf{m}) + 41(\mathbf{m}_k + \mathbf{s}_k) + 9(14\mathbf{m}_{k/2} + 2\mathbf{s}_{k/2} + k\mathbf{m} + \mathbf{m}_k) = 50\mathbf{m}_k + 41\mathbf{s}_k + 210\mathbf{m}_{k/2} + 312\mathbf{s}_{k/2} + 51k\mathbf{m}$. From [32], we consider the lower (Karatsuba) bound $\mathbf{m}_7 = \mathbf{s}_7 = 22\mathbf{m}$, and $\mathbf{m}_{14} = 3\mathbf{m}_7 = 66\mathbf{m}$, $\mathbf{s}_{14} = 2\mathbf{m}_7 = 44\mathbf{m}$. We obtain $17312\mathbf{m}$, where $\mathbf{m}$ is a multiplication in $\mathbb{F}_p$ of 340 bits. This is not competitive compared to KSS16 curves (over a 339-bit prime field).

For Curve 1 ($k = 10$, $D = 15$, p of 446 bits, $u =$ `0xfbfe03ff`), the optimal ate Miller loop has length $x - p^2 + x^2p^3$. We have u of 32 bits, u^2 of 64 bits, $\mathrm{HW}_{\text{2-NAF}}(u^2) = 13$, $a = -3$, and a quadratic twist is available. The optimisation of line and tangent computation focused on curves with twists of degrees 3, 4 and 6 in [14]. We refer to the former paper [12] for pairing formulas on curves with quadratic twists only. The Miller loop of ate pairing, of length u^2, costs

$63(6\mathbf{m}_{k/2} + 4\mathbf{s}_{k/2} + k\mathbf{m}) + 62(\mathbf{m}_k + \mathbf{s}_k) + 12(10\mathbf{m}_{k/2} + 3\mathbf{s}_{k/2} + k\mathbf{m}) + 11(\mathbf{m}_k) = 498\mathbf{m}_{k/2} + 288\mathbf{s}_{k/2} + 73\mathbf{m}_k + 62\mathbf{s}_k + 75k\mathbf{m}$. We have $\mathbf{m}_{k/2} = \mathbf{m}_5$, a schoolbook implementation of a multiplication in $\mathbb{F}_{p^5}$ would need $\mathbf{m}_5 = k^2\mathbf{m} = 25\mathbf{m}$, and a square $\mathbf{s}_5 = k(k-1)\mathbf{m} = 20\mathbf{m}$, then with a quadratic extension, $\mathbb{F}_{p^{10}}$ would have $\mathbf{m}_{10} = 3\mathbf{m}_5 = 75\mathbf{m}$ (with Karatsuba) and $\mathbf{s}_{10} = 2\mathbf{m}_5 = 50\mathbf{m}$. The total count would be $27535\mathbf{m}$. With optimised Karatsuba-like formulas [32], we would have the lower bound $\mathbf{m}_5 = \mathbf{s}_5 = 13\mathbf{m}$, and $\mathbf{m}_{10} = 39\mathbf{m}$, $\mathbf{s}_{10} = 26\mathbf{m}$, and the final count would be $15427\mathbf{m}$. Again, the curve is not competitive (by a factor two) compared to KSS16 (over 339-bit field, Miller loop in $7691\mathbf{m}$) and BLS12 (over 446-bit field, Miller loop in $7685\mathbf{m}$), because it has only a quadratic twist, whereas KSS16 curves have a quartic twist and BLS12 curves have a sextic twist.

5 Overview of the 192-bit security level

At the 192-bit security level, we set the constrain

$$7168 \leq 384\rho k \leq 14336 \tag{6}$$

With $\rho = 1$ we obtain $k \leq 37$, and with $\rho = 2$ we obtain $k \geq 10$. Curves like Fotiadis–Konstantinou with exactly $\rho = 2$ satisfy (6) for $10 \leq k \leq 18$. Table 8 lists the curves. No family of embedding degree above 32 satisfying (6) was found.

5.1 Security Estimate

We consider the families of [4, § 3.4]. These are families from Construction 6.2 (for odd k and $D = 1$), 6.3 (for $k = 2 \bmod 4$ and $D = 1$), 6.4 (for $k = 4 \bmod 8$ and $D = 1$), 6.6 (for $18 \nmid k$ and $D = 3$), 6.7 (for $D = 2$). Curves with $k = 9, 12, 15, 24, 27$ and trace given by the linear polynomial $t(x) = x + 1$ fall in the BLS family [6]. We exclude the families that do not satisfy the constraint (6) (embedding degrees above 32). For some families, it was not possible to generate parameters such that r is a 384-bit prime (or larger prime), because the degree of the polynomial giving the characteristic, $p(x)$, is excessive, and there are too few possible seeds u (this is the case in particular for $k = 17, 19, 23, 25, 29, 31, 32$). In that case, the security estimate is for parameters such that p is prime but r is not prime, and the largest prime factor of r is smaller than 384 bits. For $k = 26$, there is only one or two possibilities ($u =$ `0x12407` with construction 6.3 ($D = 1$) gives a 389-bit prime r and a 484-bit prime p, and $u =$ `-0xf527`, $u =$ `0x102a3` with construction 6.6 give a 383, resp. 385-bit prime r and 445, resp. 447-bit prime p). We report in Table 9 the curves of [4, § 3.4], the size of p^k, the required size of p, and the lowest security estimate (sometimes from the online database instead of the paper). Then we report our security estimates. We obtain three levels of improvements on the previous work in [4]:

- We consider the same polynomials and optimise the parameters A, B to get a (much) smaller security estimate ;

- We consider the same polynomial selection but with another parameter η (the degree of h is k/η) and optimised A, B ;
- When the Conjugation-Tower method gives a lower security estimate, we also try with the Sarkar–Singh-Tower method ;
- We consider alternative polynomials given by our improvements listed in Section 2.2.

We highlight in blue in Table 9 our new lowest security estimate when it is larger than 192. We highlight in orange bold our new security estimate when it is lower than 192, and in that case we put in red the size of p^k, which should be increased. We stress that 7168 is the lower bound on the size of p^k. It is the size for prime fields, extrapolating from the NFS algorithm complexity $L_p(1/3, 1.923)$. Since all the pairing-friendly curves are special because $p = p(u)$ has a special sparse form, the Special variant of NFS and TNFS apply. No family will provide 192 bits of security with a finite field p^k smaller than 7168 bits.

We list 53 families in Table 9. Our results show that 23 families (43%) from [4] of embedding degree from 9 to 22 are insecure: the size of p^k should be significantly increased.

Remark 3. The numbers for $k = 12(6.4)$ from [4] where unexpected. One reads that a 24460-bit field p^{12} is needed to ensure 192 bits of security. This family has $p(x) = (x^8 - 2x^7 + x^6 + x^2 + 2x + 1)/4$ and $r(x) = \Phi_{12}(x) = x^4 - x^2 + 1$. The polynomial $p(x)$ has an automorphism $\sigma(x) = -1/x$ and the minimal polynomial of $\alpha + \sigma(\alpha)$ is $m(x) = x^4 - 2x^3 + 5x^2 - 6x + 4$. The pair $(m(x), ux - u^2 + 1 = u(x - (u - 1/u))$ can be used for a Special polynomial selection. In the Python script, one reads $x - 1/x$, meaning the pair $(m(x), x - (u - 1/u))$ is used. We were not able to reproduce the results.

Finally we estimate in Table 10 the security of the seeds recommended at the 192-bit security level in [4, Table 25].

Remark 4. In Table 10 we reproduce the seeds from [4, Table 25] recommended at the 192-bit security level. The seed for $k = 14$ is not valid (it was generated with Construction 6.2 which is not suited to even k), we give another one of same size with Construction 6.3. The seeds for KSS16 and KSS18 do not give valid parameters ($p(u)$ is not integer), we give alternative seeds of the same size. KSS16 curves require $u = \pm 25 \bmod 70$ and KSS18 curves require $u = \pm 7 \bmod 21$ for $p(u)$ being an integer. The seed for $k = 24$ and 6.6 is the seed of BLS24 from [3]. The seed for $k = 27$ does not give $r(u)$ prime: we have $r_{27} = 109 \cdot 431947 \cdot r_0$ where r_0 is a 370-bit prime. The seed for $k = 28$ gives a 373-bit prime r, smaller than the required length of 384 bits. The sizes of p for $k = 14, 15, 16$ are too small, the fields $\mathbb{F}_{p^k}$ do not offer 192 bits of security.

5.2 A short-list of STNFS-secure pairing-friendly curves

For BN, BLS12, BLS24, KSS16, KSS18, we reproduce in Table 11 the results of Guillevic and Singh [24]: BN with a 1022-bit p, BLS12 with a 1150-bit p, KSS16

k	Construction	D	m	e_0	ρ	deg $p(x)$	$\sigma_p(x)$	$\lceil 384\rho \rceil$	$\lceil 384\rho k \rceil$
10	6.6	3	3	1	$2.00 = 2$	16	Id	768	7680
14	Cyclo (6.3)	1	2	1	$1.50 = 3/2$	18	$-x$	576	8064
15	Cyclo (BLS)	3	1	1	$1.50 = 3/2$	12	Id	576	8640
15	Cyclo (6.6)	3	1	11	$1.50 = 3/2$	12	Id	576	8640
15	Cyclo	1	4	7	$1.87 = 15/8$	30	$-x$	720	10800
15	Cyclo (6.2)	1	4	8	$2.12 = 17/8$	34	$-x$	816	12240
15	Cyclo	2	8	7	$1.75 = 7/4$	**56**	$-x$	672	10080
15	Cyclo (6.7)	2	8	8	$2.06 = 33/16$	**66**	$-x$	792	11880
16	KSS16 (6.11)	1	–	–	$1.25 = 5/4$	10	Id	480	7680
16	Cyclo (6.6)	3	3	1	$1.37 = 11/8$	22	Id	528	8448
18	KSS18 (6.12)	3	–	–	$1.33 = 4/3$	8	Id	512	9216
18	KSS [35]	3	–	–	$1.66 = 5/3$	10	Id	640	11520
18	Cyclo (6.3)	1	2	1	$1.83 = 11/6$	22	$-x$	704	12672
18	Cyclo (6.7)	2	4	1	$1.58 = 19/12$	38	$-x$	608	10944
19	Cyclo (6.2)	1	4	10	$1.16 = 7/6$	**42**	$-x$	448	8512
19	Cyclo (6.6)	3	3	13	$1.11 = 10/9$	**40**	Id	427	8107
20	Cyclo (6.4,FM27)	1	1	1	$1.50 = 3/2$	12	$-1/x$	576	11520
20	Cyclo (6.6)	3	3	7	$1.37 = 11/8$	22	Id	528	10560
21	Cyclo (6.2)	1	4	11	$1.92 = 23/12$	46	$-x$	736	*15456*
21	Cyclo (6.6)	3	1	8	$1.33 = 4/3$	16	$1/x$	512	10752
21	Cyclo (6.7)	2	8	1	$1.79 = 43/24$	**86**	$-x$	688	*14448*
22	Cyclo (6.3)	1	2	1	$1.30 = 13/10$	26	$-x$	500	10982
22	Cyclo (6.6)	3	3	1	$1.40 = 7/5$	28	Id	537	11828
23	Cyclo (6.2)	1	4	12	$1.13 = 25/22$	**50**	$-x$	437	10037
23	Cyclo (6.6)	3	3	8	$1.09 = 12/11$	**48**	$1/x$	419	9635
24	Cyclo (6.7)	2	1	1	$1.50 = 3/2$	12	Id	576	13824
24	Cyclo (BLS,6.6)	3	1	1	$1.25 = 5/4$	10	Id	480	11520
25	Cyclo (6.2)	1	4	13	$1.35 = 27/20$	**54**	$-x$	519	12960
25	Cyclo (6.6)	3	3	17	$1.30 = 13/10$	**52**	Id	500	12480
26	Cyclo (6.3)	1	2	1	$1.25 = 5/4$	**30**	$-x$	480	12480
26	Cyclo (6.6)	3	3	9	$1.16 = 7/6$	**28**	Id	448	11648
27	Cyclo (BLS)	3	1	1	$1.11 = 10/9$	20	$s_0(x)/s_1(x)$	427	11520
27	Cyclo	3	1	10	$1.11 = 10/9$	20	$1/x$	427	11520
27	Cyclo (6.6)	3	1	19	$1.11 = 10/9$	20	$s_2(x)/s_3(x)$	427	11520
28	Cyclo (6.4)	1	1	1	$1.33 = 4/3$	16	$-1/x$	512	14336
29	Cyclo (6.2)	1	4	15	$1.10 = 31/28$	**62**	$-x$	426	12330
29	Cyclo (6.6)	3	3	10	$1.07 = 15/14$	**60**	$1/x$	412	11932
31	Cyclo (6.2)	1	4	16	$1.10 = 11/10$	**66**	$-x$	423	13095
31	Cyclo (6.6)	3	3	21	$1.06 = 16/15$	**64**	Id	410	12698
32	Cyclo (6.6)	3	3	11	$1.06 = 17/16$	**34**	Id	408	13056

Table 8. Pairing-friendly Constructions for $10 \leq k \leq 28$ such that $7144 \leq 384\rho k \leq 14288$. The parameters m and e_0 correspond to the parameters m and e_0 in the Cyclo construction of Algorithm 3.1. The value 384ρ is an approximation of the minimal bit-size of p required to ensure r to be of 384 bits, so that the curve $E(\mathbb{F}_p)$ offers 192 bits of security. We include the constructions referred in [4].

k	Construction,	D, m, e_0	deg $p(x)$	p bits	p^k bits	η,	(S)TNFS variant [4]	deg $P(x)$	DL cost $\mathbb{F}_{p^k}$ [4]	DL cost newA,B	DL cost newh,f,g	deg $P(x)$	our polynomials, new η
9	Cyclo (BLS)	3,1,1	8	1279	*11510*	9,	$P(x) = 3p(x)$	8	192	**187**			
9	Cyclo (6.6)	3,1,7	8	687	*6178*	9,	$P(x) = 3p(x)$	8	196	**139**			
9	Cyclo (6.2)	1,4,5	22	1496	*13460*	9,	$P(x^2) = 4p(x)$	11	194	**188**			
9	Cyclo (6.7)	2,8,1	46	734	*6598*	9,	$P(x^2) = 8p(x)$	23	$\emptyset$	210	**160**	11	$P(u^4) = 8p(u)$
10	Cyclo (6.3)	1,2,1	14	1258	*12580*	5,	$P(x^2) = 4p(x)$	7	192	**188**			
10	6.6	3,3,1	16	1278	12780	10,	$P(x) = 3p(x)$	16	192	192	227	8	$P(u^2) = 3p(u), \eta = 5$
11	Cyclo (6.2)	1,4,6	26	648	*7128*	11,	$P(x^2) = 4p(x)$	13	192	**148**			
11	Cyclo (6.6)	3,3,4	24	479	*5263*	11,	$P(x) = 3p(x)$	24	$\emptyset$		**135**	12	$x^{12}P(x+1/x)=3p(x)$
12	BN (6.8)	3,–,–	4	1094	13120	4,	$P(x) = p(x)$	4	192	195			
12	Cyclo (BLS)	3,1,1	6	1200	14390	6,	$P(x) = 3p(x)$	4	199	199			
12	Cyclo (6.6)	3,1,11	6	1049	*12580*	6,	$P(x) = 3p(x)$	4	192	**187**			
12	6.7	2,2,1	14	669	*8028*	6,	$P(x^2) = 8p(x)$	7	199	**160**			
12	6.4	1,1,1	8	2039	**24460**	6,	$P(x-1/x)x^4=4p(x)$	4	192				
13	Cyclo (6.2)	1,4,7	**30**	479	*6216*	13,	$P(x^2) = 4p(x)$	15	210	**152**	**167**	7	$P(u^4) = 4p(u)$
13	Cyclo (6.6)	3,3,9	28	447	*5806*	13,	$P(x) = 3p(x)$	28	$\emptyset$		**165**	9	$P(u^3) = 3p(u)$
14	Cyclo (6.3)	1,2,1	18	574	*8036*	14,	$P(x^2) = 4p(x)$	9	206	**155**			
14	Cyclo (6.6)	3,3,5	16	543	*7594*	14,	$P(x) = 3p(x)$	16	$\emptyset$	**170**	**184**	8	$P(u^2) = 3p(u)$
15	Cyclo (6.6)	3,1,11	12	575	*8616*	15,	$P(x) = 3p(x)$	12	192	**160**	**178**		Conj-Tower $\eta = 3$
15	Cyclo (BLS)	3,1,1	12	599	*8985*	15,	$P(x) = 3p(x)$	12	$\emptyset$	**162**			
15	Cyclo (6.2)	1,4,8	34	814	12210	15,	$P(x^2) = 4p(x)$	17	263	200	205		Conj-Tower $\eta = 3$
15	Cyclo (6.7)	2,8,8	**66**	968	14520	15,	$P(x^2) = 8p(x)$	33	217	234	220		Conj-Tower $\eta = 3$
16	KSS16 (6.11)	1,–,–	10	511	*8161*	16,	$P(x)=980p(x-1)$	10	192	**165**			
16	Cyclo (6.6)	3,3,1	22	527	*8422*	4,	Conj-Tower		202	**175**	**186**	11	$P(u^2) = 3p(u), \eta = 16$
17	Cyclo (6.2)	1,4,9	**38**	458	*7776*	17,	$P(x^2) = 4p(x)$	19	291		**181**	9	$P(u^4) = 4p(u)$
17	Cyclo (6.6)	3,3,6	**36**	437	*7426*	17,	base-m-Tower		237		**185**	9	$u^{36}P((u+1/u)^2)=3p(u)$
18	KSS18 (6.12)	3,–,–	10	652	11730	9,	$P(x)=21p(x-2)$	8	195	196			
18	Cyclo (6.3)	1,2,1	22	703	*12640*	18,	$P(x^2) = 4p(x)$	11	275	**188**			
18	Cyclo (6.7)	2,4,1	38	606	10900	6,	Conj-Tower		246	201 $\eta = 3$	211	9	$P(u^4) = 8p(u), \eta = 18$
19	Cyclo (6.2)	1,4,10	**42**	460	8740	19,	$P(x^2) = 4p(x)$	21	329		192	10	$P(u^4) = 4p(u)$
19	Cyclo (6.6)	3,3,13	**40**	442	8397	19,	base-m-Tower		233		204	10	$P(u^4) = 3p(u)$
20	Cyclo (6.4)	1,1,1	12	574	*11480*	20,	$P(x-1/x)x^6=4p(x)$	6	227	**184**	**180**	6	same P, $\eta = 10$
20	Cyclo (6.6)	3,3,7	22	703	14050	5,	Conj-Tower		244	224	218		Conj-Tower $\eta = 4$
21	Cyclo (6.2)	1,4,11	**46**	735	15420	7,	Conj-Tower		294	238 $\eta=3$			
21	Cyclo (6.6)	3,1,8	16	511	*10720*	7,	$P(x) = 3p(x)$	16	227	211	**187**	8	$P(x+1/x)x^8=3p(x), \eta=21$
21	Cyclo (6.7)	2,8,1	**86**	724	15190	7,	Conj-Tower		276	238 $\eta=3$			
22	Cyclo (6.3)	1,2,1	26	519	*11400*	11,	Conj-Tower		273	255	**183**	13	$P(x^2) = 4p(x), \eta = 11$
22	Cyclo (6.6)	3,3,1	28	560	12320	11,	Conj-Tower		269	266	222	14	$P(u^2) = 3p(u), \eta = 22$
23	Cyclo (6.2)	1,4,12	**50**	451	10370	23,	Base-m-Tower		289		209	12	$P(u^4) = 4p(u)$
23	Cyclo (6.6)	3,3,8	**48**	486	11160	23,	Base-m-Tower		293		225	12	$P((u+1/u)^2)u^{12}=3p(u)$
24	Cyclo (6.7)	2,1,1	12	573	13750	24,	$P(x) = 8p(x)$	12	274	262	205	12	same P, $\eta = 24$
24	Cyclo (BLS,6.6)	3,1,1	10	519	12440	24,	$P(x) = 3p(x)$	10	195	195			
25	Cyclo (6.2)	1,4,13	**54**	540	13490	5,	Conj-Tower		303	214	239	13	$P(u^4) = 4p(u), \eta = 25$
25	Cyclo (6.6)	3,3,17	**52**	569	14220	5,	Conj-Tower		257	219	233	10	$P(u^5) = 3p(u), \eta = 25$
26	Cyclo (6.3)	1,2,1	**30**	479	12440	13,	Conj-Tower		288	268	206	15	$P(x^2) = 4p(x)$, $\eta = 26$
26	Cyclo (6.6)	3,3,9	**28**	447	11610	13,	Conj-Tower		267	259	232	14	$P(u^2) = 3p(u)$
27	Cyclo (BLS)	3,1,1	20	427	11520	9,	Conj-Tower		$\emptyset$	219	212		Sarkar-Singh-Tower $\eta = 9$
27	Cyclo (6.6)	3,1,19	20	439	11840	9,	Conj-Tower		259	222	216		Sarkar-Singh-Tower $\eta = 9$
28	Cyclo (6.4)	1,1,1	16	510	14280	28,	$P(x-1/x)x^8=4p(x)$	8	306	218	208	8	same P, $\eta = 14$
29	Cyclo (6.2)	1,4,15	**62**	551	15960	29,	Base-m-Tower		372		261	15	$P(u^4) = 4p(u)$
29	Cyclo (6.6)	3,3,10	**60**	644	18650	29,	Base-m-Tower		382		289	15	$P(u^4) = 3p(u)$
31	Cyclo (6.2)	1,4,16	**66**	530	16430	31,	Base-m-Tower		384		273	16	$P(u^4) = 4p(u)$
31	Cyclo (6.6)	3,3,21	**64**	471	14600	31,	Base-m-Tower		362		249	10	$P(u^6) = 3p(u)$
32	Cyclo (6.6)	3,3,11	**34**	407	13010	16,	Conj-Tower		414	216 $\eta = 8$	219	4	$P(u^8) = 3p(u), \eta = 16$

Table 9. Pairing-friendly Constructions for $9 \le k \le 32$ from Table 2 and their security estimate in [4]. The parameter η is the degree of the base extension in TNFS, and η divides k. In most of the cases the data in [4] was unexpected (and sometimes missing).

k	family	min p bits [4]	seed u from [4, Tab. 25], [20,3]	$r(u)$ bits	$p(u)$ bits	$p^k(u)$ bits	DL cost estim.	η, STNFS variant
14	6.3	*574*	[4] $1-2^3+2^7+2^8+2^{11}+2^{40}$, `0xffffffde6b`	480	718	10052	**172**	14 $P(x^2) = 4p(x)$
15	6.6	*575*	[4] $1 + 2^7 + 2^8 + 2^{12} + 2^{15} + 2^{48}$	385	575	8617	**160**	15 $P(x) = 3p(x)$
15	BLS (DCC)	*599*	[4] $2^{10} + 2^{11} + 2^{13} + 2^{15} - 2^{40} + 2^{50}$	400	599	8976	**162**	15 $P(x) = 3p(x)$
15	BLS (DCC)	*599*	[20, §5.2] $2^{48} + 2^{41} + 2^9 + 2^8 + 1$	385	575	8619	**158**	15 $P(x) = 3p(x)$
15	BLS (DCC)	*599*	[20, §8.2] $2^{72} + 2^{40} + 2^9 + 2^5 + 1$	577	863	12937	**188**	15 $P(x) = 3p(x)$
16	KSS	*511*	[4] $2^2+2^5-2^9+2^{22}-2^{23}+2^{51}$, $-1+2^5-2^9-2^{15}-2^{23}-2^{51}$	393	501	8002	**164**	10 $P(x)=980p(x-1)$
18	KSS	652	[4] $2-2^5+2^9+2^{11}+2^{14}+2^{82}$, $2^2-2^4+2^7+2^9+2^{11}-2^{82}$	484	652	11729	195	9 $P(x)=21p(x-2)$
24	BLS, 6.6	519	[3] $-2^{56} - 2^{43} + 2^9 - 2^6$	449	559	13403	202	24 $P(x) = 3p(x)$
24	6.7	573	[4] $-2^{48} + 2^{42} + 2^{12} + 1$	384	573	13746	205	24 $P(x) = 8p(x)$
27	BLS (DCC)	427	[20, §8.2] $2^{25}+2^{14}+2^{17}+2^4+1$, `0x1ffffbd`	449	499	13458	229	9 Sarkar–Singh-Tower
27	BLS (DCC)	427	[4] $2^{22} + 2^{14} + 2^9 + 2^8 + 2^4 + 2^3 + 2$	*370*	439	11841	216	9 Sarkar–Singh-Tower
28	6.4	510	[4] $-2^{31} - 2^{13} - 2 - 1$	*373*	495	13833	200	14 $P(x^2) = 4p(x)$

Table 10. Seeds provided in [4, Table 25] and alternatives in [20,3]. See Remark 4.

with a 766-bit prime p, KSS18 with a 638-bit prime p, BLS24 with a 509-bit prime p. We list in Table 12 seeds for $k \in \{14, 15, 27, 28\}$. We also refer to [19] for alternative curves with $\rho = 2$. We leave to future work a complete study of pairing-friendly curves at the 192-bit security level.

k	curve	r bits	p bits	p^k bits	seed u	DL cost
12	BN	1022	1022	12255	$-2^{254} + 2^{33} + 2^6$	191
12	BLS12	768	1150	13799	$-2^{192} + 2^{188} - 2^{115} - 2^{110} - 2^{44} - 1$	193
16	KSS16	605	766	12255	$2^{78} - 2^{76} - 2^{28} + 2^{14} + 2^7 + 1$	194
18	KSS18	474	638	11477	$2^{80} + 2^{77} + 2^{76} - 2^{61} - 2^{53} - 2^{14}$	193
24	BLS24	409	509	12202	$-2^{51} - 2^{28} + 2^{11} - 1$ [15]	193

Table 11. Seeds at the 192-bit security level from [24].

k	curve	r bits	p bits	p^k bits	seed u	DL cost
14	Cyclo 1,2,1 (6.3)	620	928	12979–12992	`0xc382fe8f05eaf` $\le u \le$ `0xcb2ff529e85b5`	194
15	Cyclo 3,1,1 (BLS)	620	928	13906–13920	`-0x29b3f997f573d609c26f` $\ge u \ge$ `-0x2c2ecd2df12c9d54ec07` `0x29b3f997f573d6097e04` $\le u \le$ `0x2c2ecd2df12c9d52b8c9`	193
27	Cyclo 3,1,1 (BLS)	384	426–427	11496–11524	`-0x29487b` $\ge u \ge$ `-0x2ac5ea` `0x2955f1` $\le u \le$ `0x2ac66d`	212
28	Cyclo 1,1,1 (6.2)	384	510	14243–14280	`0xf1a202f1` $\le u \le$ `0xffffd341`	208

Table 12. Seeds at the 192-bit security level for $k \in \{14, 15, 27, 28\}$. For $k = 14, 15$ the range of u is such that p is 928-bit long (a smaller p of 920 to 928 bits is possible). For $k = 27, 28$, the given range of u is such that r is 384-bit long.

6 Conclusion

Because of the Special Tower Number Field Sieve algorithm, the security of pairing-friendly curves should be reconsidered. We presented a new variant of STNFS for pairing-friendly curves constructed with the Brezing–Weng method, where the characteristic has a polynomial form. It does not apply to the modified Cocks–Pinch curves of [23]. We refine the analysis of Barbulescu, El Mrabet and Ghammam and present an updated short-list of secure pairing-friendly curves at the 128-bit security level. For embedding degrees from 10 to 16, we obtain curves so that the size of p^k is at least 3663 bits ($k = 11$) and at most 5376 bits (for BLS12 curves). The estimated cost of a DL computation with STNFS for these finite fields is between 2^{128} and 2^{148}. The fastest pairings are obtained with a BLS12 curve or a Fotiadis–Martindale curve of embedding degree 12, discriminant 3 and twist of degree 6 over a 446-bit prime. The additional curves of this paper have embedding degrees 10, 11, 13 and 14 and a twist of degree 2 for even embedding degrees. It was not sure by how much a prime embedding degree k allows to reduce the total size of p^k: for $k = 11$ the smallest possible p is 333 bit long, and for $k = 13$ p is 310 bit long. Although p is smaller than 446 bits, no twist is available with a prime embedding degree. For this reason, the efficiency of pairings on prime embedding degree curves does not seem competitive compared to BLS12 curves.

References

1. Aranha, D.F., Gouvêa, C.P.L.: RELIC is an Efficient LIbrary for Cryptography. `https://github.com/relic-toolkit/relic`
2. Aranha, D.F., Karabina, K., Longa, P., Gebotys, C.H., López-Hernández, J.C.: Faster explicit formulas for computing pairings over ordinary curves. In: Paterson, K.G. (ed.) EUROCRYPT 2011. LNCS, vol. 6632, pp. 48–68. Springer, Heidelberg (May 2011), `https://eprint.iacr.org/2010/526`
3. Barbulescu, R., Duquesne, S.: Updating key size estimations for pairings. Journal of Cryptology 32(4), 1298–1336 (Oct 2019), `https://doi.org/10.1007/s00145-018-9280-5`, `http://eprint.iacr.org/2017/334`
4. Barbulescu, R., El Mrabet, N., Ghammam, L.: A taxonomy of pairings, their security, their complexity. Cryptology ePrint Archive, Report 2019/485, revision of September 24 (2019), `https://eprint.iacr.org/2019/485`
5. Barbulescu, R., Gaudry, P., Kleinjung, T.: The tower number field sieve. In: Iwata, T., Cheon, J.H. (eds.) ASIACRYPT 2015, Part II. LNCS, vol. 9453, pp. 31–55. Springer, Heidelberg (Nov / Dec 2015), `https://eprint.iacr.org/2015/505`
6. Barreto, P.S.L.M., Lynn, B., Scott, M.: Constructing elliptic curves with prescribed embedding degrees. In: Cimato, S., Galdi, C., Persiano, G. (eds.) SCN 02. LNCS, vol. 2576, pp. 257–267. Springer, Heidelberg (Sep 2003)
7. Barreto, P.S.L.M., Naehrig, M.: Pairing-friendly elliptic curves of prime order. In: Preneel, B., Tavares, S. (eds.) SAC 2005. LNCS, vol. 3897, pp. 319–331. Springer, Heidelberg (Aug 2006)
8. Bowe, S.: BLS12-381: New zk-SNARK elliptic curve construction. Zcash blog (March 11 2017), `https://blog.z.cash/new-snark-curve/`

9. Bowe, S., Chiesa, A., Green, M., Miers, I., Mishra, P., Wu, H.: Zexe: Enabling decentralized private computation. Cryptology ePrint Archive, Report 2018/962 (2018), https://eprint.iacr.org/2018/962
10. Brezing, F., Weng, A.: Elliptic curves suitable for pairing based cryptography. Des. Codes Cryptogr. 37(1), 133–141 (2005), https://doi.org/10.1007/s10623-004-3808-4, https://doi.org/10.1007/s10623-004-3808-4, http://eprint.iacr.org/2003/143
11. Chatterjee, S., Menezes, A., Rodríguez-Henríquez, F.: On instantiating pairing-based protocols with elliptic curves of embedding degree one. IEEE Trans. Computers 66(6), 1061–1070 (2017), https://eprint.iacr.org/2016/403
12. Chatterjee, S., Sarkar, P., Barua, R.: Efficient computation of Tate pairing in projective coordinate over general characteristic fields. In: Park, C., Chee, S. (eds.) ICISC 04. LNCS, vol. 3506, pp. 168–181. Springer, Heidelberg (Dec 2005)
13. Chiesa, A., Chua, L., Weidner, M.: On cycles of pairing-friendly elliptic curves. SIAM Journal on Applied Algebra and Geometry 3(2), 175–192 (2019), https://doi.org/10.1137/18M1173708
14. Costello, C., Lange, T., Naehrig, M.: Faster pairing computations on curves with high-degree twists. In: Nguyen, P.Q., Pointcheval, D. (eds.) PKC 2010. LNCS, vol. 6056, pp. 224–242. Springer, Heidelberg (May 2010), https://eprint.iacr.org/2009/615
15. Costello, C., Lauter, K., Naehrig, M.: Attractive subfamilies of BLS curves for implementing high-security pairings. In: Bernstein, D.J., Chatterjee, S. (eds.) INDOCRYPT 2011. LNCS, vol. 7107, pp. 320–342. Springer, Heidelberg (Dec 2011), https://cryptosith.org/papers/blsfriendly-20111018.pdf
16. Euthereum, Go implementation. https://github.com/ethereum/go-ethereum/tree/master/crypto/bn256
17. Foster, K.: HT90 and "simplest" number fields. Illinois J. Math. 55(4), 1621–1655 (2011), http://arxiv.org/abs/1207.6099
18. Fotiadis, G., Konstantinou, E.: TNFS resistant families of pairing-friendly elliptic curves. Theoretical Computer Science (2019), https://doi.org/10.1016/j.tcs.2019.10.017, in press, preprint https://eprint.iacr.org/2018/1017
19. Fotiadis, G., Martindale, C.: Optimal TNFS-secure pairings on elliptic curves with composite embedding degree. Cryptology ePrint Archive, Report 2019/555 (2019), https://eprint.iacr.org/2019/555
20. Fouotsa, E., El Mrabet, N., Pecha, A.: Computing optimal ate pairings on elliptic curves with embedding degree 9, 15 and 27. Cryptology ePrint Archive, Report 2016/1187 (2016), https://eprint.iacr.org/2016/1187
21. Freeman, D., Scott, M., Teske, E.: A taxonomy of pairing-friendly elliptic curves. Journal of Cryptology 23(2), 224–280 (Apr 2010), https://eprint.iacr.org/2006/372
22. Galbraith, S.: Pairings. In: Blake, I.F., Seroussi, G., Smart, N.P. (eds.) Advances in Elliptic Curve Cryptography, p. 183–214. London Mathematical Society Lecture Note Series, Cambridge University Press (2005)
23. Guillevic, A., Masson, S., Thomé, E.: Cocks-Pinch curves of embedding degrees five to eight and optimal ate pairing computation. Cryptology ePrint Archive, Report 2019/431 (2019), https://eprint.iacr.org/2019/431
24. Guillevic, A., Singh, S.: On the alpha value of polynomials in the tower number field sieve algorithm. Cryptology ePrint Archive, Report 2019/885 (2019), https://eprint.iacr.org/2019/885

25. ISO: ISO/IEC 15946-5:2017 Information technology – Security techniques – Cryptographic techniques based on elliptic curves – Part 5: Elliptic curve generation, 2 edn. (August 2017), https://www.iso.org/standard/69726.html
26. Joux, A., Pierrot, C.: The special number field sieve in $\mathbb{F}_{p^n}$ - application to pairing-friendly constructions. In: Cao, Z., Zhang, F. (eds.) PAIRING 2013. LNCS, vol. 8365, pp. 45–61. Springer, Heidelberg (Nov 2014), https://eprint.iacr.org/2013/582
27. Kachisa, E.J., Schaefer, E.F., Scott, M.: Constructing Brezing-Weng pairing-friendly elliptic curves using elements in the cyclotomic field. In: Galbraith, S.D., Paterson, K.G. (eds.) PAIRING 2008. LNCS, vol. 5209, pp. 126–135. Springer, Heidelberg (Sep 2008)
28. Kim, T., Barbulescu, R.: Extended tower number field sieve: A new complexity for the medium prime case. In: Robshaw, M., Katz, J. (eds.) CRYPTO 2016, Part I. LNCS, vol. 9814, pp. 543–571. Springer, Heidelberg (Aug 2016), https://eprint.iacr.org/2015/1027
29. Kim, T., Jeong, J.: Extended tower number field sieve with application to finite fields of arbitrary composite extension degree. In: Fehr, S. (ed.) PKC 2017, Part I. LNCS, vol. 10174, pp. 388–408. Springer, Heidelberg (Mar 2017), https://eprint.iacr.org/2016/526
30. Lenstra, A.K., Verheul, E.R.: Selecting cryptographic key sizes. Journal of Cryptology 14(4), 255–293 (Sep 2001)
31. Menezes, A., Sarkar, P., Singh, S.: Challenges with assessing the impact of NFS advances on the security of pairing-based cryptography. In: Phan, R.C., Yung, M. (eds.) Mycrypt Conference, Revised Selected Papers. LNCS, vol. 10311, pp. 83–108. Springer, Kuala Lumpur, Malaysia (December 1-2 2016), http://eprint.iacr.org/2016/1102
32. Montgomery, P.L.: Five, six, and seven-term Karatsuba-like formulae. IEEE Transactions on Computers 54, 362–369 (March 2005), https://doi.org/10.1109/TC.2005.49
33. Pereira, G.C., Simplício, M.A., Naehrig, M., Barreto, P.S.: A family of implementation-friendly BN elliptic curves. Journal of Systems and Software 84(8), 1319 – 1326 (2011), http://www.sciencedirect.com/science/article/pii/S0164121211000914, https://eprint.iacr.org/2010/429
34. S. Yonezawa, T. Kobayashi, T.S.: Pairing-friendly curves, draft-yonezawa-pairing-friendly-curves-02. https://tools.ietf.org/html/draft-yonezawa-pairing-friendly-curves-02 (July 8 2019), IETF draft
35. Scott, M., Guillevic, A.: A new family of pairing-friendly elliptic curves. In: Budaghyan, L., Rodríguez-Henríquez, F. (eds.) Arithmetic of Finite Fields. pp. 43–57. Springer, Cham (2018), https://eprint.iacr.org/2018/193
36. Smith, B.: Easy scalar decompositions for efficient scalar multiplication on elliptic curves and genus 2 Jacobians. Contemporary mathematics 637, 15 (May 2015), https://hal.inria.fr/hal-00874925
37. Wahby, R.S., Boneh, D.: Fast and simple constant-time hashing to the BLS12-381 elliptic curve. IACR Transactions on Cryptographic Hardware and Embedded Systems 2019(4), 154–179 (Aug 2019), https://tches.iacr.org/index.php/TCHES/article/view/8348, https://doi.org/10.13154/tches.v2019.i4.154-179