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The topological correctness of PL-approximations of isomanifolds

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Abstract

Isomanifolds are the generalization of isosurfaces to arbitrary dimension and codimension, i.e. manifolds defined as the zero set of some multivariate multivalued function $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$. A natural (and efficient) way to approximate an isomanifold is to consider its Piecewise-Linear (PL) approximation based on a triangulation $\mathcal{T}$ of the ambient space $\mathbb{R}^d$. In this paper, we give conditions under which the PL-approximation of an isomanifold is topologically equivalent to the isomanifold. The conditions are easy to satisfy in the sense that they can always be met by taking a sufficiently fine triangulation $\mathcal{T}$. This contrasts with previous results on the triangulation of manifolds where, in arbitrary dimensions, delicate perturbations are needed to guarantee topological correctness, which leads to strong limitations in practice. We further give a bound on the Fréchet distance between the original isomanifold and its PL-approximation. Finally we show analogous results for the PL-approximation of an isomanifold with boundary.

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1 Introduction

Isosurfacing especially in low dimensions is used in many fields of application, such as CT scans in medicine, biochemistry, biomedicine, deformable modeling, digital sculpting, environmental science, and mechanics and dynamics [45] and the references mentioned there.

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The marching cube approach [41] being the most popular approach taken. However the result of the marching cube algorithm is not necessarily topologically correct.

Some results on provable correctness were achieved within the computational geometry community [10, 47] in three dimensions. In case the isosurfacing is based on simplices instead of cubes, such as in the marching tetrahedra algorithm [28], some bounds can be given [1, 2], on for example the one-sided Hausdorff distance.

In general homeomorphism proofs in higher dimensions rely on some perturbation scheme to prove that a triangulation is correct [52, 8, 11, 12, 15]. This is a major difference with one and two dimensional surfaces where no such requirements exist [30], [49, Section 10.2]. In this paper we shall see that no perturbations are necessary for isomanifolds as well.

The techniques used here are also different from many of the standard tools. Manifold triangulation/reconstruction algorithms use often Delaunay triangulations [50, 24, 19] and use the closed ball property [32], see for example [3, 18]. Others use Whitney's lemma [13] or are based on collapses [4]. While the current paper mainly relies on the non-smooth implicit function theorem [22] with some Morse theory.

We stress that this does not solve the general problem of triangulating manifolds with boundary, because there exist manifolds that cannot be written as the zero set of a smooth function, because of the existence of manifolds with a non-trivializable normal bundle (generalizations of non-orientable manifolds). For an excellent introduction to the topology of such bundles and manifolds we refer to [44]. We note that locally we can always write an embedded smooth manifold as the zero set of a smooth function, because it can be parametrized as a function from the tangent space to the manifold itself as a consequence of the implicit function theorem. Non-continuous results have been recently obtained in the neighbourhood of the manifold [20]. We stress that there are now obstructions to global closest point projection on manifold [33], and this can even be approximately reconstructed from a point sample [34].

We also emphasize that because we do not use a perturbation scheme, we cannot give lower bounds on the quality of the linear pieces in the Piecewise-Linear (PL) approximation. This is a clear difference with previous methods [52, 15, 14, 12] whose output is a thick triangulation. Although this is an appealing property, it complicates the analysis further and requires perturbation schemes that work fine in theory, but the constants are miserable and the methods do not work in practice in high dimension (see a more detailled discussion in Section 2). Here we ask for less but still provide guarantees on the Fréchet distance and the approximation of the gradients. Perturbation techniques could be used to improve the simplex quality (although only to some very limited extent).

This rest of the paper is subdivided in two sections. In the first section, we treat closed isomanifolds, that is compact manifolds without boundary. In the second section, we prove similar results for isomanifolds with boundary. Extension to general isostratifolds is briefly discussed in Section 4. Apart from some Delaunay based work on triangulations of stratifolds in three dimensions [46, 48, 27, 26, 25], we are not aware of similar results on manifolds with boundary. Significant effort also went in the detection of strata, in this case in arbitrary dimension, see for example [6, 5].

We will often give the simplified statement of results in the main text while complete versions with all constants will be given in the appendix. All proofs can also be found in the appendix.

This paper is part of a collection of closely related papers. The data structure needed to efficiently propagate along the manifold is presented in [16], while details of the implementation will be discussed in an upcoming paper.

2 Isomanifolds (without boundary)

Let $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$ be a smooth function and suppose that 0 is a regular value of f, meaning that at every point x such that $f(x) = 0$, the Jacobian of f is non-degenerate. Then the zero set of f is a manifold as a direct consequence of the implicit function theorem, see for example [29, Section 3.5]. We further assume that $f^{-1}(0)$ is compact. As in [1] we consider a triangulation $\mathcal{T}$ of $\mathbb{R}^d$. The function f_{PL} is a linear interpolation of the values of f at the vertices if restricted to a single simplex $\sigma \in \mathcal{T}$. *For any function $g : \mathbb{R}^d \to \mathbb{R}^{d-n}$ we write g^i for the components of g.*

We prove that under certain conditions that there is an isotopy from the zero set of f to the zero set of f_{PL}. The proof will be using the Piecewise-Linear (PL) map

$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x), \tag{1}$$

which interpolates between f and f_{PL} and is based on the generalized implicit function theorem. The isotopy is in fact stronger than just the existence of a homeomorphism from the zero set of f to that of f_{PL}.

Our result in particular implies that the zero set f_{PL} is a manifold. So this significantly improves on the result of Allgower and Georg [2, Theorem 15.4.1]. The conditions are also weaker, because we do not require that the zero set avoids simplices that have dimension less than the codimension, see [2, Definition 12.2.2] and the text above [2, Theorem 15.4.1]. The idea to avoid these low dimensional simplices originates from Whitney [52], which remarkably was not cited by Allgower and George [2, 1]. Very heavy perturbation schemes for the vertices of the ambient triangulation $\mathcal{T}$ are required for the manifold to be sufficiently far from simplices in the ambient triangulations that have dimension less than the codimension of the manifold [52, 15]. Various techniques have been developed to compute such perturbations with guarantees. They typically consist in perturbing the position of the sample points or in assigning weights to the points. Complexity bounds are then obtained using volume arguments. See, for example [18, 14, 11, 9]. However, these techniques suffer from several drawbacks. The constants in the complexity depend exponentially on the ambient dimension. Moreover the analysis assumes that the probability of the simplices of dimension less than the codimension to intersect the manifold is zero, which is not true when dealing with finite precision. As a result, the actual implementations we are aware of fail to work well in practice except in very simple cases.

We are, by definition, only interested in $f^{-1}(0)$ so we can ignore points that are sufficiently far from this zero set. More precisely, we observe the following: If $f^i(x)$ is positive on a simplex σ then so is f^i_{PL} and thus is F^i_{PL} positive on $\sigma \times [0, 1]$. The same argument holds for negative values. So we see that

▶ Remark. Write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i. Then for all τ, $\{x \mid F_{PL}(x, \tau) = 0\} \subset \Sigma_0$.

The results will be formulated in terms of constants defined in terms of f and the ambient triangulation $\mathcal{T}$.

▶ **Definition 1.** We define

$$\gamma_0 = \inf_{x\in\Sigma_0} |\det(\mathrm{grad}(f^i)\cdot\mathrm{grad}(f^j))_{i,j}| \tag{2}$$

$$\gamma_1 = \sup_{x\in\Sigma_0} \max_i |\mathrm{grad}(f^i)| \tag{3}$$

$$\alpha = \sup_{x\in\Sigma_0} \max_i \|\mathrm{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k\partial_l f^i)_{k,l}\|_2, \tag{4}$$

$$D : \text{the longest edge length of a simplex in } \Sigma_0 \tag{5}$$

$$T : \text{the smallest thickness of a simplex in } \Sigma_0. \tag{6}$$

Here $\mathrm{grad}(f^i) = (\partial_j f_i)_j$ denotes the gradient of component f^i, $\det(\mathrm{grad}(f^i)\cdot\mathrm{grad}(f^j))_{i,j}$ denotes the determinant of the matrix with entries $\mathrm{grad}(f^i)\cdot\mathrm{grad}(f^j)$, $\|\cdot\|_2$ the operator 2-norm, and $(\partial_k\partial_l f_i)_{k,l}$ the matrix of second order derivatives, that is the Hessian (Hes). We recall the definition of the operator norm: $\|A\|_p = \max_{x\in\mathbb{R}^n} \frac{|Ax|_p}{|x|_p}$, with $|\cdot|_p$ the p-norm on $\mathbb{R}^n$.

We will assume that $\gamma_0, \gamma_1, \alpha, D, T \in (0,\infty)$. The thickness is the ratio of the height over the longest edge length.

A good choice for $\mathcal{T}$ is the Coxeter triangulation of type A_d, see [23, 21], or the related Freudenthal triangulations, see [35, 40, 31, 51], which can be defined for different values of D while keeping T constant. In this paper, we will thus think of all the above quantities as well as d and n as constants except D and our results will hold for D small enough.

The result

We are going to construct an ambient isotopy based on (1). The zero set of $F_{PL}(x,0)$ (or $f(x)$) gives the smooth isosurface, while the zero set of $F_{PL}(x,1)$ (or $f_{PL}(x)$) gives the PL approximation, that is the triangulation of the isosurface after possible barycentric subdivision. The map $\tau \mapsto \{x \mid F_{PL}(x,\tau) = 0\}$ in fact gives an isotopy. Without too much extra work we'll also bound the Fréchet distance between $f(x)$ and $f_{PL}(x)$.

Proving isotopy consists of two technical steps, which consume most of the space in the proof below, as well as the use of a standard observation from Morse theory/gradient flow in the third step. The technical steps are

- Let $\sigma \in \mathcal{T}$. We first show that $\{x \mid F_{PL}(x,\tau) = 0\} \cap \sigma \times [0,1]$ is a smooth manifold, under certain conditions.
- We prove that $F_{PL}^{-1}(0)$ is a manifold, under certain conditions, using techniques from nonsmooth analysis.

Along the way we shall also see that $F_{PL}^{-1}(0)$ is never tangent to the $\tau = c$ planes, where c is a constant. This implies that we the gradient field of τ restricted to $F_{PL}^{-1}(0)$, is piecewise smooth and never vanishes.

Now we arrive at the third step, which is similar to a standard observation in Morse theory [42, 43], with the exception that we now consider piecewise-smooth instead of smooth vector fields. We refer to Milnor [42] for an excellent introduction, see Lemma 2.4 and Theorem 3.1 in particular.

▶ **Lemma 2** (Gradient flow induced homeomorphisms). *The flow of a non-vanishing piecewise-smooth gradient vector field of a function τ on a compact manifold generates a homeomorphism form $\tau = c_1$ to $\tau = c_2$, where c_1 and c_2 are constants.*

Bounds on the gradient of τ on the manifold give a bound on the Fréchet distance.

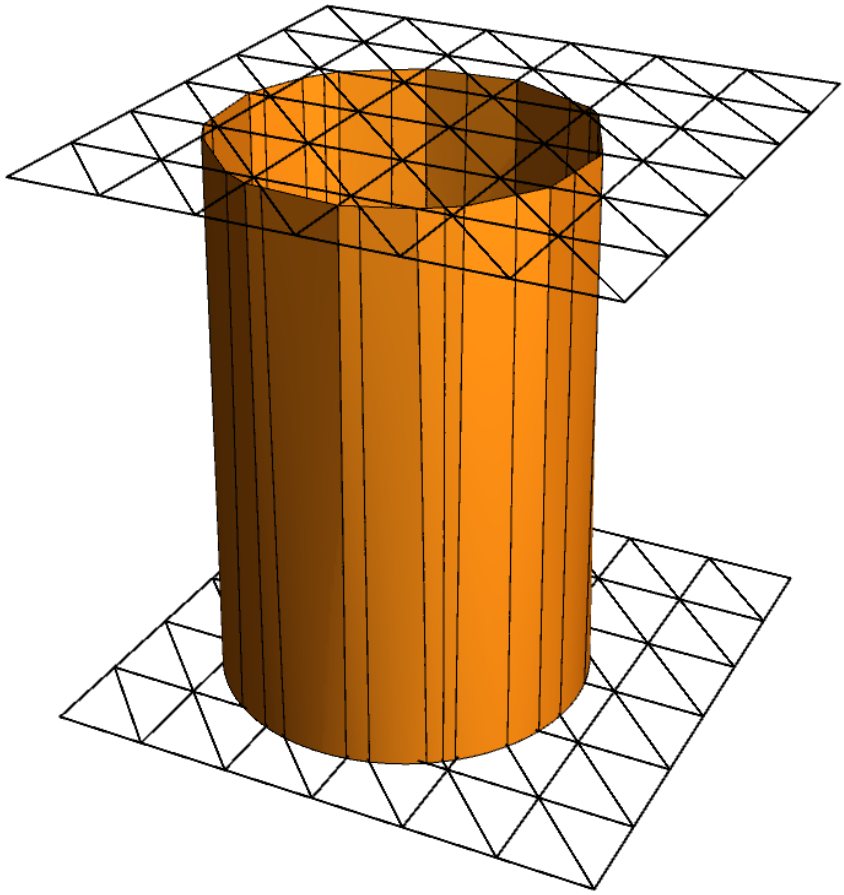

Figure 1 A pictorial overview of the proof. The τ-direction goes upwards. Similarly to Morse theory we find that $f_{PL}^{-1}(0)$ (top) and $f^{-1}(0)$ (bottom) are homeomorphic if the function τ restricted to $F_{PL}^{-1}(0)$ does not encounter a Morse critical point.

2.1 Estimates for a single simplex

We now first concentrate on a single simplex σ and write f_L for the linear function whose values on the vertices of σ coincide with f, that is f_L is the linear extension of the interpolation of f.

2.1.1 Preliminaries and variations of know results

We need a simple estimate similar to Proposition 2.1 of Allgower and George [1].

▶ **Lemma 3.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then $|f_L^i(x) - f^i(x)| \leq 2D^2\alpha$ for all $x \in \sigma$.*

We will also be using an estimate similar to Proposition 2.2 of Allgower and George [1].

▶ **Proposition 4.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then*

$$|\mathrm{grad} f_L^i - \mathrm{grad} f^i| = \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T},$$

for all $x \in \sigma = \{v_k\}$.

2.1.2 Estimates on the gradient inside a single simplex

We write

$$F_L(x,\tau) = (1-\tau)f(x) + \tau f_L(x). \tag{7}$$

We note that F_L extends smoothly outside σ. Here and throughout we restrict ourselves to the setting where $\tau \in [0,1]$. The function F_L has $\mathbb{R}^{d-n}$ as image.

We can now state the following

▶ **Lemma 5.** *If we write* $\mathrm{grad}_{(x,\tau)}$ *for the gradient that includes the* τ *component, we have*

$$|\det(\mathrm{grad}_{(x,\tau)}(F_L^i)\cdot \mathrm{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \tag{8}$$

with $g_1(D) = \mathcal{O}(D)$*. See* (18) *in Appendix A for the exact expression of* g_1*.*

From the previous statement we immediately have that

▶ **Corollary 6** ($F_L^{-1}(0)$ is a manifold in a neighbourhood of $\sigma \times [0,1]$)**.** *If* $\gamma_0 > g_1(D)$ *the implicit function theorem applies to* $F_L(x,\tau)$ *inside* $\sigma \times [0,1]$*. (In fact it applies to an open neighbourhood of this set). In particular, we have proven the first of our two technical steps,* $\{x \mid F_{PL}(x,\tau) = 0\} \cap \sigma \times [0,1]$ *is a smooth manifold.*

2.1.3 Transversality with regard to the τ-direction

We will also prove the main result which we need for the third step, that is the gradient of τ restricted to $F_{PL}(x,\tau) = 0$, is piecewise smooth and never vanishes. We now prove inside each $\sigma \times [0,1]$ the gradient of τ on $F_L = 0$ is smooth and does not vanish.

We first give a simple lower bound on the lengths of vectors $v^1, \ldots, v^{d-n}$, assuming that the norms $|v^i|$ are upper bounded and the determinant of the Gram matrix is lower bounded.

▶ **Lemma 7.** *Let* $v^1, \ldots, v^{d-n} \in \mathbb{R}^d$*,* $|v^i| \le \gamma_1$*, for all* i*, and assume that* $\det(v^i \cdot v^j)_{i,j} > \gamma_0$*. Then* $|v^i| \ge \sqrt{\gamma_0}/\gamma_1^{d-n-1}$*.*

We also need to bound the angle of the vectors $\mathrm{grad}_{(x,\tau)}(F_L^i)$ and the x plane, that is $\mathbb{R}^d \subset \mathbb{R}^{d+1}$. We recall the definition. If $v \in \mathbb{R}^{d+1}$ is a vector and $\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$, is the space spanned by the d basis vectors corresponding to the x-directions, the angle between v and Ξ is $\angle(v,\Xi) = \inf_{w\in\Xi} \angle(v,w)$.

▶ **Lemma 8.** *Let* Ξ *be as above. We have*

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_L^i),\Xi) \le \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

In particular the manifold $F_L^{-1}(0)$ *inside* $\sigma \times [0,1]$ *is never tangent to the* $\tau = c$ *planes, where* c *is a constant.*

Combining Lemma 8 and Corollary 6 gives:

▶ **Corollary 9.** *If* $\gamma_0 > g_1(D)$*, and* $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$ *, then inside each* $\sigma \times [0,1]$ *the gradient of* τ *on* $F_L^{-1}(0)$ *is smooth and does not vanish.*

2.2 Global result

We are now going to prove the global result. For this, we need to recall some definitions and results from non-smooth analysis. We refer to [22] for an extensive introduction.

▶ **Definition 10** (Generalized Jacobian, Definition 2.6.1 of [22]). Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d-n}$, where F is assumed to be just Lipschitz. The generalized Jacobian of F at x_0 denoted by $J_F(x_0)$, is the convex hull of all $(d-n) \times (d+1)$-matrices B obtained as the limit of a sequence of the form $J_F(x_i)$, where $x_i \to x_0$ and F is differentiable at x_i.

Following [22, page 253] we also define:

▶ **Definition 11.** The generalized Jacobian $J_F(x_0)$ is said to be of maximal rank provided every matrix in $J_F(x_0)$ is of maximal rank.

Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of $\mathbb{R}^{d+1}$ by (x, y) accordingly. Fix a point (a, b), with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. We now write:

▶ **Notation 12** ([22, page 256]). *$J_F(x_0, y_0)|_y$ is the set of all $(n+1) \times (n+1)$-matrices M such that, for some $(n+1) \times (d-n)$-matrix N, the $(n+1) \times (d+1)$-matrix $[N, M]$ belongs to $J_F(x_0, y_0)$.*

With these definitions and notations we now have:

▶ **Theorem 13** (The generalized implicit function theorem [22, page 256]). *Suppose that $J_F(a, b)|_y$ is of maximal rank. Then there exists an open set $U \subset \mathbb{R}^{n+1}$ containing a such that there exists a Lipschitz function $g : U \to \mathbb{R}^{d-n}$, such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.*

We recall the definition of F_{PL},

$$F_{PL}(x, \tau) = (1-\tau)f(x) + \tau f_{PL}(x). \tag{1}$$

Because of the definition of α, see (4), and Proposition 4, we have that $\mathrm{grad}_{(x,\tau)} F_{PL}(x, \tau)$ and $\mathrm{grad}_{(x,\tau)} F_{PL}(\tilde{x}, \tau)$ are close if x and $\tilde{x}$ are. In particular,

▶ **Lemma 14.** *Let v be a vertex in $\mathcal{T}$, $x_1, x_2 \in star(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that $\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_1, \tau_1)$ and $\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_2, \tau_2)$ are well defined, then*

$$|\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_1, \tau_1) - \mathrm{grad}_{(x,\tau)} F^i_{PL}(x_2, \tau_2)| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha$$

We now immediately have the same bound on points in the convex hull of a number of such vectors:

▶ **Corollary 15.** *Suppose we are in the setting of Lemma 14 and $x_0, x_1, \ldots, x_m \in star(v)$, $\tau_0, \ldots, \tau_m \in [0, 1]$, and suppose that $\mu_1, \ldots, \mu_m$ are positive weights such that $\mu_1 + \cdots + \mu_m = 1$ then,*

$$\left| \mathrm{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \mathrm{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha.$$

Using Lemma 5 we see

▶ **Lemma 16.** *Let v be a vertex in $\mathcal{T}$, $x_1, \dots, x_m \in star(v)$, and $\tau_1, \dots, \tau_m \in [0,1]$, such that $\operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k)$, $k = 0, \dots, m$ are well defined. If we moreover assume $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$ we have that*

$$\left| \det \left(\left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}(x_k, \tau_k) \right) \right)_{i,j} \right| \geq \gamma_0 - g_2(D),$$

with $g_2(D) = \mathcal{O}(D)$. See (19) in Appendix A for the exact expression of g_2.

From the previous lemma we immediately have that

▶ **Corollary 17** ($\{x \mid F_{PL}(x,\tau) = 0\}$ is a manifold)**.** *If $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, and $\gamma_0 > g_2(D)$ the generalized implicit function theorem, Theorem 13, applies to $F_{PL}(x,\tau) = 0$. In particular, $\{x \mid F_{PL}(x,\tau) = 0\}$ is a manifold.*

We notice that this bound is stronger than the bound in Corollary 6. This means that $\{x \mid F_{PL}(x,\tau) = 0\}$ is a Piecewise-Smooth manifold if the conditions of Corollary 17 hold. The second technical step of the proof is now also completed.

The fact that $F_L(x,\tau) = 0$ is a Piecewise-Smooth manifold and Corollary 9 give that the gradient of τ is a Piecewise-Smooth vector field whose flow we can integrate to give a homeomorphism from the zero set of f to that of f_{PL}.

We summarize in a theorem:

▶ **Theorem 18.** *If, $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, and $\gamma_0 > g_2(D)$ then the zero set of f is isotopic to the zero set of f_{PL}. We stress that one can satisfy all conditions by choosing D sufficiently small.*

2.2.1 Fréchet distance

To bound the Fréchet distance (d_F) between the zero sets of $f(x)$ and f_{PL}, it suffices to bound the angle that the gradient of τ (as restricted to $F_L(x,\tau) = 0$) makes with the (ambient) τ-direction.

For this we will use the angle bound of Lemma 8, together with some estimates that are similar in spirit to those in [7, Lemma C.13].

▶ **Lemma 19.** *Let $v^1, \dots, v^{d-n} \in \mathbb{R}^{d+1}$, $|v^i| \leq \tilde{\gamma}_1$, for all i, and assume that $\det(v^i \cdot v^j)_{i,j} > \tilde{\gamma}_0 > 0$. Let e_τ a unit vector. If for all i, $\cos(\angle v^i, e_\tau) \leq \phi_0$, then for any $w \in \operatorname{span}(v^1, \dots, v^{d-n})$*

$$\cos \angle(w, e_\tau) \leq \frac{(d-n) d^{d-n-1} \phi_0 \tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

Let e_τ be the τ direction and let g_τ be the gradient of τ restricted to $F_{PL}^{-1}(0)$, whenever it exists. We want to bound the angle of g_τ and the τ-direction. Because the homeomorphism is given by the gradient flow, this bounds the Fréchet distance.

There is one subtlety, because the manifold is only Piecewise-Smooth, we need to take into account the points where g_τ is not uniquely defined. Because for each simplex σ, F_L extends to a neighbourhood of $\sigma \times [0,1]$, there exists a limit of $g_\tau(x_i, \tau_i)$ for any sequence (x_i, τ_i) that lies in $\operatorname{int}(\sigma) \times [0,1]$, where int denotes the interior. This means that if we bound g_τ for each simplex we also bound its limits, where the limits are as just described.

We are now ready to combine Lemmas 8, 19, and Theorem 18.

▶ **Corollary 20** (Bound on the Fréchet distance). *Suppose that the conditions of Theorem 18 are satisfied. Then,* $d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D)$, *with* $g_3(D) = \mathcal{O}(D^2)$, *where we think of* γ_0, γ_1, d, n, T *and* α *as constants. See* (19) *in Appendix A for the exact expression of* g_2.

The most important thing to observe is that $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$, so that we find that $d_F(f^{-1}(0), f_{PL}^{-1}(0)) = \mathcal{O}(D^2)$, where we think of γ_0, γ_1, d, n, T and α as constants.

3 Isomanifolds with boundary

We'll now consider isomanifolds with boundary. By this we mean that on top of the function $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$, we'll have another function $f_\partial : \mathbb{R}^d \to \mathbb{R}$ and the set we consider is $M = f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$. This is a manifold with boundary if the gradients of f^i span a $(d-n)$-dimensional space at each point of $f^{-1}(0)$ and the gradients of f^i and f_∂ span a $(d-n+1)$-dimensional space at each point of $\partial M = f^{-1}(0) \cap f_\partial^{-1}(0)$, as a consequence of the submersion theorem.

We'll again write f_{PL} for the PL interpolation of f. Similarly we write $f_{\partial,PL}$ for the PL interpolation of f_∂.

We prove that, under certain conditions, there is an isotopy from $f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$ to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}([0, \infty))$. The conditions are very similar to the conditions we have before, but of course we need to include bounds on the gradient of $f_{\partial,PL}$.

Overview of the proof

We will again construct an isotopy, but in this case it will consist of two steps.

- In the **first step**, we isotope the part of $f^{-1}(0)$ that is far from $f_\partial^{-1}(0)$ to its piecewise linear approximation, while leaving the part of $f^{-1}(0)$ that is close to $f_\partial^{-1}(0)$ smooth. We will denote the result by $M_1 = (F_{PL,1}(\cdot, 1))^{-1}(0)$.
- In the **second step**, we consider a (small) tubular neighbourhood around $f_\partial^{-1}(0)$ as restricted to M_1 by looking at all $f_\partial^{-1}(\epsilon)$ for $|\epsilon|$ sufficiently small.[1] We then isotope $M_1 \cap f_\partial^{-1}(\epsilon)$ to its piecewise linear approximation. Again the isotopy is chosen in such a way that for ϵ relatively large (for the points such that M_1 is already Piecewise-Linear) it leaves $M_1 \cap f_\partial^{-1}(\epsilon)$ invariant. This gives an isotopy of a tubular neighbourhood of $M_1 \cap f_\partial^{-1}(0)$ to its Piecewise-Linear approximation.

We will first partition the manifold in two parts using a smooth bump function $\phi : \mathbb{R} \to [0, 1]$ that is zero in a neighbourhood of zero and $\phi(y) = 1$ if $|y| > y_0$, for some $y_0 > 0$. Such bump functions can be easily constructed, see for example [39, Section 2.2]. We will be using the function $\phi\left(\sum_i (f^i)^2 + f_\partial^2\right)$.

The first step will be using the zero set of the following function:

$$F_{PL,1}(x, \tau) = \left(1 - \tau\phi\left(\sum_i (f^i)^2 + f_\partial^2\right)\right) f(x) + \tau\phi\left(\sum_i (f^i)^2 + f_\partial^2\right) f_{PL}(x), \tag{9}$$

on which we'll apply the same gradient flow argument as before.

[1] We stress that ϵ may be negative.

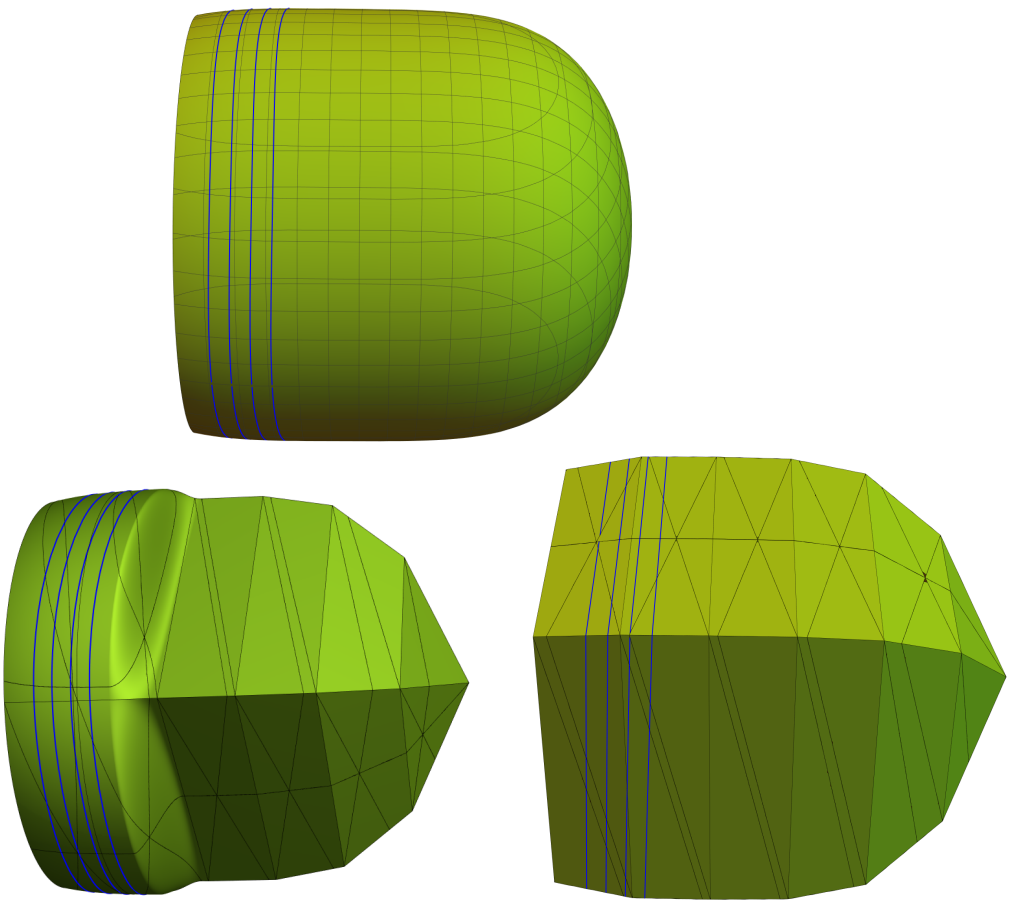

Figure 2 Top: we see the original isosurface with $f_\partial^{-1}(-1/10)$, $f_\partial^{-1}(0)$, $f_\partial^{-1}(1/10)$, and $f_\partial^{-1}(2/10)$ indicated in blue. Bottom left: we see that at the end of Step 1 the neighbourhood of the boundary is intact, while the rest has been isotoped to a Piecewise-Linear approximation. Bottom right: we have also isotoped the neighbourhood of the boundary to a Piecewise-Linear approximation by isotoping $f_\partial^{-1}(\epsilon)$, to its Piecewise-Linear approximation for all sufficiently small ϵ.

The resulting set M_1 is the same zero set of f_{PL} as before if we stay sufficiently far away from ∂M and the isotopy leaves the manifold invariant close to ∂M. In particular, $\partial M_1 = \partial M$.

In the second step, we define an isotopy that will act only on a small neighbourhood of ∂M. Consider the sets $B_1(\epsilon) = M_1 \cap f_\partial^{-1}(\epsilon)$ and, for each of them, define the function

$$\begin{aligned} F_{PL,2,\epsilon}(x,\tau) = &\left(1 - \tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)\right)(F_{PL,1}(x,1), f_\partial(x) - \epsilon) \\ &+ \tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)(f_{PL}(x), f_{\partial,PL}(x) - \epsilon), \end{aligned} \tag{10}$$

where $\psi : \mathbb{R} \to [0,1]$ is now a smooth bump function that is 1 in a sufficiently large neighbourhood of zero (somewhat larger than y_0) and zero outside some compact set. We stress that $F_{PL,2,\epsilon}$ is a mapping from $\mathbb{R}^d \times [0,1]$ to $\mathbb{R}^{d-n+1}$. Using the result for isomanifolds (with some modifications), we can prove that each individual set $B_1(\epsilon)$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}(\epsilon)$

for small ϵ while, for sufficiently large ϵ, it leaves the set invariant.

3.1 Step 1

The proof closely follows the proof for the case without boundary in Section 2. The main technical difficulty will be to provide bounds that serve as the counterparts of Lemmas 5 and 16. To be able to do so we first need to discuss bounds on the bump functions ϕ and ψ.

3.1.1 Bump functions

Following [39, Section 2.2] we write,

$$\zeta_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

For $0 < y_1 < y_2$ we write $\zeta_2(x) = \zeta_1(x - y_1)\zeta_1(y_2 - x)$. Then we define $\phi_l : \mathbb{R} \to [0, 1]$ by $\phi_l(x) = \int_x^{y_2} \zeta_2(x')\mathrm{d}x' \Big/ \int_{y_1}^{y_2} \zeta_2(x')\mathrm{d}x'$. Finally define $\phi_b : \mathbb{R} \to [0, 1]$ by $\phi_b(x) = \phi_l(|x|)$, and let $\phi(x) = 1 - \phi_b(x)$.

▶ **Lemma 21.** *We have $\phi_b(x) \in [0, 1]$ and, writing $2y_1 = y_2 = y_0$,*

$$\partial_x(\phi_l(x)) \leq 2\frac{e^{\frac{4}{3(y_2-y_1)}}}{y_2 - y_1} = 4\frac{e^{\frac{2}{3y_0}}}{y_0} = \gamma_\phi. \tag{11}$$

3.1.2 Inside a single simplex

Similarly to Lemma 5, we now give a condition that ensures that the zero set of $F^i_{PL,1}(x, \tau)$ is smooth inside $\sigma \times [0, 1]$. In fact, similarly to (7), we define

$$\begin{aligned} F^i_{L,1}(x, \tau) &= \left(1 - \tau\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right) f^i(x) + \tau\phi\left(\sum_l (f^l)^2 + f_\partial^2\right) f^i_L(x) \\ &= f(x) + \tau\phi\left(\sum_i (f^i)^2 + f_\partial^2\right)(f^i_L(x) - f^i(x)), \end{aligned}$$

where ϕ is as defined above. Observe that $F^i_{L,1}(x, \tau)$ can be extended to a neighbourhood of $\sigma \times [0, 1]$.

▶ Remark. For the constants, it is better if y_0 can be chosen as large as possible, but we need y_1 to be quite a bit larger than y_0. In turn, we cannot choose y_1 arbitrarily large because this would mean that the gradient field $\mathrm{grad} f_\partial|_{f^{-1}(0)}$ (seen as restricted on $f^{-1}(0)$) would never vanish. The latter is in general impossible thanks to the hairy ball theorem [17].

We introduce the following definition that complements Definition 1:

▶ **Definition 22.**

$$\gamma_2 = \sup_{x \in \Sigma_0} \left|\mathrm{grad}\left(\sum_l (f^l)^2 + f_\partial^2\right)\right| = 2 \sup_{x \in \Sigma_0} \left|\sum_l f^l \mathrm{grad} f^l + f_\partial \,\mathrm{grad} f_\partial\right| \tag{12}$$

We have then the analog of Lemma 5:

▶ **Lemma 23.** *We have :*

$$|\det(\mathrm{grad}_{(x,\tau)}F^i_{L,1}(x,\tau)\cdot \mathrm{grad}_{(x,\tau)}F^j_{L,1}(x,\tau))_{i,j}| > \gamma_0 - g_4(D),$$

with $g_4(D) = \mathcal{O}(D)$. The exact expression of g_4 is given in (21).

The following corollary is then the analog of Corollary 6:

▶ **Corollary 24** ($F^{-1}_{L,1}(0)$ is a manifold)**.** *If $\gamma_0 > g_4(D)$, where $g_4(D) = O(D)$ is as in Lemma 23, then $F^{-1}_{L,1}(0)$ is a smooth manifold inside an ϵ neighbourhood of $\sigma \times [0,1]$.*

3.1.3 Transversality with regard to the τ-direction

We note that, similarly to Lemma 8, we have

▶ **Lemma 25.** *Let Ξ be as in Lemma 8 and γ_ϕ as in* (11).

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_{L,1}),\Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \gamma_2\gamma_\phi 2D^2\alpha - \frac{4dD\alpha}{T}}.$$

In particular, if $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$, $F^{-1}_{L,1}(0)$ (if it is a manifold) is never tangent to the $\tau = c$ planes, where c is a constant.

Now, similarly to Corollary 9, we find that

▶ **Corollary 26** (Transversality with respect to τ for Step 1)**.** *Suppose that $\gamma_0 > g_4(D)$ and that $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$. Then, inside each $\sigma \times [0,1]$, the gradient of τ on $F^{-1}_{L,1}(0)$ is smooth and does not vanish.*

3.1.4 Global result

We now have to prove that $F^{-1}_{PL,1}(0)$ is a manifold. For this, we shall use a bound similar to the one given in Lemma 16, so that we are able to apply the generalized implicit function theorem if this bound is satisfied. But first of all, we need the following bound, which is similar to Lemma 14.

▶ **Lemma 27.** *Assuming that the gradients are well defined, we have*

$$|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_1,\tau_1) - \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_2,\tau_2)| \leq g_5(D),$$

with $g_5(D) = \mathcal{O}(D)$. The exact expression for g_5 is given in (22).

Just as in Corollary 15, we immediately have the same bound on points in the convex hull of a number of such vectors:

▶ **Corollary 28.** *Suppose we are in the setting of Lemma 27 and $x_0, x_1, \ldots, x_m \in star(v)$, $\tau_0, \ldots, \tau_m \in [0,1]$, such that $\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_i,\tau_i)$ is well defined for all i. Further assume that $\mu_1, \ldots, \mu_m$ are positive weights such that $\mu_1 + \cdots + \mu_m = 1$. Then,*

$$\left|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_0,\tau_0) - \sum_{k=1}^{m}\mu_k \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_k,\tau_k)\right| \leq g_5(D).$$

▶ **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$\det\left(\left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_k,\tau_k)\right)\right)_{i,j}$$
$$\geq \gamma_0 - g_4(D) - g_6(D),$$

with $g_6(D) = \mathcal{O}(D)$. The exact expression of g_6 is given in (23).

Lemma 29 immediately yields that

▶ **Corollary 30** ($F^{-1}_{PL,1}(0)$ is a manifold)**.** *If, $\gamma_0 > g_4(D) + g_6(D)$ the generalized implicit function theorem, Theorem 13, applies to $F_{PL,1}(x,\tau) = 0$. In particular $F^{-1}_{PL,1}(0)$ is a manifold.*

We stress again that inside the set $\{x|\phi\left(\sum_i (f^i)^2(x) + f_\partial^2(x)\right) = 1\}$ the zero set of $F_{PL,1}(x,1)$ coincides with the zero set of $f_{PL}(x)$.

3.2 Step 2

Before we can proceed we have to specify the bump function ψ. We suppose that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{101}{100}y_0 \\ 0 & \text{if } |x| \geq 2y_0. \end{cases}$$

In particular we pick $\psi(x) = \phi_b(x)$, with the choice $y_1 = \frac{101}{100}y_0$ and $y_2 = 2y_0$.

First we stress thatthe zero set of $F_{PL,2,\epsilon}(x,1)$ coincides with the zero set of $(f_{PL}(x), f_{\partial,PL}(x) - \epsilon)$, provided that $\psi(\sum_i f_i(x)^2 + f_\partial(x)^2) = 1$.

Secondly, we now claim the following:

▶ **Lemma 31.** *The zero set of $F_{PL,2,\epsilon}(x,1)$ is a subset of the zero set of $f_{PL}(x)$, for each ϵ.*

The technical result that remains to be proven is the counterpart of Theorem 18 for $F_{PL,2,\epsilon}(x,\tau)$ and for each sufficiently small ϵ. To be precise it suffices for $\epsilon \leq 2y_0$. We remark that it is likely that this bound on ϵ can be improved.

We again follow the same path to prove this result. That is we first concentrate on a single simplex and prove that inside that simplex the zero set of $F_{PL,2,\epsilon}$ is a smooth manifold on which the gradient of τ as restricted to the manifold does not vanish. We then prove that is the zero set of $F_{PL,2,\epsilon}$ is globally a manifold.

3.2.1 Assumptions and notations

Because we are now faced with both $f(x)$ and $f_\partial(x)$ we need to introduce a bound on how far the gradients of all there are from being colinear. We write

$$f_B(x) = (f(x), f_\partial(x)). \tag{13}$$

Before we were only interested in the set Σ_0, similarly here we sometimes concentrate on a neighbourhood of the zero set of both f_∂ and f. Therefore we write B_ν for all $\sigma \in \mathcal{T}$ such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$.

We define γ_0^B in terms of the determinant of the Gram matrix of the gradients, that is

$$\gamma_0^B = \inf_{x\in B_\nu\cap\Sigma_0} |\det(\operatorname{grad}(f_B^i)\cdot\operatorname{grad}(f_B^j))_{i,j}|. \tag{14}$$

We note that because we take the gradients we can just ignore the ϵ constant. For the lengths of the gradients of f_B we define,

$$\gamma_1^B = \sup_{x\in\Sigma_0} \max_i |\mathrm{grad}(f_B^i)|, \tag{15}$$

for all $1 \le i \le d-n+1$. Similarly to α, we define β as the bound on the operator 2-norm of all Hessians of f_B, that is

$$\beta = \sup_{x\in\Sigma_0} \max_i \|\mathrm{Hes}(f_B^i)\|_2 = \sup_{x\in\Sigma_0} \max_i \|(\partial_k\partial_l f_B^i)_{k,l}\|_2. \tag{16}$$

We stress that we have chosen our definitions such that $\alpha \le \beta$.

We use the same notation for the ambient triangulation $\mathcal{T}$, the lower bound on the thickness of the simplices T and upper bound on the longest edge length D. We also need to introduce a bound on the differential of the bump function ψ. Similarly to (11) we define,

$$\gamma_\psi = 2\frac{e^{\frac{4}{3(y_2-y_1)}}}{y_2-y_1} = 2\frac{e^{\frac{4}{3(2y_0-\frac{101}{100}y_0)}}}{2y_0-\frac{101}{100}y_0} = \frac{200}{99}\frac{e^{\frac{400}{297y_0}}}{y_0}, \tag{17}$$

because we picked $y_1 = \frac{101}{100}y_0$ and $y_2 = 2y_0$, for ψ.

3.2.2 Inside a single simplex

Similarly to Lemma 23 we now give a conditions that ensure that the zero set of $F_{PL,2,\epsilon}(x,\tau)$ is smooth inside $\sigma\times[0,1]$. In fact similarly to (7) we define

$$\begin{aligned} F_{L,2,\epsilon}(x,\tau) = &\left(1-\tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)\right)(F_{L,1}(x,1), f_\partial(x)-\epsilon) \\ &+ \tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)(f_L(x), f_{\partial,L}(x)-\epsilon), \end{aligned}$$

which can be extended to a neighbourhood of $\sigma\times[0,1]$.

▶ **Lemma 32.** *For all ϵ,*

$$\det(\mathrm{grad}_{(x,\tau)}F^i_{L,2,\epsilon}(x,\tau)\cdot\mathrm{grad}_{(x,\tau)}F^j_{L,2,\epsilon}(x,\tau))_{i,j} \ge \gamma_0^B - g_7(D)$$

with $g_7(D) = \mathcal{O}(D)$. The exact expression of g_7 is given in (24).

We again have the following corollary.

▶ **Corollary 33** ($F^{-1}_{L,2,\epsilon}(0)$ is a manifold)**.** *We have that $F^{-1}_{L,2,\epsilon}(0)$ is a smooth manifold inside an small neighbourhood of $\sigma\times[0,1]$ provided $\gamma_0^B > g_7(D)$, with $g_7(D)$ as in Lemma 32. As usual this can always be satisfied by choosing the triangulation fine enough, that is D sufficiently small.*

3.2.3 Transversality with regard to the τ-direction

Once more similarly to Lemma 8 we have

▶ **Lemma 34.** *Let Ξ be as in Lemma 8. We have*

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_{L,2,\epsilon}),\Xi) \le \frac{2D^2\beta}{\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta - \frac{12dD\beta}{T}}.$$

In particular the manifold $F^{-1}_{L,2,\epsilon}(0)$ inside $\sigma \times [0,1]$, if well defined, is never tangent to the $\tau = c$ planes, where c is a constant, if

$$\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T}.$$

Now similary to Corollary 9 we find that

▶ **Corollary 35** (Transversality with respect to τ for Step 2)**.** *Suppose that the conditions of Corollary 33 are satisfied. If moreover*

$$\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T},$$

then inside each $\sigma \times [0,1]$ the gradient of τ on $F^{-1}_{L,2,\epsilon}(0)$ is smooth and does not vanish.

3.2.4 Global result

We now have to prove that $F^{-1}_{PL,2,\epsilon}(0)$ is a manifold, for all sufficiently small ϵ. For this we shall use a bound similar to the one given in Lemma 16, so that we are able to apply the generalized implicit function theorem if this bound is satisfied. For this we first need the following bound, which is similar to Lemma 27.

▶ **Lemma 36.** *Let v be a vertex in $\mathcal{T}$, $x_1, x_2 \in star(v)$, and $\tau_1, \tau_2 \in [0,1]$, such that $\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_1,\tau_1)$ and $\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_2,\tau_2)$ are well defined, then*

$$|\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_1,\tau_1) - \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_2,\tau_2)| \leq g_8(D),$$

with $g_8(D) = \mathcal{O}(D)$.The exact expression of g_8 is given in (25)*.*

Just as in Corollary 15, we immediately have the same bound on points in the convex hull of a number of such vectors:

▶ **Corollary 37.** *Suppose $x_0, x_1, \ldots, x_m \in star(v)$, $\tau_0, \ldots, \tau_m \in [0,1]$, such that $\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_i,\tau_i)$ is well defined for all i. Further assume that $\mu_1, \ldots, \mu_m$ are positive weights such that $\mu_1 + \cdots + \mu_m = 1$. Then,*

$$\left| \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_0,\tau_0) - \sum_{k=1}^m \mu_k \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_k,\tau_k) \right| \leq g_8(D).$$

▶ **Lemma 38.** *Under the same conditions as in Corollary 37,*

$$\det\left(\left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_k,\tau_k)\right) \cdot \left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)} F^j_{PL,2,\epsilon}(x_k,\tau_k)\right)\right)_{i,j}$$
$$\geq \gamma_0^B - g_7(D) - g_9(D),$$

where $g_9(D) = \mathcal{O}(D)$. The exact expression of g_9 is given in (26)*.*

Lemma 38 immediately yields that

▶ **Corollary 39** (The generalized implicit function theorem in Step 2)**.** *If, $\gamma_0^B > g_7(D) + g_9(D)$ the generalized implicit function theorem, Theorem 13, applies to $F_{PL,1}(x,\tau) = 0$. In particular $F^{-1}_{PL,1}(0)$ is a manifold.*

We stress that this condition only needs to be satisfied in a when $\sum_l (f^l)^2 + (f_\partial)^2$ is small, outside this neighbourhood the isotopy leaves the zero set invariant.

▶ **Theorem 40.** *If,*

$$\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T} \qquad \text{(Corollary 26)}$$

$$\gamma_0 > g_4(D) + g_6(D) \qquad \text{(Corollaries 24 and 30)}$$

$$\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \qquad \text{(Corollary 35)}$$

$$\gamma_0^B > g_7(D) + g_9(D), \qquad \text{(Corollaries 33 and 39)}$$

then $f^{-1}(0) \cap f_\partial^{-1}([0,\infty))$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}([0,\infty))$. We stress that one can satisfy all conditions by choosing D sufficiently small. We refer to Appendix A for the definitions of $g_i(D)$.

4 Isostratifolds

There is no obstruction in principle that prevents us from extending the approach above to isostratifolds. By isostratifolds we mean stratifolds that are given by the zero sets of functions. For example suppose that we want to find a PL approximation of the unit sphere centred at 0 in $\mathbb{R}^3$ including the PL approximations of the intersections of the sphere with slightly deformed $x = 0$, $y = 0$, and $z = 0$-planes, as depicted in Figure 3. This would also give PL approximations of the respective 'octants' of the sphere.

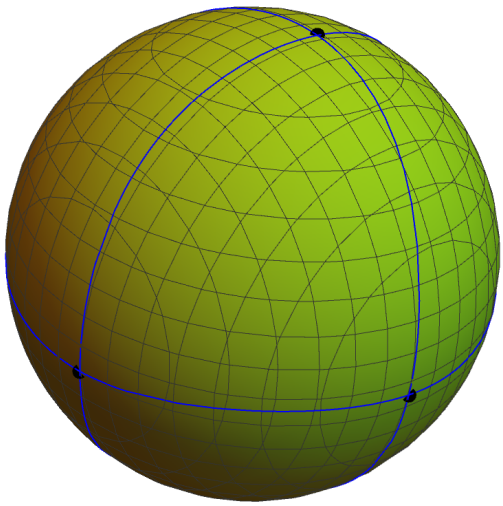

Figure 3 An example of an isostratifold.

We could follow the same procedure as for a manifold with boundary to give density conditions that ensure that the PL approximation is correct. However this would mean that we have to introduce an extra bump function for each stratum as well as an extra isotopy. Even though that in theory this should be possible the analysis becomes prohibitively lengthy.

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A Overview of constants

We give an overview. We write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq$ for all i. We write B_ν for all $\sigma \in \mathcal{T}$ such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$. We write

$$f_B(x) = (f(x), f_\partial(x)) \tag{13}$$

$$\gamma_0 = \inf_{x \in \Sigma_0} |\det(\mathrm{grad}(f^i) \cdot \mathrm{grad}(f^j))_{i,j}| \tag{2}$$

$$\gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\mathrm{grad}(f_B^i) \cdot \mathrm{grad}(f_B^j))_{i,j}| \tag{14}$$

$$\gamma_1 = \sup_{x \in \Sigma_0} \max_i |\mathrm{grad}(f^i)| \tag{3}$$

$$\gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\mathrm{grad}(f_B^i)| \tag{15}$$

$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \mathrm{grad} \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| = \sup_{x \in \Sigma_0} \left| \sum_l f^l \mathrm{grad} f^l + f_\partial \, \mathrm{grad} f_\partial \right| \tag{12}$$

$$\alpha = \sup_{x \in \Sigma_0} \max_i \|\mathrm{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k \partial_l f^i)_{k,l}\|_2 \tag{4}$$

$$\beta = \sup_{x \in \Sigma_0} \max_i \|\mathrm{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2 \tag{16}$$

$$D : \text{the longest edge length of a simplex in } \Sigma_0$$

$$T : \text{the smallest thickness of a simplex in } \Sigma_0.$$

$\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$ is the space spanned by the d basis vectors corresponding to the x-directions. The bump functions give rise to the following:

$$\gamma_\phi = 4 \frac{e^{\frac{2}{3y_0}}}{y_0} \tag{11}$$

$$\gamma_\psi = \frac{200}{99} \frac{e^{\frac{400}{297 y_0}}}{y_0}. \tag{17}$$

The precise expressions for the $g_i(D)$ are:

$$g_1(D) = n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right) \tag{18}$$

$$g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2 D\alpha}{T} + 9\gamma_1 D \right) \right) \tag{19}$$

$$g_3(D) = \left(\frac{(d-n) d^{d-n-1} \sqrt{\gamma_0} 2D^2 \alpha \gamma_1^{d-n-1} \left(\gamma_1 + 2D^2\alpha\right)^{d-n}}{\sqrt{\gamma_0 + \left(2D^2\alpha\gamma_1^{d-n-1}\right)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right) \tag{20}$$

$$g_4(D) = n \left(\left(n \left(\gamma_1 + (1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right.$$
$$\left. \cdot n \left(2 \left((1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right) \tag{21}$$

$$g_5(D) = 2\sqrt{d} d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right) \tag{22}$$

$$g_6(D) = n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1}$$

$$\cdot \left(2\left(2\sqrt{d}d\alpha D + 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\right)\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\right.$$
$$\left. + \left(2\sqrt{d}d\alpha D + 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\right)^2\right). \tag{23}$$

$$g_7(D) = (n+1)\left((n+1)\left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)^2\right)^n$$
$$\cdot (n+1)\left(2\gamma_1^B\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right.$$
$$\left. + \left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)^2\right) \tag{24}$$

$$g_8(D) = 2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right) \tag{25}$$

$$g_9(D) = (n+1)^{n+1}\left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)^2\right)^n$$
$$\cdot \left(2\left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right)\right.$$
$$\cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)$$
$$\left. + \left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right)^2\right) \tag{26}$$

If $\frac{4dD\alpha}{T} \leq \gamma_1$, $g_1(D)$ can be replaced by the simpler $34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}$.

B Recap of essential standard results

We now quickly recall two essential results, namely the smooth implicit function theorem and Friedland's determinant bound.

▶ **Theorem 41** (Smooth implicit function theorem). *Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d-n}$ be a continuously differentiable function. Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of $\mathbb{R}^{d+1}$ by (x, y) accordingly. Fix a point (a, b), with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. If the Jacobian $J_{F,y}(a, b) = (\frac{\partial F^i}{\partial y^j}(a, b))_{i,j}$ is of maximal rank (or equivalently invertible), then there exists an open set $U \subset \mathbb{R}^{n+1}$ containing a such that there exists a unique continuously differentiable function $g : U \to \mathbb{R}^{d-n}$ such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.*

The following result by Friedland [36] will also be used throughout the proofs:

$$|\det(A + E) - \det(A)| \leq n \max\{\|A\|_p, \|A + E\|_p\}^{n-1} \|E\|_p, \tag{27}$$

where A and E are $n \times n$-matrices and $\|\cdot\|_p$ is the p-norm, with $1 \leq p \leq \infty$.

C Proofs

C.1 Proofs of variations of known results in Section 2.1

The following proofs are not exactly the same as the original, but do not differ greatly from known results. We have included the results for completeness.

Proof of Lemma 2. This is a straightforward consequence of the existence and uniqueness of the solution to a differential equation. ◀

The following two proofs are similar to the proofs in Allgower and George [1], however because the statement are a bit different and for completeness we have included these.

Proof of Lemma 3. Taylor's theorem, see for example [29, Theorem 2.8.4], yields that

$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{28}$$

with

$$\begin{aligned} R(v_k) &= 2\sum_{j\neq l}(v_k - x)^j(v_k - x)^l \int_0^1 (1-t)^2 \partial_j \partial_l f^i(v_k - t(v_k - x))\mathrm{d}t \\ &\quad + 2\sum_j \frac{((v_k - x)^j)^2}{2} \int_0^1 (1-t)^2 \partial_j^2 f^i(v_k - t(v_k - x))\mathrm{d}t \\ &\leq 2|v_k - x|^2\alpha && \text{(by (4) and Cauchy-Schwarz)} \\ &\leq 2D^2\alpha && (x \in \sigma) \end{aligned}$$

The function f_L at the point $x = \sum_k \lambda_k v_k$, where $\sum_k \lambda_k = 1$, is by construction

$$\begin{aligned} f_L^i(x) &= \sum_k \lambda_k f^i(v_k) \\ &= \sum_k \lambda_k \left(f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k) \right) && \text{(by (28))} \\ &= \sum_k \lambda_k f^i(x) + \sum_j \partial_j f^i(x)(\sum_k \lambda_k v_k - x)^j + \sum_k \lambda_k R(v_k) \\ &= f^i(x) + 0 + \sum_k \lambda_k R(v_k) \end{aligned}$$

Thanks to the bounds on $R(v_k)$ and Cauchy-Schwarz one has

$$|f_L^i(x) - f^i(x)| \leq 2D^2\alpha$$

◀

Proof of Proposition 4. We again use that

$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{28}$$

with

$$|R(v_k)| \leq 2D^2\alpha \tag{29}$$

Subtracting $f^i(v_l)$ from $f^i(v_k)$ now yields

$$f^i(v_k) - f^i(v_l) = \sum_j \partial_j f^i(x)(v_k - v_l)^j + R(v_k) - R(v_l).$$

Because f_L is the linear interpolation of f, we have $f^i(v_k) - f^i(v_l) = \sum_j \partial_j f^i_L(x)(v_k - v_l)$, and thus

$$\left| \sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))(v_k - v_l)^j \right| \leq |R(v_k) - R(v_l)| \leq 4D^2\alpha.$$

Let now $u = \mu_k v^k$ and $w = \tilde{\mu}_k v_k$, with $\sum \mu_k = \sum \tilde{\mu}_k = 1$, that is $u, w \in \sigma$. Clearly $\sum \mu_k - \tilde{\mu}_k = 0$, so $u - w$ can be developed in a sum such that the m-th term in the sum is $c_m(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})$, with $c_m > 0$, and $\sum |c_m| = 1$. We stress that $v_{\tilde{k}(m)}$ may be the same as $v_{\tilde{k}(m')}$, with $m \neq m'$. We now see that

$$\begin{aligned}
\left| \sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))(u - w)^j \right| &= \left| \sum_{j,m} (\partial_j f^i_L(x) - \partial_j f^i(x)) c_m (v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right| \\
&\leq \sum_m c_m \left| \sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right| \\
&\qquad \text{(by the triangle inequality and } c_m > 0) \\
&\leq \sum_m 4c_m D^2\alpha \qquad \text{(because } \textstyle\sum |c_m| = 1) \\
&= 4D^2\alpha. \qquad (30)
\end{aligned}$$

Because the simplex σ contains a ball of radius the smallest altitude over d centred at its barycentre, that is TD/d with T the thickness, the vector $u - w$ can be chosen to be any vector of length less than tD/d. In particular we can choose

$$(u - w)^j = \frac{tD}{d} \frac{(\partial_j f^i_L(x) - \partial_j f^i(x))}{\sqrt{\sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))^2}}.$$

Plugging this choice into (30) gives

$$\frac{tD}{d} \sqrt{\sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))^2} \leq 4D^2\alpha$$

So that

$$\sqrt{\sum_j (\partial_j f^i_L(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T}.$$

◀

C.2 Proofs of new results in of Section 2.1

▶ **Lemma 5.** *Let* $\mathrm{grad}(f^i) = (\partial_j f_i)_j$ *denote the gradients of the components* f^i*. Suppose that the absolute value of the determinant of the Gram matrix of these gradients is lower bounded, that is*

$$|\det(\mathrm{grad}(f^i) \cdot \mathrm{grad}(f^j))_{i,j}| > \gamma_0,$$

and $|\mathrm{grad}(f^i)| \leq \gamma_1$*, for all* i*. Assume further that the conditions of Proposition 4 are satisfied, then*

$$|\det(\mathrm{grad}(F_L^i)\cdot\mathrm{grad}(F_L^j))_{i,j}| > \gamma_0 - n^{n+1}\left(\gamma_1 + \frac{4dD\alpha}{T}\right)^{2(n-1)}\left(2\gamma_1\frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T}\right)^2\right). \tag{31}$$

If we write $\mathrm{grad}_{(x,\tau)}$ *for the gradient that includes the* τ *component, we find that*

$$|\det(\mathrm{grad}_{(x,\tau)}(F_L^i)\cdot\mathrm{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \tag{8}$$

where $g_1(D) = \mathcal{O}(D)$*, precisely*

$$g_1(D) = n^{n+1}\left(\gamma_1 + \frac{6dD\alpha}{T}\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T}\right)^2\right). \tag{32}$$

Proof of Lemma 5. In this proof we shall be using the following notation:

$$(\mathrm{grad}(f^i)\cdot\mathrm{grad}(f^j))_{i,j} = (Y)_{i,j},$$

and write $Y_{i,j}$ for the entries of the matrix.

By definition of F_L, we have that

$$\begin{aligned}
&\det(\mathrm{grad}(F_L^i)\cdot\mathrm{grad}(F_L^j))_{i,j}\\
&\quad= \det(\mathrm{grad}((1-\tau)f^i(x) + \tau f_L(x)^i)\cdot\mathrm{grad}((1-\tau)f^j(x) + \tau f_L(x)^j))_{i,j}\\
&\quad= \det(\mathrm{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x)))\cdot\mathrm{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x))))_{i,j}
\end{aligned}$$

We shall write

$$(X)_{i,j} = (\mathrm{grad}(F_L^i)\cdot\mathrm{grad}(F_L^j))_{i,j}$$

Proposition 4 now yields that

$$\begin{aligned}
&|X_{i,j} - Y_{i,j}|\\
&\quad=|\mathrm{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x)))\cdot\mathrm{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x)))\\
&\qquad- \mathrm{grad}(f^i(x))\cdot\mathrm{grad}(f^j(x))|\\
&\quad=|\mathrm{grad}(\tau(f_L(x)^i - f^i(x)))\cdot\mathrm{grad}(f^j(x)) + \mathrm{grad}(\tau(f_L(x)^i - f^i(x)))\cdot\mathrm{grad}(f^j(x))\\
&\qquad+ \mathrm{grad}(\tau(f_L(x)^i - f^i(x)))\cdot\mathrm{grad}(\tau(f_L(x)^j - f^j(x)))|\\
&\quad\leq 2\gamma_1\frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T}\right)^2.
\end{aligned} \tag{33}$$

This estimate is used to bound the $\|E\|_p$ term in Friedland's bound, for $p = \infty$. In particular,

$$\|(X-Y)_{i,j}\|_\infty \leq n\left(2\gamma_1\frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T}\right)^2\right). \tag{34}$$

Furthermore,

$$\begin{aligned}
|X_{i,j}| &= |\mathrm{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x)))\cdot\mathrm{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x)))|\\
&\leq \left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2,
\end{aligned} \tag{35}$$

where we used Cauchy-Schwarz. Note that the latter estimate is used to bound the $\max\{\|A\|_p, \|A+E\|_p\}$ term in Friedland's bound, for $p = \infty$. In particular,

$$\|(X)_{i,j}\|_\infty \leq n \left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2. \tag{36}$$

This means that Friedland's bound now yields

$$|\det(X)_{i,j} - \det(Y)_{i,j}| \leq n(\|(X)_{i,j}\|_\infty)^{n-1} \|(X-Y)_{i,j}\|_\infty. \tag{37}$$

Thus,

$$\begin{aligned} |\det(\mathrm{grad}(F_L^i) \cdot \mathrm{grad}(F_L^j))_{i,j}| \geq{}& |\det\left(\mathrm{grad}(f^i(x)) \cdot \mathrm{grad}(f^j(x))\right)_{i,j}| \\ &- n^{n+1}\left(\gamma_1 + \frac{4dD\alpha}{T}\right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T}\right)^2\right) \\ >{}& \gamma_0 - n^{n+1}\left(\gamma_1 + \frac{4dD\alpha}{T}\right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T}\right)^2\right). \end{aligned}$$

The addition of the τ component gives a small extra contribution to the inner product to the gradients

$$\begin{aligned} \tilde{X}_{i,j} ={}& \mathrm{grad}_{(x,\tau)} F_L^i \cdot \mathrm{grad}_{(x,\tau)} F_L^j \\ ={}& \mathrm{grad}(f^i(x) + \tau(f_L^i(x) - f^i(x))) \cdot \mathrm{grad}(f^j(x) + \tau(f_L^j(x) - f^j(x))) \\ &+ (f_L^i(x) - f^i(x))(f_L^j(x) - f^j(x)). \end{aligned}$$

Combining this with (33) and Lemma 3 yields

$$\begin{aligned} |\tilde{X}_{i,j} - Y_{i,j}| &= |\mathrm{grad}_{(x,\tau)} F_L^i \cdot \mathrm{grad}_{(x,\tau)} F_L^j - \mathrm{grad}(f^i(x)) \cdot \mathrm{grad}(f^j(x))| \\ &\leq 2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2 \end{aligned}$$

Similarly to (35), we also have that

$$|\tilde{X}_{i,j}| = |\mathrm{grad}_{(x,\tau)} F_L^i \cdot \mathrm{grad}_{(x,\tau)} F_L^j| \leq \left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2. \tag{38}$$

Applying Friedland's bound again gives

$$\begin{aligned} |\det(\tilde{X})_{i,j}| &= \left|\det(\mathrm{grad}_{(x,\tau)}(F_L^i) \cdot \mathrm{grad}_{(x,\tau)}(F_L^j))_{i,j}\right| \\ &> \gamma_0 - n^{n+1}\left(\left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2\right)^{n-1} \cdot \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2\right) \\ &\geq \gamma_0 - n^{n+1}\left(\gamma_1 + \frac{6dD\alpha}{T}\right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T}\right)^2\right). \end{aligned}$$

(because $T \leq 1$)

◀

▶ **Corollary 42.** *If on top of the conditions of Lemma 5, we also assume that* $\frac{4dD\alpha}{T} \leq \gamma_1$*, the bound* (31) *can be simplified, at the cost of weakening the bound, to*

$$\det(\mathrm{grad}(F_L^i) \cdot \mathrm{grad}(F_L^j))_{i,j} > \gamma_0 - 3 \cdot 2^{2n+1} n^{n+1} (\gamma_1)^{2n-1} \frac{dD\alpha}{T}.$$

The bound (8) *can be simplified, at the cost of weakening the bound, to*

$$|\det(\mathrm{grad}_{(x,\tau)}(F_L^i)\cdot \mathrm{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - 34\cdot\left(\frac{5}{2}\right)^{2n-1} n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}.$$

Proof of Corollary 42. If we assume $\frac{4dD\alpha}{T}\leq\gamma_1$, a short calculation shows that (31) reduces to

$$\begin{aligned}|\det(\mathrm{grad}(F_L^i)\cdot \mathrm{grad}(F_L^j))_{i,j}| &> \gamma_0 - n^{n+1}\left(2\gamma_1\right)^{2(n-1)}\left(3\gamma_1\frac{4dD\alpha}{T}\right)\\ &= \gamma_0 - 3\cdot 2^{2n+1}n^{n+1}(\gamma_1)^{2n-1}\frac{dD\alpha}{T}.\end{aligned}$$

A straightforward calculation shows (8) reduces to

$$\begin{aligned}\det(\tilde{X})_{i,j} &\geq \gamma_0 - n^{n+1}\left(\frac{5}{2}\gamma_1\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T}+\frac{3}{2}\gamma_1\frac{6dD\alpha}{T}\right)\\ &= \gamma_0 - 34\cdot\left(\frac{5}{2}\right)^{2n-1} n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}.\end{aligned}$$

◀

Proof of Lemma 7. Let $A_1 = \mathrm{span}(v^1,\dots,v^{d-n})$. Write v_A^i for the projection of v^i on A_1. Note that the vector v_A^i is $d-n$-dimensional. By construction,

$$\det(v^i\cdot v^j)_{i,j} = \det(v_A^i\cdot v_A^j)_{i,j} = \det((v_A^i)_i^t(v_A^j)_j) = \det((v_A^i)_i)^2,$$

where $(v_A^i)_i$ denotes the $(d-n)\times(d-n)$- matrix whose columns are v_A^i, and $(v_A^i)_i^t$ its transposed. Moreover, $|v^i| = |v_A^i|$. Hadamard's inequality [38] now gives

$$\det((v_A^j)_j) \leq \prod_j |v_A^j| \leq \gamma_1^{d-n-1}|v_A^i|.$$

Using that $\det(v^i\cdot v^j)_{i,j} > \gamma_0$ now yields $\sqrt{\gamma_0}/\gamma_1^{d-n-1}\leq |v_A^i|$. ◀

Proof of Lemma 8. The statement follows fairly straightforwardly from Lemma 7 and Lemma 3. In particular the τ component of $\mathrm{grad}_{(x,\tau)}(F_L^i)$ is bounded in absolute value by $|f_L(x)^i - f^i(x)|\leq 2D^2\alpha$. On the other hand the x component is lower bounded by

$$\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T},$$

as a consequence of Proposition 4 and Lemma 7. This gives that

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_L^i),\Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

◀

C.3 Proofs of Section 2.2

Proof of Lemma 14. Using the standard bounds on matrix norm, see for example [37, Equation (2.3.11)], and (4), we have that $\sqrt{d}\alpha \geq |\partial_k\partial_l f^i|$, for all k, l and i. Because the

supremum of the absolute value of a derivative of a function bounds the Lipschitz constant of the function, we have that

$$|\partial_l f^i(x_1) - \partial_l f^i(x_2)| \leq \sqrt{d}\alpha|x_1 - x_2| \tag{39}$$

$$|f^i(x_1) - f^i(x_2)| \leq \gamma_1|x_1 - x_2|. \tag{40}$$

Note furthermore that

$$|x_1 - x_2| \leq 2D, \tag{41}$$

by the triangle inequality.

Lemma 3 gives that $|f^i_{PL}(x) - f^i(x)| \leq 2D^2\alpha$. Using the triangle inequality we now find that

$$\begin{aligned}|f^i_{PL}(x_1) - f^i_{PL}(x_2)| &\leq |f^i_{PL}(x_1) - f^i(x_1) + f^i(x_1) - f^i(x_2) + f^i(x_2) - f^i_{PL}(x_2)|. \\ &\leq 2\gamma_1 D + 4D^2\alpha \end{aligned} \tag{42}$$

Because

$$\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_1, \tau_1) = \begin{pmatrix} \mathrm{grad}(f^i(x) + \tau_1(f^i_{PL}(x) - f^i(x))) \\ f^i_{PL}(x) - f^i(x) \end{pmatrix},$$

we find that

$$\begin{aligned}
&|\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_1, \tau_1) - \mathrm{grad}_{(x,\tau)} F^i_{PL}(x_2, \tau_2)| \\
&\qquad \leq \sum_l |\partial_l f^i(x_1) - \partial_l f^i(x_2)| \\
&\qquad\quad + \sum_l \tau_1 |\partial_l f^i_{PL}(x_1) - \partial_l f^i(x_1)| + \sum_l \tau_2 |\partial_l f^i_{PL}(x_2) - \partial_l f^i(x_2)| \\
&\qquad\quad + |f^i_{PL}(x_1) - f^i(x_1) - f^i_{PL}(x_2) + f^i(x_2)| \\
&\qquad \leq 2\sqrt{d}d\alpha D && \text{(by (39), and (41))} \\
&\qquad\quad + d\frac{8dD\alpha}{T} && \text{(Proposition 4, and } \tau_1, \tau_2 \in [0,1]) \\
&\qquad\quad + 2\gamma_1 D + 4D^2\alpha + 2\gamma_1 D && \text{(by (40), (41) and (42))} \\
&\qquad \leq 2\sqrt{d}d\alpha D + d\frac{8dD\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \\
&\qquad \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha,
\end{aligned}$$

where we used the triangle inequality several times. ◀

▶ **Lemma 16.** *Suppose that*

$$|\det(\mathrm{grad}(f^i) \cdot \mathrm{grad}(f^j))_{i,j}| > \gamma_0,$$

and $|\mathrm{grad}(f^i)| \leq \gamma_1$, *for all* i. *Let now* v *be a vertex in* $\mathcal{T}$, $x_1, \ldots, x_m \in$ *star*(v), *and* $\tau_1, \ldots, \tau_m \in [0,1]$, *such that* $\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k)$, $k = 0, \ldots, m$ *are well defined. Then,*

$$\left| \det\left(\left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)} F^j_{PL}(x_k, \tau_k) \right) \right)_{i,j} \right|$$

$$\geq \gamma_0 - n^{n+1}\left(\gamma_1 + \frac{6dD\alpha}{T}\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T}\right)^2\right)$$
$$- n^{n+1}\left(\left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2\right)^{n-1}$$
$$\cdot\left(2\left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha\right)\left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha\right)\right.$$
$$\left. + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha\right)^2\right).$$

If we moreover assume $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$, the expression can be simplified, at the cost of weakening the bound, to

$$\left|\det\left(\left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)}F^i_{PL}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)}F^j_{PL}(x_k,\tau_k)\right)\right)_{i,j}\right| \geq \gamma_0 - g_2(D),$$

with $g_2(D) = \mathcal{O}(D)$, in fact

$$g_2(D) = n^{n+1}\left(2^{2n-1}\gamma_1^{2n}\left(\frac{14dD\alpha}{T}\right) + 5^{n-1}\gamma_1^{2n-1}(2d+5)\left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D\right)\right).$$

Proof of Lemma 16. Let $x_0 \in \operatorname{star}(v)$ and $\tau_0 \in [0,1]$, be such that $\operatorname{grad}_{(x,\tau)}F^i_L(x_0,\tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior of a d-simplex in $\mathcal{T}$. Lemma 5 gives that

$$|\det(\tilde{X})_{i,j}| = |\det(\operatorname{grad}_{(x,\tau)}(F^i_L(x_0,\tau_0)\cdot\operatorname{grad}_{(x,\tau)}(F^j_L(x_0,\tau_0)))_{i,j}|$$
$$> \gamma_0 - n^{n+1}\left(\gamma_1 + \frac{6dD\alpha}{T}\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T}\right)^2\right).$$

We want to use this together with Corollary 15 and Friedland's bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (38) also holds for the convex hull, that is

$$\left|\left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)}F^i_{PL}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)}F^j_{PL}(x_k,\tau_k)\right)\right| \leq \left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2.$$

We write

$$Z_{i,j} = \left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)}F^i_{PL}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^{m}\mu_k \operatorname{grad}_{(x,\tau)}F^j_{PL}(x_k,\tau_k)\right).$$

Similarly to (36), we have that

$$\|(Z)_{i,j}\|_\infty \leq n\left(\left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2\right) \tag{43}$$

We note that

$$\left|\operatorname{grad}_{(x,\tau)}F^i_{PL}(x_0,\tau_0)\right| = \left|\begin{pmatrix}\operatorname{grad}(f^i(x) + \tau(f^i_{PL}(x) - f^i(x)))\\ f^i_{PL}(x) - f^i(x)\end{pmatrix}\right|$$

$$\leq \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \qquad \text{(because } \operatorname{grad}(f^i) \leq \gamma_1\text{, Proposition 4, and Lemma 3)} \tag{44}$$

Corollary 15 now gives

$$\begin{aligned}
|Z_{i,j} - \tilde{X}_{i,j}| =& \left| \left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_k, \tau_k) \right) \right. \\
& \left. - \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_0, \tau_0) \right| \\
=& \left| \left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) - \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) + \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) \right) \right. \\
& \cdot \left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_k, \tau_k) - \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_0, \tau_0) + \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_0, \tau_0) \right) \\
& \left. - \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_0, \tau_0) \right| \\
\leq& \left| \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_0, \tau_0) \right| \\
& + \left| \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) \right| \\
& + \left| \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right| \\
& \cdot \left| \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_k, \tau_k) \right| \\
\leq& 2 \left(\frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
& \qquad \text{(by Corollary 15 and (44))} \\
& + \left(\frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right)^2 \qquad \text{(by Corollary 15)}
\end{aligned} \tag{45}$$

Similarly to (34), we have that

$$\begin{aligned}
\|(Z - \tilde{X})_{i,j}\|_\infty \leq& n2 \left(\frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
& + n \left(\frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right)^2
\end{aligned} \tag{46}$$

Indeed, Friedland now gives,

$$|\det(Z)_{i,j}| = \left| \det \left(\left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL}(x_k, \tau_k) \right) \right)_{i,j} \right|$$

$$
\begin{aligned}
&\geq \gamma_0 - n^{n+1}\left(\gamma_1 + \frac{6dD\alpha}{T}\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T}\right)^2\right) && \text{(by Lemma 5)}\\
&\quad - n^{n+1}\left(\left(\gamma_1 + \frac{4dD\alpha}{T}\right)^2 + (2D^2\alpha)^2\right)^{n-1} && \text{(by 43)}\\
&\quad \cdot\left(2\left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha\right)\left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha\right)\right.\\
&\quad \left. + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha\right)^2\right)\\
& && \text{(by 46, note that } n \text{ was factored out)}
\end{aligned}
$$

If we further assume that $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$, we can simplify the bound to

$$
\begin{aligned}
|\det(Z)_{i,j}| \geq &\gamma_0 - n^{n+1}\left(2\gamma_1\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T} + \gamma_1\left(\frac{6dD\alpha}{T}\right)\right)\\
&- n^{n+1}\left(\left(2\gamma_1\right)^2 + \left(2D\alpha\right)^2\right)^{n-1}\\
&\cdot\left(2\left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D\right)\left(2\gamma_1 + 2dD\alpha\right) + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha\right)^2\right)\\
& && \text{(because } D \leq 1\text{, and } T \leq 1\text{)}\\
\geq &\gamma_0 - n^{n+1}\left(2\gamma_1\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T} + \gamma_1\left(\frac{6dD\alpha}{T}\right)\right)\\
&- n^{n+1}\left(5\gamma_1^2\right)^{n-1}\\
&\cdot\left(2\left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D\right)\left(2\gamma_1 + \gamma_1\right) + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + \gamma_1 D\right)^2\right)\\
\geq &\gamma_0 - n^{n+1}\left(2\gamma_1\right)^{2n-1}\left(2\gamma_1\frac{4dD\alpha}{T} + \gamma_1\left(\frac{6dD\alpha}{T}\right)\right)\\
&- n^{n+1}\left(5\gamma_1^2\right)^{n-1}\left(6\gamma_1\left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D\right) + \left(\frac{10d^2D\alpha}{T} + 5\gamma_1 D\right)^2\right)\\
\geq &\gamma_0 - n^{n+1}\left(2\gamma_1\right)^{2n-1}\gamma_1\left(\frac{14dD\alpha}{T}\right)\\
&- n^{n+1}\left(5\gamma_1^2\right)^{n-1}\left(6\gamma_1\left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D\right) + \left(2d\gamma_1 + 5\gamma_1 D\right)\left(\frac{10d^2D\alpha}{T} + 5\gamma_1 D\right)\right)\\
\geq &\gamma_0 - n^{n+1}2^{2n-1}\gamma_1^{2n}\left(\frac{14dD\alpha}{T}\right)\\
&- n^{n+1}5^{n-1}\gamma_1^{2n-2}\left(\left(2d\gamma_1 + 5\gamma_1\right)\left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D\right)\right)\\
\geq &\gamma_0 - n^{n+1}\left(2^{2n-1}\gamma_1^{2n}\left(\frac{14dD\alpha}{T}\right) + 5^{n-1}\gamma_1^{2n-1}\left(2d+5\right)\left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D\right)\right)
\end{aligned}
$$

◀

▶ **Corollary 17.** *If $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, and*

$$
\gamma_0 > g_2(D) = n^{n+1}\left(2^{2n-1}\gamma_1^{2n}\left(\frac{14dD\alpha}{T}\right) + 5^{n-1}\gamma_1^{2n-1}\left(2d+5\right)\left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D\right)\right)
$$

the generalized implicit function theorem, Theorem 13, applies to $F_L(x,\tau) = 0$.

▶ **Theorem 18.** *Let $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$ be a smooth function. Write f_{PL} for the Piecewise-Linear function that is the linear interpolation of the values of f at the vertices, if restricted to a single simplex $\sigma \in \mathcal{T}$. Write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq$ for all i. Suppose that Σ_0 is compact. Suppose that for in each simplex in Σ_0 we have that*

- $|\mathrm{grad}(f^i)| \leq \gamma_1$, *for all i.*
- $|\det(\mathrm{grad}(f^i) \cdot \mathrm{grad}(f^j))_{i,j}| > \gamma_0$, *for all i, j.*

If moreover, $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, and

$$\gamma_0 > g_2(D) = n^{n+1}\left(2^{2n-1}\gamma_1^{2n}\left(\frac{14d\alpha}{T}\right) + 5^{n-1}\gamma_1^{2n-1}(2d+5)\left(\frac{24d^2\alpha}{T} + 9\gamma_1\right)\right)D,$$

then the zero set of f is isotopic to the zero set of f_{PL}.

Proof of Lemma 19. Write

$$w = \mu_1 v^1 + \cdots + \mu_{d-n} v^{d-n},$$

with $\mu_1, \ldots, \mu_{d-n} \in \mathbb{R}$. By definition,

$$\cos\angle(w, e_\tau) = \frac{\sum_i \mu_i v^i \cdot e_\tau}{|w|} = \frac{\sum_i \mu_i v^i \cdot e_\tau}{\sqrt{\sum_{i,j} \mu_i \mu_j v^i \cdot v^j}}.$$

We note that $|\sum_i \mu_i v^i \cdot e_\tau| \leq (d-n)|\mu|\tilde{\gamma}_1\phi_0$, where μ denotes the vector (μ_i).

The matrix $(v^i \cdot v^j)_{i,j}$ is a symmetric positive definite matrix, because it is a Gram matrix and its determinant strictly positive, by assumption. This means that the smallest eigenvalue $\lambda_{\min}$ of $(v^i \cdot v^j)_{i,j}$, gives a lower bound, that is

$$\sum_{i,j} \mu_i \mu_j v^i \cdot v^j \geq \lambda_{\min}|\mu|^2.$$

The determinant of a square matrix is the product of its eigenvalues. The largest eigenvalue is bounded by the operator norm, which in turn is bounded by the Frobenius norm [37, (2.3.7)], denoted by $\|\cdot\|_F$. This gives that the largest eigenvalue $\lambda_{\max}$ of $(\langle v^i, v^j\rangle)$ is bounded:

$$\lambda_{\max} \leq \|(v^i \cdot v^j)\|_2 \leq \|(v^i)_i^t\|_2 \|(v^i)_i\|_2 \leq \|(v^i)_i\|_F^2 \leq d^2\tilde{\gamma}_1^2.$$

For the smallest eigenvalue we now find that

$$\tilde{\gamma}_0 < \det(v^i \cdot v^j)_{i,j} \leq \lambda_{\max}^{d-n-1}\lambda_{\min} \leq d^{2(d-n-1)}\tilde{\gamma}_1^{2(d-n-1)}\lambda_{\min}.$$

Combining these results gives

$$\cos\angle(w, e_\tau) = \frac{\sum_i \mu_i \langle v^i, e_\tau\rangle}{\sqrt{\sum_{i,j} \mu_i \mu_j \langle v^i, v^j\rangle}} \leq \frac{(d-n)|\mu|\tilde{\gamma}_1\phi_0}{|\mu|\frac{\sqrt{\tilde{\gamma}_0}}{d^{d-n-1}\tilde{\gamma}_1^{d-n-1}}} = \frac{(d-n)d^{d-n-1}\phi_0\tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

◀

▶ **Corollary 20.** *Suppose that the conditions of Theorem 18 are satisfied. Then,*

$$d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan\arcsin g_3(D),$$

with $g_3(D) = \mathcal{O}(D^2)$, in fact

$$g_3(D) = \left(\frac{(d-n)d^{d-n-1}\sqrt{\gamma_0}2D^2\alpha\gamma_1^{d-n-1}\left(\gamma_1 + 2D^2\alpha\right)^{d-n}}{\sqrt{\gamma_0 + \left(2D^2\alpha\gamma_1^{d-n-1}\right)^2}\sqrt{\gamma_0 - 34\cdot\left(\frac{5}{2}\right)^{2n-1} n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}}}\right).$$

Proof of Corollary 20. Lemma 8 gives

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F^i),\Xi) \leq \frac{2D^2\alpha\gamma_1^{d-n-1}}{\sqrt{\gamma_0}},$$

and thus using the notation of Lemma 19,

$$\begin{aligned}\cos(\angle\mathrm{grad}_{(x,\tau)}(F^i),e_\tau) &\leq \sin\arctan\left(\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1}-\frac{4dD\alpha}{T}}\right)\\ &= \frac{\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1}-\frac{4dD\alpha}{T}}}{\sqrt{1+\left(\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1}-\frac{4dD\alpha}{T}}\right)^2}}\\ &= \frac{2D^2\alpha}{\sqrt{\left(\sqrt{\gamma_0}/\gamma_1^{d-n-1}-\frac{4dD\alpha}{T}\right)^2+(2D^2\alpha)^2}}.\end{aligned}$$

By Corollary 42

$$\tilde{\gamma}_0 = \gamma_0 - 34\cdot\left(\frac{5}{2}\right)^{2n-1} n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}.$$

Lemma 3 and $|\mathrm{grad}(f^i)| \leq \gamma_1$ give

$$\tilde{\gamma}_1 = \gamma_1 + 2D^2\alpha.$$

Plugging this into Lemma 19 gives

$$\cos\angle(w,e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha\left(\gamma_1+2D^2\alpha\right)^{d-n}}{\sqrt{\left(\sqrt{\gamma_0}/\gamma_1^{d-n-1}-\frac{4dD\alpha}{T}\right)^2+(2D^2\alpha)^2}\sqrt{\gamma_0-34\cdot\left(\frac{5}{2}\right)^{2n-1}n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}}},$$

with $w \in \mathrm{span}_i(\mathrm{grad}_{(x,\tau)}(F^i))$. Because the tangent space to $F_L = 0$ is normal to $\mathrm{span}_i(\mathrm{grad}_{(x,\tau)}(F_L^i))$, we see that

$$\sin\angle(g_\tau,e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha\left(\gamma_1+2D^2\alpha\right)^{d-n}}{\sqrt{\left(\sqrt{\gamma_0}/\gamma_1^{d-n-1}-\frac{4dD\alpha}{T}\right)^2+(2D^2\alpha)^2}\sqrt{\gamma_0-34\cdot\left(\frac{5}{2}\right)^{2n-1}n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}}}.$$

This means that the angle between g_τ and e_τ is very small (we can choose D arbitrarily small). This means that as $\tau \in [0,1]$ the distance between begin and end point of the gradient flow, and thus the Fréchet distance, by $\tan\angle(g_\tau,e_\tau)$, that is

$$\begin{aligned}&d_F(f^{-1}(0),f_{PL}^{-1}(0))\\ &\quad\leq \tan\arcsin\left(\frac{(d-n)d^{d-n-1}2D^2\alpha\left(\gamma_1+2D^2\alpha\right)^{d-n}}{\sqrt{\left(\sqrt{\gamma_0}/\gamma_1^{d-n-1}-\frac{4dD\alpha}{T}\right)^2+(2D^2\alpha)^2}\sqrt{\gamma_0-34\cdot\left(\frac{5}{2}\right)^{2n-1}n^{n+1}\gamma_1^{2n}\frac{dD\alpha}{T}}}\right)\end{aligned}$$

◀

C.4 Proofs of Section 3.1

Proof of Lemma 21. As mentioned, by construction $\phi_b(x) \in [0,1]$. Because $\partial_x\beta\left(\frac{y_1+y_2}{2}\right) = 0$ and this is the only zero of the derivative in the open interval (y_1,y_2), we see that

$$\beta(x) \leq \beta\left(\frac{y_1+y_2}{2}\right) = e^{\frac{4}{y_1-y_2}}.$$

Hence, because $\beta(x) \geq 0$,

$$\int_{y_1}^{y_2} \beta(x) \leq (y_2 - y_1) e^{\frac{4}{y_1 - y_2}}$$

Because $\beta(x)$ is monotone on $[y_1, \frac{y_1+y_2}{2}]$ we also have

$$\int_{y_1}^{y_2} \beta(x) \geq \frac{y_2 - y_1}{2} \beta\left(\frac{3}{4}y_1 + \frac{1}{4}y_2\right) = \frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}$$

We now have

$$\begin{aligned}
\partial_x(\phi_l(x)) &= \partial_x \left(\int_x^{y_2} \beta(x')\mathrm{d}x' \Big/ \int_{y_1}^{y_2} \beta(x')\mathrm{d}x' \right) \\
&= \beta(x) / \int_{y_1}^{y_2} \beta(x')\mathrm{d}x' \\
&\leq \frac{e^{\frac{4}{y_1 - y_2}}}{\frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}} \\
&= 2 \frac{e^{\frac{4}{y_1 - y_2} - \frac{16}{3(y_1 - y_2)}}}{y_2 - y_1} \\
&= 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} \qquad \text{(DefCphi)}
\end{aligned}$$

◀

▶ **Lemma 23.** *Suppose that*

$$|\det(\mathrm{grad}(f^i) \cdot \mathrm{grad}(f^j))_{i,j}| > \gamma_0,$$

and $|\mathrm{grad}(f^i)| \leq \gamma_1$, *for all* i *and that the bound* (12) *is satisfied. Then,*

$$\det(\mathrm{grad}_{(x,\tau)} F^i_{L,1}(x,\tau) \cdot \mathrm{grad}_{(x,\tau)} F^j_{L,1}(x,\tau))_{i,j} > \gamma_0 - g_4(D),$$

with $g_4(D) = \mathcal{O}(D)$, *in fact,*

$$\begin{aligned}
g_4(D) =& n \left(\left(n \left(\gamma_1 + (1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right. \\
& \left. \cdot n \left(2 \left((1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right),
\end{aligned}$$

Proof of Lemma 23. We start with an estimate on the individual $\mathrm{grad}_{(x,\tau)} F^i_{L,1}(x,\tau)$

$$\begin{aligned}
& \left| \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)} F^i_{L,1}(x,\tau) \right| \\
&= \left| \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)} \left(f(x) + \tau\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f^i_L(x) - f^i(x)) \right) \right| \\
&= \left| -\mathrm{grad}_{(x,\tau)} \left(\tau\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f^i_L(x) - f^i(x)) \right) \right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \begin{pmatrix} -\tau \mathrm{grad}\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L^i(x) - f^i(x)) - \tau\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\mathrm{grad}(f_L^i(x) - f^i(x)) \\ \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L^i(x) - f^i(x)) \end{pmatrix} \right| \\
&= \left(\left| \tau \mathrm{grad}\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L^i(x) - f^i(x)) + \tau\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\mathrm{grad}(f_L^i(x) - f^i(x)) \right|^2 \right. \\
&\quad \left. + \left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L^i(x) - f^i(x))\right)^2 \right)^{1/2} \\
&\leq \left| \tau \mathrm{grad}\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L^i(x) - f^i(x)) \right| + \left| \tau\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\mathrm{grad}(f_L^i(x) - f^i(x)) \right| \\
&\quad + \left| \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L^i(x) - f^i(x)) \right| && \text{(by the triangle inequality)} \\
&\leq \gamma_2\gamma_\phi |f_L^i(x) - f^i(x)| + \left| \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\mathrm{grad}(f_L^i(x) - f^i(x)) \right| \\
& && \text{(because } \tau \leq 1\text{, (12), (11))} \\
&\quad + |f_L^i(x) - f^i(x)| && \text{(because } \phi \in [0,1]\text{)} \\
&\leq \gamma_2\gamma_\phi |f_L^i(x) - f^i(x)| + \left|\mathrm{grad}(f_L^i(x) - f^i(x))\right| && \text{(because } \phi \in [0,1]\text{)} \\
&\quad + |f_L^i(x) - f^i(x)| \\
&\leq (1 + \gamma_2\gamma_\phi) 2D^2\alpha + \frac{4dD\alpha}{T} && \text{(by Lemma 3 and Proposition 4)}
\end{aligned}
\tag{47}
$$

We now write

$$(X')_{i,j} = (\mathrm{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) \cdot \mathrm{grad}_{(x,\tau)} F_{L,1}^j(x,\tau))_{i,j}.$$

Expanding yields

$$
\begin{aligned}
(X')_{i,j} &= \left(\left(\mathrm{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} \right) \right. \\
&\quad \left. \cdot \left(\mathrm{grad}_{(x,\tau)} F_{L,1}^j(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
&= \left(\left(\mathrm{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} \right) \right. \\
&\quad \left. \cdot \left(\mathrm{grad}_{(x,\tau)} F_{L,1}^j(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
&= (Y)_{i,j} \\
&\quad + \left(\left(\mathrm{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \mathrm{grad} f^j(x) \\ 0 \end{pmatrix} \right)_{i,j} \\
&\quad + \left(\begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} \cdot \left(\mathrm{grad}_{(x,\tau)} F_{L,1}^j(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
&\quad + \left(\left(\mathrm{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i(x) \\ 0 \end{pmatrix} \right) \right.
\end{aligned}
$$

$$\cdot \left(\mathrm{grad}_{(x,\tau)} F^j_{L,1}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j(x) \\ 0 \end{pmatrix}\right)\Bigg)_{i,j}, \tag{48}$$

where we again used the notation

$$(\mathrm{grad}(f^i) \cdot \mathrm{grad}(f^j))_{i,j} = (Y)_{i,j}.$$

We now see by Cauchy-Schwarz and the triangle inequality that

$$|X'_{i,j} - Y_{i,j}| \leq 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\gamma_1 + \left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2,$$

by (47). This also means that for the operator norm of the matrix we find

$$\|(X'-Y)_{i,j}\|_\infty \leq n\left(2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\gamma_1 + \left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2\right) \tag{49}$$

Similarly to (35) and (36), we also see that (47) gives

$$|X'_{i,j}| \leq \left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2, \tag{50}$$

and thus,

$$\|X'_{i,j}\|_\infty \leq n\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2. \tag{51}$$

Friedland's bound (37) now gives

$$\begin{aligned}
&\det(\mathrm{grad}_{(x,\tau)} F^i_{L,1}(x,\tau) \cdot \mathrm{grad}_{(x,\tau)} F^j_{L,1}(x,\tau))_{i,j} \\
&\quad > \gamma_0 - n\left(\left(n\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2\right)^{n-1}\right. \\
&\qquad \left.\cdot\, n\left(2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\gamma_1 + \left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2\right)\right),
\end{aligned}$$

where we used (49), and (51). ◀

Proof of Lemma 25. As we have seen in the calculation in (47), the τ component of $\mathrm{grad}_{(x,\tau)}(F^i)$ is bounded in absolute value by $|f_L(x)^i - f^i(x)| \leq 2D^2\alpha$. On the other hand the x component is lower bounded by

$$\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \gamma_2\gamma_\phi 2D^2\alpha - \frac{4dD\alpha}{T},$$

as a consequence of Proposition 4 and the calculation in (47). This gives that

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_{L,1}), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \gamma_2\gamma_\phi 2D^2\alpha - \frac{4dD\alpha}{T}}.$$

◀

▶ **Lemma 27.** *Under the same conditions as Lemma 14 we have*

$$|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_1,\tau_1) - \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_2,\tau_2)|$$
$$\leq g_5(D) = 2\sqrt{d}d\alpha D + 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)$$

Proof of Lemma 27. By expansion we see that

$$\begin{aligned}
&|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_1,\tau_1) - \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_2,\tau_2)| \\
&\quad= \left|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_1,\tau_1) - \begin{pmatrix} \mathrm{grad} f^i(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \mathrm{grad} f^i(x_2) \\ 0 \end{pmatrix}\right. \\
&\qquad \left. + \begin{pmatrix} \mathrm{grad} f^i(x_2) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_2,\tau_2)\right| \\
&\quad\leq \left|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_1,\tau_1) - \begin{pmatrix} \mathrm{grad} f^i(x_1) \\ 0 \end{pmatrix}\right| + \left|\begin{pmatrix} \mathrm{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \mathrm{grad} f^i(x_2) \\ 0 \end{pmatrix}\right| \\
&\qquad + \left|\begin{pmatrix} \mathrm{grad} f^i(x_2) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_2,\tau_2)\right| \qquad \text{(by the triangle inequality)} \\
&\quad\leq 2\sqrt{d}d\alpha D + 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right) \qquad \text{(by (39), (41) and (47) twice)}
\end{aligned}$$

◀

▶ **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$\det\left(\left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)}F^j_{PL,1}(x_k,\tau_k)\right)\right)_{i,j}$$
$$\geq \gamma_0 - g_4(D) - g_6(D),$$

with $g_6(D) = \mathcal{O}(D)$, where as usual we regard, α, γ_1, γ_2, γ_ϕ, d, n and T as constants. In fact,

$$\begin{aligned}
g_4(D) =& n^{n+1}\left(\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^{2(n-1)}\right. \\
&\left.\cdot\left(2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\gamma_1 + \left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2\right)\right) \\
g_6(D) =& n^{n+1}\left(\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2\right)^{n-1} \\
&\cdot\left(2\left(2\sqrt{d}d\alpha D + 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\right)\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\right. \\
&\left. + \left(2\sqrt{d}d\alpha D + 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\right)^2\right).
\end{aligned}$$

Proof of Lemma 29. The proof is more or less the same as the proof of Lemma 16, but with more complicated bounds. Let $x_0 \in \mathrm{star}(v)$ and $\tau_0 \in [0,1]$, be such that $\mathrm{grad}_{(x,\tau)}F^i(x_0,\tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior of a d-simplex in $\mathcal{T}$. Lemma 23 gives that

$$|\det(\tilde{X}')_{i,j}| = |\det(\mathrm{grad}_{(x,\tau)}(F^i(x_0,\tau_0)\cdot\mathrm{grad}_{(x,\tau)}(F^j(x_0,\tau_0)))_{i,j}|$$

$$>\gamma_0 - n^{n+1}\left(\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^{2(n-1)}\right.$$
$$\left.\cdot\left(2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\gamma_1 + \left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2\right)\right).$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (50) also holds for the convex hull, that is

$$\left|\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_k,\tau_k)\right)\right|$$
$$\leq \left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2.$$

We write

$$Z'_{i,j} = \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_k,\tau_k)\right).$$

Similarly to (36), we have that

$$\|(Z')_{i,j}\|_\infty \leq n\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)^2 \tag{52}$$

We note that

$$\left|\operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_0,\tau_0)\right| \leq \gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}$$
$$\text{(because } \operatorname{grad}(f^i) \leq \gamma_1\text{, and (47))} \tag{53}$$

Similarly to (45), we find that

$$|Z'_{i,j} - \tilde{X}'_{i,j}| = \left|\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_k,\tau_k)\right)\right.$$
$$\left. - \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_0,\tau_0)\cdot \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_0,\tau_0)\right|$$
$$\leq \left|\operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_0,\tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_k,\tau_k)\right|\left|\operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_0,\tau_0)\right|$$
$$+ \left|\operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_0,\tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_k,\tau_k)\right|\left|\operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_0,\tau_0)\right|$$
$$+ \left|\operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_0,\tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_k,\tau_k)\right|$$
$$\cdot\left|\operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_0,\tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_k,\tau_k)\right|$$
$$\leq 2\left(2\sqrt{d}d\alpha D + 2\left((1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)\right)\left(\gamma_1 + (1+\gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T}\right)$$
$$\text{(by Corollary 28 and (53))}$$

$$+\left(2\sqrt{d}d\alpha D+2\left((1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\right)^2 \qquad \text{(by Corollary 28)}$$

Similarly to (34), we have that

$$\begin{aligned}\|(Z'-\tilde{X}')_{i,j}\|_\infty \leq & 2n\left(2\sqrt{d}d\alpha D+2\left((1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\right)\left(\gamma_1+(1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\\ &+n\left(2\sqrt{d}d\alpha D+2\left((1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\right)^2 \qquad (54)\end{aligned}$$

Indeed, Friedland's bound now gives,

$$\begin{aligned}|\det(Z')_{i,j}| = & \left|\det\left(\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL,1}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F^j_{PL,1}(x_k,\tau_k)\right)\right)_{i,j}\right|\\ \geq & \gamma_0 - n^{n+1}\left(\left(\gamma_1+(1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)^{2(n-1)}\right.\\ & \left.\cdot\left(2\left((1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\gamma_1+\left((1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)^2\right)\right)\\ & \qquad\qquad \text{(by Lemma 23)}\\ & -n^{n+1}\left(\left(\gamma_1+(1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)^2\right)^{n-1} \qquad \text{(by 52)}\\ & \cdot\left(2\left(2\sqrt{d}d\alpha D+2\left((1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\right)\left(\gamma_1+(1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\right.\\ & \left.+\left(2\sqrt{d}d\alpha D+2\left((1+\gamma_2\gamma_\phi)2D^2\alpha+\frac{4dD\alpha}{T}\right)\right)^2\right)\\ & \qquad\qquad \text{(by 54, note that } n \text{ was factored out)}\end{aligned}$$

◀

C.5 Proofs of Section 3.2

Proof of Lemma 31. We focus on the first n coordinates of (10). We see that

$$\begin{aligned}&\left(1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)F_{PL,1}(x,1)+\psi\left(\sum_i (f^i)^2+f_\partial^2\right)f_{PL}(x)\\ =&\left(1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)\left(\left(1-\phi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)f(x)+\phi\left(\sum_i (f^i)^2+f_\partial^2\right)f_{PL}(x)\right)\\ &+\psi\left(\sum_i (f^i)^2+f_\partial^2\right)f_{PL}(x)\\ =&\left(1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)\left(1-\phi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)f(x)\\ &+\left(\left(1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)\phi\left(\sum_i (f^i)^2+f_\partial^2\right)+\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)f_{PL}(x)\\ =&\left(\left(1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)\phi\left(\sum_i (f^i)^2+f_\partial^2\right)+\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)f_{PL}(x), \qquad (55)\end{aligned}$$

where we used that

$$1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)=0,$$

if

$$\sum_i (f^i)^2+f_\partial^2 \leq \frac{101}{100}y_0,$$

and

$$\left(1-\phi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)=0,$$

if

$$\sum_i (f^i)^2+f_\partial^2 \geq y_0.$$

We can further rewrite (55),

$$\begin{aligned}
&\left(\left(1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)\left(\phi\left(\sum_i (f^i)^2+f_\partial^2\right)-1+1\right)+\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right) f_{PL}(x)\\
&=\left(\left(1-\psi\left(\sum_i (f^i)^2+f_\partial^2\right)\right)\left(\phi\left(\sum_i (f^i)^2+f_\partial^2\right)-1\right)+1\right) f_{PL}(x)\\
&=f_{PL}(x),
\end{aligned}$$

where we used the same argument as before. ◀

▶ **Lemma 32.** *For all ϵ,*

$$\det(\mathrm{grad}_{(x,\tau)}F^i_{L,2,\epsilon}(x,\tau)\cdot \mathrm{grad}_{(x,\tau)}F^j_{L,2,\epsilon}(x,\tau))_{i,j} \geq \gamma_0^B - g_7(D)$$

with $g_7(D)=\mathcal{O}(D)$. In fact,

$$\begin{aligned}
g_7(D) =&(n+1)\left((n+1)\left(\gamma_1^B+(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2\right)^n\\
&\cdot(n+1)\left(2\gamma_1^B\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)\right.\\
&\left.+\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2\right)
\end{aligned}$$

Proof of Lemma 32. We start with an estimate on the individual $\mathrm{grad}_{(x,\tau)}F^i_{L,2,\epsilon}(x,\tau)$. We will write $(v,w)^i$ for the i-th coordinate of the composed vector (v,w). We now see that

$$\begin{aligned}
&\left|\begin{pmatrix}\mathrm{grad} f_B^i(x)\\ 0\end{pmatrix}-\mathrm{grad}_{(x,\tau)}F^i_{L,2,\epsilon}(x,\tau)\right|\\
&=\left|\begin{pmatrix}\mathrm{grad} f_B^i(x)\\ 0\end{pmatrix}-\mathrm{grad}_{(x,\tau)}\left(\left(1-\tau\psi\left(\sum_l (f^l)^2+f_\partial^2\right)\right)(F_{L,1}(x,1),f_\partial(x)-\epsilon)^i\right.\right.
\end{aligned}$$

$$+\tau\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x), f_{\partial,L}(x) - \epsilon)^i\Bigg)\Bigg|$$

$$= \Bigg|\begin{pmatrix} \mathrm{grad} f_B^i(x) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)}\left(\left(1 - \tau\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)\right.$$

$$\cdot\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right) f(x) + \phi\left(\sum_l (f^l)^2 + f_\partial^2\right) f_L(x), f_\partial(x) - \epsilon\right)^i$$

(by definition of $F_{L,1}$)

$$+\tau\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x), f_{\partial,L}(x) - \epsilon)^i\Bigg)\Bigg|$$

$$= \Bigg|\begin{pmatrix} \mathrm{grad} f_B^i(x) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)}\left(\left(1 - \tau\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)\right.$$

$$\cdot\left(f(x) + \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x) - f(x)), f_\partial(x) - \epsilon\right)^i$$

$$+\tau\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x), f_{\partial,L}(x) - \epsilon)^i\Bigg)\Bigg|$$

$$= \Bigg|\begin{pmatrix} \mathrm{grad} f_B^i(x) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)}\left(\left(f(x) + \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x) - f(x)), f_\partial(x) - \epsilon\right)^i\right.$$

$$\left.+\tau\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\left(f_L(x) - f(x) - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\Bigg|$$

$$= \Bigg|-\mathrm{grad}_{(x,\tau)}\left(\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x) - f(x)), 0\right)^i\right.$$ (by definition of $f_B^i(x)$)

$$\left.+\tau\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\Bigg|$$

$$= \Bigg|\begin{pmatrix} -\mathrm{grad}\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x) - f(x)), 0\right)^i \\ 0 \end{pmatrix}$$

$$+\begin{pmatrix} \tau\mathrm{grad}\left(\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right) \\ \psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i \end{pmatrix}\Bigg|$$

$$\leq \max_j \left|\mathrm{grad}\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L^j(x) - f^j(x))\right)\right|$$ (by the triangle inequality)

$$+\left|\mathrm{grad}\left(\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\right|$$

(because $\tau \in [0,1]$)

$$+\left|((f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x))^i\right|$$ (because $\phi(y), \psi(y) \in [0,1]$, for all y)

$$\leq \max_j \left|\mathrm{grad}\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right| |f_L^j(x) - f^j(x)|$$

$$+ \max_j \left|\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right| |\mathrm{grad}(f_L^j(x) - f^j(x))|$$

(by the Leibniz rule, Cauchy-Schwarz, and the triangle inequality)

$$+ \left|\mathrm{grad}\left(\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)\right| \left|\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right|$$

$$+ \left|\psi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right| \left|\mathrm{grad}\left(\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\right|$$

(by the Leibniz rule, and the triangle inequality)

$$+ 2D^2\beta \qquad \text{(by Lemma 3)}$$

$$\leq \max_j \gamma_\phi \gamma_2 |f_L^j(x) - f^j(x)| + \max_j |\mathrm{grad}(f_L^j(x) - f^j(x))|$$

(by Lemma 21, (11), (12), and since $\phi(y) \in [0,1]$)

$$+ \gamma_\psi \gamma_2 \left|\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right|$$

(by Lemma 21, (12), (17))

$$+ \left|\mathrm{grad}\left(\left(\left(1 - \phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right)(f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x)\right)^i\right)\right|$$

(because $\psi(y) \in [0,1]$)

$$+ 2D^2\beta$$

$$\leq \gamma_\phi \gamma_2 2D^2\alpha + \frac{4dD\alpha}{T} \qquad \text{(by Lemma 3 and Proposition 4)}$$

$$+ \gamma_\psi \gamma_2 2D^2\beta \qquad \text{(by Lemma 3 and since } \phi(y) \in [0,1])$$

$$+ \left|\mathrm{grad}\,(f_L(x) - f(x), f_{\partial,L}(x) - f_\partial(x))^i\right|$$

$$+ \left|\mathrm{grad}\left(\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L(x) - f(x)), 0\right)^i\right)\right| \qquad \text{(by the triangle inequality)}$$

$$+ 2D^2\beta$$

$$\leq \gamma_\phi \gamma_2 2D^2\beta + \frac{4dD\beta}{T} \qquad \text{(because by definition } \alpha \leq \beta)$$

$$+ \gamma_\psi \gamma_2 2D^2\beta$$

$$+ \frac{4dD\beta}{T} \qquad \text{(by Proposition 4)}$$

$$+ \max_j \left|\mathrm{grad}\left(\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)(f_L^j(x) - f^j(x))\right)\right|$$

$$+ 2D^2\beta$$

$$\leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{8dD\beta}{T}$$

$$+ \max_j \left|\mathrm{grad}\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right| \left|(f_L^j(x) - f^j(x))\right|$$

$$+ \max_j \left| \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| \left| \mathrm{grad}(f_L^j(x) - f^j(x)) \right|$$

(By the Leibniz rule and the triangle inequality)

$$\leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{8dD\beta}{T}$$

$$+ \gamma_\phi \gamma_2 2D^2 \alpha$$ (by Lemma 21, (11), (12), and Lemma 3)

$$+ \frac{4dD\beta}{T}$$ (because $\phi(y) \in [0,1]$, Proposition 4, and $\alpha \leq \beta$)

$$= (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \tag{56}$$

We now write

$$(X'_B)_{i,j} = (\mathrm{grad}_{(x,\tau)} F^i_{L,2,\epsilon}(x,\tau) \cdot \mathrm{grad}_{(x,\tau)} F^j_{L,2,\epsilon}(x,\tau))_{i,j}.$$

Similarly to (48), we see that

$$\begin{aligned}
(X'_B)_{i,j} = &\left(\left(\mathrm{grad}_{(x,\tau)} F^i_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i_B(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^i_B(x) \\ 0 \end{pmatrix} \right) \right. \\
&\left. \cdot \left(\mathrm{grad}_{(x,\tau)} F^j_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j_B(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^j_B(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
= &\left(\left(\mathrm{grad}_{(x,\tau)} F_{L,2,\epsilon} F^i_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i_B(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^i_B(x) \\ 0 \end{pmatrix} \right) \right. \\
&\left. \cdot \left(\mathrm{grad}_{(x,\tau)} F^j_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j_B(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^j_B(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
= &(Y_B)_{i,j} \\
&+ \left(\left(\mathrm{grad}_{(x,\tau)} F^i_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i_B(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \mathrm{grad} f^j_B(x) \\ 0 \end{pmatrix} \right)_{i,j} \\
&+ \left(\begin{pmatrix} \mathrm{grad} f^i_B(x) \\ 0 \end{pmatrix} \cdot \left(\mathrm{grad}_{(x,\tau)} F^j_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j_B(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
&+ \left(\left(\mathrm{grad}_{(x,\tau)} F^i_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^i_B(x) \\ 0 \end{pmatrix} \right) \right. \\
&\left. \cdot \left(\mathrm{grad}_{(x,\tau)} F^j_{L,2,\epsilon}(x,\tau) - \begin{pmatrix} \mathrm{grad} f^j_B(x) \\ 0 \end{pmatrix} \right) \right)_{i,j},
\end{aligned}$$

where we used the notation

$$(\mathrm{grad}(f^i_B) \cdot \mathrm{grad}(f^j_B))_{i,j} = (Y_B)_{i,j}.$$

We now see by (56), (15), Cauchy-Schwarz and the triangle inequality that

$$\begin{aligned}
|(X'_B)_{i,j} - (Y_B)_{i,j}| \leq &2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right) \\
&+ \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right)^2.
\end{aligned}$$

This also means that for the operator norm of the matrix we find

$$\|(X'_B - Y_B)_{i,j}\|_\infty \leq (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right) \right.$$

$$+\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2\Bigg) \tag{57}$$

Similarly to (35) and (36), we also see that (56) gives

$$|X'_{i,j}|\leq\left(\gamma_1^B+(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2, \tag{58}$$

and thus,

$$\|X'_{i,j}\|_\infty\leq(n+1)\left(\gamma_1^B+(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2 \tag{59}$$

Friedland's bound (37) now gives

$$\begin{aligned}
&\det(\mathrm{grad}_{(x,\tau)}F^i_{L,2,\epsilon}(x,\tau)\cdot\mathrm{grad}_{(x,\tau)}F^j_{L,2,\epsilon}(x,\tau))_{i,j}\\
&\geq\gamma_0^B-(n+1)\left((n+1)\left(\gamma_1^B+(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2\right)^n\\
&\quad\cdot(n+1)\left(2\gamma_1^B\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)\right.\\
&\quad\left.+\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2\right)
\end{aligned}$$

where we used (57), and (59). ◀

Proof of Lemma 34. As we have seen in the calculation in (56), the τ component of $\mathrm{grad}_{(x,\tau)}(F^i_{L,2,\epsilon})$ is bounded in absolute value by $\left|((f_L(x)-f(x)),f_{\partial,L}(x)-f_\partial(x))^i\right|\leq 2D^2\beta$. On the other hand the x component is lower bounded by

$$\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2}-(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta-\frac{12dD\beta}{T},$$

as a consequence of Lemma 7 and the calculation in (56). This gives that

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_{L,2,\epsilon}),\Xi)\leq\frac{2D^2\beta}{\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2}-(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta-\frac{12dD\beta}{T}}.$$

◀

▶ **Lemma 36.** *Let v be a vertex in $\mathcal{T}$, $x_1,x_2\in star(v)$, and $\tau_1,\tau_2\in[0,1]$, such that $\mathrm{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_1,\tau_1)$ and $\mathrm{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_2,\tau_2)$ are well defined, then*

$$|\mathrm{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_1,\tau_1)-\mathrm{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_2,\tau_2)|\leq g_8(D),$$

with $g_8(D)=\mathcal{O}(D)$. In fact,

$$g_8(D)=2\sqrt{d}d\beta D+2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)$$

Proof of Lemma 36. The proof follows the same steps as the proof of Lemma 27. By expansion we see that

$$|\mathrm{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_1,\tau_1)-\mathrm{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_2,\tau_2)|$$

$$
\begin{aligned}
&= \left| \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_1,\tau_1) - \begin{pmatrix} \mathrm{grad} f^i_B(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \mathrm{grad} f^i_B(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \mathrm{grad} f^i_B(x_2) \\ 0 \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} \mathrm{grad} f^i_B(x_2) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_2,\tau_2) \right| \\
&\leq \left| \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_1,\tau_1) - \begin{pmatrix} \mathrm{grad} f^i_B(x_1) \\ 0 \end{pmatrix} \right| + \left| \begin{pmatrix} \mathrm{grad} f^i_B(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \mathrm{grad} f^i_B(x_2) \\ 0 \end{pmatrix} \right| \\
&\quad + \left| \begin{pmatrix} \mathrm{grad} f^i_B(x_2) \\ 0 \end{pmatrix} - \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_2,\tau_2) \right| \qquad \text{(by the triangle inequality)} \\
&\leq 2\sqrt{d} d\beta D + 2\left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (39), (41) and (56) twice)}
\end{aligned}
$$

◀

▶ **Lemma 38.** *Under the same conditions as in Corollary 28,*

$$
\begin{aligned}
&\det\left(\left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_k,\tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \,\mathrm{grad}_{(x,\tau)} F^j_{PL,2,\epsilon}(x_k,\tau_k) \right) \right)_{i,j} \\
&\quad \geq \gamma_0^B - g_7(D) - g_9(D),
\end{aligned}
$$

where $g_9(D) = \mathcal{O}(D)$. In fact,

$$
\begin{aligned}
g_7(D) =& (n+1)\left((n+1)\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
&\cdot (n+1)\left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
&\left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \\
g_9(D) =& (n+1)^{n+1} \left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
&\cdot \left(2\left(2\sqrt{d} d\beta D + 2\left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right. \\
&\cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right) \\
&\left. + \left(2\sqrt{d} d\beta D + 2\left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \right)
\end{aligned}
$$

Proof of Lemma 38. The proof is more or less the same as the proofs of Lemmas 16 and 28, but with even more complicated bounds. Let $x_0 \in \mathrm{star}(v)$ and $\tau_0 \in [0,1]$, be such that $\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_0,\tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior of a d-simplex in $\mathcal{T}$. Lemma 23 gives that

$$
\begin{aligned}
|\det(\tilde{X}'_B)_{i,j}| =& |\det(\mathrm{grad}_{(x,\tau)}(F^i_{PL,2,\epsilon}(x_0,\tau_0) \cdot \mathrm{grad}_{(x,\tau)}(F^j_{PL,2,\epsilon}(x_0,\tau_0)))_{i,j}| \\
>& \gamma_0^B - (n+1)\left((n+1)\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n
\end{aligned}
$$

$$\cdot (n+1)\left(2\gamma_1^B\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)\right.$$
$$\left.+\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2\right).$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give a bound on the determinant of

$$(Z'_B)_{i,j}=\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)}F^j_{PL,2,\epsilon}(x_k,\tau_k)\right).$$

As before we also need a bound on the operator norm $\|(Z'_B)_{i,j}\|_\infty$. This bound for this we notice that the bound (58) also holds for the convex hull, that is

$$\left|\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)}F^j_{PL,2,\epsilon}(x_k,\tau_k)\right)\right|$$
$$\leq\left(\gamma_1^B+(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2.$$

And thus, similarly to (52), we have that

$$\|(Z'_B)_{i,j}\|_\infty\leq(n+1)\left(\gamma_1^B+(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}\right)^2 \tag{60}$$

We note that

$$\left|\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_0,\tau_0)\right|\leq\gamma_1^B+(\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta+\frac{12dD\beta}{T}$$
$$\text{(because } \operatorname{grad}(f^i_B)\leq\gamma_1^B\text{, and (56))} \tag{61}$$

Similarly to (45), we find that

$$\begin{aligned}
&|(Z'_B)_{i,j}-(\tilde{X}'_B)_{i,j}|\\
&\quad=\left|\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)}F^j_{PL,2,\epsilon}(x_k,\tau_k)\right)\right.\\
&\qquad\left.-\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_0,\tau_0)\cdot\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_0,\tau_0)\right|\\
&\quad\leq\left|\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_0,\tau_0)-\sum_{k=1}^m\mu_k\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_k,\tau_k)\right|\\
&\qquad\cdot\left|\operatorname{grad}_{(x,\tau)}F^j_{PL,2,\epsilon}(x_0,\tau_0)\right|\\
&\qquad+\left|\operatorname{grad}_{(x,\tau)}F^j_{PL,2,\epsilon}(x_0,\tau_0)-\sum_{k=1}^m\mu_k\operatorname{grad}_{(x,\tau)}F^j_{PL,2,\epsilon}(x_k,\tau_k)\right|\\
&\qquad\cdot\left|\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_0,\tau_0)\right|\\
&\qquad+\left|\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_0,\tau_0)-\sum_{k=1}^m\mu_k\operatorname{grad}_{(x,\tau)}F^i_{PL,2,\epsilon}(x_k,\tau_k)\right|
\end{aligned}$$

$$\cdot \left| \mathrm{grad}_{(x,\tau)} F^j_{PL,2,\epsilon}(x_0,\tau_0) - \sum_{k=1}^{m} \mu_k \mathrm{grad}_{(x,\tau)} F^j_{PL,2,\epsilon}(x_k,\tau_k) \right|$$

$$\leq 2\left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right) \qquad \text{(by Corollary 37)}$$

$$\cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right) \qquad \text{(by (61))}$$

$$+ \left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right)^2 \qquad \text{(by Corollary 37)}$$

Similarly to (34), we have that

$$\begin{aligned}\|(Z'_B - \tilde{X}'_B)_{i,j}\|_\infty \leq & 2(n+1)\left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right) \\ & \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right) \\ & + (n+1)\left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right)^2 \end{aligned} \tag{62}$$

Indeed, Friedland's bound now gives,

$$\begin{aligned}|\det(Z'_B)_{i,j}| = & \left|\det\left(\sum_{k=1}^{m} \mu_k \,\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_k,\tau_k)\right) \cdot \left(\sum_{k=1}^{m} \mu_k \,\mathrm{grad}_{(x,\tau)} F^j_{PL,2,\epsilon}(x_k,\tau_k)\right)\right| \\ \geq & \gamma_0^B - (n+1)\left((n+1)\left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)^2\right)^n \\ & \cdot (n+1)\left(2\gamma_1^B\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right. \\ & \left. + \left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)^2\right) \qquad \text{(by Lemma 32)} \\ & - (n+1)^{n+1}\left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)^2\right)^n \qquad \text{(by (60))} \\ & \cdot \left(2\left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right)\right. \\ & \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right) \\ & \left. + \left(2\sqrt{d}d\beta D + 2\left((\gamma_2(2\gamma_\phi+\gamma_\psi)+1)2D^2\beta + \frac{12dD\beta}{T}\right)\right)^2\right) \\ & \qquad \text{(by (62) and factoring out } (n+1)) \end{aligned}$$

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