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## - To cite this version:

Jean-Bastien Grill, Omar D Domingues, Pierre Ménard, Rémi Munos, Michal Valko. Planning in entropy-regularized Markov decision processes and games. Neural Information Processing Systems, 2019, Vancouver, Canada. hal-02387515

HAL Id: hal-02387515
https://inria.hal.science/hal-02387515
Submitted on 29 Nov 2019

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# Planning in entropy-regularized Markov decision processes and games 

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#### Abstract

We propose SmoothCruiser, a new planning algorithm for estimating the value function in entropy-regularized Markov decision processes and two-player games, given a generative model of the environment. SmoothCruiser makes use of the smoothness of the Bellman operator promoted by the regularization to achieve problem-independent sample complexity of order $\widetilde{\mathcal{O}}\left(1 / \varepsilon^{4}\right)$ for a desired accuracy $\varepsilon$, whereas for non-regularized settings there are no known algorithms with guaranteed polynomial sample complexity in the worst case.


## 1 Introduction

Planning with a generative model is thinking before acting. An agent thinks using a world model that it has built from prior experience [Sutton, 1991, Sutton and Barto, 2018]. In the present paper, we study planning in two types of environments, Markov decision processes (MDPs) and two-player turn-based zero-sum games. In both settings, agents interact with an environment by taking actions and receiving rewards. Each action changes the state of the environment and the agent aims to choose actions to maximize the sum of rewards. We assume that we are given a generative model of the environment, that takes as input a state and an action and returns a reward and a next state as output. Such generative models, called oracles, are typically built from known data and involve simulations, for example, a physics simulation. In many cases, simulations are costly. For example, simulations may require the computation of approximate solutions of differential equations or the discretization of continuous state spaces. Therefore, a smart algorithm makes only a small the number of oracles calls required to estimate the value of a state. The total number of oracle calls made by an algorithm is referred to as sample complexity.
The value of a state $s$, denoted by $V(s)$, the maximum of the sum of discounted rewards that can be obtained from that state. We want an algorithm that returns an estimate of precision $\varepsilon$ of the $V(s)$ for any fixed $s$ and has a low sample complexity, which should naturally be a function of $\varepsilon$. An agent can then use this algorithm to predict the value of the possible actions at any given state and choose the best one. The main advantage in estimating the value of a single given state $s$ at a time instead of the complete value function ${ }^{2} s \mapsto V(s)$ is that we can have algorithms whose sample complexity does not depend on the size of the state space, which is important when our state space is very large or continuous. On the other hand, the disadvantage is that the algorithm must be run each time a new state is encountered.

[^0]Our main contribution is an algorithm that estimates the value function in a given state in planning problems that satisfy specific smoothness conditions, which is the case when the rewards are regularized by adding an entropy term. We exploit this smoothness property to obtain a polynomial sample complexity of order $\widetilde{\mathcal{O}}\left(1 / \varepsilon^{4}\right)$ that is problem independent.

Related work Kearns et al. [1999] came up with a sparse sampling algorithm (SSA) for planning in MDPs with finite actions and arbitrary state spaces. SSA estimates the value of a state $s$ by building a sparse look-ahead tree starting from $s$. However, SSA achieves a sample complexity of $\mathcal{O}\left((1 / \varepsilon)^{\log (1 / \varepsilon)}\right)$, which is non-polynomial in $1 / \varepsilon$. SSA is slow since its search is uniform, i.e., it does not select actions adaptively. Walsh et al. [2010] gave an improved version of SSA with adaptive action selection, but its sample complexity is still non-polynomial. The UCT algorithm [Kocsis and Szepesvári, 2006], used for planning in MDPs and games, selects actions based on optimistic estimates of their values and has good empirical performance in several applications. However, the sample complexity of UCT can be worse than exponential in $1 / \varepsilon$ for some environments, which is mainly due to exploration issues [Coquelin and Munos, 2007]. Algorithms with sample complexities of order $\mathcal{O}\left(1 / \varepsilon^{d}\right)$, where $d$ is a problem-dependent quantity, have been proposed for deterministic dynamics [Hren and Munos, 2008], and in an open-loop ${ }^{3}$ setting [Bubeck and Munos, 2010, Leurent and Maillard, 2019, Bartlett et al., 2019], for bounded number of next states and a full MDP model is known [Buşoniu and Munos, 2012], for bounded number of next states in a finite-horizon setting [Feldman and Domshlak, 2014], for bounded number of next states [Szörényi et al., 2014], and for general MDPs [Grill et al., 2016]. In general, when the state space is infinite and the transitions are stochastic, the problem-dependent quantity $d$ can make the sample complexity guarantees exponential. For a related setting, when rewards are only obtained in the leaves of a fixed tree, Kaufmann and Koolen [2017] and Huang et al. [2017] present algorithms to identify the optimal action in a game based on best-arm identification tools.

Entropy regularization in MDPs and reinforcement learning have been employed in several commonly used algorithms. In the context of policy gradient algorithms, common examples are the TRPO algorithm [Schulman et al., 2015] which uses the Kullback-Leibler divergence between the current and the updated policy to constrain the gradient step sizes, the A3C algorithm [Mnih et al., 2016] that penalizes policies with low entropy to improve exploration, and the work of Neu et al. [2017] presenting a theoretical framework for entropy regularization using the joint state-action distribution. Formulations with entropy-augmented rewards, which is the case in our work, have been used to learn multi-modal policies to improve exploration and robustness [Haarnoja et al., 2017, 2018] and can also be related to policy gradient methods [Schulman et al., 2017]. Furthermore, Geist et al. [2019] propose a theory of regularized MDPs which includes entropy as a special case. Summing up, reinforcement learning knows how to employ entropy regularization. In this work, we tasked ourselves to give insights on why.

## 2 Setting and motivation

Both MDPs and two-player games can be formalized as a tuple $(\mathcal{S}, \mathcal{A}, P, R, \gamma)$, where $\mathcal{S}$ is the set of states, $\mathcal{A}$ is the set of actions, $P \triangleq\{P(\cdot \mid s, a)\}_{s, a \in \mathcal{S} \times \mathcal{A}}$ is a set of probability distributions over $\mathcal{S}, R: \mathcal{S} \times \mathcal{A} \rightarrow[0,1]$ is a (possibly random) reward function and $\gamma \in[0,1[$ is the known discount factor. In the MDP case, at each round $t$, an agent is at state $s$, chooses action $a$ and observes a reward $R(s, a)$ and a transition to a next state $z \sim P(\cdot \mid s, a)$. In the case of turn-based two-player games, there are two agents and, at each round $t$, an agent chooses an action, observes a reward and a transition; at round $t+1$ it's the other player's turn. This is equivalent to an MDP with an augmented state space $\mathcal{S}^{+} \triangleq \mathcal{S} \times\{1,2\}$ and transition probabilities such that $P((z, j) \mid(s, i), a)=0$ if $i=j$. We assume that the action space $\mathcal{A}$ is a finite with cardinality $K$ and the state space $\mathcal{S}$ has arbitrary (possibly infinite) cardinality.
Our objective is to find an algorithm that outputs a good estimate of the value $V(s)$ for any given state $s$ as quickly as possible. An agent can then use this algorithm to choose the best action in an MDP or a game. More precisely, for a state $s \in \mathcal{S}$ and given $\varepsilon>0$ and $\delta>0$, our goal is to compute an estimate $\widehat{V}(s)$ of $V(s)$ such that $\mathbb{P}[|\widehat{V}(s)-V(s)|>\varepsilon] \leq \delta$ with small number of oracle calls

[^1]required to compute this estimate. In our setting, we consider the case of entropy-regularized MDPs and games, where the objective is augmented with an entropy term.

### 2.1 Value functions

Markov decision process The policy $\pi$ of an agent is a function from $\mathcal{S}$ to $\mathcal{P}(\mathcal{A})$, the set of probability distributions over $\mathcal{A}$. We denote by $\pi(a \mid s)$ the probability of the agent choosing action $a$ at state $s$. In MDPs, the value function at a state $s, V(s)$, is defined as the supremum over all possible policies of the expected sum of discounted rewards obtained starting from $s$, which satisfies the Bellman equations [Puterman, 1994],

$$
\begin{equation*}
\forall s \in \mathcal{S}, V(s)=\max _{\pi(\cdot \mid s) \in \mathcal{P}(\mathcal{A})} \mathbb{E}[R(s, a)+\gamma V(z)], a \sim \pi(\cdot \mid s), z \sim P(\cdot \mid s, a) \tag{1}
\end{equation*}
$$

Two-player turn-based zero-sum games In this case, there are two agents (1 and 2), each one with its own policy and different goals. If the policy of Agent 2 is fixed, Agent 1 aims to find a policy that maximizes the sum of discounted rewards. Conversely, if the policy of Agent 1 is fixed, Agent 2 aims to find a policy that minimizes this sum. Optimal strategies for both agents can be shown to exist and for any $(s, i) \in \mathcal{S}^{+} \triangleq \mathcal{S} \times\{1,2\}$, the value function is defined as [Hansen et al., 2013]

$$
V(s, i) \triangleq \begin{cases}\max _{\pi(\cdot \mid s) \in \mathcal{P}(\mathcal{A})} \mathbb{E}[R((s, i), a)+\gamma V(z, j)], & \text { if } i=1  \tag{2}\\ \min _{\pi(\cdot \mid s) \in \mathcal{P}(\mathcal{A})} \mathbb{E}[R((s, i), a)+\gamma V(z, j)], & \text { if } i=2\end{cases}
$$

with $a \sim \pi(\cdot \mid s)$ and $(z, j) \sim P(\cdot \mid(s, i), a)$. In this case, the function $s \mapsto V(s, i)$ is the optimal value function for Agent $i$ when the other agent follows its optimal strategy.

Entropy-regularized value functions Consider a regularization factor $\lambda>0$. In the case of MDPs, when rewards are augmented by an entropy term, the value function at state $s$ is given by [Haarnoja et al., 2017, Dai et al., 2018, Geist et al., 2019]

$$
\begin{align*}
V(s) & \triangleq \max _{\pi(\cdot \mid s) \in \mathcal{P}(\mathcal{A})}\{\mathbb{E}[R(s, a)+\gamma V(z)]+\lambda \mathcal{H}(\pi(\cdot \mid s))\}, a \sim \pi(\cdot \mid s), z \sim P(\cdot \mid s, a) \\
& =\lambda \log \sum_{a \in \mathcal{A}} \exp \left(\frac{1}{\lambda} \mathbb{E}[R(s, a)+\gamma V(z)]\right), z \sim P(\cdot \mid s, a) \tag{3}
\end{align*}
$$

where $\mathcal{H}(\pi(\cdot \mid s))$ is the entropy of the probability distribution $\pi(\cdot \mid s) \in \mathcal{P}(\mathcal{A})$.
The function $\operatorname{LogSumExp} \operatorname{En}_{\lambda}: \mathbb{R}^{K} \rightarrow \mathbb{R}$, defined as $\operatorname{LogSumExp}{ }_{\lambda}(x) \triangleq \lambda \log \sum_{i=1}^{K} \exp \left(x_{i} / \lambda\right)$, is a smooth approximation of the max function, since $\left\|\max -\log \operatorname{SumExp}_{\lambda}\right\|_{\infty} \leq \lambda \log K$. Similarly, the function $-\operatorname{LogSumExp}{ }_{-\lambda}$ is a smooth approximation of the min function. This allows us to define the regularized version of the value function for turn-based two player games, in which both players have regularized rewards, by replacing the max and the min in Equation 2 by their smooth approximations.
For any state $s$, let $F_{s} \triangleq \operatorname{LogSumExp}_{\lambda}$ or $F_{s} \triangleq-\operatorname{LogSumExp}_{-\lambda}$ depending on $s$. Both for MDPs and games, we can write the entropy-regularized value functions as

$$
\begin{equation*}
V(s)=F_{s}\left(Q_{s}\right), \text { with } Q_{s}(a) \triangleq \mathbb{E}[R(s, a)+\gamma V(z)], z \sim P(\cdot \mid s, a) \tag{4}
\end{equation*}
$$

where $Q_{s} \triangleq\left(Q_{s}(a)\right)_{a \in \mathcal{A}}$, the $Q$ function at state $s$, is a vector in $\mathbb{R}^{K}$ representing the value of each action. The function $F_{s}$ is the Bellman operator at state $s$, which becomes smooth due to the entropy regularization.

Useful properties Our algorithm exploits the smoothness property of $F_{s}$ defined above. In particular, these functions are $L$-smooth, that is, for any $Q, Q^{\prime} \in \mathbb{R}^{K}$, we have

$$
\begin{equation*}
\left|F_{s}(Q)-F_{s}\left(Q^{\prime}\right)-\left(Q-Q^{\prime}\right)^{\top} \nabla F_{s}\left(Q^{\prime}\right)\right| \leq L\left\|Q-Q^{\prime}\right\|_{2}^{2}, \text { with } L=1 / \lambda \tag{5}
\end{equation*}
$$

Furthermore, the functions $F_{s}$ have two important properties: $\nabla F_{s}(Q)^{4} \succeq 0$ and $\left\|\nabla F_{s}(Q)\right\|_{1}=1$ for all $Q \in \mathbb{R}^{K}$. This implies that the gradient $\nabla F_{s}(Q)$ defines a probability distribution.

[^2]Assumptions We assume that $\mathcal{S}, \mathcal{A}, \lambda$, and $\gamma$ are given to the learner. Moreover, we assume that we can access a generative model, the oracle, from which we can get reward and transition samples from arbitrary state-action pairs. Formally, when called with parameter $(s, a) \in \mathcal{S} \times \mathcal{A}$, the oracle outputs a new random variable $(R, Z)$ independent from any other outputs received from the generative model so far such that $Z \sim P(\cdot \mid s, a)$ and $R$ has same distribution as $R(s, a)$. We denote a call to the oracle as $R, Z \leftarrow \operatorname{oracle}(s, a)$.

### 2.2 Using regularization for the polynomial sample complexity

To pave the road for SmoothCruiser, we consider two extreme cases, based on the strength of the regularization:

1. Strong regularization In this case, $\lambda \rightarrow \infty$ and $L=0$, that is, $F_{s}$ is linear for all $s$ : $F_{s}(x)=w_{s}^{\top} x$, with $\left\|w_{s}\right\|_{1}=1, w_{s} \in \mathbb{R}^{k}$ and $w_{s} \succeq 0$,
2. No regularization In this case, $\lambda=0$ and $L \rightarrow \infty$, that is, $F_{s}$ cannot be well approximated by a linear function. ${ }^{6}$

In the strongly regularized case, we can approximate the value $V(s)$ with $\widetilde{\mathcal{O}}\left(1 / \varepsilon^{2}\right)$ oracle calls. This the linearity of $F_{s}$, since the value function can be written as $V(s)=\mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^{t} R\left(S_{t}, A_{t}\right) \mid S_{0}=s\right]$ where $A_{t}$ is distributed according to the probability vector $w_{S_{t}}$. As a result, $V(s)$ can be estimated by Monte-Carlo sampling of trajectories.
With no regularization, we can apply a simple adaptation of the sparse sampling algorithm of Kearns et al. [1999] that we briefly describe. Assume that we have an subroutine that provides an approximation of the value function with precision $\varepsilon / \sqrt{\gamma}$, denoted by $\widehat{V}_{\varepsilon / \sqrt{\gamma}}(s)$, for any $s$. We can call this subroutine several times as well as the oracle to get improved estimate $\widehat{V}$ defined as

$$
\widehat{V}(s)=F_{s}\left(\widehat{Q}_{s}\right) \quad \text { with } \quad \widehat{Q}_{s}(a) \leftarrow \frac{1}{N} \sum_{i=1}^{N}\left[r_{i}(s, a)+\gamma \widehat{V}_{\varepsilon / \sqrt{\gamma}}\left(z_{i}\right)\right]
$$

where $r_{i}(s, a)$ and $z_{i}$ are rewards and next states sampled by calling the oracle with parameters $(s, a)$. By Hoeffding's inequality, we can choose $N=\mathcal{O}\left(1 / \varepsilon^{2}\right)$ such that $\widehat{V}(s)$ is an approximation of $V(s)$ with precision $\varepsilon$ with high probability. By applying this idea recursively, we start with $\widehat{V}=0$, which is an approximation of the value function with precision $1 /(1-\gamma)$, and progressively improve the estimates towards a desired precision $\varepsilon$, which can be reached at a recursion depth of $H=\mathcal{O}(\log (1 / \varepsilon))$. Following the same reasoning as Kearns et al. [1999], this approach has a sample complexity of $\mathcal{O}\left((1 / \varepsilon)^{\log (1 / \varepsilon)}\right)$ : to estimate the value at a given recursion depth, we make $\mathcal{O}\left(1 / \varepsilon^{2}\right)$ recursive calls and stop once we reach the maximum depth, resulting in a sample complexity of

$$
\underbrace{\frac{1}{\varepsilon^{2}} \times \cdots \times \frac{1}{\varepsilon^{2}}}_{\mathcal{O}(\log (1 / \varepsilon)) \text { times }}=\left(\frac{1}{\varepsilon}\right)^{\mathcal{O}\left(\log \left(\frac{1}{\varepsilon}\right)\right)}
$$

In the next section, we provide SmoothCruiser (Algorithm 1), that uses the assumption that the functions $F_{s}$ are $L$-smooth with $0<L<\infty$ to interpolate between the two cases above and obtain a sample complexity of $\widetilde{\mathcal{O}}\left(1 / \varepsilon^{4}\right)$.

## 3 SmoothCruiser

We now describe our planning algorithm. Its building blocks are two procedures, sampleV (Algorithm 2) and estimateQ (Algorithm 3) that recursively call each other. The procedure sampleV returns a noisy estimate of $V(s)$ with a bias bounded by $\varepsilon$. The procedure estimateQ averages the outputs of several calls to sampleV to obtain an estimate $\widehat{Q}_{s}$ that is an approximation of $Q_{s}$ with precision $\varepsilon$ with high probability. Finally, SmoothCruiser calls estimateQ $(s, \varepsilon)$ to obtain $\widehat{Q}_{s}$ and outputs $\widehat{V}(s)=F_{s}\left(\widehat{Q}_{s}\right)$. Using the assumption that $F_{s}$ is 1-Lipschitz, we can show that $\widehat{V}(s)$ is an approximation of $V(s)$ with precision $\varepsilon$. Figure 1 illustrates a call to SmoothCruiser.

[^3]

Figure 1: Visualization of a call to SmoothCruiser $\left(s_{0}, \varepsilon_{0}, \delta^{\prime}\right)$.

### 3.1 Smooth sailing

The most important part of the algorithm is the procedure sampleV, that returns a low-bias estimate of the value function. Having the estimate of the value function, the procedure estimateQ averages the outputs of sampleV to obtain a good estimate of the $Q$ function with high probability. The main idea of sampleV is to first compute an estimate of preci$\operatorname{sion} \mathcal{O}(\sqrt{\varepsilon})$ of the value of each action $\left\{\widehat{Q}_{s}(a)\right\}_{a \in \mathcal{A}}$

```
Algorithm 1 SmoothCruiser
    Input: \(\left(s, \varepsilon, \delta^{\prime}\right) \in \mathcal{S} \times \mathbb{R}_{+} \times \mathbb{R}_{+}\)
    \(M_{\lambda} \leftarrow \sup _{s \in \mathcal{S}}\left|F_{s}(0)\right|=\lambda \log K\)
    \(\kappa \leftarrow(1-\sqrt{\gamma}) /(K L)\)
    Set \(\delta^{\prime}, \kappa\), and \(M_{\lambda}\) as global parameters
    \(\widehat{Q}_{s} \leftarrow \operatorname{estimateQ}(s, \varepsilon)\)
    Output: \(F_{s}\left(\widehat{Q}_{s}\right)\)
``` to linearly approximate the function \(F_{s}\) around \(\widehat{Q}_{s}\). The local approximation of \(F_{s}\) around \(\widehat{Q}_{s}\) is subsequently used to estimate the value of \(s\) with a better precision, of order \(\mathcal{O}(\varepsilon)\), which is possible due to the smoothness of \(F_{s}\).
```

Algorithm 2 sampleV
Input: $(s, \varepsilon) \in \mathcal{S} \times \mathbb{R}_{+}$
if $\varepsilon \geq\left(1+M_{\lambda}\right) /(1-\gamma)$ then
Output: 0
else if $\varepsilon \geq \kappa$ then
$\widehat{Q}_{s} \leftarrow \operatorname{estimateQ}(s, \varepsilon)$
Output: $F_{s}\left(\widehat{Q}_{s}\right)$
else if $\varepsilon<\kappa$ then
$\widehat{Q}_{s} \leftarrow \operatorname{estimateQ}(s, \sqrt{\kappa \varepsilon})$
$A \leftarrow$ action drawn from $\nabla F_{s}\left(\widehat{Q}_{s}\right)$
$(R, Z) \leftarrow \operatorname{oracle}(s, A)$
$\widehat{V} \leftarrow \operatorname{sampleV}(Z, \varepsilon / \sqrt{\gamma})$
Output:
$F_{s}\left(\widehat{Q}_{s}\right)-\widehat{Q}_{s}^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)+(R+\gamma \widehat{V})$
end if

```
```

Algorithm 3 estimateQ
Input: $(s, \varepsilon)$
$N(\varepsilon) \leftarrow\left[\frac{18\left(1+M_{\lambda}\right)^{2}}{(1-\gamma)^{4}(1-\sqrt{\gamma})^{2}} \frac{\log \left(2 K / \delta^{\prime}\right)}{\varepsilon^{2}}\right]$
for $a \in \mathcal{A}$ do
$q_{i} \leftarrow 0$ for $i \in 1, \ldots, N(\varepsilon)$
for $i \in 1, \ldots, N(\varepsilon)$ do
$(R, Z) \leftarrow \operatorname{oracle}(s, a)$.
$\widehat{V} \leftarrow \operatorname{sampleV}(Z, \varepsilon / \sqrt{\gamma})$
$q_{i} \leftarrow R+\gamma \widehat{V}$
end for
$\widehat{Q}_{s}(a) \leftarrow \operatorname{mean}\left(q_{1}, \ldots, q_{N}\right)$
$\operatorname{clip} \widehat{Q}_{s}(a)$ to $\left[0,\left(1+M_{\lambda}\right) /(1-\gamma)\right]$
$\widehat{Q}_{s}(a) \leftarrow \max \left(0, \widehat{Q}_{s}(a)\right)$
$\widehat{Q}_{s}(a) \leftarrow \min \left(\left(1+M_{\lambda}\right) /(1-\gamma), \widehat{Q}_{s}(a)\right)$
end for
Output: $\widehat{Q}_{s}(a)$

```

For a target accuracy \(\varepsilon\) at state \(s\), sampleV distinguishes three cases, based on a reference threshold \(\kappa \triangleq(1-\sqrt{\gamma}) /(K L)\), which is the maximum value of \(\varepsilon\) for which we can compute a good estimate of the value function using linear approximations of \(F_{s}\).
- First, if \(\varepsilon \geq(1+\lambda \log K) /(1-\gamma)\), then 0 is a valid output, since \(V(s)\) is bounded by \((1+\lambda \log K) /(1-\gamma)\). This case furthermore ensures that our algorithm terminates, since the recursive calls are made with increasing values of \(\varepsilon\).
- Second, if \(\kappa \leq \varepsilon \leq(1+\lambda \log K) /(1-\gamma)\), we run \(F_{s}(\) estimateQ \((s, \varepsilon))\) in which for each action, both the oracle and sampleV are called \(\mathcal{O}\left(1 / \varepsilon^{2}\right)\) times in order to return \(\widehat{V}(s)\) which is with high probability an \(\varepsilon\)-approximation of \(V(s)\).
- Finally, if \(\varepsilon<\kappa\), we take advantage of the smoothness of \(F_{s}\) to compute an \(\varepsilon\)-approximation of \(V(s)\) in a more efficient way than calling the oracle and sampleV \(\mathcal{O}\left(1 / \varepsilon^{2}\right)\) times. We achieve it by calling estimateQ with a precision \(\sqrt{\kappa \varepsilon}\) instead of \(\varepsilon\), which requires \(\mathcal{O}(1 / \varepsilon)\) calls instead.

\subsection*{3.2 Smoothness guarantee an improved sample complexity}

In this part, we describe the key ideas that allows us to exploit the smoothness of the Bellman operator to obtain a better sample complexity. Notice that when \(\varepsilon<\kappa\), the procedure estimateQ is called to obtain an estimate \(\widehat{Q}_{s}\) such that
\[
\left\|\widehat{Q}_{s}-Q_{s}\right\|_{2}=\mathcal{O}(\sqrt{\varepsilon / L})
\]

The procedure sampleV then continues with computing a linear approximation of \(F_{s}\left(Q_{s}\right)\) around \(\widehat{Q}_{s}\). Using the \(L\)-smoothness of \(F_{s}\), we guarantee the \(\varepsilon\)-approximation,
\[
\left|F_{s}\left(Q_{s}\right)-\left\{F_{s}\left(\widehat{Q}_{s}\right)+\left(Q_{s}-\widehat{Q}_{s}\right)^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)\right\}\right| \leq L\left\|\widehat{Q}_{s}-Q_{s}\right\|_{2}^{2}=\mathcal{O}(\varepsilon)
\]

We wish to output this linear approximation, but we need to handle the fact that the vector \(Q_{s}\) (the true \(Q\)-function at \(s\) ) is unknown. Notice that the vector \(\nabla F_{s}\left(\widehat{Q}_{s}\right)\) represents a probability distribution. The term \(Q_{s}^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)\) in the linear approximation of \(F_{s}\left(Q_{s}\right)\) above can be expressed as
\[
Q_{s}^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)=\mathbb{E}\left[Q_{s}(A) \mid \widehat{Q}_{s}\right], \text { with } A \sim \nabla F_{s}\left(\widehat{Q}_{s}\right)
\]

Therefore, we can build a low-bias estimate of \(Q_{s}^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)\) from estimating only \(Q_{s}(A)\) :
- sample action \(A \sim \nabla F_{s}\left(\widehat{Q}_{s}\right)\)
- call the generative model to sample a reward and a next state \(R_{s, A}, Z_{s, A} \leftarrow \operatorname{oracle}(s, A)\)
- obtain an \(\mathcal{O}(\varepsilon)\)-approximation of \(Q_{s}(A): \widetilde{Q}(A)=R_{s, A}+\gamma \operatorname{sampleV}\left(Z_{s, A}, \varepsilon / \sqrt{\gamma}\right)\)
- output \(\widehat{V}(s)=F_{s}\left(\widehat{Q}_{s}\right)-\widehat{Q}_{s}^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)+\widetilde{Q}(A)\)

We show that \(\widehat{V}(s)\) is an \(\varepsilon\)-approximation of the true value function \(V(s)\). The benefit of such approach is that we can call estimateQ with a precision \(\mathcal{O}(\sqrt{\varepsilon})\) instead of \(\mathcal{O}(\varepsilon)\), which thanks to the smoothness of \(F_{s}\), reduces the sample complexity. In particular, one call to \(\operatorname{sampleV}(s, \varepsilon)\) will make \(\mathcal{O}(1 / \varepsilon)\) recursive calls to sampleV \((s, \mathcal{O}(\sqrt{\varepsilon}))\), and the total number of calls to sampleV behaves as
\[
\frac{1}{\varepsilon} \times \frac{1}{\varepsilon^{1 / 2}} \times \frac{1}{\varepsilon^{1 / 4}} \times \cdots \leq \frac{1}{\varepsilon^{2}}
\]

Therefore, the number of sampleV calls made by SmoothCruiser is of order \(\mathcal{O}\left(1 / \varepsilon^{2}\right)\), which implies that the total sample complexity is of \(\mathcal{O}\left(1 / \varepsilon^{4}\right)\).

\subsection*{3.3 Comparison to Monte-Carlo tree search}

Several planning algorithms are based on MonteCarlo tree search (MCTS, Coulom, 2007, Kocsis and Szepesvári, 2006). Algorithm 4 gives a template for MCTS, which uses the procedure search that calls selectAction and evaluateLeaf. Algorithm 5, search, returns an estimate of the value function;
\begin{tabular}{l} 
Algorithm 4 genericMCTS \\
\begin{tabular}{l} 
Input: state \(s\) \\
repeat search \((s, 0)\) \\
until timeout \\
Output: estimate of best action or value.
\end{tabular} \\
\hline
\end{tabular} selectAction selects the action to be executed (also called tree policy); and evaluateLeaf returns an estimate of the value of a leaf. We now provide the analogies that make it possible to see SmoothCruiser as an MCTS algorithm:
- sampleV corresponds to the function search
- selectAction is implemented by calling estimateQ to compute \(\widehat{Q}_{s}\), followed by sampling an action with probability proportional to \(\nabla F_{s}\left(\widehat{Q}_{s}\right)\)
- evaluateLeaf is implemented using the sparse sampling strategy of Kearns et al. [1999], if we see leaves as the nodes reached when \(\varepsilon \geq \kappa\)

\section*{4 Theoretical guarantees}

In Theorem 1 we bound the sample complexity Note that SmoothCruiser is non-adaptive, hence its sample complexity is deterministic and problem independent. Indeed, since our algorithm is agnostic to the output of the oracle, it performs the same number of oracle calls for any given \(\varepsilon\) and \(\delta^{\prime}\), regardless of the random outcome of these calls.
```

Algorithm 5 search
Input: state $s$, depth $d$
if $d>d_{\text {max }}$ then
Output: evaluateLeaf $(s)$
end if
$a \leftarrow \operatorname{selectAction}(s, d)$
$R, Z \leftarrow \operatorname{oracle}(s, a)$
Output: $R+\gamma \operatorname{search}(Z, d+1)$

```

Theorem 1. Let \(n\left(\varepsilon, \delta^{\prime}\right)\) be the number of oracle calls before SmoothCruiser terminates. For any state \(s \in \mathcal{S}\) and \(\varepsilon, \delta^{\prime}>0\),
\[
n\left(\varepsilon, \delta^{\prime}\right) \leq \frac{c_{1}}{\varepsilon^{4}} \log \left(\frac{c_{2}}{\delta^{\prime}}\right)\left[c_{3} \log \left(\frac{c_{4}}{\varepsilon}\right)\right]^{\log _{2}\left(c_{5}\left(\log \left(\frac{c_{2}}{\delta^{\prime}}\right)\right)\right)}=\widetilde{\mathcal{O}}\left(\frac{1}{\varepsilon^{4}}\right)
\]
where \(c_{1}, c_{2}, c_{3}, c_{4}\), and \(c_{5}\) are constants that depend only on \(K, L\), and \(\gamma\).
The proof of Theorem 1 with the exact constants is in the appendix. In Theorem 2, we provide our consistency result, stating that the output of SmoothCruiser applied to a state \(s \in \mathcal{S}\) is a good approximation of \(V(s)\) with high probability.
Theorem 2. For any \(s \in \mathcal{S}, \varepsilon>0\), and \(\delta>0\), there exists a \(\delta^{\prime}\) that depends on \(\varepsilon\) and \(\delta\) such that the output \(\widehat{V}(s)\) of SmoothCruiser \(\left(s, \varepsilon, \delta^{\prime}\right)\) satisfies
\[
\mathbb{P}[|\widehat{V}(s)-V(s)|>\varepsilon] \leq \delta
\]
and such that \(n\left(\varepsilon, \delta^{\prime}\right)=\mathcal{O}\left(1 / \varepsilon^{4+c}\right)\) for any \(c>0\).

More precisely, in the proof of Theorem 2, we establish that
\[
\mathbb{P}[|\widehat{V}(s)-V(s)|>\varepsilon] \leq \delta^{\prime} n\left(\varepsilon, \delta^{\prime}\right)
\]

Therefore, for any parameter \(\delta^{\prime}\) satisfying \(\delta^{\prime} n\left(\varepsilon, \delta^{\prime}\right) \leq \delta\), SmoothCruiser with parameters \(\varepsilon\) and \(\delta^{\prime}\) provides an approximation of \(V(s)\) which is \((\varepsilon, \delta)\) correct.

Impact of regularization constant For a regularization constant \(\lambda\), the smoothness constant is \(L=1 / \lambda\). in Theorem 1 we did not make the dependence on \(L\) explicit to preserve simplicity. However, it easy to analyze the sample complexity in the two limits:

> strong regularization \(L \rightarrow 0\) and \(F_{s}\) is linear
> no regularization \(L \rightarrow \infty\) and \(F_{s}\) is not smooth

As \(L \rightarrow 0\), the condition \(\kappa \leq \varepsilon \leq(1+\lambda \log K) /(1-\gamma)\) will be met less and eventually the algorithm will sample \(N=\mathcal{O}\left(1 / \varepsilon^{2}\right)\) trajectories, which implies a sample complexity of order \(\mathcal{O}\left(1 / \varepsilon^{2}\right)\). On the other hand, as \(L\) goes to \(\infty\), the condition \(\varepsilon<\kappa\) will be met less and the algorithm eventually runs a uniform sampling strategy of Kearns et al. [1999], which results in a sample complexity of order \(\mathcal{O}\left((1 / \varepsilon)^{\log (1 / \varepsilon)}\right)\), which is non-polynomial in \(1 / \varepsilon\).

Let \(V_{\lambda}(s)\) be the entropy regularized value function and \(V_{0}(s)\) be its non-regularized version. Since \(F_{s}\) is 1-Lipschitz and \(\left\|\operatorname{LogSumExp}{ }_{\lambda}-\max \right\|_{\infty} \leq \lambda \log K\), we can prove that \(\sup _{s}\left|V_{\lambda}(s)-V_{0}(s)\right| \leq\) \(\lambda \log K /(1-\gamma)\). Thus, we can interpret \(V_{\lambda}(s)\) as an approximate value function which we can estimate faster.

Comparison to lower bound For non-regularized problems, Kearns et al. [1999] prove a sample complexity lower bound of \(\Omega\left((1 / \varepsilon)^{1 / \log (1 / \gamma)}\right)\), which is polynomial in \(1 / \varepsilon\), but its exponent grows as \(\gamma\) approaches 1. For regularized problems, Theorem 1 shows that the sample complexity is polynomial with an exponent that is independent of \(\gamma\). Hence, when \(\gamma\) is close to 1 , regularization gives us a better asymptotic behavior with respect to \(1 / \varepsilon\) than the lower bound for the non-regularized case, although we are not estimating the same value.

\section*{5 Generalization of SmoothCruiser}

Consider the general definition of value functions in Equation 4. Although we focused on the case where \(F_{s}\) is the LogSumExp function, which arises as a consequence of entropy regularization, our theoretical results hold for any set of functions \(\left\{F_{s}\right\}_{s \in \mathcal{S}}\) that for any \(s\) satisfy the following conditions:
1. \(F_{s}\) is differentiable
2. \(\forall Q \in \mathbb{R}^{K}, 0<\left\|\nabla F_{s}(Q)\right\|_{1} \leq 1\)
3. (nonnegative gradient) \(\forall Q \in \mathbb{R}^{K}, \nabla F_{s}(Q) \succeq 0\)
4. ( \(L\)-smooth) there exists \(L \geq 0\) such that for any \(Q, Q^{\prime} \in \mathbb{R}^{K}\)
\[
\left|F_{s}(Q)-F_{s}\left(Q^{\prime}\right)-\left(Q-Q^{\prime}\right)^{\top} \nabla F_{s}\left(Q^{\prime}\right)\right| \leq L\left\|Q-Q^{\prime}\right\|_{2}^{2}
\]

For the more general definition above, we need to make two simple modifications of the procedure sampleV. When \(\varepsilon<\kappa\), the action \(A\) in sampleV is sampled according to
\[
A \sim \frac{\nabla F_{s}\left(\widehat{Q}_{s}\right)}{\left\|\nabla F_{s}\left(\widehat{Q}_{s}\right)\right\|_{1}}
\]
and its output is modified to
\[
F_{s}\left(\widehat{Q}_{s}\right)-\widehat{Q}_{s}^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)+(R+\gamma \widehat{v})\left\|\nabla F_{s}(\widehat{Q})\right\|_{1} .
\]

In particular, SmoothCruiser can be used for more general regularization schemes, as long as the Bellman operators satisfy the assumptions above. One such example is presented in Appendix E.

\section*{6 Conclusion}

We provided SmoothCruiser that estimates the value function of MDPs and discounted games defined through smooth approximations of the optimal Bellman operator, which is the case in entropyregularized value functions. More generally, our algorithm can also be used when value functions are defined through any smooth Bellman operator with nonnegative gradients. We showed that our algorithm has a polynomial sample complexity of \(\widetilde{\mathcal{O}}\left(1 / \varepsilon^{4}\right)\), where \(\varepsilon\) is the desired precision. This guarantee is problem independent and holds for state spaces of arbitrary cardinality.
One interesting interpretation of our results is that computing entropy-regularized value functions, which are commonly employed for reinforcement learning, can be seen as a smooth relaxation of a planning problem for which we can obtain a much better sample complexity in terms of the required precision \(\varepsilon\). Unsurprisingly, when the regularization tends to zero, we recover the well-known non-polynomial bound \(\mathcal{O}\left((1 / \varepsilon)^{\log (1 / \varepsilon)}\right)\) of Kearns et al. [1999]. Hence, an interesting direction for future work is to study adaptive regularization schemes in order to accelerate planning algorithms. Although SmoothCruiser makes large amount of recursive calls, which makes it impractical in most situations, we believe it might help us to understand how regularization speeds planning and inspire more practical algorithms. This might be possible by exploiting its similarities to Monte-Carlo tree search that we have outlined above.

Acknowledgments The research presented was supported by European CHIST-ERA project DELTA, French Ministry of Higher Education and Research, Nord-Pas-de-Calais Regional Council, Inria and Otto-von-Guericke-Universität Magdeburg associated-team north-European project Allocate, and French National Research Agency project BoB (grant n.ANR-16-CE23-0003), FMJH Program PGMO with the support of this program from Criteo.

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\section*{A Preliminaries}

\section*{A. 1 General definition of value functions}

We consider the general definition of value functions in Equation 4 and we assume that all the functions \(F_{s}\) satisfy
1. \(F_{s}\) is differentiable,
2. \(\forall x \in \mathbb{R}^{K}, 0<\left\|\nabla F_{s}(x)\right\|_{1} \leq 1\),
3. (nonnegative gradient) \(\forall x \in \mathbb{R}^{K}, \nabla F_{s}(x) \succeq 0\),
4. ( \(L\)-smooth) There exists \(L \geq 0\) such that for any \(x_{0}, x \in \mathbb{R}^{K}\),
\[
\left|F_{s}(x)-F_{s}\left(x_{0}\right)-\left(x-x_{0}\right)^{\top} \nabla F_{s}\left(x_{0}\right)\right| \leq L\left\|x-x_{0}\right\|_{2}^{2}
\]
which is the case for the functions \(\operatorname{LogSumExp}{ }_{\lambda}\) and \(-\operatorname{LogSumExp}_{-\lambda}\) that we study in the present paper. In particular, the second requirement implies that \(F_{s}\) is 1-Lipschitz,
\[
\forall x, y \in \mathbb{R}^{K},\left|F_{s}(x)-F_{s}(y)\right| \leq\|x-y\|_{\infty}
\]

For this more general definition, we modify the output of sampleV when \(\varepsilon<\kappa\) to
\[
\text { output }=F_{s}\left(\widehat{Q}_{s}\right)-\left(\widehat{Q}_{s}\right)^{\top} \nabla F_{s}\left(\widehat{Q}_{s}\right)+(R+\gamma \widehat{v})\left\|\nabla F_{s}(\widehat{Q})\right\|_{1}
\]
and the action sampled in sampleV is sampled according to
\[
A \sim \frac{\nabla F_{s}\left(\widehat{Q}_{s}\right)}{\left\|\nabla F_{s}\left(\widehat{Q}_{s}\right)\right\|_{1}} \quad \text { instead of } \quad A \sim \nabla F_{s}\left(\widehat{Q}_{s}\right)
\]

\section*{A. 2 Other definitions}

The constant \(M_{\lambda}\) is defined as
\[
M_{\lambda} \triangleq \sup _{s \in \mathcal{S}}\left|F_{s}(0)\right|
\]

For any \(c \in \mathbb{R}\), the function \(\operatorname{clip}_{c}: \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}\) is defined component-wise as
\[
\operatorname{clip}_{c}(x)_{i}= \begin{cases}0 & \text { if } x_{i} \leq 0 \\ x_{i} & \text { if }-c<x_{i}<c \\ c & \text { if } x \geq c\end{cases}
\]

\section*{B Sample complexity}

Theorem 1. Let \(n\left(\varepsilon, \delta^{\prime}\right)\) be the number of oracle calls before SmoothCruiser terminates. For any state \(s \in \mathcal{S}\) and \(\varepsilon, \delta^{\prime}>0\),
\[
n\left(\varepsilon, \delta^{\prime}\right) \leq \frac{c_{1}}{\varepsilon^{4}} \log \left(\frac{c_{2}}{\delta^{\prime}}\right)\left[c_{3} \log \left(\frac{c_{4}}{\varepsilon}\right)\right]^{\log _{2}\left(c_{5}\left(\log \left(\frac{c_{2}}{\delta^{\prime}}\right)\right)\right)}=\widetilde{\mathcal{O}}\left(\frac{1}{\varepsilon^{4}}\right)
\]
where \(c_{1}, c_{2}, c_{3}, c_{4}\), and \(c_{5}\) are constants that depend only on \(K, L\), and \(\gamma\).
To bound the sample complexity, we make the following steps.
- Proposition 1 bounds the number of recursive calls of sampleV in the uniform sampling phase \((\varepsilon \geq \kappa)\) and is similar to the results of Kearns et al. [1999].
- Lemma 1 bounds the number of recursive calls of sampleV when \(\varepsilon<\kappa\).
- By noticing that the number of recursive calls of sampleV is equal to the number of oracle calls, we bound the sample complexity of SmoothCruiser in Theorem 1.

Let \(n_{\text {samplev }}\left(s, \varepsilon, \delta^{\prime}\right)\) be the total number of recursive calls to sampleV after an initial call with parameters \((s, \varepsilon)\), and including the initial call. Since this number does not depend on the state \(s\), we denote it by \(n_{\text {sampleV }}\left(\varepsilon, \delta^{\prime}\right)\).
Proposition 1. Let \(\varepsilon \geq \kappa\). For all \(h \in \mathbb{N}, \forall \varepsilon\) such that \(\frac{\left(1+M_{\lambda}\right) \sqrt{\gamma}^{h}}{1-\gamma} \leq \varepsilon \leq \frac{1+M_{\lambda}}{1-\gamma}\), we have
\[
\begin{aligned}
n_{\text {sampleV }}\left(\varepsilon, \delta^{\prime}\right) & \leq \gamma^{\frac{1}{2} H(\varepsilon)(H(\varepsilon)-1)}\left(\frac{2 \alpha\left(\delta^{\prime}\right)}{\varepsilon^{2}}\right)^{H(\varepsilon)} \\
& \leq \gamma^{\frac{1}{2} H(\kappa)(H(\kappa)-1)}\left(\frac{2 \alpha\left(\delta^{\prime}\right)}{\kappa^{2}}\right)^{H(\kappa)}
\end{aligned}
\]
where
\[
H(\varepsilon)=\left\lceil 2 \log _{\gamma}\left(\frac{\varepsilon(1-\gamma)}{1+M_{\lambda}}\right)\right\rceil
\]
and
\[
\alpha\left(\delta^{\prime}\right)=\frac{18\left(1+M_{\lambda}\right)^{2} K}{(1-\gamma)^{4}(1-\sqrt{\gamma})^{2}} \log \left(\frac{2 K}{\delta^{\prime}}\right)
\]

Proof. We want to prove that \(n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right) \leq G(\varepsilon)\), where
\[
G(\varepsilon)=\gamma^{\frac{1}{2} H(\varepsilon)(H(\varepsilon)-1)}\left(\frac{2 \alpha\left(\delta^{\prime}\right)}{\varepsilon^{2}}\right)^{H(\varepsilon)}
\]

We proceed by induction on \(h\).
Base case Let \(h=0\). We have \(\varepsilon=\frac{1+M_{\lambda}}{1-\gamma}\), which implies \(n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)=1\) and \(G(\varepsilon)=1\) (since \(H(\varepsilon)=0\) ). Hence, the proposition is true for \(h=0\).

Induction hypothesis Assume true for \(h\).

Induction step Let \(\varepsilon \geq \frac{\left(1+M_{\lambda}\right) \sqrt{\gamma}^{h+1}}{1-\gamma}\). Since \(\frac{\varepsilon}{\sqrt{\gamma}} \geq \frac{\left(1+M_{\lambda}\right) \sqrt{\gamma}^{h}}{1-\gamma}\), we use the induction hypothesis to obtain
\[
\begin{aligned}
n_{\text {sampleV }}\left(\varepsilon, \delta^{\prime}\right) & =\underbrace{1}_{\text {current call }}+\underbrace{K N(\varepsilon) n_{\text {samplev }}\left(\frac{\varepsilon}{\sqrt{\gamma}}, \delta^{\prime}\right)}_{\text {calls in estimateQ }} \\
& \leq \frac{2 \alpha\left(\delta^{\prime}\right)}{\varepsilon^{2}} n_{\text {samplev }}\left(\frac{\varepsilon}{\sqrt{\gamma}}, \delta^{\prime}\right) \\
& \leq \frac{2 \alpha\left(\delta^{\prime}\right)}{\varepsilon^{2}} \gamma^{\frac{1}{2}(H(\varepsilon)-1)(H(\varepsilon)-2)\left(\frac{\gamma 2 \alpha\left(\delta^{\prime}\right)}{\varepsilon^{2}}\right)^{H(\varepsilon)-1}, \quad \text { since } H\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)=H(\varepsilon)-1} \\
& =\gamma^{\frac{1}{2} H(\varepsilon)(H(\varepsilon)-1)}\left(\frac{2 \alpha\left(\delta^{\prime}\right)}{\varepsilon^{2}}\right)^{H(\varepsilon)},
\end{aligned}
\]
which completes the proof.
Lemma 1. Let \(\varepsilon \leq \kappa\). For all \(h \in \mathbb{N}, \forall \varepsilon \geq \kappa \sqrt{\gamma}^{h}\), we have
\[
n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right) \leq \eta_{1}\left[\log _{\frac{1}{\gamma}}\left(\frac{\kappa / \gamma}{\varepsilon}\right)\right]^{\eta_{2}\left(\delta^{\prime}\right)} \frac{1}{\varepsilon^{2}}
\]
where
\[
\begin{aligned}
& \kappa=\frac{1-\sqrt{\gamma}}{K L} \\
& \eta_{1}=\kappa^{2} n_{\text {samplev }}\left(\kappa, \delta^{\prime}\right) \\
& \eta_{2}\left(\delta^{\prime}\right)=\log _{2}\left(\frac{\gamma}{1-\gamma} \frac{2 \beta\left(\delta^{\prime}\right)}{\kappa}\right) \\
& \beta\left(\delta^{\prime}\right)=\frac{18\left(1+M_{\lambda}\right)^{2} K^{2} L}{(1-\gamma)^{4}(1-\sqrt{\gamma})^{3}} \log \left(\frac{2 K}{\delta^{\prime}}\right)
\end{aligned}
\]
under the condition that
\[
\begin{equation*}
\log _{2}\left(\frac{\gamma}{1-\gamma} \frac{2 \beta\left(\delta^{\prime}\right)}{\kappa}\right) \geq 0, \quad \text { i.e., } \quad \beta\left(\delta^{\prime}\right) \geq \frac{(1-\gamma)(1-\sqrt{\gamma})}{2 \gamma K L} \tag{6}
\end{equation*}
\]
which is satisfied by choosing \(\delta^{\prime}\) small enough.

Proof. First, let us define some auxiliary quantities,
\[
\begin{align*}
& B_{1}(\varepsilon) \triangleq\left[\log _{\frac{1}{\gamma}}\left(\frac{\kappa / \gamma}{\varepsilon}\right)\right]^{\eta_{2}\left(\delta^{\prime}\right)}  \tag{7}\\
& B_{2}(\varepsilon) \triangleq \frac{\eta_{1}}{\varepsilon^{2}} \quad \text { and }  \tag{8}\\
& B(\varepsilon) \triangleq B_{1}(\varepsilon) B_{2}(\varepsilon) \tag{9}
\end{align*}
\]

We want to prove that \(n_{\operatorname{samplev}}\left(\varepsilon, \delta^{\prime}\right) \leq B(\varepsilon)\) and we proceed by induction on \(h\).
Base case For \(h=0\), we have \(\varepsilon \geq \kappa\) and, by assumption, \(\varepsilon \leq \kappa\). Therefore, \(\varepsilon=\kappa\). It can be easily verified that \(B(\kappa)=n_{\text {samplev }}\left(\kappa, \delta^{\prime}\right)\), hence the lemma is true for \(h=0\).

Induction hypothesis Assume that the lemma is true for \(h\).
Induction step Let \(\varepsilon \geq \kappa \sqrt{\gamma}^{h+1}\). We have that
\[
\begin{aligned}
n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right) & =\underbrace{1}_{\text {current call }}+\underbrace{n_{\text {samplev }}\left(\frac{\varepsilon}{\sqrt{\gamma}}, \delta^{\prime}\right)}_{\text {call in line } 11 \text { of sampleV }}+\underbrace{K N(\sqrt{\kappa \varepsilon}) n_{\text {samplev }}\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}, \delta^{\prime}\right)}_{\text {calls in estimateQ }} \\
& =1+n_{\text {samplev }}\left(\frac{\varepsilon}{\sqrt{\gamma}}, \delta^{\prime}\right)+\frac{\beta\left(\delta^{\prime}\right)}{\varepsilon} n_{\text {samplev }}\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}, \delta^{\prime}\right) \\
& \leq n_{\text {samplev }}\left(\frac{\varepsilon}{\sqrt{\gamma}}, \delta^{\prime}\right)+\frac{2 \beta\left(\delta^{\prime}\right)}{\varepsilon} n_{\text {samplev }}\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}, \delta^{\prime}\right)
\end{aligned}
\]

Since \(\varepsilon \geq \kappa \sqrt{\gamma}^{h+1}\) and \(\varepsilon \leq \kappa\), we have \(\sqrt{\frac{\kappa \varepsilon}{\gamma}} \geq \frac{\varepsilon}{\sqrt{\gamma}} \geq \kappa \sqrt{\gamma}^{h}\). This allows us to use our induction hypothesis to get
\[
n_{\mathrm{samplev}}\left(\varepsilon, \delta^{\prime}\right) \leq B\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)+\frac{2 \beta\left(\delta^{\prime}\right)}{\varepsilon} B\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}\right)
\]

We will need the equation bellow, which is easily verified as
\[
\begin{equation*}
\log \left(\frac{\kappa}{\sqrt{\frac{\kappa \varepsilon}{\gamma}} \gamma}\right)=\frac{1}{2} \log \left(\frac{\kappa / \gamma}{\varepsilon}\right) \tag{10}
\end{equation*}
\]

We have that
\[
\begin{aligned}
\frac{B\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)}{B(\varepsilon)} & =\frac{B_{1}\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)}{B_{1}(\varepsilon)} \frac{B_{2}\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)}{B_{2}(\varepsilon)} \\
& =\gamma \underbrace{\left[\frac{\log \left(\frac{\kappa / \gamma}{\varepsilon}\right)-\frac{1}{2} \log \frac{1}{\gamma}}{\log \left(\frac{\kappa / \gamma}{\varepsilon}\right)}\right]}_{<1}
\end{aligned}
\]
\(\leq \gamma\),
where we used the assumption that \(\eta_{2}\left(\delta^{\prime}\right) \geq 0\).
Also we get that
\[
\begin{aligned}
\frac{B\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}\right)}{B(\varepsilon)} & =\frac{\varepsilon \gamma}{\kappa} \frac{B_{1}\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}\right)}{B_{1}(\varepsilon)} \\
& =\frac{\varepsilon \gamma}{\kappa}\left[\frac{\log _{\frac{1}{\gamma}}\left(\frac{\kappa}{\sqrt{\frac{\kappa \varepsilon}{\gamma}}}\right)}{\log _{\frac{1}{\gamma}}\left(\frac{\kappa / \gamma}{\varepsilon}\right)}\right]^{\eta_{2}\left(\delta^{\prime}\right)} \\
& =\frac{\varepsilon \gamma}{\kappa}\left[\frac{\frac{1}{2} \log _{\frac{1}{\gamma}}\left(\frac{\kappa / \gamma}{\varepsilon}\right)}{\log _{\frac{1}{\gamma}}\left(\frac{\kappa / \gamma}{\varepsilon}\right)}\right]^{\eta_{2}\left(\delta^{\prime}\right)} \\
& =\frac{\varepsilon \gamma}{\kappa}\left(\frac{1}{2}\right)^{\eta_{2}\left(\delta^{\prime}\right)}=\frac{\varepsilon \gamma}{\kappa} \frac{(1-\gamma)}{\gamma} \frac{\kappa}{2 \beta\left(\delta^{\prime}\right)}=\frac{(1-\gamma) \varepsilon}{2 \beta\left(\delta^{\prime}\right)}
\end{aligned}
\]

Finally, we obtain
\[
\begin{aligned}
n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right) & \leq B\left(\frac{\varepsilon}{\sqrt{\gamma}}\right)+\frac{2 \beta\left(\delta^{\prime}\right)}{\varepsilon} B\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}\right) \\
& \leq \gamma B(\varepsilon)+\frac{2 \beta\left(\delta^{\prime}\right)}{\varepsilon} \frac{(1-\gamma) \varepsilon}{2 \beta\left(\delta^{\prime}\right)} B(\varepsilon) \\
& =B(\varepsilon)
\end{aligned}
\]
which proves the lemma.
Now we can prove Theorem 1, which is restated below.
Theorem. Let \(n\left(\varepsilon, \delta^{\prime}\right)\) be the number of calls to the generative model (oracle) before the algorithm terminates. For any state \(s \in \mathcal{S}\) and \(\varepsilon, \delta^{\prime}>0\),
\[
n\left(\varepsilon, \delta^{\prime}\right) \leq \frac{c_{1}}{\varepsilon^{4}} \log \left(\frac{c_{2}}{\delta^{\prime}}\right)\left[c_{3} \log \left(\frac{c_{4}}{\varepsilon}\right)\right]^{\log _{2}\left(c_{5}\left(\log \left(\frac{c_{2}}{\delta^{\prime}}\right)\right)\right)}=\widetilde{\mathcal{O}}\left(\frac{1}{\varepsilon^{4}}\right)
\]
where \(c_{1}, c_{2}, c_{3}, c_{4}\) and \(c_{5}\) are constants that depend only on \(K, L\) and \(\gamma\).
Proof. First, notice that the number of calls to the generative model is smaller than the total number of calls to sampleV. SmoothCruiser makes one call to estimateQ, which makes \(N(\varepsilon)\) calls to sampleV. If \(\varepsilon \geq \kappa\), Proposition 1 shows that the sample complexity is bounded by a constant. Lemma 1 bounds the sample complexity for \(\varepsilon \leq \kappa\), and we use it to bound \(n\left(\varepsilon, \delta^{\prime}\right)\) :
\[
\begin{aligned}
n\left(\varepsilon, \delta^{\prime}\right) & =N(\varepsilon) n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right) \\
& \leq N(\varepsilon) \eta_{1}\left[\log _{\frac{1}{\gamma}}\left(\frac{\kappa / \gamma}{\varepsilon}\right)\right]^{\eta_{2}\left(\delta^{\prime}\right)} \frac{1}{\varepsilon^{2}} \\
& \leq \frac{c_{1}}{\varepsilon^{4}} \log \left(\frac{c_{2}}{\delta^{\prime}}\right)\left[c_{3} \log \left(\frac{c_{4}}{\varepsilon}\right)\right]^{\log _{2}\left(c_{5}\left(\log \left(\frac{c_{2}}{\delta^{\prime}}\right)\right)\right)}=\widetilde{\mathcal{O}}\left(\frac{1}{\varepsilon^{4}}\right)
\end{aligned}
\]
by using the definition of \(N(\varepsilon)\) for \(\varepsilon \leq \kappa\) and the definition of \(\eta_{2}\left(\delta^{\prime}\right)\) in Lemma 1 .
The constants are given by:
- \(c_{1}=\frac{18\left(1+M_{\lambda}\right)^{2} n_{\text {samplev }}\left(\kappa, \delta^{\prime}\right)}{K^{2} L^{2}(1-\gamma)^{4}}\);
- \(c_{2}=2 K\);
- \(c_{3}=[\log (1 / \gamma)]^{-1}\);
- \(c_{4}=(1-\sqrt{\gamma}) /(\gamma K L)\);
- \(c_{5}=\frac{36\left(1+M_{\lambda}\right)^{2} \gamma K^{3} L^{2}}{(1-\gamma)^{5}(1-\sqrt{\gamma})^{4}}\).

\section*{C Consistency}

Theorem 2. For any \(s \in \mathcal{S}, \varepsilon>0\), and \(\delta>0\), there exists a \(\delta^{\prime}\) that depends on \(\varepsilon\) and \(\delta\) such that the output \(\widehat{V}(s)\) of SmoothCruiser \(\left(s, \varepsilon, \delta^{\prime}\right)\) satisfies
\[
\mathbb{P}[|\widehat{V}(s)-V(s)|>\varepsilon] \leq \delta
\]
and such that \(n\left(\varepsilon, \delta^{\prime}\right)=\mathcal{O}\left(1 / \varepsilon^{4+c}\right)\) for any \(c>0\).
To prove that our algorithm outputs a good estimate of the value function with high probability, we proceed as follows:
- In Lemma 2, we prove that the output of sampleV, conditioned on an event \(\mathcal{A}\), is a low-bias estimate of the true value function, and that \(\mathcal{A}\) happens with high probability;
- Given Lemma 2, the proof of Theorem 2 is straightforward.

Throughout the proof, we will make distinctions between two cases:
- Case 1: \(\kappa \leq \varepsilon<\frac{1+M_{\lambda}}{1-\gamma}\)
- Case 2: \(\varepsilon<\kappa\)

\section*{C. 1 Definitions}

We define the function \(\zeta(\varepsilon)\) as
\[
\zeta(\varepsilon)= \begin{cases}\varepsilon, & \text { if } \kappa \leq \varepsilon<\frac{1+M_{\lambda}}{1-\gamma}  \tag{11}\\ \sqrt{\kappa \varepsilon}, & \text { if } \varepsilon<\kappa \\ \infty, & \text { otherwise }\end{cases}
\]

Define params \((s, \varepsilon)\) as the (random) set of parameters used to call sampleV after a call to sampleV \((s, \varepsilon)\), that is
\[
\begin{equation*}
\operatorname{params}(s, \varepsilon)=\left\{\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right) \text { for } k=1, \ldots, N(\varepsilon) ; a \in \mathcal{A}\right\} \tag{12}
\end{equation*}
\]
in case 1 and
\[
\begin{equation*}
\operatorname{params}(s, \varepsilon)=\left\{\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right) \text { for } k=1, \ldots, N(\varepsilon) ; a \in \mathcal{A}\right\} \bigcup\left\{\left(Z_{s, A}, \frac{\varepsilon}{\sqrt{\gamma}}\right)\right\} \tag{13}
\end{equation*}
\]
in case 2 , where \(Z_{s, a}^{(k)}\) are the next states sampled in estimateQ and \(Z_{s, A}\) is the next state sampled sampleV \((s, \varepsilon)\).
A call to sampleV \((s, \varepsilon)\) makes one call to estimateQ. Denote the output of this call to estimateQ by \(\widehat{Q}_{s}^{\varepsilon}\). We define the event \(\mathcal{A}(s, \varepsilon)\) as follows:
\[
\mathcal{A}(s, \varepsilon)=\left\{\begin{array}{l}
\left\{\left\|\widehat{Q}_{s}^{\varepsilon}-Q_{s}\right\|_{\infty} \leq \zeta(\varepsilon)\right\} \bigcap \mathcal{B}(s, \varepsilon), \quad \text { if } 0<\varepsilon<\frac{1+M_{\lambda}}{1-\gamma}  \tag{14}\\
\Omega, \quad \text { if } \varepsilon \geq \frac{1+M_{\lambda}}{1-\gamma}
\end{array}\right.
\]
where \(\Omega\) is the whole sample space and
\[
\begin{equation*}
\mathcal{B}(s, \varepsilon)=\bigcap_{(z, e) \in \operatorname{params}(s, \varepsilon)} \mathcal{A}(z, e) \tag{15}
\end{equation*}
\]

Define \(C_{\gamma}\) as:
\[
\begin{equation*}
C_{\gamma}=\frac{3\left(1+M_{\lambda}\right)}{(1-\gamma)^{2}} \tag{16}
\end{equation*}
\]

\section*{C. 2 Proofs}

Lemma 2. Let \(\widehat{V}_{\varepsilon}(s)=\operatorname{sampleV}(s, \varepsilon)\). For all \(h \in \mathbb{N}, s \in \mathcal{S}, \varepsilon \geq \frac{\left(1+M_{\lambda}\right) \sqrt{\gamma}^{h}}{1-\gamma}\), we have:
(i) \(\left|\mathbb{E}\left[\widehat{V}_{\varepsilon}(s) \mid \mathcal{A}(s, \varepsilon)\right]-V(s)\right| \leq \varepsilon\), and
(ii) \(\mathbb{P}\left[\left|\widehat{V}_{\varepsilon}(s)\right| \leq C_{\gamma} \mid \mathcal{A}(s, \varepsilon)\right]=1\)
(iii) \(\mathbb{P}[\mathcal{A}(s, \varepsilon)] \geq 1-\delta^{\prime} n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)\)
where
\[
\begin{equation*}
n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)=1+\sum_{(z, e) \in \operatorname{params}(s, \varepsilon)} n_{\text {samplev }}\left(e, \delta^{\prime}\right) \tag{17}
\end{equation*}
\]
is the total number of recursive calls to sampleV after an initial call with parameters \((s, \varepsilon)\).
Proof. We proceed by induction over \(h\).
(1) Base case. If \(h=0, \varepsilon \geq \frac{1+M_{\lambda}}{1-\gamma}\) and \(\mathcal{A}(s, \varepsilon)=\Omega\). The output is then \(\widehat{V}_{\varepsilon}(s)=0\). Point (i) is verified by using the fact that \(|V(s)| \leq \frac{1+M_{\lambda}}{1-\gamma} \leq \varepsilon\); points (ii) and (iii) are trivially verified.
(2) Induction hypothesis. Assume that (i), (ii) and (iii) are true for \(h\).
(3) Induction step. Let \(\varepsilon \geq \frac{\left(1+M_{\lambda}\right) \sqrt{\gamma}^{h+1}}{1-\gamma}\). This implies that \(\varepsilon / \sqrt{\gamma}\) and \(\zeta(\varepsilon) / \sqrt{\gamma}\) are both greater than \(\frac{\left(1+M_{\lambda}\right) \sqrt{\gamma}^{h}}{1-\gamma}\), which will allow us to use our induction hypothesis.
We start by proving (iii).
Let \(\widehat{Q}_{s}^{\varepsilon}=\operatorname{estimateQ}(s, \zeta(\varepsilon))\). Let the reward \(R_{s, a}^{(k)}\) and state \(Z_{s, a}^{(k)}\) be the random variables associated to the \(k\)-th call to the generative model used to compute \(\widehat{Q}_{s}\) in estimateQ, for \(k \in\{1, \cdots, N(\varepsilon)\}\). Let
\[
\begin{equation*}
q_{s}^{k}(a):=R_{s, a}^{(k)}+\gamma \operatorname{sampleV}\left(Z_{s, a}^{(k)}, \zeta(\varepsilon) / \sqrt{\gamma}\right) \tag{18}
\end{equation*}
\]
and let
\[
\begin{equation*}
\bar{Q}_{s}^{\varepsilon}(a)=\frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} q_{s}^{k}(a) \tag{19}
\end{equation*}
\]
so that:
\[
\begin{equation*}
\widehat{Q}_{s}^{\varepsilon}=\operatorname{clip}_{\left(1+M_{\lambda}\right)(1-\gamma)^{-1}}\left(\bar{Q}_{s}^{\varepsilon}(a)\right) \tag{20}
\end{equation*}
\]

Using Fact 2, we have:
\[
\begin{align*}
\left|\widehat{Q}_{s}^{\varepsilon}(a)-Q_{s}(a)\right| & \leq\left|\bar{Q}_{s}^{\varepsilon}(a)-Q_{s}(a)\right|  \tag{21}\\
& \leq \underbrace{\left|\bar{Q}_{s}^{\varepsilon}(a)-\mathbb{E}\left[\bar{Q}_{s}^{\varepsilon}(a) \mid \mathcal{B}(s, \varepsilon)\right]\right|}_{(\mathbf{I})}+\underbrace{\left|\mathbb{E}\left[\bar{Q}_{s}^{\varepsilon}(a) \mid \mathcal{B}(s, \varepsilon)\right]-Q_{s}(a)\right|}_{(\mathbf{I I})} \tag{22}
\end{align*}
\]

We'd like to use Hoeffding's inequality to bound (I) in probability. For that, we need to verify that the random variables \(\left\{q_{s}^{k}(a)\right\}_{k=1}^{N(\varepsilon)}\) are bounded and independent conditionally on \(\mathcal{B}(s, \varepsilon)\).
Boundedness. By induction hypothesis (ii) In the event \(\mathcal{B}(s, \varepsilon)\), the random variables sampleV \(\left(Z_{s, a}^{(k)}, \zeta(\varepsilon) / \sqrt{\gamma}\right)\), for all \(k\), are bounded by \(C_{\gamma}\). Using the fact that the rewards are in \([0,1]\) and that \(C_{\gamma} \geq 1 /(1-\gamma)\), we obtain \(q_{s}^{k}(a)\) is also bounded by \(C_{\gamma}\).

Independence. Let \(E_{k}=\mathcal{A}\left(Z_{s, a}^{k}, \zeta(\varepsilon) / \sqrt{\gamma}\right)\). For any \(t \in \mathbb{R}^{N(\varepsilon)}\), the characteristic function of \(\left\{q_{s}^{k}(a)\right\}_{k=1}^{N(\varepsilon)}\) conditionally on \(\mathcal{B}(s, \varepsilon)\) is given by
\[
\begin{aligned}
\mathbb{E}\left[\exp \left(\mathrm{i} \sum_{k} t_{k} q_{s}^{k}(a)\right) \mid \mathcal{B}(s, \varepsilon)\right] & \stackrel{(\mathbf{a})}{=} \mathbb{E}\left[\exp \left(\mathrm{i} \sum_{k} t_{k} q_{s}^{k}(a)\right) \mid \bigcap_{k} E_{k}\right] \\
& =\frac{\mathbb{E}\left[\exp \left(\mathrm{i} \sum_{k} t_{k} q_{s}^{k}(a)\right) \prod_{k} \mathbb{I}_{\left\{E_{k}\right\}}\right]}{\mathbb{E}\left[\prod_{k} \mathbb{I}_{\left\{E_{k}\right\}}\right]} \\
& =\frac{\mathbb{E}\left[\prod_{k} \exp \left(\mathrm{i} t_{k} q_{s}^{k}(a)\right) \mathbb{I}_{\left\{E_{k}\right\}}\right]}{\mathbb{E}\left[\prod_{k} \mathbb{I}_{\left\{E_{k}\right\}}\right]} \\
& \stackrel{(\mathbf{b})}{=} \frac{\prod_{k} \mathbb{E}\left[\exp \left(\mathrm{i} t_{k} q_{s}^{k}(a)\right) \mathbb{I}_{\left\{E_{k}\right\}}\right]}{\prod_{k} \mathbb{E}\left[\mathbb{I}_{\left\{E_{k}\right\}}\right]} \\
& =\prod_{k} \mathbb{E}\left[\exp \left(\mathrm{i} t_{k} q_{s}^{k}(a)\right) \mid E_{k}\right] \\
& \stackrel{(\mathbf{c})}{=} \prod_{k} \mathbb{E}\left[\exp \left(\mathrm{i} t_{k} q_{s}^{k}(a)\right) \mid \mathcal{B}(s, \varepsilon)\right]
\end{aligned}
\]
which is justified by
(a) Definition of \(\mathcal{B}(s, \varepsilon)\) and the fact that \(\left\{q_{s}^{k}(a)\right\}_{k=1}^{N(\varepsilon)}\) are independent of \(\mathcal{A}\left(Z_{s, A}, \frac{\varepsilon}{\sqrt{\gamma}}\right)\);
(b) The random variables \(\left\{q_{s}^{k}(a)\right\}_{k=1}^{N(\varepsilon)}\) are independent and the events \(\left\{E_{k}\right\}_{i=1}^{N(\varepsilon)}\) are also independent;
(c) The random variable \(q_{s}^{k}(a)\) is independent of every \(E_{j}\) for \(j \neq k\).

Since the characteristic function of \(\left\{q_{s}^{k}(a)\right\}_{k=1}^{N(\varepsilon)}\) is the product of their characteristic functions, these random variables are independent given \(\mathcal{B}(s, \varepsilon)\).

Now we can use Hoeffding's inequality:
\[
\begin{aligned}
& \mathbb{P}\left[\left|\bar{Q}_{s}^{\varepsilon}(a)-\mathbb{E}\left[\bar{Q}_{s}^{\varepsilon}(a) \mid \mathcal{B}(s, \varepsilon)\right]\right| \geq(1-\sqrt{\gamma}) \zeta(\varepsilon) \mid \mathcal{B}(s, \varepsilon)\right] \\
& =\mathbb{P}\left[\left.\left|\frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} q_{s}^{k}(a)-\mathbb{E}\left[q_{s}^{k}(a) \mid \mathcal{B}(s, \varepsilon)\right]\right| \geq(1-\sqrt{\gamma}) \zeta(\varepsilon) \right\rvert\, \mathcal{B}(s, \varepsilon)\right] \\
& \leq 2 \exp \left(-\frac{N(\varepsilon)(1-\sqrt{\gamma})^{2} \zeta(\varepsilon)^{2}}{2 C_{\gamma}^{2}}\right) \\
& \leq \frac{\delta^{\prime}}{K}
\end{aligned}
\]

And (II) is bounded by using the induction hypothesis (i):
\[
\begin{aligned}
& \left|\mathbb{E}\left[q_{s}^{k}(a) \mid \mathcal{B}(s, \varepsilon)\right]-Q_{s}(a)\right| \\
& \stackrel{(\mathbf{a})}{=} \gamma\left|\mathbb{E}\left[\left.\operatorname{sampleV}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right) \right\rvert\, \mathcal{B}(s, \varepsilon)\right]-\mathbb{E}\left[V\left(Z_{s, a}^{(k)}\right) \mid \mathcal{B}(s, \varepsilon)\right]\right| \\
& \stackrel{(\mathbf{b})}{=} \gamma\left|\mathbb{E}\left[\left.\operatorname{sampleV}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right) \right\rvert\, \mathcal{A}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right)\right]-\mathbb{E}\left[V\left(Z_{s, a}^{(k)}\right) \left\lvert\, \mathcal{A}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right)\right.\right]\right| \\
& \stackrel{(\mathbf{c})}{=} \gamma\left|\mathbb{E}\left[\left.\mathbb{E}\left[\left.\operatorname{sampleV}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right) \right\rvert\, Z_{s, a}^{(k)}, \mathcal{A}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right)\right]-V\left(Z_{s, a}^{(k)}\right) \right\rvert\, \mathcal{A}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right)\right]\right| \\
& \stackrel{(\mathbf{d})}{\leq} \gamma \frac{\zeta(\varepsilon)}{\sqrt{\gamma}} \\
& =\sqrt{\gamma} \zeta(\varepsilon)
\end{aligned}
\]
which is justified by the following:
(a) \(\mathbb{E}\left[R_{s, a}^{(k)} \mid \mathcal{B}(s, \varepsilon)\right]=\mathbb{E}\left[R_{s, a}^{(k)}\right]\), since the reward depends only on \(s, a\);
(b) The term \(\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right)\) depends on \(\mathcal{B}(s, \varepsilon)\) only through \(\mathcal{A}\left(Z_{s, a}^{(k)}, \frac{\zeta(\varepsilon)}{\sqrt{\gamma}}\right)\);
(c) Law of total expectation;
(d) Consequence of induction hypothesis (i).

Putting together the bounds for (I) and (II) and doing an union bound over all actions, we obtain:
\[
\mathbb{P}\left[\left\|\widehat{Q}_{s}^{\varepsilon}-Q_{s}\right\|_{\infty} \geq \zeta(\varepsilon) \mid \mathcal{B}(s, \varepsilon)\right] \leq \delta^{\prime}
\]

We can now give a lower bound to the probability of the event \(\mathcal{A}(s, \varepsilon)\). Let
\[
\begin{equation*}
\mathcal{E}=\left\{\left\|\widehat{Q}_{s}^{\varepsilon}-Q_{s}\right\|_{\infty}<\zeta(\varepsilon)\right\} \tag{23}
\end{equation*}
\]

We have:
\[
\begin{aligned}
\mathbb{P}[\mathcal{A}(s, \varepsilon)] & \geq \mathbb{P}[\mathcal{E} \cap \mathcal{B}(s, \varepsilon)] \\
& =\mathbb{P}[\mathcal{E} \mid \mathcal{B}(s, \varepsilon)] \mathbb{P}[\mathcal{B}(s, \varepsilon)] \\
& =\left(1-\mathbb{P}\left[\mathcal{E}^{\mathcal{C}} \mid \mathcal{B}(s, \varepsilon)\right]\right) \mathbb{P}[\mathcal{B}(s, \varepsilon)] \\
& \geq \mathbb{P}[\mathcal{B}(s, \varepsilon)]-\delta^{\prime} \\
& \geq 1-\delta^{\prime} n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)
\end{aligned}
\]
since
\[
\begin{aligned}
\mathbb{P}[\mathcal{B}(s, \varepsilon)] & =1-\mathbb{P}\left[\mathcal{B}(s, \varepsilon)^{\complement}\right] \\
& =1-\mathbb{P}\left[\bigcup_{(z, e) \in \operatorname{params}(s, \varepsilon)} \mathcal{A}(z, e)^{\complement}\right] \\
& \geq 1-\sum_{(z, e) \in \operatorname{params}(s, \varepsilon)} \mathbb{P}\left[\mathcal{A}(z, e)^{\complement}\right] \\
& \geq 1-\delta^{\prime} \sum_{(z, e) \in \operatorname{params}(s, \varepsilon)} n_{\text {sampleV }}\left(e, \delta^{\prime}\right) \quad \text { by induction hypothesis (iii) } \\
& =1-\delta^{\prime}\left(n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)-1\right)
\end{aligned}
\]

This proves (iii). Now, let's prove (i).
For any event \(\mathcal{E}\), we write
\[
\mathbb{E}_{\mathcal{E}}[\cdot]=\mathbb{E}[\cdot \mid \mathcal{E}]
\]

Case 1. We start with case \(1, \kappa \leq \varepsilon<\frac{1+M_{\lambda}}{1-\gamma}\), where \(\zeta(\varepsilon)=\varepsilon\) and
\[
\begin{equation*}
\widehat{V}_{\varepsilon}(s)=F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right) \tag{24}
\end{equation*}
\]

We have:
\[
\begin{aligned}
\left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\widehat{V}_{\varepsilon}(s)\right]-V(s)\right| & =\left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)-F_{s}\left(Q_{s}\right)\right]\right| \\
& \leq \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left|F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)-F_{s}\left(Q_{s}\right)\right|\right] \\
& \leq \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left\|\widehat{Q}_{s}^{\varepsilon}(a)-Q_{s}(a)\right\|_{\infty}\right] \\
& \leq \zeta(\varepsilon)=\varepsilon
\end{aligned}
\]
and (i) is verified for case 1.
Case 2. Consider now the case \(2, \varepsilon<\kappa\), where \(\zeta(\varepsilon)=\sqrt{\kappa \varepsilon}\).
Let \(A\) be the action following the distribution \(\frac{\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)}{\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1}}\), and let the reward \(R_{s, A}\) and the state \(Z_{s, A}\) be the random variables associated to the call to the generative model with parameters \((s, A)\). Let \(\widehat{v}=\operatorname{sampleV}\left(Z_{s, A}, \varepsilon / \sqrt{\gamma}\right)\). The output in this case is given by
\[
\begin{equation*}
\widehat{V}_{\varepsilon}(s)=F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)-\left(\widehat{Q}_{s}^{\varepsilon}\right)^{\top} \nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)+(R+\gamma \widehat{v})\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1} \tag{25}
\end{equation*}
\]

Let
\[
\begin{aligned}
Q_{s}(A) & =\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[R_{s, A}+\gamma V\left(Z_{s, A}\right) \mid A, \widehat{Q}_{s}^{\varepsilon}\right] \\
& =\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[R_{s, A}+\gamma V\left(Z_{s, A}\right) \mid A\right]
\end{aligned}
\]
and let
\[
\begin{equation*}
\tilde{V}(s)=\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)-\left(\widehat{Q}_{s}^{\varepsilon}\right)^{\top} \nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)+Q_{s}(A)\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1}\right] \tag{26}
\end{equation*}
\]

We have
\[
\begin{aligned}
& \left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\widehat{V}_{\varepsilon}(s)\right]-\tilde{V}(s)\right| \\
& \stackrel{(\mathbf{a})}{=} \gamma\left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left.\operatorname{sampleV}\left(Z_{s, A}, \frac{\varepsilon}{\sqrt{\gamma}}\right)-V\left(Z_{s, A}\right) \right\rvert\, A, \widehat{Q}_{s}^{\varepsilon}, Z_{s, A}\right]\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1}\right]\right| \\
& \stackrel{(\mathbf{b})}{=} \gamma\left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left(\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left.\operatorname{sampleV}\left(Z_{s, A}, \frac{\varepsilon}{\sqrt{\gamma}}\right) \right\rvert\, A, \widehat{Q}_{s}^{\varepsilon}, Z_{s, A}\right]-V\left(Z_{s, A}\right)\right)\right]\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1}\right| \\
& \stackrel{(\mathbf{c})}{\leq} \gamma \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left.\operatorname{sampleV}\left(Z_{s, A}, \frac{\varepsilon}{\sqrt{\gamma}}\right) \right\rvert\, A, \widehat{Q}_{s}^{\varepsilon}, Z_{s, A}\right]-V\left(Z_{s, A}\right)\right|\right] \\
& \stackrel{(\mathbf{d})}{=} \gamma \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left|\mathbb{E}_{\mathcal{A}\left(Z_{s, A}, \varepsilon / \sqrt{\gamma}\right)}\left[\left.\operatorname{sampleV}\left(Z_{s, A}, \frac{\varepsilon}{\sqrt{\gamma}}\right) \right\rvert\, Z_{s, A}\right]-V\left(Z_{s, A}\right)\right|\right] \\
& \stackrel{(\mathbf{e})}{\leq} \gamma \frac{\varepsilon}{\sqrt{\gamma}}=\sqrt{\gamma} \varepsilon
\end{aligned}
\]
which is justified by the following points:
(a) The reward depend only on \(s, a\) and law of total expectation;
(b) \(V\left(Z_{s, A}\right)\) is a function of \(Z_{s, A}\) and no other random variable;
(c) Jensen's inequality and the fact that \(\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1} \leq 1\);
(d) Given \(Z_{s, A}\), the term samplev \(\left(Z_{s, A}, \frac{\varepsilon}{\sqrt{\gamma}}\right)\) depends on \(\mathcal{A}(s, \varepsilon)\) only through \(\mathcal{A}\left(Z_{s, A}, \varepsilon / \sqrt{\gamma}\right) ;\)
(e) Induction hypothesis (i).

Now, \(\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[Q_{s}(A)\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1}\right]\) can be written as
\[
\begin{aligned}
& \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[Q_{s}(A)\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1}\right] \\
& =\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[Q_{s}(A) \mid \widehat{Q}_{s}^{\varepsilon}\right]\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1}\right] \\
& =\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[Q_{s}^{\top} \nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right]
\end{aligned}
\]
so that \(\widetilde{V}(s)\) is given by
\[
\begin{equation*}
\tilde{V}(s)=\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)+\left(Q_{s}-\widehat{Q}_{s}^{\varepsilon}\right)^{\top} \nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right] \tag{27}
\end{equation*}
\]

Finally, we bound the difference between \(\widetilde{V}(s)\) and \(V(s)\) :
\[
\begin{aligned}
|\widetilde{V}(s)-V(s)| & \leq \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left|F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)+\left(Q_{s}-\widehat{Q}_{s}^{\varepsilon}\right)^{\top} \nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)-V(s)\right|\right] \\
& \leq L \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left\|Q_{s}-\widehat{Q}_{s}^{\varepsilon}\right\|_{2}^{2}\right] \\
& \leq K L \mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\left\|Q_{s}-\widehat{Q}_{s}^{\varepsilon}\right\|_{\infty}^{2}\right] \\
& \leq K L \zeta(\varepsilon)^{2} \\
& =K L \kappa \varepsilon \\
& =(1-\sqrt{\gamma}) \varepsilon
\end{aligned}
\]
by using the fact that we are on \(\mathcal{A}(s, \varepsilon)\) and (a) uses the fact that for all \(x \in \mathbb{R}^{K},\|x\|_{2}^{2} \leq K\|x\|_{\infty}^{2}\). We can now prove (i) for case 2 :
\[
\begin{align*}
\left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\widehat{V}_{\varepsilon}(s)\right]-V(s)\right| & \leq\left|\mathbb{E}_{\mathcal{A}(s, \varepsilon)}\left[\widehat{V}_{\varepsilon}(s)\right]-\widetilde{V}(s)\right|+|\widetilde{V}(s)-V(s)|  \tag{28}\\
& \leq \sqrt{\gamma} \varepsilon+(1-\sqrt{\gamma}) \varepsilon=\varepsilon \tag{29}
\end{align*}
\]

Finally, let's prove (ii).
Case 1. In this case, \(\widehat{V}_{\varepsilon}(s)=F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\) with \(\left\|\widehat{Q}_{s}^{\varepsilon}\right\|_{\infty} \leq\left(1+M_{\lambda}\right) /(1-\gamma)\), since each component of \(\widehat{Q}_{s}^{\varepsilon}\) is clipped and lie in the interval \(\left[0, \frac{1+M_{\lambda}}{1-\gamma}\right]\). The assumptions on \(F_{s}\) imply that \(\left|\widehat{V}_{\varepsilon}(s)\right| \leq\) \(\frac{1+M_{\lambda}}{1-\gamma} \leq C_{\gamma}\).

Case 2. In this case, we have:
\[
\begin{aligned}
\left|\widehat{V}_{\varepsilon}(s)\right| & \leq\left|F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)-\left(\widehat{Q}_{s}^{\varepsilon}\right)^{\top} \nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right|+|R+\gamma \widehat{v}|\left\|\nabla F_{s}\left(\widehat{Q}_{s}^{\varepsilon}\right)\right\|_{1} \\
& \leq 2\left\|\widehat{Q}_{s}^{\varepsilon}\right\|_{\infty}+M_{\lambda}+1+\gamma C_{\gamma} \\
& \leq \frac{2\left(1+M_{\lambda}\right)}{1-\gamma}+M_{\lambda}+1+\gamma C_{\gamma} \\
& \leq C_{\gamma}
\end{aligned}
\]
since \(|\hat{v}| \leq C_{\gamma}\) by induction hypothesis (ii).
This proves (ii) for case 2:
\[
\begin{equation*}
\mathbb{P}\left[|\widehat{V}(s)| \leq C_{\gamma} \mid \mathcal{A}(s, \varepsilon)\right]=1 \tag{30}
\end{equation*}
\]

Now, we can prove Theorem 2, which is restated as follows:
Theorem. Let \(\widehat{V}(s)\) be the output of SmoothCruiser \(\left(s, \varepsilon, \delta^{\prime}\right)\). For any state \(s \in \mathcal{S}\) and \(\varepsilon, \delta^{\prime}>0\),
\[
\mathbb{P}[|\widehat{V}(s)-V(s)|>\varepsilon] \leq \delta^{\prime} n\left(\varepsilon, \delta^{\prime}\right)
\]

Proof. Let \(\widehat{Q}_{s}=\) estimate \((s, \varepsilon)\). We have \(\widehat{V}(s)=F_{s}\left(\widehat{Q}_{s}\right)\). As in the proof of Lemma 2, let the reward \(R_{s, a}^{(k)}\) and state \(Z_{s, a}^{(k)}\) be the random variables associated to the \(k\)-th call to the generative model used to compute \(\widehat{Q}_{s}(a)\) in estimateQ, for \(k \in\{1, \cdots, N(\varepsilon)\}\).
We have:
\[
\begin{equation*}
\widehat{Q}_{s}(a)=\frac{1}{N(\varepsilon)} \sum_{k=1}^{N(\varepsilon)} R_{s, a}^{(k)}+\gamma \operatorname{sampleV}\left(Z_{s, a}^{(k)}, \varepsilon / \sqrt{\gamma}\right) \tag{31}
\end{equation*}
\]

Consider the event \(\mathcal{E}\) defined by:
\[
\begin{equation*}
\mathcal{E}=\bigcap_{k=1}^{N(\varepsilon)} \mathcal{A}\left(Z_{s, a}^{(k)}, \frac{\varepsilon}{\sqrt{\gamma}}\right) \tag{32}
\end{equation*}
\]

By the same arguments as in the proof of Lemma 2, we have:
- In \(\mathcal{E}\), we have \(\left\|\widehat{Q}_{s}-Q_{s}\right\|_{\infty} \leq \varepsilon\);
- \(\mathbb{P}[\mathcal{E}] \geq 1-\delta^{\prime} N(\varepsilon) n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)=1-\delta^{\prime} n\left(\varepsilon, \delta^{\prime}\right)\).

This implies the result, since \(|\widehat{V}(s)-V(s)| \leq\left\|\widehat{Q}_{s}-Q_{s}\right\|_{\infty}\).
Now, for every \(\varepsilon>0\) and every \(\delta>0\), we need to be able to find a value of \(\delta^{\prime}\) such that \(\delta^{\prime} n\left(\varepsilon, \delta^{\prime}\right) \leq \delta\). That is, given \(\varepsilon\) and \(\delta\), we need to find \(\delta^{\prime}\) such that
\[
\begin{equation*}
\delta^{\prime} \frac{c_{1}}{\varepsilon^{4}} \log \left(\frac{c_{2}}{\delta^{\prime}}\right)\left[c_{3} \log \left(\frac{c_{4}}{\varepsilon}\right)\right]^{\log _{2}\left(c_{5}\left(\log \left(\frac{c_{2}}{\delta^{\prime}}\right)\right)\right)} \leq \delta \tag{33}
\end{equation*}
\]

Such value exists, since the term on the LHS tends to 0 as \(\delta^{\prime} \rightarrow 0\), and it depends on \(\varepsilon\). We will show that this dependence is polynomial when \(\varepsilon \rightarrow 0\).
Let \(\delta^{\prime}=\varepsilon^{5}\). There exists a value \(\widetilde{\varepsilon}\) that depends on \(\delta\) such that:
\[
\begin{equation*}
\forall \varepsilon \leq \widetilde{\varepsilon}, \quad \varepsilon^{5} \frac{c_{1}}{\varepsilon^{4}} \log \left(\frac{c_{2}}{\varepsilon^{5}}\right)\left[c_{3} \log \left(\frac{c_{4}}{\varepsilon}\right)\right]^{\log _{2}\left(c_{5}\left(\log \left(\frac{c_{2}}{\varepsilon^{5}}\right)\right)\right)} \leq \delta \tag{34}
\end{equation*}
\]
since the term on the LHS tends to 0 as \(\varepsilon \rightarrow 0\), as a consequence of Proposition 2.
Putting it all together, we can choose \(\delta^{\prime}\) as folllows:
\[
\delta^{\prime}=\left\{\begin{array}{l}
\widetilde{\delta} \text { such that } \widetilde{\delta} \frac{c_{1}}{\varepsilon^{4}} \log \left(\frac{c_{2}}{\delta}\right)\left[c_{3} \log \left(\frac{c_{4}}{\varepsilon}\right)\right]^{\log _{2}\left(c_{5}\left(\log \left(\frac{c_{2}}{\delta}\right)\right)\right)} \leq \delta, \quad \text { if } \varepsilon>\widetilde{\varepsilon},  \tag{35}\\
\varepsilon^{5}, \quad \text { if } \varepsilon \leq \widetilde{\varepsilon}
\end{array}\right.
\]
which is \(\mathcal{O}\left(\varepsilon^{5}\right)\).
Proposition 3 implies that, for this choice of \(\delta^{\prime}\), the sample complexity is still of order \(\mathcal{O}\left(1 / \varepsilon^{4+c}\right)\) for any \(c>0\).

\section*{D Auxiliary results}

Fact 1. For all \(s \in \mathcal{S}\) and all \(x \in \mathbb{R}^{K}\), we have \(F_{s}(x) \leq\|x\|_{\infty}+\sup _{s}\left|F_{s}(0)\right|\).
Proof. By the assumptions on \(F_{s}\), we have:
\[
\begin{align*}
\left|F_{s}(x)\right| & =\left|F_{s}(x)-F_{s}(0)+F_{s}(0)\right| \leq\left|F_{s}(x)-F_{s}(0)\right|+\left|F_{s}(0)\right|  \tag{36}\\
& \leq\|x-0\|_{\infty}+\left|F_{s}(0)\right| \leq\|x\|_{\infty}+\sup _{s}\left|F_{s}(0)\right| \tag{37}
\end{align*}
\]

Fact 2. Let \(x, q \in \mathbb{R}^{d}\) be such that \(0 \leq q_{i} \leq c\) for all \(i\). Let \(\widetilde{x}=\operatorname{clip}_{c}(x)\). Then, \(\|\widetilde{x}-q\|_{\infty} \leq\) \(\|x-q\|_{\infty}\).

Proof. For any \(i \in\{1, \ldots, d\}\), we have \(\left|\widetilde{x}_{i}-q_{i}\right| \leq\left|x_{i}-q_{i}\right|\), since \(0 \leq q_{i} \leq c\). The result follows.

Proposition 2. \(\forall a, b, c>0\)
\[
\lim _{x \rightarrow \infty} \frac{1}{x^{c}} \exp \left(a\left[\log \log \left(x^{b}\right)\right]^{2}\right)=0
\]

Proof. We have
\[
\begin{aligned}
\frac{1}{x^{c}} \exp \left(a\left[\log \log \left(x^{b}\right)\right]^{2}\right) & =\exp \left(a\left[\log \log \left(x^{b}\right)\right]^{2}-c \log x\right) \\
& =\exp \left(a[\log u]^{2}-\frac{c}{b} u\right), \quad \text { by setting } u=\log \left(x^{b}\right)
\end{aligned}
\]

And, for any \(k>0\), we have
\[
\begin{equation*}
\lim _{u \rightarrow \infty} \log ^{2} u-k u=-\infty \tag{38}
\end{equation*}
\]
which allows us to conclude.
Proposition 3. If we set \(\delta^{\prime}=\delta^{\prime}(\varepsilon)=\varepsilon^{5}\), we have:
\[
n\left(\varepsilon, \delta^{\prime}(\varepsilon)\right)=\mathcal{O}\left(\frac{1}{\varepsilon^{4+c}}\right), \quad \forall c>0
\]

Proof. We have:
\[
\begin{aligned}
n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}(\varepsilon)\right) & \leq \eta_{1}\left[\log _{\frac{1}{\gamma}}\left(\frac{\bar{\varepsilon} / \gamma}{\varepsilon}\right)\right]^{\eta_{2}\left(\varepsilon^{3}\right)} \frac{1}{\varepsilon^{2}} \\
& =\underbrace{\left[\log _{\frac{1}{\gamma}}\left(\frac{\bar{\varepsilon} / \gamma}{\varepsilon}\right)\right]^{\log _{2}\left(k \log \left(\frac{2 K}{\varepsilon^{3}}\right)\right)}}_{(\text {A) }} \frac{1}{\varepsilon^{2}}
\end{aligned}
\]
where \(k\) is a constant that does not depend on \(\varepsilon\). The term (A) can be rewritten as:
\[
\begin{aligned}
{\left[\log _{\frac{1}{\gamma}}\left(\frac{\bar{\varepsilon} / \gamma}{\varepsilon}\right)\right]^{\log _{2}\left(k \log \left(\frac{2 K}{\varepsilon^{3}}\right)\right)} } & =\left[c_{1} \log \left(\frac{c_{2}}{\varepsilon}\right)\right]^{c_{3} \log \left[k \log \left(\frac{c_{4}}{\varepsilon^{3}}\right)\right]} \\
& =\exp \left\{c_{3} \log \left[k \log \left(\frac{c_{4}}{\varepsilon^{3}}\right)\right] \log \left(c_{1} \log \left(\frac{c_{2}}{\varepsilon}\right)\right)\right\}
\end{aligned}
\]
which can be shown to be \(\mathcal{O}\left(\frac{1}{\varepsilon^{c}}\right)\) for any \(c>0\) by applying proposition 2 after some algebraic manipulations.
Hence,
\[
n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}(\varepsilon)\right)=\frac{1}{\varepsilon^{2}} \mathcal{O}\left(\frac{1}{\varepsilon^{c}}\right)=\mathcal{O}\left(\frac{1}{\varepsilon^{2+c}}\right), \quad \forall c>0
\]

Since we have
\[
n\left(\varepsilon, \delta^{\prime}\right)=N(\varepsilon) n_{\mathrm{samplev}}\left(\varepsilon, \delta^{\prime}\right)
\]
with \(N(\varepsilon)=\widetilde{\mathcal{O}}\left(1 / \varepsilon^{2}\right)\), this proves the result.

Corollary 1. If we set \(\delta^{\prime}=\delta^{\prime}(\varepsilon)=\varepsilon^{5}\), we have:
\[
\lim _{\varepsilon \rightarrow 0} \delta^{\prime}(\varepsilon) n\left(\varepsilon, \delta^{\prime}(\varepsilon)\right)=0
\]

Proof. It is an immediate consequence of proposition 3 by taking \(c \in] 0,1[\).

\section*{E On other smooth approximations of the max}

In this paper we focus on the \(\operatorname{LogSumExp}{ }_{\lambda}\) function as a smooth approximation to the maximum function. Yet our proof is more general and can handle any approximation of the max function which verifies the properties listed in Section 5. For instance let's consider the following regularization of the Bellman equation:
\[
\begin{equation*}
F(Q)=\max _{\left(\pi_{a}\right)_{a \in \mathcal{A}}} \sum_{a \in \mathcal{A}}\left(Q_{a} \cdot \pi_{a}+\lambda \sqrt{\pi_{a}}\right) \tag{39}
\end{equation*}
\]

This smooth function is particularly interesting because it approximates the distribution of pulled armed of the UCB algorithm by taking \(\lambda=2 c \cdot \sqrt{\frac{\ln (n)}{n}}\) (see 41 and notice that \(\pi_{a}^{\star} \cdot n\) approximates \(n_{a}\) ). We show that this smooth approximation of the maximum verifies the assumptions made in Section 5. We have
\[
\begin{equation*}
F(Q)=\sum_{a \in \mathcal{A}}\left(Q_{a} \cdot \pi_{a}^{\star}+\lambda \sqrt{\pi_{a}^{\star}}\right) \tag{40}
\end{equation*}
\]
and we can show that \(\nabla_{Q} F(Q)=\pi^{\star}\). Therefore point 1,2 and 3 of Section 5 are verified. Now by differentiating with respect to \(\pi\) this time:
\[
\begin{equation*}
\forall a \in \mathcal{A} \quad Q_{a}+\frac{\lambda}{2 \sqrt{\pi_{a}^{\star}}}=U \tag{41}
\end{equation*}
\]
where \(U\) is the Lagrange multiplier. Using the fact that \(\sum_{a \in \mathcal{A}} \pi_{a}^{\star}=1\), we get
\[
\begin{equation*}
\sum_{a \in \mathcal{A}}\left(\frac{\lambda / 2}{U-Q_{a}}\right)^{2}=1 \tag{42}
\end{equation*}
\]

Because \(U>\max _{a} \pi_{a}^{\star}\) the derivative of the left side with respect to \(U\) is positive for all \(Q_{a} \in\) \(\left[0,\left(1+M_{\lambda}\right) /(1-\gamma)\right]^{|\mathcal{A}|}\). Using the inverse function theorem we get that \(U\) is differentiable with respect to Q and that \(\pi_{a}^{\star}=\left(\frac{\lambda / 2}{U-Q_{a}}\right)^{2}\) is also differentiable with respect to Q . Finally because \(\left[0,\left(1+M_{\lambda}\right) /(1-\gamma)\right]^{|\mathcal{A}|}\) is compact we can conclude that \(F\) is \(L\)-smooth for some \(L \geq 0\) verifying point 4 of Section 5.

\section*{F Experimental validation of the theoretical results}

In this section, we present the experiments we made to verify the correctness of our sample complexity bounds (Theorem 1) and of our consistency results (Theorem 2).

\section*{F. 1 Checking the sample complexity guarantee}

The key step for proving Theorem 1 is using Lemma 1, that bounds the number of calls to the generative model made by a call to sampleV \((s, \varepsilon)\).
Figure 2 shows the simulated number of calls to the generative model made by \(n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)\) as a function of \(1 / \varepsilon\) and compares it to our theoretical bound in Lemma 1 and to the number of calls that would be required by a Sparse Sampling strategy, which corresponds to the bound in Proposition 1
extrapolated to all values of \(\varepsilon\). The simulated values where obtained by computing the following recurrence for several values of \(\varepsilon\) :
\[
n_{\text {samplev }}^{\operatorname{sim}}\left(\varepsilon, \delta^{\prime}\right)=\left\{\begin{array}{l}
1+n_{\text {samplev }}^{\operatorname{sim}}\left(\frac{\varepsilon}{\sqrt{\gamma}}, \delta^{\prime}\right)+K N(\sqrt{\kappa \varepsilon}) n_{\text {samplev }}^{\operatorname{sim}}\left(\sqrt{\frac{\kappa \varepsilon}{\gamma}}, \delta^{\prime}\right), \text { if } \varepsilon<\kappa, \\
\gamma^{\frac{1}{2} H(\varepsilon)(H(\varepsilon)-1)}\left(\frac{2 \alpha\left(\delta^{\prime}\right)}{\varepsilon^{2}}\right)^{H(\varepsilon)}, \text { otherwise } .
\end{array}\right.
\]

Figure 3 shows the mumber of calls to the generative model made by \(n_{\text {samplev }}\) as a function of the regularization parameter \(\lambda\) in order to achieve a relative error of \(0.01^{7}\) and its ratio with respect to the number of calls that would be required by Sparse Sampling in the same setting. We see that fewer samples are required as the regularization increases. We also see that, for small \(\lambda\), there is no advantage with respect to Sparse Sampling, but SmoothCruiser has a very large advantage when the regularization \(\lambda\) grows.


Figure 2: Simulated number of calls to the generative model made by \(n_{\text {samplev }}\left(\varepsilon, \delta^{\prime}\right)\) as a function of \(1 / \varepsilon\) compared to our theoretical bound (Lemma 1 ) and to the number of calls that would be required by a Sparse Sampling strategy. The parameters used were: \(\gamma=0.2, \delta^{\prime}=0.1, K=2\) and \(\lambda=0.1\).

\section*{F. 2 Checking the consistency guarantee}

Using our MCTS analogy in Section 3.3, the two most computationally costly operations of SmoothCruiser are the selectAction and the evaluateLeaf functions. They both rely on estimates of the \(Q\) function with some required accuracy. Hence, for a sanity-check, we implemented the function sampleV by replacing its calls to estimate \(\mathrm{Q}(s\), accuracy) by the true Q function at state \(s\) plus some accuracy-dependent noise, and we denote this simplified version of sampleV by sampleV \({ }_{\text {check }}\). This allowed us to verify that our bounds for the bias of the sampleV outputs (Lemma 2) are correct. After \(N_{\text {sim }}\) calls to sampleV \({ }_{\text {check }}\left(\varepsilon, \delta^{\prime}\right)\), we compute the error
\[
\begin{equation*}
\widehat{\Delta}(s, \varepsilon)=\frac{1}{N_{\mathrm{sim}}} \sum_{i=1}^{N_{\mathrm{sim}}}\left(\widehat{V}_{i}(s, \varepsilon)-V(s)\right) \tag{43}
\end{equation*}
\]

\footnotetext{
\({ }^{7}\) We set \(\varepsilon=0.01 V_{\lambda}^{\max }\), where \(V_{\lambda}^{\max }=(1+\lambda \log K) /(1-\gamma)\) is an upper bound on the regularized value function.
}


Figure 3: Number of calls to the generative model made by \(n_{\text {samplev }}\) as a function of the regularization parameter \(\lambda\) in order to achieve a relative error of 0.01 (left) and its ratio with respect to the number of calls that would be required by Sparse Sampling in the same setting (right). The parameters used were: \(\gamma=0.2, \delta^{\prime}=0.1\) and \(K=2\).
where \(s\) is a reference state and \(\widehat{V}_{i}(s, \varepsilon)\) is the output of the \(i\)-th call to \(\operatorname{sampleV}_{\text {check }}(s, \varepsilon)\). Lemma 2 states that, for some high probability event \(B\), we have \(-\varepsilon \leq \mathbb{E}[\widehat{\Delta}(s, \varepsilon) \mid B] \leq \varepsilon\). Hence, for large \(N_{\text {sim }}\), we should have \(-\varepsilon \leq \widehat{\Delta}(s, \varepsilon) \leq \varepsilon\) approximately.

Table 1 shows simulated values of \(\widehat{\Delta}(s, \varepsilon)\) and their standard deviations for different environments. The value of \(N_{\text {sim }}\) was chosen so that \(\widehat{\Delta}(s, \varepsilon)\) is close to its mean, by using Hoeffdings's inequality and assuming that \(\widehat{V}_{i}(s, \varepsilon)\) is bounded by \(C_{\gamma}\) (which holds with high probability, by Lemma 2).
\begin{tabular}{ll}
\hline Environment & \(\widehat{\Delta}(s, \varepsilon)\) \\
\hline 5-Chain & \((-1.21 \pm 1.65) \times 10^{-2}\) \\
10-Chain & \((-1.20 \pm 1.63) \times 10^{-2}\) \\
5x5-GridWorld & \((-0.71 \pm 2.04) \times 10^{-2}\) \\
10x10-GridWorld & \((-0.71 \pm 2.03) \times 10^{-2}\) \\
\hline
\end{tabular}

Table 1: Simulated values of \(\widehat{\Delta}(s, \varepsilon)\) and its standard deviation for different environments, for \(\varepsilon=0.35\). The value of \(\varepsilon\) was chosen such that \(\varepsilon \leq \kappa / 4\) in all environments. The parameters used were: \(N_{\operatorname{sim}}=32723, \gamma=0.2\) and \(\lambda=10\). The \(n\)-Chain environments have \(K=2\) and \(n\) states and the \(n \times n\)-GridWorld environments have \(K=4\) and \(n^{2}\) states.

The code for the experiments is at https://github.com/omardrwch/smoothcruiser-check.```


[^0]:    *equal contribution
    ${ }^{2}$ as done by approximate dynamic programming

[^1]:    ${ }^{3}$ This means that the policy is seen as a function of time, not the states. The open-loop setting is particularly adapted to environments with deterministic transitions.

[^2]:    ${ }^{4} \nabla F_{s}(Q)$ is the gradient of $F_{s}(Q)$ with respect to $Q$.
    ${ }^{5} \mathrm{It}$ is a Boltzmann distribution with temperature $\lambda$.

[^3]:    ${ }^{6}$ This is the case of the max and min functions.

