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# The topological correctness of PL-approximations of isomanifolds

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## 1 Abstract

2 Isomanifolds are the generalization of isosurfaces to arbitrary dimension and codimension, i.e.  
3 manifolds defined as the zero set of some multivariate multivalued function  $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ .  
4 A natural (and efficient) way to approximate an isomanifold is to consider its Piecewise-Linear  
5 (PL) approximation based on a triangulation  $\mathcal{T}$  of the ambient space  $\mathbb{R}^d$ . In this paper, we  
6 give conditions under which the PL-approximation of an isomanifold is topologically equivalent  
7 to the isomanifold. The conditions are easy to satisfy in the sense that they can always be  
8 met by taking a sufficiently fine triangulation  $\mathcal{T}$ . This contrasts with previous results on the  
9 triangulation of manifolds where, in arbitrary dimensions, delicate perturbations are needed to  
10 guarantee topological correctness, which leads to strong limitations in practice. We further give  
11 a bound on the Fréchet distance between the original isomanifold and its PL-approximation.  
12 Finally we show analogous results for the PL-approximation of an isomanifold with boundary.

**2012 ACM Subject Classification** Theory of computation → Computational geometry

**Keywords and phrases** PL-approximations, isomanifolds, topological correctness

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**Lines** 489

## 13 1 Introduction

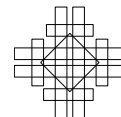
14 Isosurfacing especially in low dimensions is used in many fields of application, such as CT  
15 scans in medicine, biochemistry, biomedicine, deformable modeling, digital sculpting, en-  
16 vironmental science, and mechanics and dynamics [45] and the references mentioned there.



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17 The marching cube approach [41] being the most popular approach taken. However the  
18 result of the marching cube algorithm is not necessarily topologically correct.

19 Some results on provable correctness were achieved within the computational geometry  
20 community [10, 47] in three dimensions. In case the isosurfacing is based on simplices instead  
21 of cubes, such as in the marching tetrahedra algorithm [28], some bounds can be given [1, 2],  
22 on for example the one-sided Hausdorff distance.

23 In general homeomorphism proofs in higher dimensions rely on some perturbation scheme  
24 to prove that a triangulation is correct [52, 8, 11, 12, 15]. This is a major difference with  
25 one and two dimensional surfaces where no such requirements exist [30], [49, Section 10.2].  
26 In this paper we shall see that no perturbations are necessary for isomanifolds as well.

27 The techniques used here are also different from many of the standard tools. Manifold  
28 triangulation/reconstruction algorithms use often Delaunay triangulations [50, 24, 19] and  
29 use the closed ball property [32], see for example [3, 18]. Others use Whitney’s lemma [13] or  
30 are based on collapses [4]. While the current paper mainly relies on the non-smooth implicit  
31 function theorem [22] with some Morse theory.

32 We stress that this does not solve the general problem of triangulating manifolds with  
33 boundary, because there exist manifolds that cannot be written as the zero set of a smooth  
34 function, because of the existence of manifolds with a non-trivializable normal bundle (gen-  
35 eralizations of non-orientable manifolds). For an excellent introduction to the topology of  
36 such bundles and manifolds we refer to [44]. We note that locally we can always write an  
37 embedded smooth manifold as the zero set of a smooth function, because it can be para-  
38 metrized as a function from the tangent space to the manifold itself as a consequence of  
39 the implicit function theorem. Non-continuous results have been recently obtained in the  
40 neighbourhood of the manifold [20]. We stress that there are now obstructions to global  
41 closest point projection on manifold [33], and this can even be approximately reconstructed  
42 from a point sample [34].

43 We also emphasize that because we do not use a perturbation scheme, we cannot give  
44 lower bounds on the quality of the linear pieces in the Piecewise-Linear (PL) approximation.  
45 This is a clear difference with previous methods [52, 15, 14, 12] whose output is a thick  
46 triangulation. Although this is an appealing property, it complicates the analysis further  
47 and requires perturbation schemes that work fine in theory, but the constants are miserable  
48 and the methods do not work in practice in high dimension (see a more detailed discussion  
49 in Section 2). Here we ask for less but still provide guarantees on the Fréchet distance and  
50 the approximation of the gradients. Perturbation techniques could be used to improve the  
51 simplex quality (although only to some very limited extent).

52 This rest of the paper is subdivided in two sections. In the first section, we treat closed  
53 isomanifolds, that is compact manifolds without boundary. In the second section, we prove  
54 similar results for isomanifolds with boundary. Extension to general isostratifolds is briefly  
55 discussed in Section 4. Apart from some Delaunay based work on triangulations of stratifolds  
56 in three dimensions [46, 48, 27, 26, 25], we are not aware of similar results on manifolds with  
57 boundary. Significant effort also went in the detection of strata, in this case in arbitrary  
58 dimension, see for example [6, 5].

59 We will often give the simplified statement of results in the main text while complete  
60 versions with all constants will be given in the appendix. All proofs can also be found in  
61 the appendix.

62 This paper is part of a collection of closely related papers. The data structure needed to  
63 efficiently propagate along the manifold is presented in [16], while details of the implement-  
64 ation will be discussed in an upcoming paper.

## 2 Isomanifolds (without boundary)

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$  be a smooth function and suppose that 0 is a regular value of  $f$ , meaning that at every point  $x$  such that  $f(x) = 0$ , the Jacobian of  $f$  is non-degenerate. Then the zero set of  $f$  is a manifold as a direct consequence of the implicit function theorem, see for example [29, Section 3.5]. We further assume that  $f^{-1}(0)$  is compact. As in [1] we consider a triangulation  $\mathcal{T}$  of  $\mathbb{R}^d$ . The function  $f_{PL}$  is a linear interpolation of the values of  $f$  at the vertices if restricted to a single simplex  $\sigma \in \mathcal{T}$ . For any function  $g : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$  we write  $g^i$  for the components of  $g$ .

We prove that under certain conditions that there is an isotopy from the zero set of  $f$  to the zero set of  $f_{PL}$ . The proof will be using the Piecewise-Linear (PL) map

$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x), \quad (1)$$

which interpolates between  $f$  and  $f_{PL}$  and is based on the generalized implicit function theorem. The isotopy is in fact stronger than just the existence of a homeomorphism from the zero set of  $f$  to that of  $f_{PL}$ .

Our result in particular implies that the zero set  $f_{PL}$  is a manifold. So this significantly improves on the result of Allgower and Georg [2, Theorem 15.4.1]. The conditions are also weaker, because we do not require that the zero set avoids simplices that have dimension less than the codimension, see [2, Definition 12.2.2] and the text above [2, Theorem 15.4.1]. The idea to avoid these low dimensional simplices originates from Whitney [52], which remarkably was not cited by Allgower and George [2, 1]. Very heavy perturbation schemes for the vertices of the ambient triangulation  $\mathcal{T}$  are required for the manifold to be sufficiently far from simplices in the ambient triangulations that have dimension less than the codimension of the manifold [52, 15]. Various techniques have been developed to compute such perturbations with guarantees. They typically consist in perturbing the position of the sample points or in assigning weights to the points. Complexity bounds are then obtained using volume arguments. See, for example [18, 14, 11, 9]. However, these techniques suffer from several drawbacks. The constants in the complexity depend exponentially on the ambient dimension. Moreover the analysis assumes that the probability of the simplices of dimension less than the codimension to intersect the manifold is zero, which is not true when dealing with finite precision. As a result, the actual implementations we are aware of fail to work well in practice except in very simple cases.

We are, by definition, only interested in  $f^{-1}(0)$  so we can ignore points that are sufficiently far from this zero set. More precisely, we observe the following: If  $f^i(x)$  is positive on a simplex  $\sigma$  then so is  $f_{PL}^i$  and thus is  $F_{PL}^i$  positive on  $\sigma \times [0, 1]$ . The same argument holds for negative values. So we see that

► **Remark.** Write  $\Sigma_0$  for the set of all  $\sigma \in \mathcal{T}$ , such that  $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$  for all  $i$ . Then for all  $\tau$ ,  $\{x \mid F_{PL}(x, \tau) = 0\} \subset \Sigma_0$ .

The results will be formulated in terms of constants defined in terms of  $f$  and the ambient triangulation  $\mathcal{T}$ .

104 ► **Definition 1.** We define

$$105 \quad \gamma_0 = \inf_{x \in \Sigma_0} |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| \quad (2)$$

$$106 \quad \gamma_1 = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f^i)| \quad (3)$$

$$107 \quad \alpha = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k \partial_l f^i)_{k,l}\|_2, \quad (4)$$

$$108 \quad D : \text{the longest edge length of a simplex in } \Sigma_0 \quad (5)$$

$$109 \quad T : \text{the smallest thickness of a simplex in } \Sigma_0. \quad (6)$$

110 Here  $\text{grad}(f^i) = (\partial_j f^i)_j$  denotes the gradient of component  $f^i$ ,  $\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}$   
 111 denotes the determinant of the matrix with entries  $\text{grad}(f^i) \cdot \text{grad}(f^j)$ ,  $\|\cdot\|_2$  the operator  
 112 2-norm, and  $(\partial_k \partial_l f^i)_{k,l}$  the matrix of second order derivatives, that is the Hessian (Hes).  
 113 We recall the definition of the operator norm:  $\|A\|_p = \max_{x \in \mathbb{R}^n} \frac{|Ax|_p}{|x|_p}$ , with  $|\cdot|_p$  the  $p$ -norm  
 114 on  $\mathbb{R}^n$ .

115 We will assume that  $\gamma_0, \gamma_1, \alpha, D, T \in (0, \infty)$ . The thickness is the ratio of the height over  
 116 the longest edge length.

117 A good choice for  $\mathcal{T}$  is the Coxeter triangulation of type  $A_d$ , see [23, 21], or the related  
 118 Freudenthal triangulations, see [35, 40, 31, 51], which can be defined for different values of  
 119  $D$  while keeping  $T$  constant. In this paper, we will thus think of all the above quantities as  
 120 well as  $d$  and  $n$  as constants except  $D$  and our results will hold for  $D$  small enough.

## 121 The result

122 We are going to construct an ambient isotopy based on (1). The zero set of  $F_{PL}(x, 0)$   
 123 (or  $f(x)$ ) gives the smooth isosurface, while the zero set of  $F_{PL}(x, 1)$  (or  $f_{PL}(x)$ ) gives  
 124 the PL approximation, that is the triangulation of the isosurface after possible barycentric  
 125 subdivision. The map  $\tau \mapsto \{x \mid F_{PL}(x, \tau) = 0\}$  in fact gives an isotopy. Without too much  
 126 extra work we'll also bound the Fréchet distance between  $f(x)$  and  $f_{PL}(x)$ .

130 Proving isotopy consists of two technical steps, which consume most of the space in the  
 131 proof below, as well as the use of a standard observation from Morse theory/gradient flow  
 132 in the third step. The technical steps are

133 ■ Let  $\sigma \in \mathcal{T}$ . We first show that  $\{x \mid F_{PL}(x, \tau) = 0\} \cap \sigma \times [0, 1]$  is a smooth manifold,  
 134 under certain conditions.

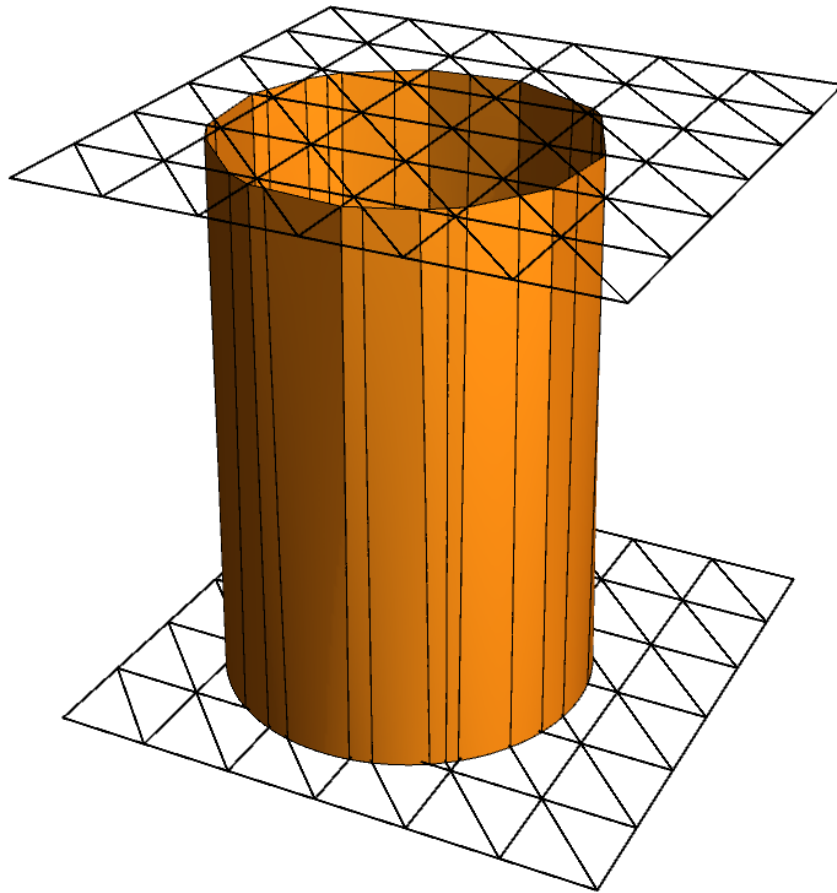
135 ■ We prove that  $F_{PL}^{-1}(0)$  is a manifold, under certain conditions, using techniques from  
 136 nonsmooth analysis.

137 Along the way we shall also see that  $F_{PL}^{-1}(0)$  is never tangent to the  $\tau = c$  planes, where  $c$   
 138 is a constant. This implies that the gradient field of  $\tau$  restricted to  $F_{PL}^{-1}(0)$ , is piecewise  
 139 smooth and never vanishes.

140 Now we arrive at the third step, which is similar to a standard observation in Morse theory  
 141 [42, 43], with the exception that we now consider piecewise-smooth instead of smooth vector  
 142 fields. We refer to Milnor [42] for an excellent introduction, see Lemma 2.4 and Theorem  
 143 3.1 in particular.

144 ► **Lemma 2** (Gradient flow induced homeomorphisms). *The flow of a non-vanishing piecewise-*  
 145 *smooth gradient vector field of a function  $\tau$  on a compact manifold generates a homeomorph-*  
 146 *ism from  $\tau = c_1$  to  $\tau = c_2$ , where  $c_1$  and  $c_2$  are constants.*

147 Bounds on the gradient of  $\tau$  on the manifold give a bound on the Fréchet distance.



127 **Figure 1** A pictorial overview of the proof. The  $\tau$ -direction goes upwards. Similarly to Morse  
 128 theory we find that  $f_{PL}^{-1}(0)$  (top) and  $f^{-1}(0)$  (bottom) are homeomorphic if the function  $\tau$  restricted  
 129 to  $F_{PL}^{-1}(0)$  does not encounter a Morse critical point.

148 **2.1 Estimates for a single simplex**

149 We now first concentrate on a single simplex  $\sigma$  and write  $f_L$  for the linear function whose val-  
 150 ues on the vertices of  $\sigma$  coincide with  $f$ , that is  $f_L$  is the linear extension of the interpolation  
 151 of  $f$ .

152 **2.1.1 Preliminaries and variations of known results**

153 We need a simple estimate similar to Proposition 2.1 of Allgower and George [1].

154 **Lemma 3.** *Let  $\sigma \subset \Sigma_0$  and let  $f_L$  be as described above. Then  $|f_L^i(x) - f^i(x)| \leq 2D^2\alpha$   
 155 for all  $x \in \sigma$ .*

156 We will also be using an estimate similar to Proposition 2.2 of Allgower and George [1].

157 **Proposition 4.** *Let  $\sigma \subset \Sigma_0$  and let  $f_L$  be as described above. Then*

158 
$$|\text{grad}f_L^i - \text{grad}f^i| = \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T},$$

159 for all  $x \in \sigma = \{v_k\}$ .

160 **2.1.2 Estimates on the gradient inside a single simplex**

161 We write

$$162 \quad F_L(x, \tau) = (1 - \tau)f(x) + \tau f_L(x). \quad (7)$$

163 We note that  $F_L$  extends smoothly outside  $\sigma$ . Here and throughout we restrict ourselves to  
 164 the setting where  $\tau \in [0, 1]$ . The function  $F_L$  has  $\mathbb{R}^{d-n}$  as image.

165 We can now state the following

166 ► **Lemma 5.** *If we write  $\text{grad}_{(x,\tau)}$  for the gradient that includes the  $\tau$  component, we have*

$$167 \quad |\det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \quad (8)$$

168 with  $g_1(D) = \mathcal{O}(D)$ . See (18) in Appendix A for the exact expression of  $g_1$ .

169 From the previous statement we immediately have that

170 ► **Corollary 6** ( $F_L^{-1}(0)$  is a manifold in a neighbourhood of  $\sigma \times [0, 1]$ ). *If  $\gamma_0 > g_1(D)$  the  
 171 implicit function theorem applies to  $F_L(x, \tau)$  inside  $\sigma \times [0, 1]$ . (In fact it applies to an open  
 172 neighbourhood of this set). In particular, we have proven the first of our two technical steps,  
 173  $\{x \mid F_{PL}(x, \tau) = 0\} \cap \sigma \times [0, 1]$  is a smooth manifold.*

174 **2.1.3 Transversality with regard to the  $\tau$ -direction**

175 We will also prove the main result which we need for the third step, that is the gradient of  
 176  $\tau$  restricted to  $F_{PL}(x, \tau) = 0$ , is piecewise smooth and never vanishes. We now prove inside  
 177 each  $\sigma \times [0, 1]$  the gradient of  $\tau$  on  $F_L = 0$  is smooth and does not vanish.

178 We first give a simple lower bound on the lengths of vectors  $v^1, \dots, v^{d-n}$ , assuming that  
 179 the norms  $|v^i|$  are upper bounded and the determinant of the Gram matrix is lower bounded.

180 ► **Lemma 7.** *Let  $v^1, \dots, v^{d-n} \in \mathbb{R}^d$ ,  $|v^i| \leq \gamma_1$ , for all  $i$ , and assume that  $\det(v^i \cdot v^j)_{i,j} > \gamma_0$ .  
 181 Then  $|v^i| \geq \sqrt{\gamma_0}/\gamma_1^{d-n-1}$ .*

182 We also need to bound the angle of the vectors  $\text{grad}_{(x,\tau)}(F_L^i)$  and the  $x$  plane, that is  
 183  $\mathbb{R}^d \subset \mathbb{R}^{d+1}$ . We recall the definition. If  $v \in \mathbb{R}^{d+1}$  is a vector and  $\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$ , is the  
 184 space spanned by the  $d$  basis vectors corresponding to the  $x$ -directions, the angle between  
 185  $v$  and  $\Xi$  is  $\angle(v, \Xi) = \inf_{w \in \Xi} \angle(v, w)$ .

186 ► **Lemma 8.** *Let  $\Xi$  be as above. We have*

$$187 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_L^i), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

188 *In particular the manifold  $F_L^{-1}(0)$  inside  $\sigma \times [0, 1]$  is never tangent to the  $\tau = c$  planes,  
 189 where  $c$  is a constant.*

190 Combining Lemma 8 and Corollary 6 gives:

191 ► **Corollary 9.** *If  $\gamma_0 > g_1(D)$ , and  $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$ , then inside each  $\sigma \times [0, 1]$  the  
 192 gradient of  $\tau$  on  $F_L^{-1}(0)$  is smooth and does not vanish.*

193 **2.2 Global result**

194 We are now going to prove the global result. For this, we need to recall some definitions  
 195 and results from non-smooth analysis. We refer to [22] for an extensive introduction.

196 ► **Definition 10** (Generalized Jacobian, Definition 2.6.1 of [22]). Let  $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d-n}$ , where  
 197  $F$  is assumed to be just Lipschitz. The generalized Jacobian of  $F$  at  $x_0$  denoted by  $J_F(x_0)$ ,  
 198 is the convex hull of all  $(d-n) \times (d+1)$ -matrices  $B$  obtained as the limit of a sequence of  
 199 the form  $J_F(x_i)$ , where  $x_i \rightarrow x_0$  and  $F$  is differentiable at  $x_i$ .

200 Following [22, page 253] we also define:

201 ► **Definition 11.** The generalized Jacobian  $J_F(x_0)$  is said to be of maximal rank provided  
 202 every matrix in  $J_F(x_0)$  is of maximal rank.

203 Write  $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$  and denote the coordinates of  $\mathbb{R}^{d+1}$  by  $(x, y)$  accordingly.  
 204 Fix a point  $(a, b)$ , with  $F(a, b) = 0 \in \mathbb{R}^{d-n}$ . We now write:

205 ► **Notation 12** ([22, page 256]).  $J_F(x_0, y_0)|_y$  is the set of all  $(n+1) \times (n+1)$ -matrices  $M$   
 206 such that, for some  $(n+1) \times (d-n)$ -matrix  $N$ , the  $(n+1) \times (d+1)$ -matrix  $[N, M]$  belongs  
 207 to  $J_F(x_0, y_0)$ .

208 With these definitions and notations we now have:

209 ► **Theorem 13** (The generalized implicit function theorem [22, page 256]). Suppose that  
 210  $J_F(a, b)|_y$  is of maximal rank. Then there exists an open set  $U \subset \mathbb{R}^{n+1}$  containing  $a$  such  
 211 that there exists a Lipschitz function  $g : U \rightarrow \mathbb{R}^{d-n}$ , such that  $g(a) = b$  and  $F(x, g(x)) = 0$   
 212 for all  $x \in U$ .

213 We recall the definition of  $F_{PL}$ ,

214 
$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x). \tag{1}$$

215 Because of the definition of  $\alpha$ , see (4), and Proposition 4, we have that  $\text{grad}_{(x,\tau)} F_{PL}(x, \tau)$   
 216 and  $\text{grad}_{(x,\tau)} F_{PL}(\tilde{x}, \tau)$  are close if  $x$  and  $\tilde{x}$  are. In particular,

217 ► **Lemma 14.** Let  $v$  be a vertex in  $\mathcal{T}$ ,  $x_1, x_2 \in \text{star}(v)$ , and  $\tau_1, \tau_2 \in [0, 1]$ , such that  
 218  $\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1)$  and  $\text{grad}_{(x,\tau)} F_{PL}^i(x_2, \tau_2)$  are well defined, then

219 
$$|\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL}^i(x_2, \tau_2)| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha$$

220 We now immediately have the same bound on points in the convex hull of a number of  
 221 such vectors:

222 ► **Corollary 15.** Suppose we are in the setting of Lemma 14 and  $x_0, x_1, \dots, x_m \in \text{star}(v)$ ,  
 223  $\tau_0, \dots, \tau_m \in [0, 1]$ , and suppose that  $\mu_1, \dots, \mu_m$  are positive weights such that  $\mu_1 + \dots + \mu_m =$   
 224  $1$  then,

225 
$$\left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha.$$

226 Using Lemma 5 we see



227 ► **Lemma 16.** *Let  $v$  be a vertex in  $\mathcal{T}$ ,  $x_1, \dots, x_m \in \text{star}(v)$ , and  $\tau_1, \dots, \tau_m \in [0, 1]$ , such*  
 228 *that  $\text{grad}_{(x, \tau)} F_{PL}^i(x_k, \tau_k)$ ,  $k = 0, \dots, m$  are well defined. If we moreover assume  $D \leq 1$ , and*  
 229  *$\frac{6dD\alpha}{T} \leq \gamma_1$  we have that*

$$230 \quad \left| \det \left( \left( \sum_{k=1}^m \mu_k \text{grad}_{(x, \tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x, \tau)} F_{PL}(x_k, \tau_k) \right) \right)_{i,j} \right| \geq \gamma_0 - g_2(D),$$

231 *with  $g_2(D) = \mathcal{O}(D)$ . See (19) in Appendix A for the exact expression of  $g_2$ .*

232 From the previous lemma we immediately have that

233 ► **Corollary 17** ( $\{x \mid F_{PL}(x, \tau) = 0\}$  is a manifold). *If  $D \leq 1$ ,  $\frac{6dD\alpha}{T} \leq \gamma_1$ , and  $\gamma_0 > g_2(D)$  the*  
 234 *generalized implicit function theorem, Theorem 13, applies to  $F_{PL}(x, \tau) = 0$ . In particular,*  
 235  *$\{x \mid F_{PL}(x, \tau) = 0\}$  is a manifold.*

236 We notice that this bound is stronger than the bound in Corollary 6. This means that  
 237  $\{x \mid F_{PL}(x, \tau) = 0\}$  is a Piecewise-Smooth manifold if the conditions of Corollary 17 hold.  
 238 The second technical step of the proof is now also completed.

239 The fact that  $F_L(x, \tau) = 0$  is a Piecewise-Smooth manifold and Corollary 9 give that  
 240 the gradient of  $\tau$  is a Piecewise-Smooth vector field whose flow we can integrate to give a  
 241 homeomorphism from the zero set of  $f$  to that of  $f_{PL}$ .

242 We summarize in a theorem:

243 ► **Theorem 18.** *If,  $D \leq 1$ ,  $\frac{6dD\alpha}{T} \leq \gamma_1$ ,  $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$ , and  $\gamma_0 > g_2(D)$  then the zero*  
 244 *set of  $f$  is isotopic to the zero set of  $f_{PL}$ . We stress that one can satisfy all conditions by*  
 245 *choosing  $D$  sufficiently small.*

## 246 2.2.1 Fréchet distance

247 To bound the Fréchet distance ( $d_F$ ) between the zero sets of  $f(x)$  and  $f_{PL}$ , it suffices to  
 248 bound the angle that the gradient of  $\tau$  (as restricted to  $F_L(x, \tau) = 0$ ) makes with the  
 249 (ambient)  $\tau$ -direction.

250 For this we will use the angle bound of Lemma 8, together with some estimates that are  
 251 similar in spirit to those in [7, Lemma C.13].

252 ► **Lemma 19.** *Let  $v^1, \dots, v^{d-n} \in \mathbb{R}^{d+1}$ ,  $|v^i| \leq \tilde{\gamma}_1$ , for all  $i$ , and assume that  $\det(v^i \cdot$   
 253  $v^j)_{i,j} > \tilde{\gamma}_0 > 0$ . Let  $e_\tau$  a unit vector. If for all  $i$ ,  $\cos(\angle v^i, e_\tau) \leq \phi_0$ , then for any  $w \in$   
 254  $\text{span}(v^1, \dots, v^{d-n})$*

$$255 \quad \cos \angle(w, e_\tau) \leq \frac{(d-n)d^{d-n-1}\phi_0\tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

256 Let  $e_\tau$  be the  $\tau$  direction and let  $g_\tau$  be the gradient of  $\tau$  restricted to  $F_{PL}^{-1}(0)$ , whenever it  
 257 exists. We want to bound the angle of  $g_\tau$  and the  $\tau$ -direction. Because the homeomorphism  
 258 is given by the gradient flow, this bounds the Fréchet distance.

259 There is one subtlety, because the manifold is only Piecewise-Smooth, we need to take  
 260 into account the points where  $g_\tau$  is not uniquely defined. Because for each simplex  $\sigma$ ,  $F_L$   
 261 extends to a neighbourhood of  $\sigma \times [0, 1]$ , there exists a limit of  $g_\tau(x_i, \tau_i)$  for any sequence  
 262  $(x_i, \tau_i)$  that lies in  $\text{int}(\sigma) \times [0, 1]$ , where  $\text{int}$  denotes the interior. This means that if we bound  
 263  $g_\tau$  for each simplex we also bound its limits, where the limits are as just described.

264 We are now ready to combine Lemmas 8, 19, and Theorem 18.

265 ► **Corollary 20** (Bound on the Fréchet distance). *Suppose that the conditions of Theorem 18*  
 266 *are satisfied. Then,  $d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D)$ , with  $g_3(D) = \mathcal{O}(D^2)$ , where we*  
 267 *think of  $\gamma_0, \gamma_1, d, n, T$  and  $\alpha$  as constants. See (19) in Appendix A for the exact expression*  
 268 *of  $g_2$ .*

269 The most important thing to observe is that  $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$ , so that we find  
 270 that  $d_F(f^{-1}(0), f_{PL}^{-1}(0)) = \mathcal{O}(D^2)$ , where we think of  $\gamma_0, \gamma_1, d, n, T$  and  $\alpha$  as constants.

### 271 3 Isomanifolds with boundary

272 We'll now consider isomanifolds with boundary. By this we mean that on top of the function  
 273  $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ , we'll have another function  $f_\partial : \mathbb{R}^d \rightarrow \mathbb{R}$  and the set we consider is  
 274  $M = f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$ . This is a manifold with boundary if the gradients of  $f^i$  span  
 275 a  $(d - n)$ -dimensional space at each point of  $f^{-1}(0)$  and the gradients of  $f^i$  and  $f_\partial$  span a  
 276  $(d - n + 1)$ -dimensional space at each point of  $\partial M = f^{-1}(0) \cap f_\partial^{-1}(0)$ , as a consequence of  
 277 the submersion theorem.

278 We'll again write  $f_{PL}$  for the PL interpolation of  $f$ . Similarly we write  $f_{\partial, PL}$  for the PL  
 279 interpolation of  $f_\partial$ .

280 We prove that, under certain conditions, there is an isotopy from  $f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$   
 281 to  $f_{PL}^{-1}(0) \cap f_{\partial, PL}^{-1}([0, \infty))$ . The conditions are very similar to the conditions we have before,  
 282 but of course we need to include bounds on the gradient of  $f_{\partial, PL}$ .

### 283 Overview of the proof

284 We will again construct an isotopy, but in this case it will consist of two steps.

- 285 ■ In the **first step**, we isotope the part of  $f^{-1}(0)$  that is far from  $f_\partial^{-1}(0)$  to its piecewise  
 286 linear approximation, while leaving the part of  $f^{-1}(0)$  that is close to  $f_\partial^{-1}(0)$  smooth.  
 287 We will denote the result by  $M_1 = (F_{PL,1}(\cdot, 1))^{-1}(0)$ .
- 289 ■ In the **second step**, we consider a (small) tubular neighbourhood around  $f_\partial^{-1}(0)$  as  
 290 restricted to  $M_1$  by looking at all  $f_\partial^{-1}(\epsilon)$  for  $|\epsilon|$  sufficiently small.<sup>1</sup> We then isotope  
 291  $M_1 \cap f_\partial^{-1}(\epsilon)$  to its piecewise linear approximation. Again the isotopy is chosen in such a  
 292 way that for  $\epsilon$  relatively large (for the points such that  $M_1$  is already Piecewise-Linear)  
 293 it leaves  $M_1 \cap f_\partial^{-1}(\epsilon)$  invariant. This gives an isotopy of a tubular neighbourhood of  
 294  $M_1 \cap f_\partial^{-1}(0)$  to its Piecewise-Linear approximation.

300 We will first partition the manifold in two parts using a smooth bump function  $\phi : \mathbb{R} \rightarrow$   
 301  $[0, 1]$  that is zero in a neighbourhood of zero and  $\phi(y) = 1$  if  $|y| > y_0$ , for some  $y_0 > 0$ .  
 302 Such bump functions can be easily constructed, see for example [39, Section 2.2]. We will  
 303 be using the function  $\phi(\sum_i (f^i)^2 + f_\partial^2)$ .

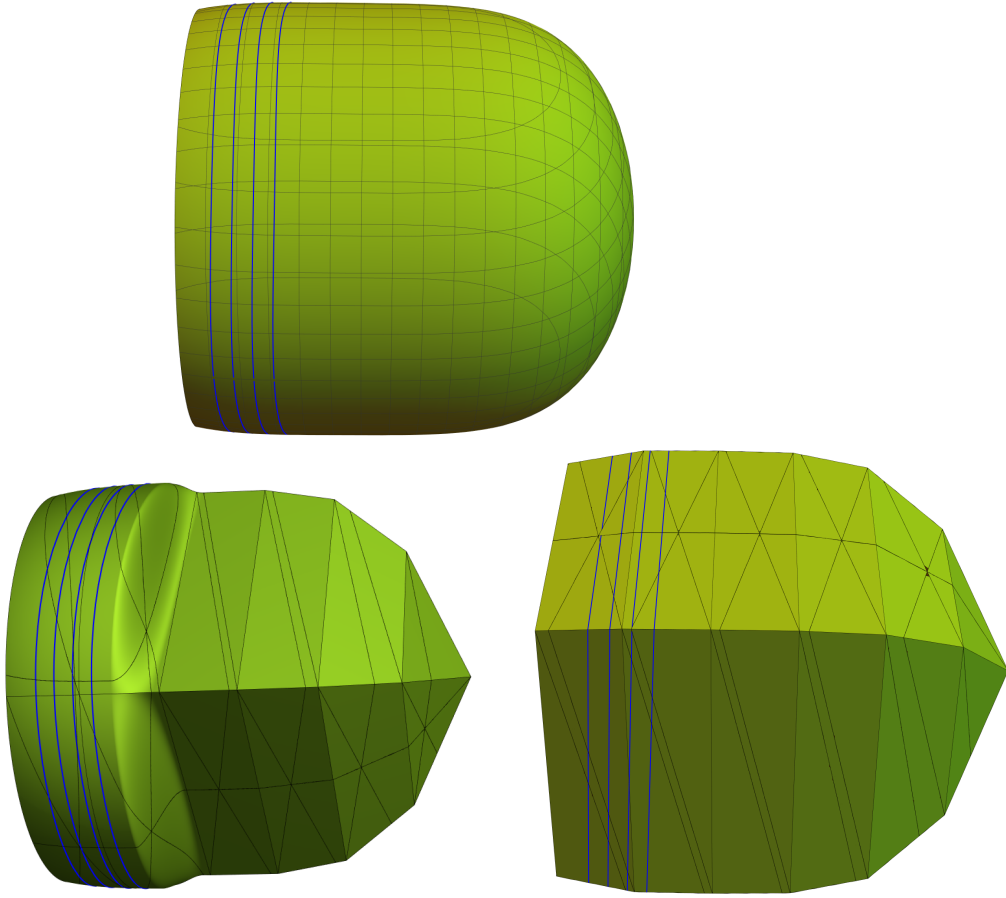
304 The first step will be using the zero set of the following function:

$$305 F_{PL,1}(x, \tau) = \left( 1 - \tau \phi \left( \sum_i (f^i)^2 + f_\partial^2 \right) \right) f(x) + \tau \phi \left( \sum_i (f^i)^2 + f_\partial^2 \right) f_{PL}(x), \quad (9)$$

306 on which we'll apply the same gradient flow argument as before.

---

288 <sup>1</sup> We stress that  $\epsilon$  may be negative.



295 ■ **Figure 2** Top: we see the original isosurface with  $f_{\partial}^{-1}(-1/10)$ ,  $f_{\partial}^{-1}(0)$ ,  $f_{\partial}^{-1}(1/10)$ , and  $f_{\partial}^{-1}(2/10)$   
 296 indicated in blue. Bottom left: we see that at the end of Step 1 the neighbourhood of the boundary  
 297 is intact, while the rest has been isotoped to a Piecewise-Linear approximation. Bottom right:  
 298 we have also isotoped the neighbourhood of the boundary to a Piecewise-Linear approximation by  
 299 isotoping  $f_{\partial}^{-1}(\epsilon)$ , to its Piecewise-Linear approximation for all sufficiently small  $\epsilon$ .

307 The resulting set  $M_1$  is the same zero set of  $f_{PL}$  as before if we stay sufficiently far  
 308 away from  $\partial M$  and the isotopy leaves the manifold invariant close to  $\partial M$ . In particular,  
 309  $\partial M_1 = \partial M$ .

310 In the second step, we define an isotopy that will act only on a small neighbourhood of  
 311  $\partial M$ . Consider the sets  $B_1(\epsilon) = M_1 \cap f_{\partial}^{-1}(\epsilon)$  and, for each of them, define the function

$$\begin{aligned}
 312 \quad F_{PL,2,\epsilon}(x,\tau) &= \left(1 - \tau\psi\left(\sum_i (f^i)^2 + f_{\partial}^2\right)\right) (F_{PL,1}(x,1), f_{\partial}(x) - \epsilon) \\
 313 \quad &\quad + \tau\psi\left(\sum_i (f^i)^2 + f_{\partial}^2\right) (f_{PL}(x), f_{\partial,PL}(x) - \epsilon), \tag{10}
 \end{aligned}$$

314 where  $\psi : \mathbb{R} \rightarrow [0, 1]$  is now a smooth bump function that is 1 in a sufficiently large neighbour-  
 315 hood of zero (somewhat larger than  $y_0$ ) and zero outside some compact set. We stress that  
 316  $F_{PL,2,\epsilon}$  is a mapping from  $\mathbb{R}^d \times [0, 1]$  to  $\mathbb{R}^{d-n+1}$ . Using the result for isomanifolds (with some  
 317 modifications), we can prove that each individual set  $B_1(\epsilon)$  is isotopic to  $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}(\epsilon)$

318 for small  $\epsilon$  while, for sufficiently large  $\epsilon$ , it leaves the set invariant.

319 **3.1 Step 1**

320 The proof closely follows the proof for the case without boundary in Section 2. The main  
 321 technical difficulty will be to provide bounds that serve as the counterparts of Lemmas 5  
 322 and 16. To be able to do so we first need to discuss bounds on the bump functions  $\phi$  and  $\psi$ .

323 **3.1.1 Bump functions**

324 Following [39, Section 2.2] we write,

325 
$$\zeta_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

326 For  $0 < y_1 < y_2$  we write  $\zeta_2(x) = \zeta_1(x - y_1)\zeta_1(y_2 - x)$ . Then we define  $\phi_l : \mathbb{R} \rightarrow [0, 1]$  by  
 327  $\phi_l(x) = \int_x^{y_2} \zeta_2(x')dx' / \int_{y_1}^{y_2} \zeta_2(x')dx'$ . Finally define  $\phi_b : \mathbb{R} \rightarrow [0, 1]$  by  $\phi_b(x) = \phi_l(|x|)$ , and  
 328 let  $\phi(x) = 1 - \phi_b(x)$ .

329 **► Lemma 21.** *We have  $\phi_b(x) \in [0, 1]$  and, writing  $2y_1 = y_2 = y_0$ ,*

330 
$$\partial_x(\phi_l(x)) \leq 2 \frac{e^{\frac{4}{3(y_2-y_1)}}}{y_2 - y_1} = 4 \frac{e^{\frac{2}{3y_0}}}{y_0} = \gamma_\phi. \tag{11}$$

331 **3.1.2 Inside a single simplex**

332 Similarly to Lemma 5, we now give a condition that ensures that the zero set of  $F_{PL,1}^i(x, \tau)$   
 333 is smooth inside  $\sigma \times [0, 1]$ . In fact, similarly to (7), we define

334 
$$F_{L,1}^i(x, \tau) = \left( 1 - \tau \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) f^i(x) + \tau \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) f_L^i(x)$$
  
 335 
$$= f(x) + \tau \phi \left( \sum_i (f^i)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)),$$

336 where  $\phi$  is as defined above. Observe that  $F_{L,1}^i(x, \tau)$  can be extended to a neighbourhood  
 337 of  $\sigma \times [0, 1]$ .

338 **► Remark.** For the constants, it is better if  $y_0$  can be chosen as large as possible, but we need  
 339  $y_1$  to be quite a bit larger than  $y_0$ . In turn, we cannot choose  $y_1$  arbitrarily large because  
 340 this would mean that the gradient field  $\text{grad} f_\partial|_{f^{-1}(0)}$  (seen as restricted on  $f^{-1}(0)$ ) would  
 341 never vanish. The latter is in general impossible thanks to the hairy ball theorem [17].

342 We introduce the following definition that complements Definition 1:

**► Definition 22.**

343 
$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \text{grad} \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right| = 2 \sup_{x \in \Sigma_0} \left| \sum_l f^l \text{grad} f^l + f_\partial \text{grad} f_\partial \right| \tag{12}$$

344 We have then the analog of Lemma 5:

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► **Lemma 23.** *We have :*

$$|\det(\text{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x,\tau))_{i,j}| > \gamma_0 - g_4(D),$$

345 with  $g_4(D) = \mathcal{O}(D)$ . The exact expression of  $g_4$  is given in (21).

346 The following corollary is then the analog of Corollary 6:

347 ► **Corollary 24** ( $F_{L,1}^{-1}(0)$  is a manifold). *If  $\gamma_0 > g_4(D)$ , where  $g_4(D) = \mathcal{O}(D)$  is as in Lemma*  
 348 *23, then  $F_{L,1}^{-1}(0)$  is a smooth manifold inside an  $\epsilon$  neighbourhood of  $\sigma \times [0, 1]$ .*

### 349 3.1.3 Transversality with regard to the $\tau$ -direction

350 We note that, similarly to Lemma 8, we have

351 ► **Lemma 25.** *Let  $\Xi$  be as in Lemma 8 and  $\gamma_\phi$  as in (11).*

$$352 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,1}), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0/\gamma_1^{d-n-1}} - \gamma_2\gamma_\phi 2D^2\alpha - \frac{4dD\alpha}{T}}.$$

353 *In particular, if  $\sqrt{\gamma_0/\gamma_1^{d-n-1}} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$ ,  $F_{L,1}^{-1}(0)$  (if it is a manifold) is never*  
 354 *tangent to the  $\tau = c$  planes, where  $c$  is a constant.*

355 Now, similarly to Corollary 9, we find that

356 ► **Corollary 26** (Transversality with respect to  $\tau$  for Step 1). *Suppose that  $\gamma_0 > g_4(D)$  and*  
 357 *that  $\sqrt{\gamma_0/\gamma_1^{d-n-1}} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$ . Then, inside each  $\sigma \times [0, 1]$ , the gradient of  $\tau$  on*  
 358  *$F_{L,1}^{-1}(0)$  is smooth and does not vanish.*

### 359 3.1.4 Global result

360 We now have to prove that  $F_{PL,1}^{-1}(0)$  is a manifold. For this, we shall use a bound similar to  
 361 the one given in Lemma 16, so that we are able to apply the generalized implicit function  
 362 theorem if this bound is satisfied. But first of all, we need the following bound, which is  
 363 similar to Lemma 14.

364 ► **Lemma 27.** *Assuming that the gradients are well defined, we have*

$$365 \quad |\text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2)| \leq g_5(D),$$

366 with  $g_5(D) = \mathcal{O}(D)$ . The exact expression for  $g_5$  is given in (22).

367 Just as in Corollary 15, we immediately have the same bound on points in the convex  
 368 hull of a number of such vectors:

369 ► **Corollary 28.** *Suppose we are in the setting of Lemma 27 and  $x_0, x_1, \dots, x_m \in \text{star}(v)$ ,*  
 370  *$\tau_0, \dots, \tau_m \in [0, 1]$ , such that  $\text{grad}_{(x,\tau)} F_{PL,1}^i(x_i, \tau_i)$  is well defined for all  $i$ . Further assume*  
 371 *that  $\mu_1, \dots, \mu_m$  are positive weights such that  $\mu_1 + \dots + \mu_m = 1$ . Then,*

$$372 \quad \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \leq g_5(D).$$

373 ▶ **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$374 \quad \det \left( \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$375 \quad \geq \gamma_0 - g_4(D) - g_6(D),$$

376 *with  $g_6(D) = \mathcal{O}(D)$ . The exact expression of  $g_6$  is given in (23).*

377 Lemma 29 immediately yields that

378 ▶ **Corollary 30** ( $F_{PL,1}^{-1}(0)$  is a manifold). *If,  $\gamma_0 > g_4(D) + g_6(D)$  the generalized implicit*  
 379 *function theorem, Theorem 13, applies to  $F_{PL,1}(x, \tau) = 0$ . In particular  $F_{PL,1}^{-1}(0)$  is a*  
 380 *manifold.*

381 We stress again that inside the set  $\{x | \phi(\sum_i (f^i)^2(x) + f_\partial^2(x)) = 1\}$  the zero set of  
 382  $F_{PL,1}(x, 1)$  coincides with the zero set of  $f_{PL}(x)$ .

### 383 3.2 Step 2

384 Before we can proceed we have to specify the bump function  $\psi$ . We suppose that

$$385 \quad \psi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{101}{100}y_0 \\ 0 & \text{if } |x| \geq 2y_0. \end{cases}$$

386 In particular we pick  $\psi(x) = \phi_b(x)$ , with the choice  $y_1 = \frac{101}{100}y_0$  and  $y_2 = 2y_0$ .

387 First we stress that the zero set of  $F_{PL,2,\epsilon}(x, 1)$  coincides with the zero set of  $(f_{PL}(x), f_{\partial,PL}(x) -$   
 388  $\epsilon)$ , provided that  $\psi(\sum_i f_i(x)^2 + f_\partial(x)^2) = 1$ .

389 Secondly, we now claim the following:

390 ▶ **Lemma 31.** *The zero set of  $F_{PL,2,\epsilon}(x, 1)$  is a subset of the zero set of  $f_{PL}(x)$ , for each  $\epsilon$ .*

391 The technical result that remains to be proven is the counterpart of Theorem 18 for  
 392  $F_{PL,2,\epsilon}(x, \tau)$  and for each sufficiently small  $\epsilon$ . To be precise it suffices for  $\epsilon \leq 2y_0$ . We  
 393 remark that it is likely that this bound on  $\epsilon$  can be improved.

394 We again follow the same path to prove this result. That is we first concentrate on a  
 395 single simplex and prove that inside that simplex the zero set of  $F_{PL,2,\epsilon}$  is a smooth manifold  
 396 on which the gradient of  $\tau$  as restricted to the manifold does not vanish. We then prove  
 397 that is the zero set of  $F_{PL,2,\epsilon}$  is globally a manifold.

#### 398 3.2.1 Assumptions and notations

399 Because we are now faced with both  $f(x)$  and  $f_\partial(x)$  we need to introduce a bound on how  
 400 far the gradients of all there are from being colinear. We write

$$401 \quad f_B(x) = (f(x), f_\partial(x)). \tag{13}$$

402 Before we were only interested in the set  $\Sigma_0$ , similarly here we sometimes concentrate  
 403 on a neighbourhood of the zero set of both  $f_\partial$  and  $f$ . Therefore we write  $B_\nu$  for all  $\sigma \in \mathcal{T}$   
 404 such that  $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$ .

405 We define  $\gamma_0^B$  in terms of the determinant of the Gram matrix of the gradients, that is

$$406 \quad \gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\operatorname{grad}(f_B^i) \cdot \operatorname{grad}(f_B^j))_{i,j}|. \tag{14}$$

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407 We note that because we take the gradients we can just ignore the  $\epsilon$  constant. For the  
408 lengths of the gradients of  $f_B$  we define,

$$409 \quad \gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f_B^i)|, \quad (15)$$

410 for all  $1 \leq i \leq d - n + 1$ . Similarly to  $\alpha$ , we define  $\beta$  as the bound on the operator 2-norm  
411 of all Hessians of  $f_B$ , that is

$$412 \quad \beta = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2. \quad (16)$$

413 We stress that we have chosen our definitions such that  $\alpha \leq \beta$ .

414 We use the same notation for the ambient triangulation  $\mathcal{T}$ , the lower bound on the  
415 thickness of the simplices  $T$  and upper bound on the longest edge length  $D$ . We also need  
416 to introduce a bound on the differential of the bump function  $\psi$ . Similarly to (11) we define,

$$417 \quad \gamma_\psi = 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} = 2 \frac{e^{\frac{4}{3(2y_0 - \frac{101}{100}y_0)}}}{2y_0 - \frac{101}{100}y_0} = \frac{200}{99} \frac{e^{\frac{400}{297y_0}}}{y_0}, \quad (17)$$

418 because we picked  $y_1 = \frac{101}{100}y_0$  and  $y_2 = 2y_0$ , for  $\psi$ .

### 419 3.2.2 Inside a single simplex

420 Similarly to Lemma 23 we now give a conditions that ensure that the zero set of  $F_{PL,2,\epsilon}(x, \tau)$   
421 is smooth inside  $\sigma \times [0, 1]$ . In fact similarly to (7) we define

$$422 \quad F_{L,2,\epsilon}(x, \tau) = \left( 1 - \tau \psi \left( \sum_i (f^i)^2 + f_\partial^2 \right) \right) (F_{L,1}(x, 1), f_\partial(x) - \epsilon)$$

$$423 \quad + \tau \psi \left( \sum_i (f^i)^2 + f_\partial^2 \right) (f_L(x), f_{\partial,L}(x) - \epsilon),$$

424 which can be extended to a neighbourhood of  $\sigma \times [0, 1]$ .

425 ► **Lemma 32.** *For all  $\epsilon$ ,*

$$426 \quad \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau))_{i,j} \geq \gamma_0^B - g_\tau(D)$$

427 *with  $g_\tau(D) = \mathcal{O}(D)$ . The exact expression of  $g_\tau$  is given in (24).*

428 We again have the following corollary.

429 ► **Corollary 33** ( $F_{L,2,\epsilon}^{-1}(0)$  is a manifold). *We have that  $F_{L,2,\epsilon}^{-1}(0)$  is a smooth manifold inside  
430 an small neighbourhood of  $\sigma \times [0, 1]$  provided  $\gamma_0^B > g_\tau(D)$ , with  $g_\tau(D)$  as in Lemma 32.  
431 As usual this can always be satisfied by choosing the triangulation fine enough, that is  $D$   
432 sufficiently small.*

### 433 3.2.3 Transversality with regard to the $\tau$ -direction

434 Once more similarly to Lemma 8 we have

435 ► **Lemma 34.** *Let  $\Xi$  be as in Lemma 8. We have*

$$436 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}), \Xi) \leq \frac{2D^2\beta}{\sqrt{\gamma_0^B / (\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T}}}.$$

437 In particular the manifold  $F_{L,2,\epsilon}^{-1}(0)$  inside  $\sigma \times [0, 1]$ , if well defined, is never tangent to the  
 438  $\tau = c$  planes, where  $c$  is a constant, if

$$439 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T}.$$

440 Now similarly to Corollary 9 we find that

441 ► **Corollary 35** (Transversality with respect to  $\tau$  for Step 2). *Suppose that the conditions of*  
 442 *Corollary 33 are satisfied. If moreover*

$$443 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T},$$

444 then inside each  $\sigma \times [0, 1]$  the gradient of  $\tau$  on  $F_{L,2,\epsilon}^{-1}(0)$  is smooth and does not vanish.

### 445 3.2.4 Global result

446 We now have to prove that  $F_{PL,2,\epsilon}^{-1}(0)$  is a manifold, for all sufficiently small  $\epsilon$ . For this we  
 447 shall use a bound similar to the one given in Lemma 16, so that we are able to apply the  
 448 generalized implicit function theorem if this bound is satisfied. For this we first need the  
 449 following bound, which is similar to Lemma 27.

450 ► **Lemma 36.** *Let  $v$  be a vertex in  $\mathcal{T}$ ,  $x_1, x_2 \in \text{star}(v)$ , and  $\tau_1, \tau_2 \in [0, 1]$ , such that*  
 451  *$\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1)$  and  $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)$  are well defined, then*

$$452 \quad |\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)| \leq g_8(D),$$

453 with  $g_8(D) = \mathcal{O}(D)$ . The exact expression of  $g_8$  is given in (25).

454 Just as in Corollary 15, we immediately have the same bound on points in the convex  
 455 hull of a number of such vectors:

456 ► **Corollary 37.** *Suppose  $x_0, x_1, \dots, x_m \in \text{star}(v)$ ,  $\tau_0, \dots, \tau_m \in [0, 1]$ , such that  $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_i, \tau_i)$*   
 457 *is well defined for all  $i$ . Further assume that  $\mu_1, \dots, \mu_m$  are positive weights such that*  
 458  *$\mu_1 + \dots + \mu_m = 1$ . Then,*

$$459 \quad \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right| \leq g_8(D).$$

460 ► **Lemma 38.** *Under the same conditions as in Corollary 37,*

$$461 \quad \det \left( \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$462 \quad \geq \gamma_0^B - g_7(D) - g_9(D),$$

463 where  $g_9(D) = \mathcal{O}(D)$ . The exact expression of  $g_9$  is given in (26).

464 Lemma 38 immediately yields that

465 ► **Corollary 39** (The generalized implicit function theorem in Step 2). *If,  $\gamma_0^B > g_7(D) +$*   
 466  *$g_9(D)$  the generalized implicit function theorem, Theorem 13, applies to  $F_{PL,1}(x, \tau) = 0$ . In*  
 467 *particular  $F_{PL,1}^{-1}(0)$  is a manifold.*



## 00:16 Topologically correct PL-approximations of isomanifolds

468 We stress that this condition only needs to be satisfied in a when  $\sum_l (f^l)^2 + (f_\partial)^2$  is small,  
 469 outside this neighbourhood the isotopy leaves the zero set invariant.

470 ► **Theorem 40.** *If,*

$$471 \quad \sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T} \quad (\text{Corollary 26})$$

$$472 \quad \gamma_0 > g_4(D) + g_6(D) \quad (\text{Corollaries 24 and 30})$$

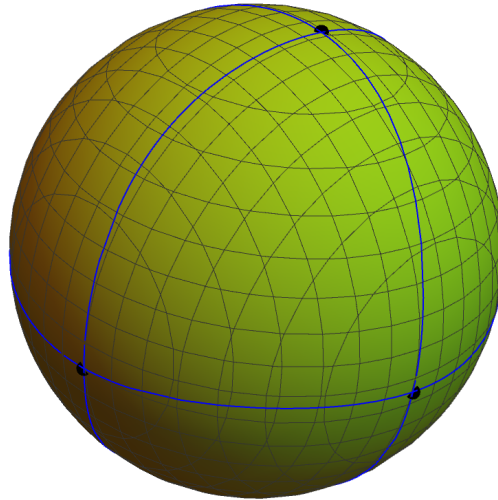
$$473 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \quad (\text{Corollary 35})$$

$$474 \quad \gamma_0^B > g_7(D) + g_9(D), \quad (\text{Corollaries 33 and 39})$$

475 *then  $f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$  is isotopic to  $f_{PL}^{-1}(0) \cap f_{\partial, PL}^{-1}([0, \infty))$ . We stress that one can satisfy*  
 476 *all conditions by choosing  $D$  sufficiently small. We refer to Appendix A for the definitions*  
 477 *of  $g_i(D)$ .*

### 478 **4 Isostratifolds**

479 There is no obstruction in principle that prevents us from extending the approach above  
 480 to isostratifolds. By isostratifolds we mean stratifolds that are given by the zero sets of  
 481 functions. For example suppose that we want to find a PL approximation of the unit sphere  
 482 centred at 0 in  $\mathbb{R}^3$  including the PL approximations of the intersections of the sphere with  
 483 slightly deformed  $x = 0$ ,  $y = 0$ , and  $z = 0$ -planes, as depicted in Figure 3. This would also  
 484 give PL approximations of the respective ‘octants’ of the sphere.



485 ■ **Figure 3** An example of an isostratifold.

486 We could follow the same procedure as for a manifold with boundary to give density con-  
 487 ditions that ensure that the PL approximation is correct. However this would mean that we  
 488 have to introduce an extra bump function for each stratum as well as an extra isotopy. Even  
 489 though that in theory this should be possible the analysis becomes prohibitively lengthy.

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**A Overview of constants**

We give an overview. We write  $\Sigma_0$  for the set of all  $\sigma \in \mathcal{T}$ , such that  $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$  for all  $i$ . We write  $B_\nu$  for all  $\sigma \in \mathcal{T}$  such that  $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$ . We write

$$f_B(x) = (f(x), f_\partial(x)) \quad (13)$$

$$\gamma_0 = \inf_{x \in \Sigma_0} |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| \quad (2)$$

$$\gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\text{grad}(f_B^i) \cdot \text{grad}(f_B^j))_{i,j}| \quad (14)$$

$$\gamma_1 = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f^i)| \quad (3)$$

$$\gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f_B^i)| \quad (15)$$

$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \text{grad} \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right| = \sup_{x \in \Sigma_0} \left| \sum_l f^l \text{grad} f^l + f_\partial \text{grad} f_\partial \right| \quad (12)$$

$$\alpha = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k \partial_l f^i)_{k,l}\|_2 \quad (4)$$

$$\beta = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2 \quad (16)$$

$D$  : the longest edge length of a simplex in  $\Sigma_0$

$T$  : the smallest thickness of a simplex in  $\Sigma_0$ .

$\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$  is the space spanned by the  $d$  basis vectors corresponding to the  $x$ -directions.

The bump functions give rise to the following:

$$\gamma_\phi = 4 \frac{e^{\frac{2}{3y_0}}}{y_0} \quad (11)$$

$$\gamma_\psi = \frac{200}{99} \frac{e^{\frac{400}{297y_0}}}{y_0}. \quad (17)$$

The precise expressions for the  $g_i(D)$  are:

$$g_1(D) = n^{n+1} \left( \gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{6dD\alpha}{T} \right)^2 \right) \quad (18)$$

$$g_2(D) = n^{n+1} \left( 2^{2n-1} \gamma_1^{2n} \left( \frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left( \frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right) \quad (19)$$

$$g_3(D) = \left( \frac{(d-n)d^{d-n-1} \sqrt{\gamma_0} 2D^2 \alpha \gamma_1^{d-n-1} (\gamma_1 + 2D^2 \alpha)^{d-n}}{\sqrt{\gamma_0 + (2D^2 \alpha \gamma_1^{d-n-1})^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right) \quad (20)$$

$$g_4(D) = n \left( \left( n \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \cdot n \left( 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right) \quad (21)$$

$$g_5(D) = 2\sqrt{d} d \alpha D + 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \quad (22)$$

$$g_6(D) = n^{n+1} \left( \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1}$$

$$\begin{aligned}
 & \cdot \left( 2 \left( 2\sqrt{d}d\alpha D + 2 \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left( \gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right. \\
 & \left. + \left( 2\sqrt{d}d\alpha D + 2 \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \right). \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 g_7(D) = & (n + 1) \left( (n + 1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 & \cdot (n + 1) \left( 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 & \left. + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \tag{24}
 \end{aligned}$$

$$g_8(D) = 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \tag{25}$$

$$\begin{aligned}
 g_9(D) = & (n + 1)^{n+1} \left( \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 & \cdot \left( 2 \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right. \\
 & \cdot \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & \left. + \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \right) \tag{26}
 \end{aligned}$$

If  $\frac{4dD\alpha}{T} \leq \gamma_1$ ,  $g_1(D)$  can be replaced by the simpler  $34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}$ .

658 **B** Recap of essential standard results

659 We now quickly recall two essential results, namely the smooth implicit function theorem  
 660 and Friedland's determinant bound.

661 ► **Theorem 41** (Smooth implicit function theorem). *Let  $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d-n}$  be a continuously*  
 662 *differentiable function. Write  $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$  and denote the coordinates of  $\mathbb{R}^{d+1}$  by*  
 663  *$(x, y)$  accordingly. Fix a point  $(a, b)$ , with  $F(a, b) = 0 \in \mathbb{R}^{d-n}$ . If the Jacobian  $J_{F,y}(a, b) =$*   
 664  *$(\frac{\partial F^i}{\partial y^j}(a, b))_{i,j}$  is of maximal rank (or equivalently invertible), then there exists an open set*  
 665  *$U \subset \mathbb{R}^{n+1}$  containing  $a$  such that there exists a unique continuously differentiable function*  
 666  *$g : U \rightarrow \mathbb{R}^{d-n}$  such that  $g(a) = b$  and  $F(x, g(x)) = 0$  for all  $x \in U$ .*

667 The following result by Friedland [36] will also be used throughout the proofs:

$$668 \quad |\det(A + E) - \det(A)| \leq n \max\{\|A\|_p, \|A + E\|_p\}^{n-1} \|E\|_p, \quad (27)$$

669 where  $A$  and  $E$  are  $n \times n$ -matrices and  $\|\cdot\|_p$  is the  $p$ -norm, with  $1 \leq p \leq \infty$ .

670 **C Proofs**

671 **C.1 Proofs of variations of known results in Section 2.1**

672 The following proofs are not exactly the same as the original, but do not differ greatly from  
 673 known results. We have included the results for completeness.

674 **Proof of Lemma 2.** This is a straightforward consequence of the existence and uniqueness  
 675 of the solution to a differential equation. ◀

676 The following two proofs are similar to the proofs in Allgower and George [1], however  
 677 because the statement are a bit different and for completeness we have included these.

678 **Proof of Lemma 3.** Taylor’s theorem, see for example [29, Theorem 2.8.4], yields that

679 
$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{28}$$

680 with

681 
$$R(v_k) = 2 \sum_{j \neq l} (v_k - x)^j (v_k - x)^l \int_0^1 (1-t)^2 \partial_j \partial_l f^i(v_k - t(v_k - x)) dt$$
  
 682 
$$+ 2 \sum_j \frac{((v_k - x)^j)^2}{2} \int_0^1 (1-t)^2 \partial_j^2 f^i(v_k - t(v_k - x)) dt$$
  
 683 
$$\leq 2|v_k - x|^2 \alpha \tag{by (4) and Cauchy-Schwarz}$$
  
 684 
$$\leq 2D^2 \alpha \tag{(x \in \sigma)}$$

685 The function  $f_L$  at the point  $x = \sum_k \lambda_k v_k$ , where  $\sum_k \lambda_k = 1$ , is by construction

686 
$$f_L^i(x) = \sum_k \lambda_k f^i(v_k)$$
  
 687 
$$= \sum_k \lambda_k \left( f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k) \right) \tag{by (28)}$$
  
 688 
$$= \sum_k \lambda_k f^i(x) + \sum_j \partial_j f^i(x) \left( \sum_k \lambda_k v_k - x \right)^j + \sum_k \lambda_k R(v_k)$$
  
 689 
$$= f^i(x) + 0 + \sum_k \lambda_k R(v_k)$$

690 Thanks to the bounds on  $R(v_k)$  and Cauchy-Schwarz one has

691 
$$|f_L^i(x) - f^i(x)| \leq 2D^2 \alpha$$

692

693 **Proof of Proposition 4.** We again use that

694 
$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{28}$$

695 with

696 
$$|R(v_k)| \leq 2D^2 \alpha \tag{29}$$



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697 Subtracting  $f^i(v_l)$  from  $f^i(v_k)$  now yields

$$698 \quad f^i(v_k) - f^i(v_l) = \sum_j \partial_j f^i(x)(v_k - v_l)^j + R(v_k) - R(v_l).$$

699 Because  $f_L$  is the linear interpolation of  $f$ , we have  $f^i(v_k) - f^i(v_l) = \sum_j \partial_j f_L^i(x)(v_k - v_l)$ ,  
700 and thus

$$701 \quad \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(v_k - v_l)^j \right| \leq |R(v_k) - R(v_l)| \leq 4D^2\alpha.$$

702 Let now  $u = \mu_k v^k$  and  $w = \tilde{\mu}_k v_k$ , with  $\sum \mu_k = \sum \tilde{\mu}_k = 1$ , that is  $u, w \in \sigma$ . Clearly  
703  $\sum \mu_k - \tilde{\mu}_k = 0$ , so  $u - w$  can be developed in a sum such that the  $m$ -th term in the sum is  
704  $c_m(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})$ , with  $c_m > 0$ , and  $\sum |c_m| = 1$ . We stress that  $v_{\tilde{k}(m)}$  may be the same as  
705  $v_{\tilde{k}(m')}$ , with  $m \neq m'$ . We now see that

$$706 \quad \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(u - w)^j \right| = \left| \sum_{j,m} (\partial_j f_L^i(x) - \partial_j f^i(x))c_m(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right|$$

$$707 \quad \leq \sum_m c_m \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right|$$

(by the triangle inequality and  $c_m > 0$ )

$$708 \quad \leq \sum_m 4c_m D^2\alpha \quad (\text{because } \sum |c_m| = 1)$$

$$709 \quad = 4D^2\alpha. \quad (30)$$

710 Because the simplex  $\sigma$  contains a ball of radius the smallest altitude over  $d$  centred at its  
711 barycentre, that is  $TD/d$  with  $T$  the thickness, the vector  $u - w$  can be chosen to be any  
712 vector of length less than  $tD/d$ . In particular we can choose

$$713 \quad (u - w)^j = \frac{tD}{d} \frac{(\partial_j f_L^i(x) - \partial_j f^i(x))}{\sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2}}.$$

714 Plugging this choice into (30) gives

$$715 \quad \frac{tD}{d} \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq 4D^2\alpha$$

716 So that

$$717 \quad \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T}.$$

718 ◀

### 719 C.2 Proofs of new results in of Section 2.1

720 ► **Lemma 5.** Let  $\text{grad}(f^i) = (\partial_j f_i)_j$  denote the gradients of the components  $f^i$ . Suppose  
721 that the absolute value of the determinant of the Gram matrix of these gradients is lower  
722 bounded, that is

$$723 \quad |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

724 and  $|\text{grad}(f^i)| \leq \gamma_1$ , for all  $i$ . Assume further that the conditions of Proposition 4 are  
 725 satisfied, then

$$726 \quad |\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| > \gamma_0 - n^{n+1} \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{4dD\alpha}{T} \right)^2 \right). \quad (31)$$

727 If we write  $\text{grad}_{(x,\tau)}$  for the gradient that includes the  $\tau$  component, we find that

$$728 \quad |\det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \quad (8)$$

729 where  $g_1(D) = \mathcal{O}(D)$ , precisely

$$730 \quad g_1(D) = n^{n+1} \left( \gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{6dD\alpha}{T} \right)^2 \right). \quad (32)$$

731 **Proof of Lemma 5.** In this proof we shall be using the following notation:

$$732 \quad (\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j} = (Y)_{i,j},$$

733 and write  $Y_{i,j}$  for the entries of the matrix.

734 By definition of  $F_L$ , we have that

$$\begin{aligned} 735 \quad & \det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j} \\ 736 \quad &= \det(\text{grad}((1-\tau)f^i(x) + \tau f_L(x)^i) \cdot \text{grad}((1-\tau)f^j(x) + \tau f_L(x)^j))_{i,j} \\ 737 \quad &= \det(\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x))))_{i,j} \end{aligned}$$

738 We shall write

$$739 \quad (X)_{i,j} = (\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}$$

740 Proposition 4 now yields that

$$\begin{aligned} 741 \quad & |X_{i,j} - Y_{i,j}| \\ 742 \quad &= |\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x))) \\ 743 \quad &\quad - \text{grad}(f^i(x)) \cdot \text{grad}(f^j(x))| \\ 744 \quad &= |\text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x)) + \text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x)) \\ 745 \quad &\quad + \text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(\tau(f_L(x)^j - f^j(x)))| \\ 746 \quad &\leq 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{4dD\alpha}{T} \right)^2. \quad (33) \end{aligned}$$

747 This estimate is used to bound the  $\|E\|_p$  term in Friedland's bound, for  $p = \infty$ . In particular,

$$748 \quad \|(X - Y)_{i,j}\|_\infty \leq n \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{4dD\alpha}{T} \right)^2 \right). \quad (34)$$

749 Furthermore,

$$\begin{aligned} 750 \quad & |X_{i,j}| = |\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x)))| \\ 751 \quad &\leq \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2, \quad (35) \end{aligned}$$

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752 where we used Cauchy-Schwarz. Note that the latter estimate is used to bound the  $\max\{\|A\|_p, \|A+$   
 753  $E\|_p\}$  term in Friedland's bound, for  $p = \infty$ . In particular,

$$754 \quad \|(X)_{i,j}\|_\infty \leq n \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2. \quad (36)$$

755 This means that Friedland's bound now yields

$$756 \quad |\det(X)_{i,j} - \det(Y)_{i,j}| \leq n(\|(X)_{i,j}\|_\infty)^{n-1} \|(X - Y)_{i,j}\|_\infty. \quad (37)$$

757 Thus,

$$758 \quad \begin{aligned} |\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| &\geq |\det(\text{grad}(f^i(x)) \cdot \text{grad}(f^j(x)))_{i,j}| \\ 759 \quad &\quad - n^{n+1} \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{4dD\alpha}{T} \right)^2 \right) \\ 760 \quad &\quad > \gamma_0 - n^{n+1} \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{4dD\alpha}{T} \right)^2 \right). \end{aligned}$$

761 The addition of the  $\tau$  component gives a small extra contribution to the inner product to  
 762 the gradients

$$763 \quad \begin{aligned} \tilde{X}_{i,j} &= \text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j \\ 764 \quad &= \text{grad}(f^i(x) + \tau(f_L^i(x) - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L^j(x) - f^j(x))) \\ 765 \quad &\quad + (f_L^i(x) - f^i(x))(f_L^j(x) - f^j(x)). \end{aligned}$$

766 Combining this with (33) and Lemma 3 yields

$$767 \quad \begin{aligned} |\tilde{X}_{i,j} - Y_{i,j}| &= |\text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j - \text{grad}(f^i(x)) \cdot \text{grad}(f^j(x))| \\ 768 \quad &\leq 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \end{aligned}$$

769 Similarly to (35), we also have that

$$770 \quad |\tilde{X}_{i,j}| = |\text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j| \leq \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2. \quad (38)$$

771 Applying Friedland's bound again gives

$$772 \quad \begin{aligned} |\det(\tilde{X})_{i,j}| &= \left| \det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j} \right| \\ 773 \quad &\geq \gamma_0 - n^{n+1} \left( \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \cdot \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right) \\ 774 \quad &\geq \gamma_0 - n^{n+1} \left( \gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{6dD\alpha}{T} \right)^2 \right). \end{aligned}$$

(because  $T \leq 1$ )

775

776 ► **Corollary 42.** *If on top of the conditions of Lemma 5, we also assume that  $\frac{4dD\alpha}{T} \leq \gamma_1$ ,*  
 777 *the bound (31) can be simplified, at the cost of weakening the bound, to*

$$778 \quad \det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j} > \gamma_0 - 3 \cdot 2^{2n+1} n^{n+1} (\gamma_1)^{2n-1} \frac{dD\alpha}{T}.$$

779 The bound (8) can be simplified, at the cost of weakening the bound, to

$$780 \quad |\det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

781 **Proof of Corollary 42.** If we assume  $\frac{4dD\alpha}{T} \leq \gamma_1$ , a short calculation shows that (31) reduces  
782 to

$$783 \quad |\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| > \gamma_0 - n^{n+1} (2\gamma_1)^{2(n-1)} \left(3\gamma_1 \frac{4dD\alpha}{T}\right)$$

$$784 \quad = \gamma_0 - 3 \cdot 2^{2n+1} n^{n+1} (\gamma_1)^{2n-1} \frac{dD\alpha}{T}.$$

785 A straightforward calculation shows (8) reduces to

$$786 \quad \det(\tilde{X})_{i,j} \geq \gamma_0 - n^{n+1} \left(\frac{5}{2}\gamma_1\right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \frac{3}{2}\gamma_1 \frac{6dD\alpha}{T}\right)$$

$$787 \quad = \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

788 ◀

789 **Proof of Lemma 7.** Let  $A_1 = \text{span}(v^1, \dots, v^{d-n})$ . Write  $v_A^i$  for the projection of  $v^i$  on  $A_1$ .  
790 Note that the vector  $v_A^i$  is  $d - n$ -dimensional. By construction,

$$791 \quad \det(v^i \cdot v^j)_{i,j} = \det(v_A^i \cdot v_A^j)_{i,j} = \det((v_A^i)_i^t (v_A^j)_j) = \det((v_A^i)_i)^2,$$

792 where  $(v_A^i)_i$  denotes the  $(d - n) \times (d - n)$ - matrix whose columns are  $v_A^i$ , and  $(v_A^i)_i^t$  its  
793 transposed. Moreover,  $|v^i| = |v_A^i|$ . Hadamard's inequality [38] now gives

$$794 \quad \det((v_A^j)_j) \leq \prod_j |v_A^j| \leq \gamma_1^{d-n-1} |v_A^i|.$$

795 Using that  $\det(v^i \cdot v^j)_{i,j} > \gamma_0$  now yields  $\sqrt{\gamma_0}/\gamma_1^{d-n-1} \leq |v_A^i|$ . ◀

796 **Proof of Lemma 8.** The statement follows fairly straightforwardly from Lemma 7 and Lemma 3.  
797 In particular the  $\tau$  component of  $\text{grad}_{(x,\tau)}(F_L^i)$  is bounded in absolute value by  $|f_L(x)^i -$   
798  $f^i(x)| \leq 2D^2\alpha$ . On the other hand the  $x$  component is lower bounded by

$$799 \quad \sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T},$$

800 as a consequence of Proposition 4 and Lemma 7. This gives that

$$801 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_L^i), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

802 ◀

### 803 C.3 Proofs of Section 2.2

804 **Proof of Lemma 14.** Using the standard bounds on matrix norm, see for example [37,  
805 Equation (2.3.11)], and (4), we have that  $\sqrt{d}\alpha \geq |\partial_k \partial_l f^i|$ , for all  $k, l$  and  $i$ . Because the

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806 supremum of the absolute value of a derivative of a function bounds the Lipschitz constant  
807 of the function, we have that

$$808 \quad |\partial_l f^i(x_1) - \partial_l f^i(x_2)| \leq \sqrt{d}\alpha |x_1 - x_2| \quad (39)$$

$$809 \quad |f^i(x_1) - f^i(x_2)| \leq \gamma_1 |x_1 - x_2|. \quad (40)$$

810 Note furthermore that

$$811 \quad |x_1 - x_2| \leq 2D, \quad (41)$$

812 by the triangle inequality.

813 Lemma 3 gives that  $|f_{PL}^i(x) - f^i(x)| \leq 2D^2\alpha$ . Using the triangle inequality we now find  
814 that

$$815 \quad |f_{PL}^i(x_1) - f_{PL}^i(x_2)| \leq |f_{PL}^i(x_1) - f^i(x_1) + f^i(x_1) - f^i(x_2) + f^i(x_2) - f_{PL}^i(x_2)|.  
816 \quad \leq 2\gamma_1 D + 4D^2\alpha \quad (42)$$

817 Because

$$818 \quad \text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) = \begin{pmatrix} \text{grad}(f^i(x) + \tau_1(f_{PL}^i(x) - f^i(x))) \\ f_{PL}^i(x) - f^i(x) \end{pmatrix},$$

819 we find that

$$820 \quad |\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL}^i(x_2, \tau_2)|  
821 \quad \leq \sum_l |\partial_l f^i(x_1) - \partial_l f^i(x_2)|  
822 \quad \quad + \sum_l \tau_1 |\partial_l f_{PL}^i(x_1) - \partial_l f^i(x_1)| + \sum_l \tau_2 |\partial_l f_{PL}^i(x_2) - \partial_l f^i(x_2)|  
823 \quad \quad + |f_{PL}^i(x_1) - f^i(x_1) - f_{PL}^i(x_2) + f^i(x_2)|  
824 \quad \leq 2\sqrt{d}d\alpha D \quad (\text{by (39), and (41)})  
825 \quad \quad + d \frac{8dD\alpha}{T} \quad (\text{Proposition 4, and } \tau_1, \tau_2 \in [0, 1])  
826 \quad \quad + 2\gamma_1 D + 4D^2\alpha + 2\gamma_1 D \quad (\text{by (40), (41) and (42)})  
827 \quad \leq 2\sqrt{d}d\alpha D + d \frac{8dD\alpha}{T} + 4\gamma_1 D + 4D^2\alpha  
828 \quad \leq \frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha,$$

829 where we used the triangle inequality several times. ◀

830 ► **Lemma 16.** *Suppose that*

$$831 \quad |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

832 and  $|\text{grad}(f^i)| \leq \gamma_1$ , for all  $i$ . Let now  $v$  be a vertex in  $\mathcal{T}$ ,  $x_1, \dots, x_m \in \text{star}(v)$ , and  
833  $\tau_1, \dots, \tau_m \in [0, 1]$ , such that  $\text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k)$ ,  $k = 0, \dots, m$  are well defined. Then,

$$834 \quad \left| \det \left( \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right) \right|_{i,j}$$

$$\begin{aligned}
 &\geq \gamma_0 - n^{n+1} \left( \gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{6dD\alpha}{T} \right)^2 \right) \\
 &- n^{n+1} \left( \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \\
 &\cdot \left( 2 \left( \frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right) \left( \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \right. \\
 &\left. + \left( \frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right)^2 \right).
 \end{aligned}$$

If we moreover assume  $D \leq 1$ , and  $\frac{6dD\alpha}{T} \leq \gamma_1$ , the expression can be simplified, at the cost of weakening the bound, to

$$\left| \det \left( \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right)_{i,j} \right| \geq \gamma_0 - g_2(D),$$

with  $g_2(D) = \mathcal{O}(D)$ , in fact

$$g_2(D) = n^{n+1} \left( 2^{2n-1} \gamma_1^{2n} \left( \frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left( \frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right).$$

**Proof of Lemma 16.** Let  $x_0 \in \operatorname{star}(v)$  and  $\tau_0 \in [0, 1]$ , be such that  $\operatorname{grad}_{(x,\tau)} F_L^i(x_0, \tau_0)$  is well defined. Note that it is sufficient for  $x_0$  to lie in the interior of a  $d$ -simplex in  $\mathcal{T}$ . Lemma 5 gives that

$$\begin{aligned}
 |\det(\tilde{X})_{i,j}| &= |\det(\operatorname{grad}_{(x,\tau)}(F_L^i(x_0, \tau_0)) \cdot \operatorname{grad}_{(x,\tau)}(F_L^j(x_0, \tau_0)))_{i,j}| \\
 &> \gamma_0 - n^{n+1} \left( \gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{6dD\alpha}{T} \right)^2 \right).
 \end{aligned}$$

We want to use this together with Corollary 15 and Friedland’s bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (38) also holds for the convex hull, that is

$$\left| \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right| \leq \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2.$$

We write

$$Z_{i,j} = \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right).$$

Similarly to (36), we have that

$$\| (Z)_{i,j} \|_\infty \leq n \left( \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right) \tag{43}$$

We note that

$$\left| \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right| = \left| \left( \begin{array}{c} \operatorname{grad}(f^i(x) + \tau(f_{PL}^i(x) - f^i(x))) \\ f_{PL}^i(x) - f^i(x) \end{array} \right) \right|$$

$$\begin{aligned}
 &\leq \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \\
 &\quad \text{(because } \text{grad}(f^i) \leq \gamma_1, \text{ Proposition 4, and Lemma 3)} \\
 &\tag{44}
 \end{aligned}$$

Corollary 15 now gives

$$\begin{aligned}
 |Z_{i,j} - \tilde{X}_{i,j}| &= \left| \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right. \\
 &\quad \left. - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
 &= \left| \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) + \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right) \right. \\
 &\quad \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) - \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) + \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right) \\
 &\quad \left. - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
 &\leq \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
 &\quad + \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right| \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right| \\
 &\quad + \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \\
 &\quad \cdot \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right| \\
 &\leq 2 \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left( \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
 &\quad \text{(by Corollary 15 and (44))} \\
 &\quad + \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \\
 &\quad \text{(by Corollary 15)} \\
 &\tag{45}
 \end{aligned}$$

Similarly to (34), we have that

$$\begin{aligned}
 \|(Z - \tilde{X})_{i,j}\|_\infty &\leq n2 \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left( \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
 &\quad + n \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \\
 &\tag{46}
 \end{aligned}$$

Indeed, Friedland now gives,

$$|\det(Z)_{i,j}| = \left| \det \left( \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right)_{i,j} \right|$$

$$\begin{aligned}
 &\geq \gamma_0 - n^{n+1} \left( \gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \left( \frac{6dD\alpha}{T} \right)^2 \right) \quad (\text{by Lemma 5}) \\
 &\quad - n^{n+1} \left( \left( \gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \quad (\text{by 43}) \\
 &\quad \cdot \left( 2 \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left( \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \right. \\
 &\quad \left. + \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \right) \\
 &\hspace{15em} (\text{by 46, note that } n \text{ was factored out})
 \end{aligned}$$

If we further assume that  $D \leq 1$ , and  $\frac{6dD\alpha}{T} \leq \gamma_1$ , we can simplify the bound to

$$\begin{aligned}
 |\det(Z)_{i,j}| &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left( \frac{6dD\alpha}{T} \right) \right) \\
 &\quad - n^{n+1} \left( (2\gamma_1)^2 + (2D\alpha)^2 \right)^{n-1} \\
 &\quad \cdot \left( 2 \left( \frac{14d^2D\alpha}{T} + 4\gamma_1D \right) (2\gamma_1 + 2dD\alpha) + \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \right) \\
 &\hspace{15em} (\text{because } D \leq 1, \text{ and } T \leq 1) \\
 &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left( \frac{6dD\alpha}{T} \right) \right) \\
 &\quad - n^{n+1} (5\gamma_1^2)^{n-1} \\
 &\quad \cdot \left( 2 \left( \frac{14d^2D\alpha}{T} + 4\gamma_1D \right) (2\gamma_1 + \gamma_1) + \left( \frac{10d^2D\alpha}{T} + 4\gamma_1D + \gamma_1D \right)^2 \right) \\
 &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left( 2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left( \frac{6dD\alpha}{T} \right) \right) \\
 &\quad - n^{n+1} (5\gamma_1^2)^{n-1} \left( 6\gamma_1 \left( \frac{14d^2D\alpha}{T} + 4\gamma_1D \right) + \left( \frac{10d^2D\alpha}{T} + 5\gamma_1D \right)^2 \right) \\
 &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \gamma_1 \left( \frac{14dD\alpha}{T} \right) \\
 &\quad - n^{n+1} (5\gamma_1^2)^{n-1} \left( 6\gamma_1 \left( \frac{14d^2D\alpha}{T} + 4\gamma_1D \right) + (2d\gamma_1 + 5\gamma_1D) \left( \frac{10d^2D\alpha}{T} + 5\gamma_1D \right) \right) \\
 &\geq \gamma_0 - n^{n+1} 2^{2n-1} \gamma_1^{2n} \left( \frac{14dD\alpha}{T} \right) \\
 &\quad - n^{n+1} 5^{n-1} \gamma_1^{2n-2} \left( (2d\gamma_1 + 5\gamma_1) \left( \frac{24d^2D\alpha}{T} + 9\gamma_1D \right) \right) \\
 &\geq \gamma_0 - n^{n+1} \left( 2^{2n-1} \gamma_1^{2n} \left( \frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d + 5) \left( \frac{24d^2D\alpha}{T} + 9\gamma_1D \right) \right)
 \end{aligned}$$

► **Corollary 17.** If  $D \leq 1$ ,  $\frac{6dD\alpha}{T} \leq \gamma_1$ , and

$$\gamma_0 > g_2(D) = n^{n+1} \left( 2^{2n-1} \gamma_1^{2n} \left( \frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d + 5) \left( \frac{24d^2D\alpha}{T} + 9\gamma_1D \right) \right)$$

the generalized implicit function theorem, Theorem 13, applies to  $F_L(x, \tau) = 0$ .



901 ► **Theorem 18.** Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$  be a smooth function. Write  $f_{PL}$  for the Piecewise-  
 902 Linear function that is the linear interpolation of the values of  $f$  at the vertices, if restricted  
 903 to a single simplex  $\sigma \in \mathcal{T}$ . Write  $\Sigma_0$  for the set of all  $\sigma \in \mathcal{T}$ , such that  $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$  for  
 904 all  $i$ . Suppose that  $\Sigma_0$  is compact. Suppose that for in each simplex in  $\Sigma_0$  we have that

- 905 ■  $|\text{grad}(f^i)| \leq \gamma_1$ , for all  $i$ .
- 906 ■  $|\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0$ , for all  $i, j$ .

907 If moreover,  $D \leq 1$ ,  $\frac{6dD\alpha}{T} \leq \gamma_1$ ,  $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$ , and

$$908 \quad \gamma_0 > g_2(D) = n^{n+1} \left( 2^{2n-1} \gamma_1^{2n} \left( \frac{14d\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left( \frac{24d^2\alpha}{T} + 9\gamma_1 \right) \right) D,$$

909 then the zero set of  $f$  is isotopic to the zero set of  $f_{PL}$ .

910 **Proof of Lemma 19.** Write

$$911 \quad w = \mu_1 v^1 + \dots + \mu_{d-n} v^{d-n},$$

912 with  $\mu_1, \dots, \mu_{d-n} \in \mathbb{R}$ . By definition,

$$913 \quad \cos \angle(w, e_\tau) = \frac{\sum_i \mu_i v^i \cdot e_\tau}{|w|} = \frac{\sum_i \mu_i v^i \cdot e_\tau}{\sqrt{\sum_{i,j} \mu_i \mu_j v^i \cdot v^j}}.$$

914 We note that  $|\sum_i \mu_i v^i \cdot e_\tau| \leq (d-n)|\mu| \tilde{\gamma}_1 \phi_0$ , where  $\mu$  denotes the vector  $(\mu_i)$ .

915 The matrix  $(v^i \cdot v^j)_{i,j}$  is a symmetric positive definite matrix, because it is a Gram  
 916 matrix and its determinant strictly positive, by assumption. This means that the smallest  
 917 eigenvalue  $\lambda_{\min}$  of  $(v^i \cdot v^j)_{i,j}$ , gives a lower bound, that is

$$918 \quad \sum_{i,j} \mu_i \mu_j v^i \cdot v^j \geq \lambda_{\min} |\mu|^2.$$

919 The determinant of a square matrix is the product of its eigenvalues. The largest eigenvalue is  
 920 bounded by the operator norm, which in turn is bounded by the Frobenius norm [37, (2.3.7)],  
 921 denoted by  $\|\cdot\|_F$ . This gives that the largest eigenvalue  $\lambda_{\max}$  of  $(\langle v^i, v^j \rangle)$  is bounded:

$$922 \quad \lambda_{\max} \leq \|(v^i \cdot v^j)\|_2 \leq \|(v^i)^t\|_2 \|(v^i)_i\|_2 \leq \|(v^i)_i\|_F^2 \leq d^2 \tilde{\gamma}_1^2.$$

923 For the smallest eigenvalue we now find that

$$924 \quad \tilde{\gamma}_0 < \det(v^i \cdot v^j)_{i,j} \leq \lambda_{\max}^{d-n-1} \lambda_{\min} \leq d^{2(d-n-1)} \tilde{\gamma}_1^{2(d-n-1)} \lambda_{\min}.$$

925 Combining these results gives

$$926 \quad \cos \angle(w, e_\tau) = \frac{\sum_i \mu_i \langle v^i, e_\tau \rangle}{\sqrt{\sum_{i,j} \mu_i \mu_j \langle v^i, v^j \rangle}} \leq \frac{(d-n)|\mu| \tilde{\gamma}_1 \phi_0}{|\mu| \frac{\sqrt{\tilde{\gamma}_0}}{d^{d-n-1} \tilde{\gamma}_1^{d-n-1}}} = \frac{(d-n) d^{d-n-1} \phi_0 \tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

927

928 ► **Corollary 20.** Suppose that the conditions of Theorem 18 are satisfied. Then,

$$929 \quad d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D),$$

930 with  $g_3(D) = \mathcal{O}(D^2)$ , in fact

$$931 \quad g_3(D) = \left( \frac{(d-n) d^{d-n-1} \sqrt{\gamma_0} 2D^2 \alpha \gamma_1^{d-n-1} (\gamma_1 + 2D^2 \alpha)^{d-n}}{\sqrt{\gamma_0 + (2D^2 \alpha \gamma_1^{d-n-1})^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right).$$

932 **Proof of Corollary 20.** Lemma 8 gives

$$933 \quad \tan \angle(\text{grad}_{(x,\tau)}(F^i), \Xi) \leq \frac{2D^2\alpha\gamma_1^{d-n-1}}{\sqrt{\gamma_0}},$$

934 and thus using the notation of Lemma 19,

$$935 \quad \begin{aligned} \cos(\angle \text{grad}_{(x,\tau)}(F^i), e_\tau) &\leq \sin \arctan \left( \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}} \right) \\ 936 \quad &= \frac{\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}}{\sqrt{1 + \left( \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}} \right)^2}} \\ 937 \quad &= \frac{2D^2\alpha}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2}}. \end{aligned}$$

938 By Corollary 42

$$939 \quad \tilde{\gamma}_0 = \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

940 Lemma 3 and  $|\text{grad}(f^i)| \leq \gamma_1$  give

$$941 \quad \tilde{\gamma}_1 = \gamma_1 + 2D^2\alpha.$$

942 Plugging this into Lemma 19 gives

$$943 \quad \cos \angle(w, e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}},$$

944 with  $w \in \text{span}_i(\text{grad}_{(x,\tau)}(F^i))$ . Because the tangent space to  $F_L = 0$  is normal to  $\text{span}_i(\text{grad}_{(x,\tau)}(F^i))$ ,  
945 we see that

$$946 \quad \sin \angle(g_\tau, e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}}.$$

947 This means that the angle between  $g_\tau$  and  $e_\tau$  is very small (we can choose  $D$  arbitrarily  
948 small). This means that as  $\tau \in [0, 1]$  the distance between begin and end point of the  
949 gradient flow, and thus the Fréchet distance, by  $\tan \angle(g_\tau, e_\tau)$ , that is

$$950 \quad \begin{aligned} &d_F(f^{-1}(0), f_{PL}^{-1}(0)) \\ 951 \quad &\leq \tan \arcsin \left( \frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right) \end{aligned}$$

952 ◀

### 953 C.4 Proofs of Section 3.1

954 **Proof of Lemma 21.** As mentioned, by construction  $\phi_b(x) \in [0, 1]$ . Because  $\partial_x \beta \left(\frac{y_1+y_2}{2}\right) =$   
955 0 and this is the only zero of the derivative in the open interval  $(y_1, y_2)$ , we see that

$$956 \quad \beta(x) \leq \beta \left(\frac{y_1 + y_2}{2}\right) = e^{\frac{4}{y_1 - y_2}}.$$

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957 Hence, because  $\beta(x) \geq 0$ ,

$$958 \int_{y_1}^{y_2} \beta(x) \leq (y_2 - y_1) e^{\frac{4}{y_1 - y_2}}$$

959 Because  $\beta(x)$  is monotone on  $[y_1, \frac{y_1 + y_2}{2}]$  we also have

$$960 \int_{y_1}^{y_2} \beta(x) \geq \frac{y_2 - y_1}{2} \beta\left(\frac{3}{4}y_1 + \frac{1}{4}y_2\right) = \frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}$$

961 We now have

$$\begin{aligned} 962 \partial_x(\phi_l(x)) &= \partial_x \left( \int_x^{y_2} \beta(x') dx' / \int_{y_1}^{y_2} \beta(x') dx' \right) \\ 963 &= \beta(x) / \int_{y_1}^{y_2} \beta(x') dx' \\ 964 &\leq \frac{e^{\frac{4}{y_1 - y_2}}}{\frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}} \\ 965 &= 2 \frac{e^{\frac{4}{y_1 - y_2} - \frac{16}{3(y_1 - y_2)}}}{y_2 - y_1} \\ 966 &= 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} \end{aligned} \quad (\text{DefCphi})$$

967

968 ► **Lemma 23.** *Suppose that*

$$969 |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

970 and  $|\text{grad}(f^i)| \leq \gamma_1$ , for all  $i$  and that the bound (12) is satisfied. Then,

$$971 \det(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j} > \gamma_0 - g_4(D),$$

972 with  $g_4(D) = \mathcal{O}(D)$ , in fact,

$$\begin{aligned} 973 g_4(D) &= n \left( \left( n \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right. \\ 974 &\quad \cdot n \left( 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \Big), \end{aligned}$$

975 **Proof of Lemma 23.** We start with an estimate on the individual  $\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau)$

$$\begin{aligned} 976 &\left| \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \right| \\ 977 &= \left| \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} \left( f(x) + \tau \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right) \right| \\ 978 &= \left| -\text{grad}_{(x,\tau)} \left( \tau \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right) \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \left( \begin{array}{c} -\tau \text{grad} \left( \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) - \tau \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \\ \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \end{array} \right) \right| \\
 &= \left( \left| \tau \text{grad} \left( \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) + \tau \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \right|^2 \right. \\
 &\quad \left. + \left( \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right)^2 \right)^{1/2} \\
 &\leq \left| \tau \text{grad} \left( \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) \right| + \left| \tau \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \right| \\
 &\quad + \left| \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right| \quad \text{(by the triangle inequality)} \\
 &\leq \gamma_2 \gamma_\phi |f_L^i(x) - f^i(x)| + \left| \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \right| \\
 &\quad \text{(because } \tau \leq 1, (12), (11)) \\
 &\quad + |f_L^i(x) - f^i(x)| \quad \text{(because } \phi \in [0, 1]) \\
 &\leq \gamma_2 \gamma_\phi |f_L^i(x) - f^i(x)| + |\text{grad}(f_L^i(x) - f^i(x))| \quad \text{(because } \phi \in [0, 1]) \\
 &\quad + |f_L^i(x) - f^i(x)| \\
 &\leq (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \quad \text{(by Lemma 3 and Proposition 4)} \\
 & \tag{47}
 \end{aligned}$$

We now write

$$(X')_{i,j} = (\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j}.$$

Expanding yields

$$\begin{aligned}
 (X')_{i,j} &= \left( \left( \text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \\
 &\quad \cdot \left( \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 &= \left( \left( \text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \\
 &\quad \cdot \left( \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 &= (Y)_{i,j} \\
 &\quad + \left( \left( \text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right)_{i,j} \\
 &\quad + \left( \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \cdot \left( \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
 &\quad + \left( \left( \text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \right)
 \end{aligned}$$

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$$1001 \quad \cdot \left( \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right)_{i,j}, \quad (48)$$

1002 where we again used the notation

$$1003 \quad (\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j} = (Y)_{i,j}.$$

1004 We now see by Cauchy-Schwarz and the triangle inequality that

$$1005 \quad |X'_{i,j} - Y_{i,j}| \leq 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2,$$

1006 by (47). This also means that for the operator norm of the matrix we find

$$1007 \quad \|(X' - Y)_{i,j}\|_\infty \leq n \left( 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \quad (49)$$

1008 Similarly to (35) and (36), we also see that (47) gives

$$1009 \quad |X'_{i,j}| \leq \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2, \quad (50)$$

1010 and thus,

$$1011 \quad \|X'_{i,j}\|_\infty \leq n \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2. \quad (51)$$

1012 Friedland's bound (37) now gives

$$1013 \quad \det(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j} \\ 1014 \quad > \gamma_0 - n \left( \left( n \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right. \\ 1015 \quad \cdot \left. n \left( 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right),$$

1016 where we used (49), and (51). ◀

1017 **Proof of Lemma 25.** As we have seen in the calculation in (47), the  $\tau$  component of  $\text{grad}_{(x,\tau)}(F^i)$   
1018 is bounded in absolute value by  $|f_L(x)^i - f^i(x)| \leq 2D^2 \alpha$ . On the other hand the  $x$  component  
1019 is lower bounded by

$$1020 \quad \sqrt{\gamma_0} / \gamma_1^{d-n-1} - \gamma_2 \gamma_\phi 2D^2 \alpha - \frac{4dD\alpha}{T},$$

1021 as a consequence of Proposition 4 and the calculation in (47). This gives that

$$1022 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,1}), \Xi) \leq \frac{2D^2 \alpha}{\sqrt{\gamma_0} / \gamma_1^{d-n-1} - \gamma_2 \gamma_\phi 2D^2 \alpha - \frac{4dD\alpha}{T}}.$$

1023 ◀

1024 ▶ **Lemma 27.** *Under the same conditions as Lemma 14 we have*

$$1025 \quad \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right|$$

$$1026 \quad \leq g_5(D) = 2\sqrt{d}d\alpha D + 2 \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)$$

1027 **Proof of Lemma 27.** By expansion we see that

$$1028 \quad \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right|$$

$$1029 \quad = \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} \right.$$

$$1030 \quad \left. + \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right|$$

$$1031 \quad \leq \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} \right| + \left| \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} \right|$$

$$1032 \quad + \left| \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right| \quad (\text{by the triangle inequality})$$

$$1033 \quad \leq 2\sqrt{d}d\alpha D + 2 \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \quad (\text{by (39), (41) and (47) twice})$$

1034

1035 ▶ **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$1036 \quad \det \left( \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$1037 \quad \geq \gamma_0 - g_4(D) - g_6(D),$$

1038 with  $g_6(D) = \mathcal{O}(D)$ , where as usual we regard,  $\alpha, \gamma_1, \gamma_2, \gamma_\phi, d, n$  and  $T$  as constants. In  
1039 fact,

$$1040 \quad g_4(D) = n^{n+1} \left( \left( \gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^{2(n-1)} \right.$$

$$1041 \quad \left. \cdot \left( 2 \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right)$$

$$1042 \quad g_6(D) = n^{n+1} \left( \left( \gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1}$$

$$1043 \quad \cdot \left( 2 \left( 2\sqrt{d}d\alpha D + 2 \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left( \gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right.$$

$$1044 \quad \left. + \left( 2\sqrt{d}d\alpha D + 2 \left( (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \right).$$

1045 **Proof of Lemma 29.** The proof is more or less the same as the proof of Lemma 16, but with  
1046 more complicated bounds. Let  $x_0 \in \text{star}(v)$  and  $\tau_0 \in [0, 1]$ , be such that  $\text{grad}_{(x,\tau)} F^i(x_0, \tau_0)$   
1047 is well defined. Note that it is sufficient for  $x_0$  to lie in the interior of a  $d$ -simplex in  $\mathcal{T}$ .  
1048 Lemma 23 gives that

$$1049 \quad \left| \det(\tilde{X}')_{i,j} \right| = \left| \det(\text{grad}_{(x,\tau)}(F^i(x_0, \tau_0)) \cdot \text{grad}_{(x,\tau)}(F^j(x_0, \tau_0)))_{i,j} \right|$$

$$\begin{aligned}
 &> \gamma_0 - n^{n+1} \left( \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^{2(n-1)} \right. \\
 &\quad \left. \cdot \left( 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right).
 \end{aligned}$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (50) also holds for the convex hull, that is

$$\begin{aligned}
 &\left| \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right| \\
 &\leq \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2.
 \end{aligned}$$

We write

$$Z'_{i,j} = \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right).$$

Similarly to (36), we have that

$$\| (Z')_{i,j} \|_\infty \leq n \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \quad (52)$$

We note that

$$\begin{aligned}
 \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \right| &\leq \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \\
 &\quad \text{(because } \operatorname{grad}(f^i) \leq \gamma_1, \text{ and (47))} \\
 &\quad (53)
 \end{aligned}$$

Similarly to (45), we find that

$$\begin{aligned}
 |Z'_{i,j} - \tilde{X}'_{i,j}| &= \left| \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right. \\
 &\quad \left. - \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) \right| \\
 &\leq \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) \right| \\
 &\quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \right| \\
 &\quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \\
 &\quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right| \\
 &\leq 2 \left( 2\sqrt{d}d\alpha D + 2 \left( (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \right) \left( \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \\
 &\quad \text{(by Corollary 28 and (53))}
 \end{aligned}$$





1092 where we used that

$$1093 \quad 1 - \psi \left( \sum_i (f^i)^2 + f_\partial^2 \right) = 0,$$

1094 if

$$1095 \quad \sum_i (f^i)^2 + f_\partial^2 \leq \frac{101}{100} y_0,$$

1096 and

$$1097 \quad \left( 1 - \phi \left( \sum_i (f^i)^2 + f_\partial^2 \right) \right) = 0,$$

1098 if

$$1099 \quad \sum_i (f^i)^2 + f_\partial^2 \geq y_0.$$

1100 We can further rewrite (55),

$$\begin{aligned} 1101 \quad & \left( \left( 1 - \psi \left( \sum_i (f^i)^2 + f_\partial^2 \right) \right) \left( \phi \left( \sum_i (f^i)^2 + f_\partial^2 \right) - 1 + 1 \right) + \psi \left( \sum_i (f^i)^2 + f_\partial^2 \right) \right) f_{PL}(x) \\ 1102 \quad & = \left( \left( 1 - \psi \left( \sum_i (f^i)^2 + f_\partial^2 \right) \right) \left( \phi \left( \sum_i (f^i)^2 + f_\partial^2 \right) - 1 \right) + 1 \right) f_{PL}(x) \\ 1103 \quad & = f_{PL}(x), \end{aligned}$$

1104 where we used the same argument as before. ◀

1105 ▶ **Lemma 32.** For all  $\epsilon$ ,

$$1106 \quad \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x,\tau))_{i,j} \geq \gamma_0^B - g_\tau(D)$$

1107 with  $g_\tau(D) = \mathcal{O}(D)$ . In fact,

$$\begin{aligned} 1108 \quad g_\tau(D) &= (n+1) \left( (n+1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\ 1109 \quad &\cdot (n+1) \left( 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\ 1110 \quad &\left. + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \end{aligned}$$

1111 **Proof of Lemma 32.** We start with an estimate on the individual  $\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau)$ . We  
1112 will write  $(v, w)^i$  for the  $i$ -th coordinate of the composed vector  $(v, w)$ . We now see that

$$\begin{aligned} 1113 \quad & \left| \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \right| \\ 1114 \quad & = \left| \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} \left( \left( 1 - \tau\psi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (F_{L,1}(x,1), f_\partial(x) - \epsilon)^i \right) \right| \end{aligned}$$



$$\begin{aligned}
 & + \max_j \left| \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right| |\text{grad}(f_L^j(x) - f^j(x))| \\
 & \quad \text{(by the Leibniz rule, Cauchy-Schwarz, and the triangle inequality)} \\
 & + \left| \text{grad} \left( \psi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) \right| \left| \left( \left( 1 - \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right| \\
 & + \left| \psi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right| \left| \text{grad} \left( \left( \left( 1 - \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right) \right| \\
 & \quad \text{(by the Leibniz rule, and the triangle inequality)} \\
 & + 2D^2\beta \quad \text{(by Lemma 3)} \\
 & \leq \max_j \gamma_\phi \gamma_2 |f_L^j(x) - f^j(x)| + \max_j |\text{grad}(f_L^j(x) - f^j(x))| \\
 & \quad \text{(by Lemma 21, (11), (12), and since } \phi(y) \in [0, 1]) \\
 & + \gamma_\psi \gamma_2 \left| \left( \left( 1 - \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right| \\
 & \quad \text{(by Lemma 21, (12), (17))} \\
 & + \left| \text{grad} \left( \left( \left( 1 - \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right) \right| \\
 & \quad \text{(because } \psi(y) \in [0, 1]) \\
 & + 2D^2\beta \\
 & \leq \gamma_\phi \gamma_2 2D^2\alpha + \frac{4dD\alpha}{T} \quad \text{(by Lemma 3 and Proposition 4)} \\
 & + \gamma_\psi \gamma_2 2D^2\beta \quad \text{(by Lemma 3 and since } \phi(y) \in [0, 1]) \\
 & + \left| \text{grad} (f_L(x) - f(x), f_{\partial,L}(x) - f_{\partial}(x))^i \right| \\
 & + \left| \text{grad} \left( \left( \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), 0 \right)^i \right) \right| \quad \text{(by the triangle inequality)} \\
 & + 2D^2\beta \\
 & \leq \gamma_\phi \gamma_2 2D^2\beta + \frac{4dD\beta}{T} \quad \text{(because by definition } \alpha \leq \beta) \\
 & + \gamma_\psi \gamma_2 2D^2\beta \\
 & + \frac{4dD\beta}{T} \quad \text{(by Proposition 4)} \\
 & + \max_j \left| \text{grad} \left( \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) (f_L^j(x) - f^j(x)) \right) \right| \\
 & + 2D^2\beta \\
 & \leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{8dD\beta}{T} \\
 & + \max_j \left| \text{grad} \phi \left( \sum_l (f^l)^2 + f_\partial^2 \right) \right| |(f_L^j(x) - f^j(x))|
 \end{aligned}$$

$$\begin{aligned}
 & + \max_j \left| \phi \left( \sum_l (f^l)^2 + f_\delta^2 \right) \right| \left| \text{grad}(f_L^j(x) - f^j(x)) \right| \\
 & \hspace{15em} \text{(By the Leibniz rule and the triangle inequality)} \\
 & \leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{8dD\beta}{T} \\
 & \quad + \gamma_\phi\gamma_2 2D^2\alpha \hspace{10em} \text{(by Lemma 21, (11), (12), and Lemma 3)} \\
 & \quad + \frac{4dD\beta}{T} \hspace{10em} \text{(because } \phi(y) \in [0, 1], \text{ Proposition 4, and } \alpha \leq \beta) \\
 & = (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \hspace{10em} (56)
 \end{aligned}$$

We now write

$$(X'_B)_{i,j} = (\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau))_{i,j}.$$

Similarly to (48), we see that

$$\begin{aligned}
 (X'_B)_{i,j} & = \left( \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \\
 & \quad \cdot \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 & = \left( \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \\
 & \quad \cdot \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 & = (Y_B)_{i,j} \\
 & \quad + \left( \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right)_{i,j} \\
 & \quad + \left( \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \cdot \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
 & \quad + \left( \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \right. \\
 & \quad \left. \cdot \left( \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j},
 \end{aligned}$$

where we used the notation

$$(\text{grad}(f_B^i) \cdot \text{grad}(f_B^j))_{i,j} = (Y_B)_{i,j}.$$

We now see by (56), (15), Cauchy-Schwarz and the triangle inequality that

$$\begin{aligned}
 |(X'_B)_{i,j} - (Y_B)_{i,j}| & \leq 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & \quad + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2.
 \end{aligned}$$

This also means that for the operator norm of the matrix we find

$$\|(X'_B - Y_B)_{i,j}\|_\infty \leq (n + 1) \left( 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)$$

$$1176 \quad + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (57)$$

1177 Similarly to (35) and (36), we also see that (56) gives

$$1178 \quad |X'_{i,j}| \leq \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2, \quad (58)$$

1179 and thus,

$$1180 \quad \|X'_{i,j}\|_\infty \leq (n+1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (59)$$

1181 Friedland's bound (37) now gives

$$1182 \quad \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x,\tau))_{i,j}$$

$$1183 \quad \geq \gamma_0^B - (n+1) \left( (n+1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n$$

$$1184 \quad \cdot (n+1) \left( 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right.$$

$$1185 \quad \left. + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)$$

1186 where we used (57), and (59). ◀

1187 **Proof of Lemma 34.** As we have seen in the calculation in (56), the  $\tau$  component of  $\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}^i)$   
 1188 is bounded in absolute value by  $\left| ((f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x))^i \right| \leq 2D^2\beta$ . On the other  
 1189 hand the  $x$  component is lower bounded by

$$1190 \quad \sqrt{\gamma_0^B} / (\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T},$$

1191 as a consequence of Lemma 7 and the calculation in (56). This gives that

$$1192 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}), \Xi) \leq \frac{2D^2\beta}{\sqrt{\gamma_0^B} / (\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T}}.$$

1193 ◀

1194 **► Lemma 36.** Let  $v$  be a vertex in  $\mathcal{T}$ ,  $x_1, x_2 \in \text{star}(v)$ , and  $\tau_1, \tau_2 \in [0, 1]$ , such that  
 1195  $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1)$  and  $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)$  are well defined, then

$$1196 \quad |\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)| \leq g_8(D),$$

1197 with  $g_8(D) = \mathcal{O}(D)$ . In fact,

$$1198 \quad g_8(D) = 2\sqrt{dd}\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)$$

1199 **Proof of Lemma 36.** The proof follows the same steps as the proof of Lemma 27. By  
 1200 expansion we see that

$$1201 \quad |\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)|$$

$$\begin{aligned}
 &= \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} \right. \\
 &\quad \left. + \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2) \right| \\
 &\leq \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} \right| + \left| \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} \right| \\
 &\quad + \left| \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2) \right| \quad (\text{by the triangle inequality}) \\
 &\leq 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 &\hspace{15em} (\text{by (39), (41) and (56) twice})
 \end{aligned}$$

1207

1208 ► **Lemma 38.** *Under the same conditions as in Corollary 28,*

$$\begin{aligned}
 &\det \left( \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right)_{i,j} \\
 &\geq \gamma_0^B - g_7(D) - g_9(D),
 \end{aligned}$$

1211 where  $g_9(D) = \mathcal{O}(D)$ . In fact,

$$\begin{aligned}
 g_7(D) &= (n+1) \left( (n+1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 &\quad \cdot (n+1) \left( 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 &\quad \left. + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \\
 g_9(D) &= (n+1)^{n+1} \left( \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 &\quad \cdot \left( 2 \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right) \\
 &\quad \cdot \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 &\quad + \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2
 \end{aligned}$$

1219 **Proof of Lemma 38.** The proof is more or less the same as the proofs of Lemmas 16 and  
 1220 28, but with even more complicated bounds. Let  $x_0 \in \text{star}(v)$  and  $\tau_0 \in [0, 1]$ , be such that  
 1221  $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0)$  is well defined. Note that it is sufficient for  $x_0$  to lie in the interior  
 1222 of a  $d$ -simplex in  $\mathcal{T}$ . Lemma 23 gives that

$$\begin{aligned}
 &|\det(\tilde{X}'_B)_{i,j}| = |\det(\text{grad}_{(x,\tau)}(F_{PL,2,\epsilon}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)}(F_{PL,2,\epsilon}^j(x_0, \tau_0)))_{i,j}| \\
 &\quad > \gamma_0^B - (n+1) \left( (n+1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n
 \end{aligned}$$

1224

$$\begin{aligned}
 & \cdot (n+1) \left( 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 & \left. + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right).
 \end{aligned}$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give a bound on the determinant of

$$(Z'_B)_{i,j} = \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right).$$

As before we also need a bound on the operator norm  $\|(Z'_B)_{i,j}\|_\infty$ . This bound for this we notice that the bound (58) also holds for the convex hull, that is

$$\begin{aligned}
 & \left| \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right| \\
 & \leq \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2.
 \end{aligned}$$

And thus, similarly to (52), we have that

$$\|(Z'_B)_{i,j}\|_\infty \leq (n+1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (60)$$

We note that

$$\begin{aligned}
 \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \right| & \leq \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \\
 & \quad \text{(because } \operatorname{grad}(f_B^i) \leq \gamma_1^B \text{, and (56))}
 \end{aligned} \quad (61)$$

Similarly to (45), we find that

$$\begin{aligned}
 & |(Z'_B)_{i,j} - (\tilde{X}'_B)_{i,j}| \\
 & = \left| \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right. \\
 & \quad \left. - \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) \right| \\
 & \leq \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right| \\
 & \quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) \right| \\
 & \quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right| \\
 & \quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \right| \\
 & \quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right|
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right| \\
 & \leq 2 \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \quad (\text{by Corollary 37}) \\
 & \cdot \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \quad (\text{by (61)}) \\
 & + \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \quad (\text{by Corollary 37})
 \end{aligned}$$

Similarly to (34), we have that

$$\begin{aligned}
 \| (Z'_B - \tilde{X}'_B)_{i,j} \|_\infty & \leq 2(n+1) \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \\
 & \cdot \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & + (n+1) \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \quad (62)
 \end{aligned}$$

Indeed, Friedland's bound now gives,

$$\begin{aligned}
 |\det(Z'_B)_{i,j}| & = \left| \det \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left( \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right| \\
 & \geq \gamma_0^B - (n+1) \left( (n+1) \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 & \cdot (n+1) \left( 2\gamma_1^B \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 & \left. + \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \quad (\text{by Lemma 32}) \\
 & - (n+1)^{n+1} \left( \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \quad (\text{by (60)}) \\
 & \cdot \left( 2 \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right. \\
 & \cdot \left( \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & \left. + \left( 2\sqrt{d}d\beta D + 2 \left( (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \right) \\
 & \quad (\text{by (62) and factoring out } (n+1))
 \end{aligned}$$

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