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► **To cite this version:**

Jean-Daniel Boissonnat, Mathijs Wintraecken. The topological correctness of PL-approximations of isomanifolds. 2019. hal-02386193v2

HAL Id: hal-02386193

<https://inria.hal.science/hal-02386193v2>

Preprint submitted on 29 Nov 2019 (v2), last revised 20 Oct 2020 (v3)

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The topological correctness of PL-approximations of isomanifolds

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1 Abstract

2 Isomanifolds are the generalization of isosurfaces to arbitrary dimension and codimension, i.e.
3 manifolds defined as the zero set of some multivariate multivalued function $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$.
4 A natural (and efficient) way to approximate an isomanifold is to consider its Piecewise-Linear
5 (PL) approximation based on a triangulation \mathcal{T} of the ambient space \mathbb{R}^d . In this paper, we
6 give conditions under which the PL-approximation of an isomanifold is topologically equivalent
7 to the isomanifold. The conditions are easy to satisfy in the sense that they can always be
8 met by taking a sufficiently fine triangulation \mathcal{T} . This contrasts with previous results on the
9 triangulation of manifolds where, in arbitrary dimensions, delicate perturbations are needed to
10 guarantee topological correctness, which leads to strong limitations in practice. We further give
11 a bound on the Fréchet distance between the original isomanifold and its PL-approximation.
12 Finally we show analogous results for the PL-approximation of an isomanifold with boundary.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases PL-approximations, isomanifolds, topological correctness

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Funding The research leading to these results has received funding from the European Research Council (ERC) under the European Union’s Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement No. 339025 GUDHI (Algorithmic Foundations of Geometry Understanding in Higher Dimensions) and the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 754411.

Acknowledgements First and foremost we acknowledge Siargey Kachanovich for discussions. We thank Herbert Edelsbrunner and all members of his group, all former and current members of the Datashape team (formerly known as Geometrica), and André Lieutier for encouragement.

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13 1 Introduction

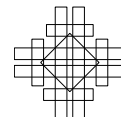
14 Isosurfacing especially in low dimensions is used in many fields of application, such as CT
15 scans in medicine, biochemistry, biomedicine, deformable modeling, digital sculpting, en-
16 vironmental science, and mechanics and dynamics [45] and the references mentioned there.



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35th International Symposium on Computational Geometry (SoCG 2020).
Editors: ???; Article No. 00; pp. 00:1–00:47



Leibniz International Proceedings in Informatics
Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



17 The marching cube approach [41] being the most popular approach taken. However the
18 result of the marching cube algorithm is not necessarily topologically correct.

19 Some results on provable correctness were achieved within the computational geometry
20 community [10, 47] in three dimensions. In case the isosurfacing is based on simplices instead
21 of cubes, such as in the marching tetrahedra algorithm [28], some bounds can be given [1, 2],
22 on for example the one-sided Hausdorff distance.

23 In general homeomorphism proofs in higher dimensions rely on some perturbation scheme
24 to prove that a triangulation is correct [52, 8, 11, 12, 15]. This is a major difference with
25 one and two dimensional surfaces where no such requirements exist [30], [49, Section 10.2].
26 In this paper we shall see that no perturbations are necessary for isomanifolds as well.

27 The techniques used here are also different from many of the standard tools. Manifold
28 triangulation/reconstruction algorithms use often Delaunay triangulations [50, 24, 19] and
29 use the closed ball property [32], see for example [3, 18]. Others use Whitney’s lemma [13] or
30 are based on collapses [4]. While the current paper mainly relies on the non-smooth implicit
31 function theorem [22] with some Morse theory.

32 We stress that this does not solve the general problem of triangulating manifolds with
33 boundary, because there exist manifolds that cannot be written as the zero set of a smooth
34 function, because of the existence of manifolds with a non-trivializable normal bundle (gen-
35 eralizations of non-orientable manifolds). For an excellent introduction to the topology of
36 such bundles and manifolds we refer to [44]. We note that locally we can always write an
37 embedded smooth manifold as the zero set of a smooth function, because it can be para-
38 metrized as a function from the tangent space to the manifold itself as a consequence of
39 the implicit function theorem. Non-continuous results have been recently obtained in the
40 neighbourhood of the manifold [20]. We stress that there are now obstructions to global
41 closest point projection on manifold [33], and this can even be approximately reconstructed
42 from a point sample [34].

43 We also emphasize that because we do not use a perturbation scheme, we cannot give
44 lower bounds on the quality of the linear pieces in the Piecewise-Linear (PL) approximation.
45 This is a clear difference with previous methods [52, 15, 14, 12] whose output is a thick
46 triangulation. Although this is an appealing property, it complicates the analysis further
47 and requires perturbation schemes that work fine in theory, but the constants are miserable
48 and the methods do not work in practice in high dimension (see a more detailed discussion
49 in Section 2). Here we ask for less but still provide guarantees on the Fréchet distance and
50 the approximation of the gradients. Perturbation techniques could be used to improve the
51 simplex quality (although only to some very limited extent).

52 This rest of the paper is subdivided in two sections. In the first section, we treat closed
53 isomanifolds, that is compact manifolds without boundary. In the second section, we prove
54 similar results for isomanifolds with boundary. Extension to general isostratifolds is briefly
55 discussed in Section 4. Apart from some Delaunay based work on triangulations of stratifolds
56 in three dimensions [46, 48, 27, 26, 25], we are not aware of similar results on manifolds with
57 boundary. Significant effort also went in the detection of strata, in this case in arbitrary
58 dimension, see for example [6, 5].

59 We will often give the simplified statement of results in the main text while complete
60 versions with all constants will be given in the appendix. All proofs can also be found in
61 the appendix.

62 This paper is part of a collection of closely related papers. The data structure needed to
63 efficiently propagate along the manifold is presented in [16], while details of the implement-
64 ation will be discussed in an upcoming paper.

2 Isomanifolds (without boundary)

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ be a smooth function and suppose that 0 is a regular value of f , meaning that at every point x such that $f(x) = 0$, the Jacobian of f is non-degenerate. Then the zero set of f is a manifold as a direct consequence of the implicit function theorem, see for example [29, Section 3.5]. We further assume that $f^{-1}(0)$ is compact. As in [1] we consider a triangulation \mathcal{T} of \mathbb{R}^d . The function f_{PL} is a linear interpolation of the values of f at the vertices if restricted to a single simplex $\sigma \in \mathcal{T}$. For any function $g : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ we write g^i for the components of g .

We prove that under certain conditions that there is an isotopy from the zero set of f to the zero set of f_{PL} . The proof will be using the Piecewise-Linear (PL) map

$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x), \quad (1)$$

which interpolates between f and f_{PL} and is based on the generalized implicit function theorem. The isotopy is in fact stronger than just the existence of a homeomorphism from the zero set of f to that of f_{PL} .

Our result in particular implies that the zero set f_{PL} is a manifold. So this significantly improves on the result of Allgower and Georg [2, Theorem 15.4.1]. The conditions are also weaker, because we do not require that the zero set avoids simplices that have dimension less than the codimension, see [2, Definition 12.2.2] and the text above [2, Theorem 15.4.1]. The idea to avoid these low dimensional simplices originates from Whitney [52], which remarkably was not cited by Allgower and George [2, 1]. Very heavy perturbation schemes for the vertices of the ambient triangulation \mathcal{T} are required for the manifold to be sufficiently far from simplices in the ambient triangulations that have dimension less than the codimension of the manifold [52, 15]. Various techniques have been developed to compute such perturbations with guarantees. They typically consist in perturbing the position of the sample points or in assigning weights to the points. Complexity bounds are then obtained using volume arguments. See, for example [18, 14, 11, 9]. However, these techniques suffer from several drawbacks. The constants in the complexity depend exponentially on the ambient dimension. Moreover the analysis assumes that the probability of the simplices of dimension less than the codimension to intersect the manifold is zero, which is not true when dealing with finite precision. As a result, the actual implementations we are aware of fail to work well in practice except in very simple cases.

We are, by definition, only interested in $f^{-1}(0)$ so we can ignore points that are sufficiently far from this zero set. More precisely, we observe the following: If $f^i(x)$ is positive on a simplex σ then so is f_{PL}^i and thus is F_{PL}^i positive on $\sigma \times [0, 1]$. The same argument holds for negative values. So we see that

► **Remark.** Write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i . Then for all τ , $\{x \mid F_{PL}(x, \tau) = 0\} \subset \Sigma_0$.

The results will be formulated in terms of constants defined in terms of f and the ambient triangulation \mathcal{T} .

104 ► **Definition 1.** We define

$$105 \quad \gamma_0 = \inf_{x \in \Sigma_0} |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| \quad (2)$$

$$106 \quad \gamma_1 = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f^i)| \quad (3)$$

$$107 \quad \alpha = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k \partial_l f^i)_{k,l}\|_2, \quad (4)$$

$$108 \quad D : \text{the longest edge length of a simplex in } \Sigma_0 \quad (5)$$

$$109 \quad T : \text{the smallest thickness of a simplex in } \Sigma_0. \quad (6)$$

110 Here $\text{grad}(f^i) = (\partial_j f^i)_j$ denotes the gradient of component f^i , $\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}$
 111 denotes the determinant of the matrix with entries $\text{grad}(f^i) \cdot \text{grad}(f^j)$, $\|\cdot\|_2$ the operator
 112 2-norm, and $(\partial_k \partial_l f^i)_{k,l}$ the matrix of second order derivatives, that is the Hessian (Hes).
 113 We recall the definition of the operator norm: $\|A\|_p = \max_{x \in \mathbb{R}^n} \frac{|Ax|_p}{|x|_p}$, with $|\cdot|_p$ the p -norm
 114 on \mathbb{R}^n .

115 We will assume that $\gamma_0, \gamma_1, \alpha, D, T \in (0, \infty)$. The thickness is the ratio of the height over
 116 the longest edge length.

117 A good choice for \mathcal{T} is the Coxeter triangulation of type A_d , see [23, 21], or the related
 118 Freudenthal triangulations, see [35, 40, 31, 51], which can be defined for different values of
 119 D while keeping T constant. In this paper, we will thus think of all the above quantities as
 120 well as d and n as constants except D and our results will hold for D small enough.

121 The result

122 We are going to construct an ambient isotopy based on (1). The zero set of $F_{PL}(x, 0)$
 123 (or $f(x)$) gives the smooth isosurface, while the zero set of $F_{PL}(x, 1)$ (or $f_{PL}(x)$) gives
 124 the PL approximation, that is the triangulation of the isosurface after possible barycentric
 125 subdivision. The map $\tau \mapsto \{x \mid F_{PL}(x, \tau) = 0\}$ in fact gives an isotopy. Without too much
 126 extra work we'll also bound the Fréchet distance between $f(x)$ and $f_{PL}(x)$.

130 Proving isotopy consists of two technical steps, which consume most of the space in the
 131 proof below, as well as the use of a standard observation from Morse theory/gradient flow
 132 in the third step. The technical steps are

133 ■ Let $\sigma \in \mathcal{T}$. We first show that $\{x \mid F_{PL}(x, \tau) = 0\} \cap \sigma \times [0, 1]$ is a smooth manifold,
 134 under certain conditions.

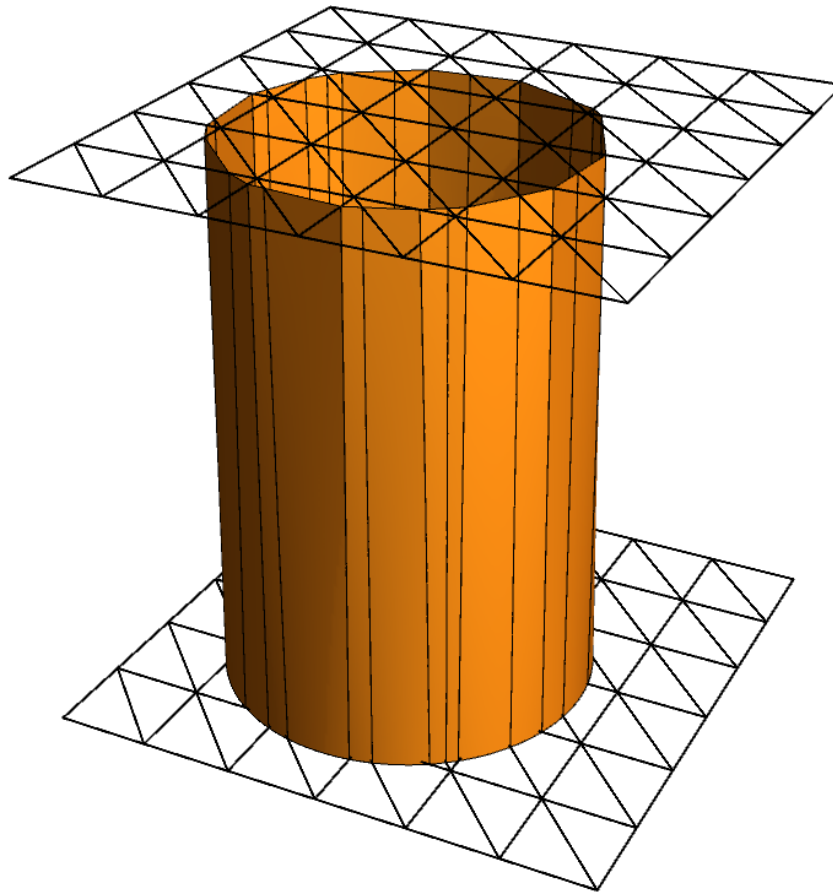
135 ■ We prove that $F_{PL}^{-1}(0)$ is a manifold, under certain conditions, using techniques from
 136 nonsmooth analysis.

137 Along the way we shall also see that $F_{PL}^{-1}(0)$ is never tangent to the $\tau = c$ planes, where c
 138 is a constant. This implies that the gradient field of τ restricted to $F_{PL}^{-1}(0)$, is piecewise
 139 smooth and never vanishes.

140 Now we arrive at the third step, which is similar to a standard observation in Morse theory
 141 [42, 43], with the exception that we now consider piecewise-smooth instead of smooth vector
 142 fields. We refer to Milnor [42] for an excellent introduction, see Lemma 2.4 and Theorem
 143 3.1 in particular.

144 ► **Lemma 2 (Gradient flow induced homeomorphisms).** *The flow of a non-vanishing piecewise-*
 145 *smooth gradient vector field of a function τ on a compact manifold generates a homeomorph-*
 146 *ism from $\tau = c_1$ to $\tau = c_2$, where c_1 and c_2 are constants.*

147 Bounds on the gradient of τ on the manifold give a bound on the Fréchet distance.



127 **Figure 1** A pictorial overview of the proof. The τ -direction goes upwards. Similarly to Morse
 128 theory we find that $f_{PL}^{-1}(0)$ (top) and $f^{-1}(0)$ (bottom) are homeomorphic if the function τ restricted
 129 to $F_{PL}^{-1}(0)$ does not encounter a Morse critical point.

148 **2.1 Estimates for a single simplex**

149 We now first concentrate on a single simplex σ and write f_L for the linear function whose val-
 150 ues on the vertices of σ coincide with f , that is f_L is the linear extension of the interpolation
 151 of f .

152 **2.1.1 Preliminaries and variations of known results**

153 We need a simple estimate similar to Proposition 2.1 of Allgower and George [1].

154 **Lemma 3.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then $|f_L^i(x) - f^i(x)| \leq 2D^2\alpha$
 155 for all $x \in \sigma$.*

156 We will also be using an estimate similar to Proposition 2.2 of Allgower and George [1].

157 **Proposition 4.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then*

158
$$|\text{grad}f_L^i - \text{grad}f^i| = \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T},$$

159 for all $x \in \sigma = \{v_k\}$.

160 **2.1.2 Estimates on the gradient inside a single simplex**

161 We write

$$162 \quad F_L(x, \tau) = (1 - \tau)f(x) + \tau f_L(x). \quad (7)$$

163 We note that F_L extends smoothly outside σ . Here and throughout we restrict ourselves to
 164 the setting where $\tau \in [0, 1]$. The function F_L has \mathbb{R}^{d-n} as image.

165 We can now state the following

166 ► **Lemma 5.** *If we write $\text{grad}_{(x,\tau)}$ for the gradient that includes the τ component, we have*

$$167 \quad |\det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \quad (8)$$

168 with $g_1(D) = \mathcal{O}(D)$. See (18) in Appendix A for the exact expression of g_1 .

169 From the previous statement we immediately have that

170 ► **Corollary 6** ($F_L^{-1}(0)$ is a manifold in a neighbourhood of $\sigma \times [0, 1]$). *If $\gamma_0 > g_1(D)$ the
 171 implicit function theorem applies to $F_L(x, \tau)$ inside $\sigma \times [0, 1]$. (In fact it applies to an open
 172 neighbourhood of this set). In particular, we have proven the first of our two technical steps,
 173 $\{x \mid F_{PL}(x, \tau) = 0\} \cap \sigma \times [0, 1]$ is a smooth manifold.*

174 **2.1.3 Transversality with regard to the τ -direction**

175 We will also prove the main result which we need for the third step, that is the gradient of
 176 τ restricted to $F_{PL}(x, \tau) = 0$, is piecewise smooth and never vanishes. We now prove inside
 177 each $\sigma \times [0, 1]$ the gradient of τ on $F_L = 0$ is smooth and does not vanish.

178 We first give a simple lower bound on the lengths of vectors v^1, \dots, v^{d-n} , assuming that
 179 the norms $|v^i|$ are upper bounded and the determinant of the Gram matrix is lower bounded.

180 ► **Lemma 7.** *Let $v^1, \dots, v^{d-n} \in \mathbb{R}^d$, $|v^i| \leq \gamma_1$, for all i , and assume that $\det(v^i \cdot v^j)_{i,j} > \gamma_0$.
 181 Then $|v^i| \geq \sqrt{\gamma_0}/\gamma_1^{d-n-1}$.*

182 We also need to bound the angle of the vectors $\text{grad}_{(x,\tau)}(F_L^i)$ and the x plane, that is
 183 $\mathbb{R}^d \subset \mathbb{R}^{d+1}$. We recall the definition. If $v \in \mathbb{R}^{d+1}$ is a vector and $\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$, is the
 184 space spanned by the d basis vectors corresponding to the x -directions, the angle between
 185 v and Ξ is $\angle(v, \Xi) = \inf_{w \in \Xi} \angle(v, w)$.

186 ► **Lemma 8.** *Let Ξ be as above. We have*

$$187 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_L^i), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

188 *In particular the manifold $F_L^{-1}(0)$ inside $\sigma \times [0, 1]$ is never tangent to the $\tau = c$ planes,
 189 where c is a constant.*

190 Combining Lemma 8 and Corollary 6 gives:

191 ► **Corollary 9.** *If $\gamma_0 > g_1(D)$, and $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, then inside each $\sigma \times [0, 1]$ the
 192 gradient of τ on $F_L^{-1}(0)$ is smooth and does not vanish.*

193 **2.2 Global result**

194 We are now going to prove the global result. For this, we need to recall some definitions
 195 and results from non-smooth analysis. We refer to [22] for an extensive introduction.

196 ► **Definition 10** (Generalized Jacobian, Definition 2.6.1 of [22]). Let $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d-n}$, where
 197 F is assumed to be just Lipschitz. The generalized Jacobian of F at x_0 denoted by $J_F(x_0)$,
 198 is the convex hull of all $(d-n) \times (d+1)$ -matrices B obtained as the limit of a sequence of
 199 the form $J_F(x_i)$, where $x_i \rightarrow x_0$ and F is differentiable at x_i .

200 Following [22, page 253] we also define:

201 ► **Definition 11.** The generalized Jacobian $J_F(x_0)$ is said to be of maximal rank provided
 202 every matrix in $J_F(x_0)$ is of maximal rank.

203 Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of \mathbb{R}^{d+1} by (x, y) accordingly.
 204 Fix a point (a, b) , with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. We now write:

205 ► **Notation 12** ([22, page 256]). $J_F(x_0, y_0)|_y$ is the set of all $(n+1) \times (n+1)$ -matrices M
 206 such that, for some $(n+1) \times (d-n)$ -matrix N , the $(n+1) \times (d+1)$ -matrix $[N, M]$ belongs
 207 to $J_F(x_0, y_0)$.

208 With these definitions and notations we now have:

209 ► **Theorem 13** (The generalized implicit function theorem [22, page 256]). Suppose that
 210 $J_F(a, b)|_y$ is of maximal rank. Then there exists an open set $U \subset \mathbb{R}^{n+1}$ containing a such
 211 that there exists a Lipschitz function $g : U \rightarrow \mathbb{R}^{d-n}$, such that $g(a) = b$ and $F(x, g(x)) = 0$
 212 for all $x \in U$.

213 We recall the definition of F_{PL} ,

214
$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x). \tag{1}$$

215 Because of the definition of α , see (4), and Proposition 4, we have that $\text{grad}_{(x,\tau)} F_{PL}(x, \tau)$
 216 and $\text{grad}_{(x,\tau)} F_{PL}(\tilde{x}, \tau)$ are close if x and \tilde{x} are. In particular,

217 ► **Lemma 14.** Let v be a vertex in \mathcal{T} , $x_1, x_2 \in \text{star}(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that
 218 $\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1)$ and $\text{grad}_{(x,\tau)} F_{PL}^i(x_2, \tau_2)$ are well defined, then

219
$$|\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL}^i(x_2, \tau_2)| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha$$

220 We now immediately have the same bound on points in the convex hull of a number of
 221 such vectors:

222 ► **Corollary 15.** Suppose we are in the setting of Lemma 14 and $x_0, x_1, \dots, x_m \in \text{star}(v)$,
 223 $\tau_0, \dots, \tau_m \in [0, 1]$, and suppose that μ_1, \dots, μ_m are positive weights such that $\mu_1 + \dots + \mu_m =$
 224 1 then,

225
$$\left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha.$$

226 Using Lemma 5 we see

227 ► **Lemma 16.** *Let v be a vertex in \mathcal{T} , $x_1, \dots, x_m \in \text{star}(v)$, and $\tau_1, \dots, \tau_m \in [0, 1]$, such*
 228 *that $\text{grad}_{(x, \tau)} F_{PL}^i(x_k, \tau_k)$, $k = 0, \dots, m$ are well defined. If we moreover assume $D \leq 1$, and*
 229 *$\frac{6dD\alpha}{T} \leq \gamma_1$ we have that*

$$230 \quad \left| \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x, \tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x, \tau)} F_{PL}(x_k, \tau_k) \right) \right)_{i,j} \right| \geq \gamma_0 - g_2(D),$$

231 *with $g_2(D) = \mathcal{O}(D)$. See (19) in Appendix A for the exact expression of g_2 .*

232 From the previous lemma we immediately have that

233 ► **Corollary 17** ($\{x \mid F_{PL}(x, \tau) = 0\}$ is a manifold). *If $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, and $\gamma_0 > g_2(D)$ the*
 234 *generalized implicit function theorem, Theorem 13, applies to $F_{PL}(x, \tau) = 0$. In particular,*
 235 *$\{x \mid F_{PL}(x, \tau) = 0\}$ is a manifold.*

236 We notice that this bound is stronger than the bound in Corollary 6. This means that
 237 $\{x \mid F_{PL}(x, \tau) = 0\}$ is a Piecewise-Smooth manifold if the conditions of Corollary 17 hold.
 238 The second technical step of the proof is now also completed.

239 The fact that $F_L(x, \tau) = 0$ is a Piecewise-Smooth manifold and Corollary 9 give that
 240 the gradient of τ is a Piecewise-Smooth vector field whose flow we can integrate to give a
 241 homeomorphism from the zero set of f to that of f_{PL} .

242 We summarize in a theorem:

243 ► **Theorem 18.** *If, $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, and $\gamma_0 > g_2(D)$ then the zero*
 244 *set of f is isotopic to the zero set of f_{PL} . We stress that one can satisfy all conditions by*
 245 *choosing D sufficiently small.*

246 2.2.1 Fréchet distance

247 To bound the Fréchet distance (d_F) between the zero sets of $f(x)$ and f_{PL} , it suffices to
 248 bound the angle that the gradient of τ (as restricted to $F_L(x, \tau) = 0$) makes with the
 249 (ambient) τ -direction.

250 For this we will use the angle bound of Lemma 8, together with some estimates that are
 251 similar in spirit to those in [7, Lemma C.13].

252 ► **Lemma 19.** *Let $v^1, \dots, v^{d-n} \in \mathbb{R}^{d+1}$, $|v^i| \leq \tilde{\gamma}_1$, for all i , and assume that $\det(v^i \cdot$
 253 $v^j)_{i,j} > \tilde{\gamma}_0 > 0$. Let e_τ a unit vector. If for all i , $\cos(\angle v^i, e_\tau) \leq \phi_0$, then for any $w \in$
 254 $\text{span}(v^1, \dots, v^{d-n})$*

$$255 \quad \cos \angle(w, e_\tau) \leq \frac{(d-n)d^{d-n-1}\phi_0\tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

256 Let e_τ be the τ direction and let g_τ be the gradient of τ restricted to $F_{PL}^{-1}(0)$, whenever it
 257 exists. We want to bound the angle of g_τ and the τ -direction. Because the homeomorphism
 258 is given by the gradient flow, this bounds the Fréchet distance.

259 There is one subtlety, because the manifold is only Piecewise-Smooth, we need to take
 260 into account the points where g_τ is not uniquely defined. Because for each simplex σ , F_L
 261 extends to a neighbourhood of $\sigma \times [0, 1]$, there exists a limit of $g_\tau(x_i, \tau_i)$ for any sequence
 262 (x_i, τ_i) that lies in $\text{int}(\sigma) \times [0, 1]$, where int denotes the interior. This means that if we bound
 263 g_τ for each simplex we also bound its limits, where the limits are as just described.

264 We are now ready to combine Lemmas 8, 19, and Theorem 18.

265 ► **Corollary 20** (Bound on the Fréchet distance). *Suppose that the conditions of Theorem 18*
 266 *are satisfied. Then, $d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D)$, with $g_3(D) = \mathcal{O}(D^2)$, where we*
 267 *think of $\gamma_0, \gamma_1, d, n, T$ and α as constants. See (19) in Appendix A for the exact expression*
 268 *of g_2 .*

269 The most important thing to observe is that $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$, so that we find
 270 that $d_F(f^{-1}(0), f_{PL}^{-1}(0)) = \mathcal{O}(D^2)$, where we think of $\gamma_0, \gamma_1, d, n, T$ and α as constants.

271 3 Isomanifolds with boundary

272 We'll now consider isomanifolds with boundary. By this we mean that on top of the function
 273 $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$, we'll have another function $f_\partial : \mathbb{R}^d \rightarrow \mathbb{R}$ and the set we consider is
 274 $M = f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$. This is a manifold with boundary if the gradients of f^i span
 275 a $(d - n)$ -dimensional space at each point of $f^{-1}(0)$ and the gradients of f^i and f_∂ span a
 276 $(d - n + 1)$ -dimensional space at each point of $\partial M = f^{-1}(0) \cap f_\partial^{-1}(0)$, as a consequence of
 277 the submersion theorem.

278 We'll again write f_{PL} for the PL interpolation of f . Similarly we write $f_{\partial, PL}$ for the PL
 279 interpolation of f_∂ .

280 We prove that, under certain conditions, there is an isotopy from $f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$
 281 to $f_{PL}^{-1}(0) \cap f_{\partial, PL}^{-1}([0, \infty))$. The conditions are very similar to the conditions we have before,
 282 but of course we need to include bounds on the gradient of $f_{\partial, PL}$.

283 Overview of the proof

284 We will again construct an isotopy, but in this case it will consist of two steps.

- 285 ■ In the **first step**, we isotope the part of $f^{-1}(0)$ that is far from $f_\partial^{-1}(0)$ to its piecewise
 286 linear approximation, while leaving the part of $f^{-1}(0)$ that is close to $f_\partial^{-1}(0)$ smooth.
 287 We will denote the result by $M_1 = (F_{PL,1}(\cdot, 1))^{-1}(0)$.
- 289 ■ In the **second step**, we consider a (small) tubular neighbourhood around $f_\partial^{-1}(0)$ as
 290 restricted to M_1 by looking at all $f_\partial^{-1}(\epsilon)$ for $|\epsilon|$ sufficiently small.¹ We then isotope
 291 $M_1 \cap f_\partial^{-1}(\epsilon)$ to its piecewise linear approximation. Again the isotopy is chosen in such a
 292 way that for ϵ relatively large (for the points such that M_1 is already Piecewise-Linear)
 293 it leaves $M_1 \cap f_\partial^{-1}(\epsilon)$ invariant. This gives an isotopy of a tubular neighbourhood of
 294 $M_1 \cap f_\partial^{-1}(0)$ to its Piecewise-Linear approximation.

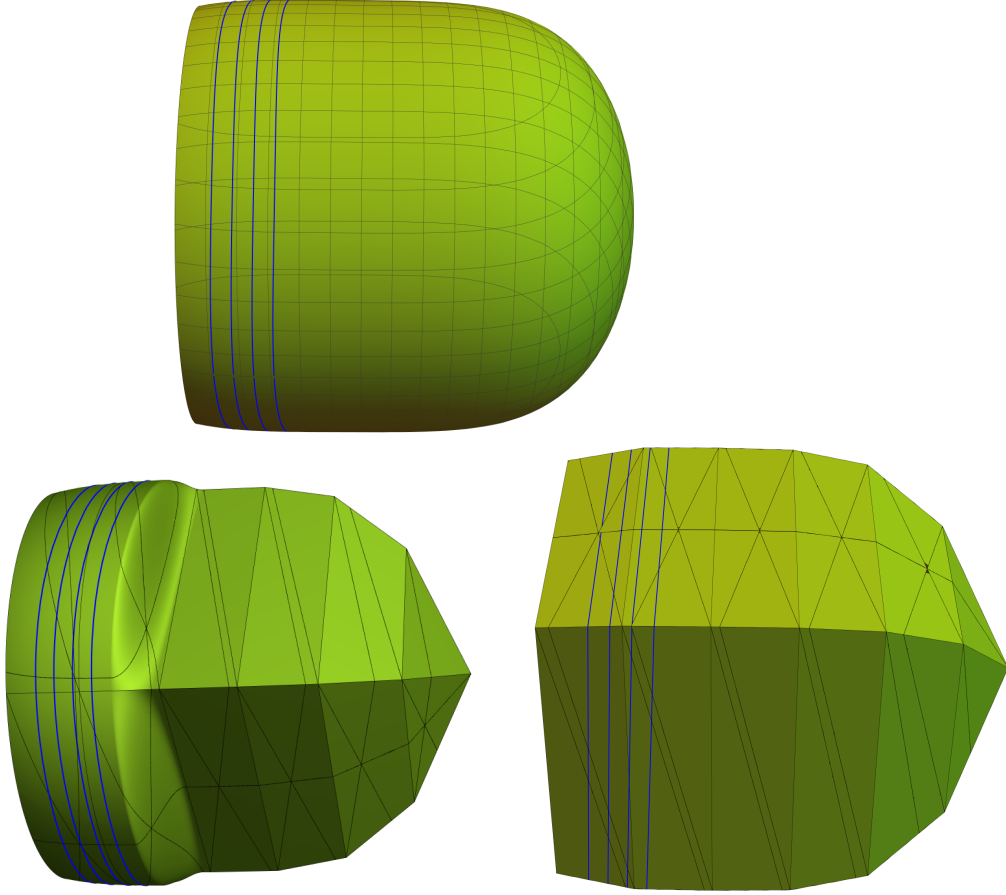
300 We will first partition the manifold in two parts using a smooth bump function $\phi : \mathbb{R} \rightarrow$
 301 $[0, 1]$ that is zero in a neighbourhood of zero and $\phi(y) = 1$ if $|y| > y_0$, for some $y_0 > 0$.
 302 Such bump functions can be easily constructed, see for example [39, Section 2.2]. We will
 303 be using the function $\phi(\sum_i (f^i)^2 + f_\partial^2)$.

304 The first step will be using the zero set of the following function:

$$305 F_{PL,1}(x, \tau) = \left(1 - \tau \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) f(x) + \tau \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) f_{PL}(x), \quad (9)$$

306 on which we'll apply the same gradient flow argument as before.

288 ¹ We stress that ϵ may be negative.



295 **Figure 2** Top: we see the original isosurface with $f_{\partial}^{-1}(-1/10)$, $f_{\partial}^{-1}(0)$, $f_{\partial}^{-1}(1/10)$, and $f_{\partial}^{-1}(2/10)$
 296 indicated in blue. Bottom left: we see that at the end of Step 1 the neighbourhood of the boundary
 297 is intact, while the rest has been isotoped to a Piecewise-Linear approximation. Bottom right:
 298 we have also isotoped the neighbourhood of the boundary to a Piecewise-Linear approximation by
 299 isotoping $f_{\partial}^{-1}(\epsilon)$, to its Piecewise-Linear approximation for all sufficiently small ϵ .

307 The resulting set M_1 is the same zero set of f_{PL} as before if we stay sufficiently far
 308 away from ∂M and the isotopy leaves the manifold invariant close to ∂M . In particular,
 309 $\partial M_1 = \partial M$.

310 In the second step, we define an isotopy that will act only on a small neighbourhood of
 311 ∂M . Consider the sets $B_1(\epsilon) = M_1 \cap f_{\partial}^{-1}(\epsilon)$ and, for each of them, define the function

$$\begin{aligned}
 312 \quad F_{PL,2,\epsilon}(x,\tau) &= \left(1 - \tau\psi\left(\sum_i (f^i)^2 + f_{\partial}^2\right)\right) (F_{PL,1}(x,1), f_{\partial}(x) - \epsilon) \\
 313 \quad &\quad + \tau\psi\left(\sum_i (f^i)^2 + f_{\partial}^2\right) (f_{PL}(x), f_{\partial,PL}(x) - \epsilon), \tag{10}
 \end{aligned}$$

314 where $\psi : \mathbb{R} \rightarrow [0, 1]$ is now a smooth bump function that is 1 in a sufficiently large neighbour-
 315 hood of zero (somewhat larger than y_0) and zero outside some compact set. We stress that
 316 $F_{PL,2,\epsilon}$ is a mapping from $\mathbb{R}^d \times [0, 1]$ to \mathbb{R}^{d-n+1} . Using the result for isomanifolds (with some
 317 modifications), we can prove that each individual set $B_1(\epsilon)$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}(\epsilon)$

318 for small ϵ while, for sufficiently large ϵ , it leaves the set invariant.

319 **3.1 Step 1**

320 The proof closely follows the proof for the case without boundary in Section 2. The main
 321 technical difficulty will be to provide bounds that serve as the counterparts of Lemmas 5
 322 and 16. To be able to do so we first need to discuss bounds on the bump functions ϕ and ψ .

323 **3.1.1 Bump functions**

324 Following [39, Section 2.2] we write,

325
$$\zeta_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

326 For $0 < y_1 < y_2$ we write $\zeta_2(x) = \zeta_1(x - y_1)\zeta_1(y_2 - x)$. Then we define $\phi_l : \mathbb{R} \rightarrow [0, 1]$ by
 327 $\phi_l(x) = \int_x^{y_2} \zeta_2(x')dx' / \int_{y_1}^{y_2} \zeta_2(x')dx'$. Finally define $\phi_b : \mathbb{R} \rightarrow [0, 1]$ by $\phi_b(x) = \phi_l(|x|)$, and
 328 let $\phi(x) = 1 - \phi_b(x)$.

329 **► Lemma 21.** *We have $\phi_b(x) \in [0, 1]$ and, writing $2y_1 = y_2 = y_0$,*

330
$$\partial_x(\phi_l(x)) \leq 2 \frac{e^{\frac{4}{3(y_2-y_1)}}}{y_2 - y_1} = 4 \frac{e^{\frac{2}{3y_0}}}{y_0} = \gamma_\phi. \tag{11}$$

331 **3.1.2 Inside a single simplex**

332 Similarly to Lemma 5, we now give a condition that ensures that the zero set of $F_{PL,1}^i(x, \tau)$
 333 is smooth inside $\sigma \times [0, 1]$. In fact, similarly to (7), we define

334
$$F_{L,1}^i(x, \tau) = \left(1 - \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) f^i(x) + \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) f_L^i(x)$$

 335
$$= f(x) + \tau \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)),$$

336 where ϕ is as defined above. Observe that $F_{L,1}^i(x, \tau)$ can be extended to a neighbourhood
 337 of $\sigma \times [0, 1]$.

338 **► Remark.** For the constants, it is better if y_0 can be chosen as large as possible, but we need
 339 y_1 to be quite a bit larger than y_0 . In turn, we cannot choose y_1 arbitrarily large because
 340 this would mean that the gradient field $\text{grad} f_\partial|_{f^{-1}(0)}$ (seen as restricted on $f^{-1}(0)$) would
 341 never vanish. The latter is in general impossible thanks to the hairy ball theorem [17].

342 We introduce the following definition that complements Definition 1:

► Definition 22.

343
$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \text{grad} \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| = 2 \sup_{x \in \Sigma_0} \left| \sum_l f^l \text{grad} f^l + f_\partial \text{grad} f_\partial \right| \tag{12}$$

344 We have then the analog of Lemma 5:

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► **Lemma 23.** *We have :*

$$|\det(\text{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x,\tau))_{i,j}| > \gamma_0 - g_4(D),$$

345 with $g_4(D) = \mathcal{O}(D)$. The exact expression of g_4 is given in (21).

346 The following corollary is then the analog of Corollary 6:

347 ► **Corollary 24** ($F_{L,1}^{-1}(0)$ is a manifold). *If $\gamma_0 > g_4(D)$, where $g_4(D) = \mathcal{O}(D)$ is as in Lemma*
 348 *23, then $F_{L,1}^{-1}(0)$ is a smooth manifold inside an ϵ neighbourhood of $\sigma \times [0, 1]$.*

349 3.1.3 Transversality with regard to the τ -direction

350 We note that, similarly to Lemma 8, we have

351 ► **Lemma 25.** *Let Ξ be as in Lemma 8 and γ_ϕ as in (11).*

$$352 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,1}), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0/\gamma_1^{d-n-1}} - \gamma_2\gamma_\phi 2D^2\alpha - \frac{4dD\alpha}{T}}.$$

353 *In particular, if $\sqrt{\gamma_0/\gamma_1^{d-n-1}} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$, $F_{L,1}^{-1}(0)$ (if it is a manifold) is never*
 354 *tangent to the $\tau = c$ planes, where c is a constant.*

355 Now, similarly to Corollary 9, we find that

356 ► **Corollary 26** (Transversality with respect to τ for Step 1). *Suppose that $\gamma_0 > g_4(D)$ and*
 357 *that $\sqrt{\gamma_0/\gamma_1^{d-n-1}} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$. Then, inside each $\sigma \times [0, 1]$, the gradient of τ on*
 358 *$F_{L,1}^{-1}(0)$ is smooth and does not vanish.*

359 3.1.4 Global result

360 We now have to prove that $F_{PL,1}^{-1}(0)$ is a manifold. For this, we shall use a bound similar to
 361 the one given in Lemma 16, so that we are able to apply the generalized implicit function
 362 theorem if this bound is satisfied. But first of all, we need the following bound, which is
 363 similar to Lemma 14.

364 ► **Lemma 27.** *Assuming that the gradients are well defined, we have*

$$365 \quad |\text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2)| \leq g_5(D),$$

366 with $g_5(D) = \mathcal{O}(D)$. The exact expression for g_5 is given in (22).

367 Just as in Corollary 15, we immediately have the same bound on points in the convex
 368 hull of a number of such vectors:

369 ► **Corollary 28.** *Suppose we are in the setting of Lemma 27 and $x_0, x_1, \dots, x_m \in \text{star}(v)$,*
 370 *$\tau_0, \dots, \tau_m \in [0, 1]$, such that $\text{grad}_{(x,\tau)} F_{PL,1}^i(x_i, \tau_i)$ is well defined for all i . Further assume*
 371 *that μ_1, \dots, μ_m are positive weights such that $\mu_1 + \dots + \mu_m = 1$. Then,*

$$372 \quad \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \leq g_5(D).$$

373 ▶ **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$374 \quad \det \left(\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$375 \quad \geq \gamma_0 - g_4(D) - g_6(D),$$

376 *with $g_6(D) = \mathcal{O}(D)$. The exact expression of g_6 is given in (23).*

377 Lemma 29 immediately yields that

378 ▶ **Corollary 30** ($F_{PL,1}^{-1}(0)$ is a manifold). *If, $\gamma_0 > g_4(D) + g_6(D)$ the generalized implicit*
 379 *function theorem, Theorem 13, applies to $F_{PL,1}(x, \tau) = 0$. In particular $F_{PL,1}^{-1}(0)$ is a*
 380 *manifold.*

381 We stress again that inside the set $\{x | \phi(\sum_i (f^i)^2(x) + f_\partial^2(x)) = 1\}$ the zero set of
 382 $F_{PL,1}(x, 1)$ coincides with the zero set of $f_{PL}(x)$.

383 3.2 Step 2

384 Before we can proceed we have to specify the bump function ψ . We suppose that

$$385 \quad \psi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{101}{100}y_0 \\ 0 & \text{if } |x| \geq 2y_0. \end{cases}$$

386 In particular we pick $\psi(x) = \phi_b(x)$, with the choice $y_1 = \frac{101}{100}y_0$ and $y_2 = 2y_0$.

387 First we stress that the zero set of $F_{PL,2,\epsilon}(x, 1)$ coincides with the zero set of $(f_{PL}(x), f_{\partial,PL}(x) -$
 388 $\epsilon)$, provided that $\psi(\sum_i f_i(x)^2 + f_\partial(x)^2) = 1$.

389 Secondly, we now claim the following:

390 ▶ **Lemma 31.** *The zero set of $F_{PL,2,\epsilon}(x, 1)$ is a subset of the zero set of $f_{PL}(x)$, for each ϵ .*

391 The technical result that remains to be proven is the counterpart of Theorem 18 for
 392 $F_{PL,2,\epsilon}(x, \tau)$ and for each sufficiently small ϵ . To be precise it suffices for $\epsilon \leq 2y_0$. We
 393 remark that it is likely that this bound on ϵ can be improved.

394 We again follow the same path to prove this result. That is we first concentrate on a
 395 single simplex and prove that inside that simplex the zero set of $F_{PL,2,\epsilon}$ is a smooth manifold
 396 on which the gradient of τ as restricted to the manifold does not vanish. We then prove
 397 that is the zero set of $F_{PL,2,\epsilon}$ is globally a manifold.

398 3.2.1 Assumptions and notations

399 Because we are now faced with both $f(x)$ and $f_\partial(x)$ we need to introduce a bound on how
 400 far the gradients of all there are from being colinear. We write

$$401 \quad f_B(x) = (f(x), f_\partial(x)). \tag{13}$$

402 Before we were only interested in the set Σ_0 , similarly here we sometimes concentrate
 403 on a neighbourhood of the zero set of both f_∂ and f . Therefore we write B_ν for all $\sigma \in \mathcal{T}$
 404 such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$.

405 We define γ_0^B in terms of the determinant of the Gram matrix of the gradients, that is

$$406 \quad \gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\operatorname{grad}(f_B^i) \cdot \operatorname{grad}(f_B^j))_{i,j}|. \tag{14}$$

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407 We note that because we take the gradients we can just ignore the ϵ constant. For the
408 lengths of the gradients of f_B we define,

$$409 \quad \gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f_B^i)|, \quad (15)$$

410 for all $1 \leq i \leq d - n + 1$. Similarly to α , we define β as the bound on the operator 2-norm
411 of all Hessians of f_B , that is

$$412 \quad \beta = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2. \quad (16)$$

413 We stress that we have chosen our definitions such that $\alpha \leq \beta$.

414 We use the same notation for the ambient triangulation \mathcal{T} , the lower bound on the
415 thickness of the simplices T and upper bound on the longest edge length D . We also need
416 to introduce a bound on the differential of the bump function ψ . Similarly to (11) we define,

$$417 \quad \gamma_\psi = 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} = 2 \frac{e^{\frac{4}{3(2y_0 - \frac{101}{100}y_0)}}}{2y_0 - \frac{101}{100}y_0} = \frac{200}{99} \frac{e^{\frac{400}{297y_0}}}{y_0}, \quad (17)$$

418 because we picked $y_1 = \frac{101}{100}y_0$ and $y_2 = 2y_0$, for ψ .

419 3.2.2 Inside a single simplex

420 Similarly to Lemma 23 we now give a conditions that ensure that the zero set of $F_{PL,2,\epsilon}(x, \tau)$
421 is smooth inside $\sigma \times [0, 1]$. In fact similarly to (7) we define

$$422 \quad F_{L,2,\epsilon}(x, \tau) = \left(1 - \tau \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) (F_{L,1}(x, 1), f_\partial(x) - \epsilon)$$

$$423 \quad + \tau \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) (f_L(x), f_{\partial,L}(x) - \epsilon),$$

424 which can be extended to a neighbourhood of $\sigma \times [0, 1]$.

425 ► **Lemma 32.** *For all ϵ ,*

$$426 \quad \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau))_{i,j} \geq \gamma_0^B - g_\tau(D)$$

427 *with $g_\tau(D) = \mathcal{O}(D)$. The exact expression of g_τ is given in (24).*

428 We again have the following corollary.

429 ► **Corollary 33** ($F_{L,2,\epsilon}^{-1}(0)$ is a manifold). *We have that $F_{L,2,\epsilon}^{-1}(0)$ is a smooth manifold inside
430 an small neighbourhood of $\sigma \times [0, 1]$ provided $\gamma_0^B > g_\tau(D)$, with $g_\tau(D)$ as in Lemma 32.
431 As usual this can always be satisfied by choosing the triangulation fine enough, that is D
432 sufficiently small.*

433 3.2.3 Transversality with regard to the τ -direction

434 Once more similarly to Lemma 8 we have

435 ► **Lemma 34.** *Let Ξ be as in Lemma 8. We have*

$$436 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}), \Xi) \leq \frac{2D^2\beta}{\sqrt{\gamma_0^B / (\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T}}}.$$

437 In particular the manifold $F_{L,2,\epsilon}^{-1}(0)$ inside $\sigma \times [0, 1]$, if well defined, is never tangent to the
 438 $\tau = c$ planes, where c is a constant, if

$$439 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T}.$$

440 Now similarly to Corollary 9 we find that

441 ► **Corollary 35** (Transversality with respect to τ for Step 2). *Suppose that the conditions of*
 442 *Corollary 33 are satisfied. If moreover*

$$443 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T},$$

444 then inside each $\sigma \times [0, 1]$ the gradient of τ on $F_{L,2,\epsilon}^{-1}(0)$ is smooth and does not vanish.

445 3.2.4 Global result

446 We now have to prove that $F_{PL,2,\epsilon}^{-1}(0)$ is a manifold, for all sufficiently small ϵ . For this we
 447 shall use a bound similar to the one given in Lemma 16, so that we are able to apply the
 448 generalized implicit function theorem if this bound is satisfied. For this we first need the
 449 following bound, which is similar to Lemma 27.

450 ► **Lemma 36.** *Let v be a vertex in \mathcal{T} , $x_1, x_2 \in \text{star}(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that*
 451 *$\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1)$ and $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)$ are well defined, then*

$$452 \quad |\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)| \leq g_8(D),$$

453 with $g_8(D) = \mathcal{O}(D)$. The exact expression of g_8 is given in (25).

454 Just as in Corollary 15, we immediately have the same bound on points in the convex
 455 hull of a number of such vectors:

456 ► **Corollary 37.** *Suppose $x_0, x_1, \dots, x_m \in \text{star}(v)$, $\tau_0, \dots, \tau_m \in [0, 1]$, such that $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_i, \tau_i)$*
 457 *is well defined for all i . Further assume that μ_1, \dots, μ_m are positive weights such that*
 458 *$\mu_1 + \dots + \mu_m = 1$. Then,*

$$459 \quad \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right| \leq g_8(D).$$

460 ► **Lemma 38.** *Under the same conditions as in Corollary 37,*

$$461 \quad \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$462 \quad \geq \gamma_0^B - g_7(D) - g_9(D),$$

463 where $g_9(D) = \mathcal{O}(D)$. The exact expression of g_9 is given in (26).

464 Lemma 38 immediately yields that

465 ► **Corollary 39** (The generalized implicit function theorem in Step 2). *If, $\gamma_0^B > g_7(D) +$*
 466 *$g_9(D)$ the generalized implicit function theorem, Theorem 13, applies to $F_{PL,1}(x, \tau) = 0$. In*
 467 *particular $F_{PL,1}^{-1}(0)$ is a manifold.*

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468 We stress that this condition only needs to be satisfied in a when $\sum_l (f^l)^2 + (f_\partial)^2$ is small,
 469 outside this neighbourhood the isotopy leaves the zero set invariant.

470 ► **Theorem 40.** *If,*

$$471 \quad \sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T} \quad (\text{Corollary 26})$$

$$472 \quad \gamma_0 > g_4(D) + g_6(D) \quad (\text{Corollaries 24 and 30})$$

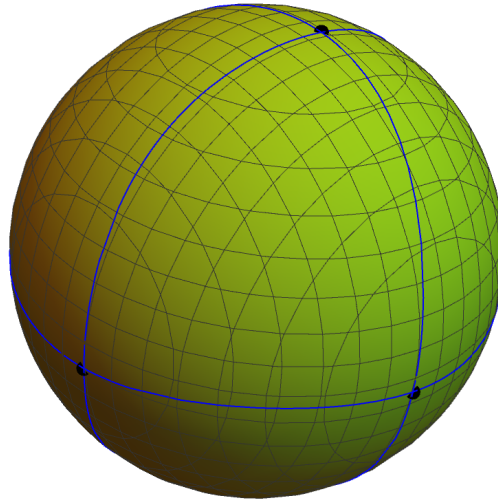
$$473 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \quad (\text{Corollary 35})$$

$$474 \quad \gamma_0^B > g_7(D) + g_9(D), \quad (\text{Corollaries 33 and 39})$$

475 *then $f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial, PL}^{-1}([0, \infty))$. We stress that one can satisfy*
 476 *all conditions by choosing D sufficiently small. We refer to Appendix A for the definitions*
 477 *of $g_i(D)$.*

478 **4 Isostratifolds**

479 There is no obstruction in principle that prevents us from extending the approach above
 480 to isostratifolds. By isostratifolds we mean stratifolds that are given by the zero sets of
 481 functions. For example suppose that we want to find a PL approximation of the unit sphere
 482 centred at 0 in \mathbb{R}^3 including the PL approximations of the intersections of the sphere with
 483 slightly deformed $x = 0$, $y = 0$, and $z = 0$ -planes, as depicted in Figure 3. This would also
 484 give PL approximations of the respective ‘octants’ of the sphere.



485 ■ **Figure 3** An example of an isostratifold.

486 We could follow the same procedure as for a manifold with boundary to give density con-
 487 ditions that ensure that the PL approximation is correct. However this would mean that we
 488 have to introduce an extra bump function for each stratum as well as an extra isotopy. Even
 489 though that in theory this should be possible the analysis becomes prohibitively lengthy.

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A Overview of constants

We give an overview. We write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i . We write B_ν for all $\sigma \in \mathcal{T}$ such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$. We write

$$f_B(x) = (f(x), f_\partial(x)) \quad (13)$$

$$\gamma_0 = \inf_{x \in \Sigma_0} |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| \quad (2)$$

$$\gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\text{grad}(f_B^i) \cdot \text{grad}(f_B^j))_{i,j}| \quad (14)$$

$$\gamma_1 = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f^i)| \quad (3)$$

$$\gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f_B^i)| \quad (15)$$

$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \text{grad} \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| = \sup_{x \in \Sigma_0} \left| \sum_l f^l \text{grad} f^l + f_\partial \text{grad} f_\partial \right| \quad (12)$$

$$\alpha = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k \partial_l f^i)_{k,l}\|_2 \quad (4)$$

$$\beta = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2 \quad (16)$$

D : the longest edge length of a simplex in Σ_0

T : the smallest thickness of a simplex in Σ_0 .

$\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$ is the space spanned by the d basis vectors corresponding to the x -directions.

The bump functions give rise to the following:

$$\gamma_\phi = 4 \frac{e^{\frac{2}{3y_0}}}{y_0} \quad (11)$$

$$\gamma_\psi = \frac{200}{99} \frac{e^{\frac{400}{297y_0}}}{y_0}. \quad (17)$$

The precise expressions for the $g_i(D)$ are:

$$g_1(D) = n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right) \quad (18)$$

$$g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right) \quad (19)$$

$$g_3(D) = \left(\frac{(d-n)d^{d-n-1} \sqrt{\gamma_0} 2D^2 \alpha \gamma_1^{d-n-1} (\gamma_1 + 2D^2 \alpha)^{d-n}}{\sqrt{\gamma_0 + (2D^2 \alpha \gamma_1^{d-n-1})^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right) \quad (20)$$

$$g_4(D) = n \left(\left(n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \cdot n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right) \quad (21)$$

$$g_5(D) = 2\sqrt{d} d \alpha D + 2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \quad (22)$$

$$g_6(D) = n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1}$$

$$\begin{aligned}
 & \cdot \left(2 \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right. \\
 & \left. + \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \right). \tag{23}
 \end{aligned}$$

$$\begin{aligned}
 g_7(D) = & (n + 1) \left((n + 1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 & \cdot (n + 1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 & \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \tag{24}
 \end{aligned}$$

$$g_8(D) = 2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \tag{25}$$

$$\begin{aligned}
 g_9(D) = & (n + 1)^{n+1} \left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 & \cdot \left(2 \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right. \\
 & \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & \left. + \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \right) \tag{26}
 \end{aligned}$$

If $\frac{4dD\alpha}{T} \leq \gamma_1$, $g_1(D)$ can be replaced by the simpler $34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}$.

658 **B** Recap of essential standard results

659 We now quickly recall two essential results, namely the smooth implicit function theorem
660 and Friedland's determinant bound.

661 ► **Theorem 41** (Smooth implicit function theorem). *Let $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d-n}$ be a continuously
662 differentiable function. Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of \mathbb{R}^{d+1} by
663 (x, y) accordingly. Fix a point (a, b) , with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. If the Jacobian $J_{F,y}(a, b) =$
664 $(\frac{\partial F^i}{\partial y^j}(a, b))_{i,j}$ is of maximal rank (or equivalently invertible), then there exists an open set
665 $U \subset \mathbb{R}^{n+1}$ containing a such that there exists a unique continuously differentiable function
666 $g : U \rightarrow \mathbb{R}^{d-n}$ such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.*

667 The following result by Friedland [36] will also be used throughout the proofs:

$$668 \quad |\det(A + E) - \det(A)| \leq n \max\{\|A\|_p, \|A + E\|_p\}^{n-1} \|E\|_p, \quad (27)$$

669 where A and E are $n \times n$ -matrices and $\|\cdot\|_p$ is the p -norm, with $1 \leq p \leq \infty$.

670 **C Proofs**

671 **C.1 Proofs of variations of known results in Section 2.1**

672 The following proofs are not exactly the same as the original, but do not differ greatly from
 673 known results. We have included the results for completeness.

674 **Proof of Lemma 2.** This is a straightforward consequence of the existence and uniqueness
 675 of the solution to a differential equation. ◀

676 The following two proofs are similar to the proofs in Allgower and George [1], however
 677 because the statement are a bit different and for completeness we have included these.

678 **Proof of Lemma 3.** Taylor’s theorem, see for example [29, Theorem 2.8.4], yields that

679
$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{28}$$

680 with

681
$$R(v_k) = 2 \sum_{j \neq l} (v_k - x)^j (v_k - x)^l \int_0^1 (1-t)^2 \partial_j \partial_l f^i(v_k - t(v_k - x)) dt$$

 682
$$+ 2 \sum_j \frac{((v_k - x)^j)^2}{2} \int_0^1 (1-t)^2 \partial_j^2 f^i(v_k - t(v_k - x)) dt$$

 683
$$\leq 2|v_k - x|^2 \alpha \tag{by (4) and Cauchy-Schwarz}$$

 684
$$\leq 2D^2 \alpha \tag{(x \in \sigma)}$$

685 The function f_L at the point $x = \sum_k \lambda_k v_k$, where $\sum_k \lambda_k = 1$, is by construction

686
$$f_L^i(x) = \sum_k \lambda_k f^i(v_k)$$

 687
$$= \sum_k \lambda_k \left(f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k) \right) \tag{by (28)}$$

 688
$$= \sum_k \lambda_k f^i(x) + \sum_j \partial_j f^i(x) \left(\sum_k \lambda_k v_k - x \right)^j + \sum_k \lambda_k R(v_k)$$

 689
$$= f^i(x) + 0 + \sum_k \lambda_k R(v_k)$$

690 Thanks to the bounds on $R(v_k)$ and Cauchy-Schwarz one has

691
$$|f_L^i(x) - f^i(x)| \leq 2D^2 \alpha$$

692 ◀

693 **Proof of Proposition 4.** We again use that

694
$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \tag{28}$$

695 with

696
$$|R(v_k)| \leq 2D^2 \alpha \tag{29}$$

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697 Subtracting $f^i(v_l)$ from $f^i(v_k)$ now yields

$$698 \quad f^i(v_k) - f^i(v_l) = \sum_j \partial_j f^i(x)(v_k - v_l)^j + R(v_k) - R(v_l).$$

699 Because f_L is the linear interpolation of f , we have $f^i(v_k) - f^i(v_l) = \sum_j \partial_j f_L^i(x)(v_k - v_l)$,
700 and thus

$$701 \quad \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(v_k - v_l)^j \right| \leq |R(v_k) - R(v_l)| \leq 4D^2\alpha.$$

702 Let now $u = \mu_k v^k$ and $w = \tilde{\mu}_k v_k$, with $\sum \mu_k = \sum \tilde{\mu}_k = 1$, that is $u, w \in \sigma$. Clearly
703 $\sum \mu_k - \tilde{\mu}_k = 0$, so $u - w$ can be developed in a sum such that the m -th term in the sum is
704 $c_m(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})$, with $c_m > 0$, and $\sum |c_m| = 1$. We stress that $v_{\tilde{k}(m)}$ may be the same as
705 $v_{\tilde{k}(m')}$, with $m \neq m'$. We now see that

$$706 \quad \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(u - w)^j \right| = \left| \sum_{j,m} (\partial_j f_L^i(x) - \partial_j f^i(x))c_m(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right|$$

$$707 \quad \leq \sum_m c_m \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right|$$

(by the triangle inequality and $c_m > 0$)

$$708 \quad \leq \sum_m 4c_m D^2\alpha \quad (\text{because } \sum |c_m| = 1)$$

$$709 \quad = 4D^2\alpha. \quad (30)$$

710 Because the simplex σ contains a ball of radius the smallest altitude over d centred at its
711 barycentre, that is TD/d with T the thickness, the vector $u - w$ can be chosen to be any
712 vector of length less than tD/d . In particular we can choose

$$713 \quad (u - w)^j = \frac{tD}{d} \frac{(\partial_j f_L^i(x) - \partial_j f^i(x))}{\sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2}}.$$

714 Plugging this choice into (30) gives

$$715 \quad \frac{tD}{d} \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq 4D^2\alpha$$

716 So that

$$717 \quad \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T}.$$

718 ◀

719 C.2 Proofs of new results in of Section 2.1

720 ► **Lemma 5.** Let $\text{grad}(f^i) = (\partial_j f_i)_j$ denote the gradients of the components f^i . Suppose
721 that the absolute value of the determinant of the Gram matrix of these gradients is lower
722 bounded, that is

$$723 \quad |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

724 and $|\text{grad}(f^i)| \leq \gamma_1$, for all i . Assume further that the conditions of Proposition 4 are
 725 satisfied, then

$$726 \quad |\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| > \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right). \quad (31)$$

727 If we write $\text{grad}_{(x,\tau)}$ for the gradient that includes the τ component, we find that

$$728 \quad |\det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \quad (8)$$

729 where $g_1(D) = \mathcal{O}(D)$, precisely

$$730 \quad g_1(D) = n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right). \quad (32)$$

731 **Proof of Lemma 5.** In this proof we shall be using the following notation:

$$732 \quad (\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j} = (Y)_{i,j},$$

733 and write $Y_{i,j}$ for the entries of the matrix.

734 By definition of F_L , we have that

$$\begin{aligned} 735 \quad & \det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j} \\ 736 \quad &= \det(\text{grad}((1-\tau)f^i(x) + \tau f_L(x)^i) \cdot \text{grad}((1-\tau)f^j(x) + \tau f_L(x)^j))_{i,j} \\ 737 \quad &= \det(\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x))))_{i,j} \end{aligned}$$

738 We shall write

$$739 \quad (X)_{i,j} = (\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}$$

740 Proposition 4 now yields that

$$\begin{aligned} 741 \quad & |X_{i,j} - Y_{i,j}| \\ 742 \quad &= |\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x))) \\ 743 \quad &\quad - \text{grad}(f^i(x)) \cdot \text{grad}(f^j(x))| \\ 744 \quad &= |\text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x)) + \text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x)) \\ 745 \quad &\quad + \text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(\tau(f_L(x)^j - f^j(x)))| \\ 746 \quad &\leq 2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2. \quad (33) \end{aligned}$$

747 This estimate is used to bound the $\|E\|_p$ term in Friedland's bound, for $p = \infty$. In particular,

$$748 \quad \|(X - Y)_{i,j}\|_\infty \leq n \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right). \quad (34)$$

749 Furthermore,

$$\begin{aligned} 750 \quad & |X_{i,j}| = |\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x)))| \\ 751 \quad & \leq \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2, \quad (35) \end{aligned}$$

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752 where we used Cauchy-Schwarz. Note that the latter estimate is used to bound the $\max\{\|A\|_p, \|A+$
 753 $E\|_p\}$ term in Friedland's bound, for $p = \infty$. In particular,

$$754 \quad \|(X)_{i,j}\|_\infty \leq n \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2. \quad (36)$$

755 This means that Friedland's bound now yields

$$756 \quad |\det(X)_{i,j} - \det(Y)_{i,j}| \leq n(\|(X)_{i,j}\|_\infty)^{n-1} \|(X - Y)_{i,j}\|_\infty. \quad (37)$$

757 Thus,

$$758 \quad |\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| \geq |\det(\text{grad}(f^i(x)) \cdot \text{grad}(f^j(x)))_{i,j}|$$

$$759 \quad \quad \quad - n^{n+1} \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right)$$

$$760 \quad \quad \quad > \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right).$$

761 The addition of the τ component gives a small extra contribution to the inner product to
 762 the gradients

$$763 \quad \tilde{X}_{i,j} = \text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j$$

$$764 \quad \quad = \text{grad}(f^i(x) + \tau(f_L^i(x) - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L^j(x) - f^j(x)))$$

$$765 \quad \quad + (f_L^i(x) - f^i(x))(f_L^j(x) - f^j(x)).$$

766 Combining this with (33) and Lemma 3 yields

$$767 \quad |\tilde{X}_{i,j} - Y_{i,j}| = |\text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j - \text{grad}(f^i(x)) \cdot \text{grad}(f^j(x))|$$

$$768 \quad \quad \leq 2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2$$

769 Similarly to (35), we also have that

$$770 \quad |\tilde{X}_{i,j}| = |\text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j| \leq \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2. \quad (38)$$

771 Applying Friedland's bound again gives

$$772 \quad |\det(\tilde{X})_{i,j}| = \left| \det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j} \right|$$

$$773 \quad \quad > \gamma_0 - n^{n+1} \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \cdot \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)$$

$$774 \quad \quad \geq \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right).$$

(because $T \leq 1$)

775

776 ► **Corollary 42.** *If on top of the conditions of Lemma 5, we also assume that $\frac{4dD\alpha}{T} \leq \gamma_1$,*
 777 *the bound (31) can be simplified, at the cost of weakening the bound, to*

$$778 \quad \det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j} > \gamma_0 - 3 \cdot 2^{2n+1} n^{n+1} (\gamma_1)^{2n-1} \frac{dD\alpha}{T}.$$

779 The bound (8) can be simplified, at the cost of weakening the bound, to

$$780 \quad |\det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

781 **Proof of Corollary 42.** If we assume $\frac{4dD\alpha}{T} \leq \gamma_1$, a short calculation shows that (31) reduces
782 to

$$783 \quad |\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| > \gamma_0 - n^{n+1} (2\gamma_1)^{2(n-1)} \left(3\gamma_1 \frac{4dD\alpha}{T}\right)$$

$$784 \quad = \gamma_0 - 3 \cdot 2^{2n+1} n^{n+1} (\gamma_1)^{2n-1} \frac{dD\alpha}{T}.$$

785 A straightforward calculation shows (8) reduces to

$$786 \quad \det(\tilde{X})_{i,j} \geq \gamma_0 - n^{n+1} \left(\frac{5}{2}\gamma_1\right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \frac{3}{2}\gamma_1 \frac{6dD\alpha}{T}\right)$$

$$787 \quad = \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

788 ◀

789 **Proof of Lemma 7.** Let $A_1 = \text{span}(v^1, \dots, v^{d-n})$. Write v_A^i for the projection of v^i on A_1 .
790 Note that the vector v_A^i is $d - n$ -dimensional. By construction,

$$791 \quad \det(v^i \cdot v^j)_{i,j} = \det(v_A^i \cdot v_A^j)_{i,j} = \det((v_A^i)_i^t (v_A^j)_j) = \det((v_A^i)_i)^2,$$

792 where $(v_A^i)_i$ denotes the $(d - n) \times (d - n)$ - matrix whose columns are v_A^i , and $(v_A^i)_i^t$ its
793 transposed. Moreover, $|v^i| = |v_A^i|$. Hadamard's inequality [38] now gives

$$794 \quad \det((v_A^j)_j) \leq \prod_j |v_A^j| \leq \gamma_1^{d-n-1} |v_A^i|.$$

795 Using that $\det(v^i \cdot v^j)_{i,j} > \gamma_0$ now yields $\sqrt{\gamma_0}/\gamma_1^{d-n-1} \leq |v_A^i|$. ◀

796 **Proof of Lemma 8.** The statement follows fairly straightforwardly from Lemma 7 and Lemma 3.
797 In particular the τ component of $\text{grad}_{(x,\tau)}(F_L^i)$ is bounded in absolute value by $|f_L(x)^i -$
798 $f^i(x)| \leq 2D^2\alpha$. On the other hand the x component is lower bounded by

$$799 \quad \sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T},$$

800 as a consequence of Proposition 4 and Lemma 7. This gives that

$$801 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_L^i), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

802 ◀

803 C.3 Proofs of Section 2.2

804 **Proof of Lemma 14.** Using the standard bounds on matrix norm, see for example [37,
805 Equation (2.3.11)], and (4), we have that $\sqrt{d}\alpha \geq |\partial_k \partial_l f^i|$, for all k, l and i . Because the

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806 supremum of the absolute value of a derivative of a function bounds the Lipschitz constant
807 of the function, we have that

$$808 \quad |\partial_l f^i(x_1) - \partial_l f^i(x_2)| \leq \sqrt{d}\alpha|x_1 - x_2| \quad (39)$$

$$809 \quad |f^i(x_1) - f^i(x_2)| \leq \gamma_1|x_1 - x_2|. \quad (40)$$

810 Note furthermore that

$$811 \quad |x_1 - x_2| \leq 2D, \quad (41)$$

812 by the triangle inequality.

813 Lemma 3 gives that $|f_{PL}^i(x) - f^i(x)| \leq 2D^2\alpha$. Using the triangle inequality we now find
814 that

$$815 \quad |f_{PL}^i(x_1) - f_{PL}^i(x_2)| \leq |f_{PL}^i(x_1) - f^i(x_1) + f^i(x_1) - f^i(x_2) + f^i(x_2) - f_{PL}^i(x_2)|.
816 \quad \leq 2\gamma_1 D + 4D^2\alpha \quad (42)$$

817 Because

$$818 \quad \text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) = \begin{pmatrix} \text{grad}(f^i(x) + \tau_1(f_{PL}^i(x) - f^i(x))) \\ f_{PL}^i(x) - f^i(x) \end{pmatrix},$$

819 we find that

$$820 \quad |\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL}^i(x_2, \tau_2)|
821 \quad \leq \sum_l |\partial_l f^i(x_1) - \partial_l f^i(x_2)|
822 \quad \quad + \sum_l \tau_1 |\partial_l f_{PL}^i(x_1) - \partial_l f^i(x_1)| + \sum_l \tau_2 |\partial_l f_{PL}^i(x_2) - \partial_l f^i(x_2)|
823 \quad \quad + |f_{PL}^i(x_1) - f^i(x_1) - f_{PL}^i(x_2) + f^i(x_2)|
824 \quad \leq 2\sqrt{d}d\alpha D \quad (\text{by (39), and (41)})
825 \quad \quad + d \frac{8dD\alpha}{T} \quad (\text{Proposition 4, and } \tau_1, \tau_2 \in [0, 1])
826 \quad \quad + 2\gamma_1 D + 4D^2\alpha + 2\gamma_1 D \quad (\text{by (40), (41) and (42)})
827 \quad \leq 2\sqrt{d}d\alpha D + d \frac{8dD\alpha}{T} + 4\gamma_1 D + 4D^2\alpha
828 \quad \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha,$$

829 where we used the triangle inequality several times. ◀

830 ► **Lemma 16.** *Suppose that*

$$831 \quad |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

832 and $|\text{grad}(f^i)| \leq \gamma_1$, for all i . Let now v be a vertex in \mathcal{T} , $x_1, \dots, x_m \in \text{star}(v)$, and
833 $\tau_1, \dots, \tau_m \in [0, 1]$, such that $\text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k)$, $k = 0, \dots, m$ are well defined. Then,

$$834 \quad \left| \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right) \right|_{i,j}$$

$$\begin{aligned}
 &\geq \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right) \\
 &- n^{n+1} \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \\
 &\cdot \left(2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \right. \\
 &\left. + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \right).
 \end{aligned}$$

If we moreover assume $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$, the expression can be simplified, at the cost of weakening the bound, to

$$\left| \det \left(\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right)_{i,j} \right| \geq \gamma_0 - g_2(D),$$

with $g_2(D) = \mathcal{O}(D)$, in fact

$$g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1D \right) \right).$$

Proof of Lemma 16. Let $x_0 \in \operatorname{star}(v)$ and $\tau_0 \in [0, 1]$, be such that $\operatorname{grad}_{(x,\tau)} F_L^i(x_0, \tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior of a d -simplex in \mathcal{T} . Lemma 5 gives that

$$\begin{aligned}
 |\det(\tilde{X})_{i,j}| &= |\det(\operatorname{grad}_{(x,\tau)}(F_L^i(x_0, \tau_0)) \cdot \operatorname{grad}_{(x,\tau)}(F_L^j(x_0, \tau_0)))_{i,j}| \\
 &> \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right).
 \end{aligned}$$

We want to use this together with Corollary 15 and Friedland’s bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (38) also holds for the convex hull, that is

$$\left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right| \leq \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2.$$

We write

$$Z_{i,j} = \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right).$$

Similarly to (36), we have that

$$\| (Z)_{i,j} \|_\infty \leq n \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right) \tag{43}$$

We note that

$$\left| \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right| = \left| \left(\begin{array}{c} \operatorname{grad}(f^i(x) + \tau(f_{PL}^i(x) - f^i(x))) \\ f_{PL}^i(x) - f^i(x) \end{array} \right) \right|$$

$$\begin{aligned}
 &\leq \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \\
 &\quad \text{(because } \text{grad}(f^i) \leq \gamma_1, \text{ Proposition 4, and Lemma 3)} \\
 &\tag{44}
 \end{aligned}$$

Corollary 15 now gives

$$\begin{aligned}
 |Z_{i,j} - \tilde{X}_{i,j}| &= \left| \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right. \\
 &\quad \left. - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
 &= \left| \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) + \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right) \right. \\
 &\quad \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) - \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) + \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right) \\
 &\quad \left. - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
 &\leq \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
 &\quad + \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right| \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right| \\
 &\quad + \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \\
 &\quad \cdot \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right| \\
 &\leq 2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
 &\quad \text{(by Corollary 15 and (44))} \\
 &\quad + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \\
 &\quad \text{(by Corollary 15)} \\
 &\tag{45}
 \end{aligned}$$

Similarly to (34), we have that

$$\begin{aligned}
 \|(Z - \tilde{X})_{i,j}\|_\infty &\leq n2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
 &\quad + n \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \\
 &\tag{46}
 \end{aligned}$$

Indeed, Friedland now gives,

$$|\det(Z)_{i,j}| = \left| \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right)_{i,j} \right|$$

$$\begin{aligned}
 &\geq \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right) \quad (\text{by Lemma 5}) \\
 &\quad - n^{n+1} \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \quad (\text{by 43}) \\
 &\quad \cdot \left(2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \right. \\
 &\quad \left. + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \right) \\
 &\hspace{15em} (\text{by 46, note that } n \text{ was factored out})
 \end{aligned}$$

If we further assume that $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$, we can simplify the bound to

$$\begin{aligned}
 |\det(Z)_{i,j}| &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left(\frac{6dD\alpha}{T} \right) \right) \\
 &\quad - n^{n+1} \left((2\gamma_1)^2 + (2D\alpha)^2 \right)^{n-1} \\
 &\quad \cdot \left(2 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1D \right) (2\gamma_1 + 2dD\alpha) + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \right) \\
 &\hspace{15em} (\text{because } D \leq 1, \text{ and } T \leq 1) \\
 &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left(\frac{6dD\alpha}{T} \right) \right) \\
 &\quad - n^{n+1} (5\gamma_1^2)^{n-1} \\
 &\quad \cdot \left(2 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1D \right) (2\gamma_1 + \gamma_1) + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + \gamma_1D \right)^2 \right) \\
 &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left(\frac{6dD\alpha}{T} \right) \right) \\
 &\quad - n^{n+1} (5\gamma_1^2)^{n-1} \left(6\gamma_1 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1D \right) + \left(\frac{10d^2D\alpha}{T} + 5\gamma_1D \right)^2 \right) \\
 &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \gamma_1 \left(\frac{14dD\alpha}{T} \right) \\
 &\quad - n^{n+1} (5\gamma_1^2)^{n-1} \left(6\gamma_1 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1D \right) + (2d\gamma_1 + 5\gamma_1D) \left(\frac{10d^2D\alpha}{T} + 5\gamma_1D \right) \right) \\
 &\geq \gamma_0 - n^{n+1} 2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) \\
 &\quad - n^{n+1} 5^{n-1} \gamma_1^{2n-2} \left((2d\gamma_1 + 5\gamma_1) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1D \right) \right) \\
 &\geq \gamma_0 - n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d + 5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1D \right) \right)
 \end{aligned}$$

► **Corollary 17.** If $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, and

$$\gamma_0 > g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d + 5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1D \right) \right)$$

the generalized implicit function theorem, Theorem 13, applies to $F_L(x, \tau) = 0$.

901 ► **Theorem 18.** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ be a smooth function. Write f_{PL} for the Piecewise-
 902 Linear function that is the linear interpolation of the values of f at the vertices, if restricted
 903 to a single simplex $\sigma \in \mathcal{T}$. Write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for
 904 all i . Suppose that Σ_0 is compact. Suppose that for in each simplex in Σ_0 we have that

- 905 ■ $|\text{grad}(f^i)| \leq \gamma_1$, for all i .
- 906 ■ $|\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0$, for all i, j .

907 If moreover, $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, and

$$908 \quad \gamma_0 > g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14d\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2\alpha}{T} + 9\gamma_1 \right) \right) D,$$

909 then the zero set of f is isotopic to the zero set of f_{PL} .

910 **Proof of Lemma 19.** Write

$$911 \quad w = \mu_1 v^1 + \dots + \mu_{d-n} v^{d-n},$$

912 with $\mu_1, \dots, \mu_{d-n} \in \mathbb{R}$. By definition,

$$913 \quad \cos \angle(w, e_\tau) = \frac{\sum_i \mu_i v^i \cdot e_\tau}{|w|} = \frac{\sum_i \mu_i v^i \cdot e_\tau}{\sqrt{\sum_{i,j} \mu_i \mu_j v^i \cdot v^j}}.$$

914 We note that $|\sum_i \mu_i v^i \cdot e_\tau| \leq (d-n)|\mu| \tilde{\gamma}_1 \phi_0$, where μ denotes the vector (μ_i) .

915 The matrix $(v^i \cdot v^j)_{i,j}$ is a symmetric positive definite matrix, because it is a Gram
 916 matrix and its determinant strictly positive, by assumption. This means that the smallest
 917 eigenvalue λ_{\min} of $(v^i \cdot v^j)_{i,j}$, gives a lower bound, that is

$$918 \quad \sum_{i,j} \mu_i \mu_j v^i \cdot v^j \geq \lambda_{\min} |\mu|^2.$$

919 The determinant of a square matrix is the product of its eigenvalues. The largest eigenvalue is
 920 bounded by the operator norm, which in turn is bounded by the Frobenius norm [37, (2.3.7)],
 921 denoted by $\|\cdot\|_F$. This gives that the largest eigenvalue λ_{\max} of $(\langle v^i, v^j \rangle)$ is bounded:

$$922 \quad \lambda_{\max} \leq \|(v^i \cdot v^j)\|_2 \leq \|(v^i)^t\|_2 \|(v^i)_i\|_2 \leq \|(v^i)_i\|_F^2 \leq d^2 \tilde{\gamma}_1^2.$$

923 For the smallest eigenvalue we now find that

$$924 \quad \tilde{\gamma}_0 < \det(v^i \cdot v^j)_{i,j} \leq \lambda_{\max}^{d-n-1} \lambda_{\min} \leq d^{2(d-n-1)} \tilde{\gamma}_1^{2(d-n-1)} \lambda_{\min}.$$

925 Combining these results gives

$$926 \quad \cos \angle(w, e_\tau) = \frac{\sum_i \mu_i \langle v^i, e_\tau \rangle}{\sqrt{\sum_{i,j} \mu_i \mu_j \langle v^i, v^j \rangle}} \leq \frac{(d-n)|\mu| \tilde{\gamma}_1 \phi_0}{|\mu| \frac{\sqrt{\tilde{\gamma}_0}}{d^{d-n-1} \tilde{\gamma}_1^{d-n-1}}} = \frac{(d-n) d^{d-n-1} \phi_0 \tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

927

928 ► **Corollary 20.** Suppose that the conditions of Theorem 18 are satisfied. Then,

$$929 \quad d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D),$$

930 with $g_3(D) = \mathcal{O}(D^2)$, in fact

$$931 \quad g_3(D) = \left(\frac{(d-n) d^{d-n-1} \sqrt{\gamma_0} 2D^2 \alpha \gamma_1^{d-n-1} (\gamma_1 + 2D^2 \alpha)^{d-n}}{\sqrt{\gamma_0 + (2D^2 \alpha \gamma_1^{d-n-1})^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right).$$

932 **Proof of Corollary 20.** Lemma 8 gives

$$933 \quad \tan \angle(\text{grad}_{(x,\tau)}(F^i), \Xi) \leq \frac{2D^2\alpha\gamma_1^{d-n-1}}{\sqrt{\gamma_0}},$$

934 and thus using the notation of Lemma 19,

$$935 \quad \begin{aligned} \cos(\angle \text{grad}_{(x,\tau)}(F^i), e_\tau) &\leq \sin \arctan \left(\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}} \right) \\ 936 \quad &= \frac{\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}}{\sqrt{1 + \left(\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}} \right)^2}} \\ 937 \quad &= \frac{2D^2\alpha}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2}}. \end{aligned}$$

938 By Corollary 42

$$939 \quad \tilde{\gamma}_0 = \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

940 Lemma 3 and $|\text{grad}(f^i)| \leq \gamma_1$ give

$$941 \quad \tilde{\gamma}_1 = \gamma_1 + 2D^2\alpha.$$

942 Plugging this into Lemma 19 gives

$$943 \quad \cos \angle(w, e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}},$$

944 with $w \in \text{span}_i(\text{grad}_{(x,\tau)}(F^i))$. Because the tangent space to $F_L = 0$ is normal to $\text{span}_i(\text{grad}_{(x,\tau)}(F^i))$,
945 we see that

$$946 \quad \sin \angle(g_\tau, e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}}.$$

947 This means that the angle between g_τ and e_τ is very small (we can choose D arbitrarily
948 small). This means that as $\tau \in [0, 1]$ the distance between begin and end point of the
949 gradient flow, and thus the Fréchet distance, by $\tan \angle(g_\tau, e_\tau)$, that is

$$950 \quad \begin{aligned} &d_F(f^{-1}(0), f_{PL}^{-1}(0)) \\ 951 \quad &\leq \tan \arcsin \left(\frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right) \end{aligned}$$

952 ◀

953 C.4 Proofs of Section 3.1

954 **Proof of Lemma 21.** As mentioned, by construction $\phi_b(x) \in [0, 1]$. Because $\partial_x \beta \left(\frac{y_1+y_2}{2}\right) =$
955 0 and this is the only zero of the derivative in the open interval (y_1, y_2) , we see that

$$956 \quad \beta(x) \leq \beta \left(\frac{y_1 + y_2}{2}\right) = e^{\frac{4}{y_1 - y_2}}.$$

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957 Hence, because $\beta(x) \geq 0$,

$$958 \int_{y_1}^{y_2} \beta(x) \leq (y_2 - y_1) e^{\frac{4}{y_1 - y_2}}$$

959 Because $\beta(x)$ is monotone on $[y_1, \frac{y_1 + y_2}{2}]$ we also have

$$960 \int_{y_1}^{y_2} \beta(x) \geq \frac{y_2 - y_1}{2} \beta\left(\frac{3}{4}y_1 + \frac{1}{4}y_2\right) = \frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}$$

961 We now have

$$\begin{aligned} 962 \partial_x(\phi_l(x)) &= \partial_x \left(\int_x^{y_2} \beta(x') dx' / \int_{y_1}^{y_2} \beta(x') dx' \right) \\ 963 &= \beta(x) / \int_{y_1}^{y_2} \beta(x') dx' \\ 964 &\leq \frac{e^{\frac{4}{y_1 - y_2}}}{\frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}} \\ 965 &= 2 \frac{e^{\frac{4}{y_1 - y_2} - \frac{16}{3(y_1 - y_2)}}}{y_2 - y_1} \\ 966 &= 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} \end{aligned} \quad (\text{DefCphi})$$

967

968 ► **Lemma 23.** *Suppose that*

$$969 |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

970 and $|\text{grad}(f^i)| \leq \gamma_1$, for all i and that the bound (12) is satisfied. Then,

$$971 \det(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j} > \gamma_0 - g_4(D),$$

972 with $g_4(D) = \mathcal{O}(D)$, in fact,

$$\begin{aligned} 973 g_4(D) &= n \left(\left(n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right. \\ 974 &\quad \cdot n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \Big), \end{aligned}$$

975 **Proof of Lemma 23.** We start with an estimate on the individual $\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau)$

$$\begin{aligned} 976 &\left| \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \right| \\ 977 &= \left| \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} \left(f(x) + \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right) \right| \\ 978 &= \left| -\text{grad}_{(x,\tau)} \left(\tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right) \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \left(\begin{array}{c} -\tau \text{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) - \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \\ \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \end{array} \right) \right| \\
 &= \left(\left| \tau \text{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) + \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \right|^2 \right. \\
 &\quad \left. + \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right)^2 \right)^{1/2} \\
 &\leq \left| \tau \text{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) \right| + \left| \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \right| \\
 &\quad + \left| \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right| \quad \text{(by the triangle inequality)} \\
 &\leq \gamma_2 \gamma_\phi |f_L^i(x) - f^i(x)| + \left| \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \text{grad}(f_L^i(x) - f^i(x)) \right| \\
 &\quad \text{(because } \tau \leq 1, (12), (11)) \\
 &\quad + |f_L^i(x) - f^i(x)| \quad \text{(because } \phi \in [0, 1]) \\
 &\leq \gamma_2 \gamma_\phi |f_L^i(x) - f^i(x)| + |\text{grad}(f_L^i(x) - f^i(x))| \quad \text{(because } \phi \in [0, 1]) \\
 &\quad + |f_L^i(x) - f^i(x)| \\
 &\leq (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \quad \text{(by Lemma 3 and Proposition 4)} \\
 &\tag{47}
 \end{aligned}$$

We now write

$$(X')_{i,j} = (\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j}.$$

Expanding yields

$$\begin{aligned}
 (X')_{i,j} &= \left(\left(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \\
 &\quad \cdot \left(\text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 &= \left(\left(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \\
 &\quad \cdot \left(\text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 &= (Y)_{i,j} \\
 &\quad + \left(\left(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right)_{i,j} \\
 &\quad + \left(\begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \cdot \left(\text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
 &\quad + \left(\left(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} \right) \right)
 \end{aligned}$$

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$$1001 \quad \cdot \left(\text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right)_{i,j}, \quad (48)$$

1002 where we again used the notation

$$1003 \quad (\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j} = (Y)_{i,j}.$$

1004 We now see by Cauchy-Schwarz and the triangle inequality that

$$1005 \quad |X'_{i,j} - Y_{i,j}| \leq 2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2,$$

1006 by (47). This also means that for the operator norm of the matrix we find

$$1007 \quad \|(X' - Y)_{i,j}\|_\infty \leq n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \quad (49)$$

1008 Similarly to (35) and (36), we also see that (47) gives

$$1009 \quad |X'_{i,j}| \leq \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2, \quad (50)$$

1010 and thus,

$$1011 \quad \|X'_{i,j}\|_\infty \leq n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2. \quad (51)$$

1012 Friedland's bound (37) now gives

$$1013 \quad \det(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j} \\ 1014 \quad > \gamma_0 - n \left(\left(n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right. \\ 1015 \quad \cdot \left. n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right),$$

1016 where we used (49), and (51). ◀

1017 **Proof of Lemma 25.** As we have seen in the calculation in (47), the τ component of $\text{grad}_{(x,\tau)}(F^i)$
1018 is bounded in absolute value by $|f_L(x)^i - f^i(x)| \leq 2D^2\alpha$. On the other hand the x component
1019 is lower bounded by

$$1020 \quad \sqrt{\gamma_0}/\gamma_1^{d-n-1} - \gamma_2 \gamma_\phi 2D^2 \alpha - \frac{4dD\alpha}{T},$$

1021 as a consequence of Proposition 4 and the calculation in (47). This gives that

$$1022 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,1}), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \gamma_2 \gamma_\phi 2D^2 \alpha - \frac{4dD\alpha}{T}}.$$

1023 ◀

1024 ▶ **Lemma 27.** *Under the same conditions as Lemma 14 we have*

$$1025 \quad \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right|$$

$$1026 \quad \leq g_5(D) = 2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)$$

1027 **Proof of Lemma 27.** By expansion we see that

$$1028 \quad \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right|$$

$$1029 \quad = \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} \right.$$

$$1030 \quad \left. + \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right|$$

$$1031 \quad \leq \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} \right| + \left| \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} \right|$$

$$1032 \quad + \left| \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right| \quad (\text{by the triangle inequality})$$

$$1033 \quad \leq 2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \quad (\text{by (39), (41) and (47) twice})$$

1034

1035 ▶ **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$1036 \quad \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$1037 \quad \geq \gamma_0 - g_4(D) - g_6(D),$$

1038 with $g_6(D) = \mathcal{O}(D)$, where as usual we regard, $\alpha, \gamma_1, \gamma_2, \gamma_\phi, d, n$ and T as constants. In
1039 fact,

$$1040 \quad g_4(D) = n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^{2(n-1)} \right.$$

$$1041 \quad \left. \cdot \left(2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right)$$

$$1042 \quad g_6(D) = n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1}$$

$$1043 \quad \cdot \left(2 \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right.$$

$$1044 \quad \left. + \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \right).$$

1045 **Proof of Lemma 29.** The proof is more or less the same as the proof of Lemma 16, but with
1046 more complicated bounds. Let $x_0 \in \text{star}(v)$ and $\tau_0 \in [0, 1]$, be such that $\text{grad}_{(x,\tau)} F^i(x_0, \tau_0)$
1047 is well defined. Note that it is sufficient for x_0 to lie in the interior of a d -simplex in \mathcal{T} .
1048 Lemma 23 gives that

$$1049 \quad \left| \det(\tilde{X}')_{i,j} \right| = \left| \det(\text{grad}_{(x,\tau)}(F^i(x_0, \tau_0)) \cdot \text{grad}_{(x,\tau)}(F^j(x_0, \tau_0)))_{i,j} \right|$$

$$\begin{aligned}
 &> \gamma_0 - n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^{2(n-1)} \right. \\
 &\quad \left. \cdot \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right).
 \end{aligned}$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (50) also holds for the convex hull, that is

$$\begin{aligned}
 &\left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right| \\
 &\leq \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2.
 \end{aligned}$$

We write

$$Z'_{i,j} = \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right).$$

Similarly to (36), we have that

$$\| (Z')_{i,j} \|_\infty \leq n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \quad (52)$$

We note that

$$\begin{aligned}
 \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \right| &\leq \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \\
 &\quad \text{(because } \operatorname{grad}(f^i) \leq \gamma_1, \text{ and (47))} \\
 &\quad (53)
 \end{aligned}$$

Similarly to (45), we find that

$$\begin{aligned}
 |Z'_{i,j} - \tilde{X}'_{i,j}| &= \left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right. \\
 &\quad \left. - \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) \right| \\
 &\leq \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) \right| \\
 &\quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \right| \\
 &\quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \\
 &\quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right| \\
 &\leq 2 \left(2\sqrt{dd}\alpha D + 2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \\
 &\quad \text{(by Corollary 28 and (53))}
 \end{aligned}$$

1092 where we used that

$$1093 \quad 1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) = 0,$$

1094 if

$$1095 \quad \sum_i (f^i)^2 + f_\partial^2 \leq \frac{101}{100} y_0,$$

1096 and

$$1097 \quad \left(1 - \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) = 0,$$

1098 if

$$1099 \quad \sum_i (f^i)^2 + f_\partial^2 \geq y_0.$$

1100 We can further rewrite (55),

$$\begin{aligned} 1101 \quad & \left(\left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \left(\phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) - 1 + 1 \right) + \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) f_{PL}(x) \\ 1102 \quad & = \left(\left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \left(\phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) - 1 \right) + 1 \right) f_{PL}(x) \\ 1103 \quad & = f_{PL}(x), \end{aligned}$$

1104 where we used the same argument as before. ◀

1105 ► **Lemma 32.** For all ϵ ,

$$1106 \quad \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x,\tau))_{i,j} \geq \gamma_0^B - g_\tau(D)$$

1107 with $g_\tau(D) = \mathcal{O}(D)$. In fact,

$$\begin{aligned} 1108 \quad g_\tau(D) &= (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\ 1109 \quad &\cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\ 1110 \quad &\left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \end{aligned}$$

1111 **Proof of Lemma 32.** We start with an estimate on the individual $\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau)$. We
1112 will write $(v, w)^i$ for the i -th coordinate of the composed vector (v, w) . We now see that

$$\begin{aligned} 1113 \quad & \left| \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \right| \\ 1114 \quad & = \left| \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} \left(\left(1 - \tau\psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (F_{L,1}(x,1), f_\partial(x) - \epsilon)^i \right) \right| \end{aligned}$$

$$\begin{aligned}
 & + \max_j \left| \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| |\text{grad}(f_L^j(x) - f^j(x))| \\
 & \quad \text{(by the Leibniz rule, Cauchy-Schwarz, and the triangle inequality)} \\
 & + \left| \text{grad} \left(\psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) \right| \left| \left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right| \\
 & + \left| \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| \left| \text{grad} \left(\left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right) \right| \\
 & \quad \text{(by the Leibniz rule, and the triangle inequality)} \\
 & + 2D^2\beta \quad \text{(by Lemma 3)} \\
 & \leq \max_j \gamma_\phi \gamma_2 |f_L^j(x) - f^j(x)| + \max_j |\text{grad}(f_L^j(x) - f^j(x))| \\
 & \quad \text{(by Lemma 21, (11), (12), and since } \phi(y) \in [0, 1]) \\
 & + \gamma_\psi \gamma_2 \left| \left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right| \\
 & \quad \text{(by Lemma 21, (12), (17))} \\
 & + \left| \text{grad} \left(\left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x) \right)^i \right) \right| \\
 & \quad \text{(because } \psi(y) \in [0, 1]) \\
 & + 2D^2\beta \\
 & \leq \gamma_\phi \gamma_2 2D^2\alpha + \frac{4dD\alpha}{T} \quad \text{(by Lemma 3 and Proposition 4)} \\
 & + \gamma_\psi \gamma_2 2D^2\beta \quad \text{(by Lemma 3 and since } \phi(y) \in [0, 1]) \\
 & + \left| \text{grad} (f_L(x) - f(x), f_{\partial,L}(x) - f_{\partial}(x))^i \right| \\
 & + \left| \text{grad} \left(\left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), 0 \right)^i \right) \right| \quad \text{(by the triangle inequality)} \\
 & + 2D^2\beta \\
 & \leq \gamma_\phi \gamma_2 2D^2\beta + \frac{4dD\beta}{T} \quad \text{(because by definition } \alpha \leq \beta) \\
 & + \gamma_\psi \gamma_2 2D^2\beta \\
 & + \frac{4dD\beta}{T} \quad \text{(by Proposition 4)} \\
 & + \max_j \left| \text{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^j(x) - f^j(x)) \right) \right| \\
 & + 2D^2\beta \\
 & \leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{8dD\beta}{T} \\
 & + \max_j \left| \text{grad} \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| |(f_L^j(x) - f^j(x))|
 \end{aligned}$$

$$\begin{aligned}
 & + \max_j \left| \phi \left(\sum_l (f^l)^2 + f_\delta^2 \right) \right| \left| \text{grad}(f_L^j(x) - f^j(x)) \right| \\
 & \hspace{15em} \text{(By the Leibniz rule and the triangle inequality)} \\
 & \leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{8dD\beta}{T} \\
 & \quad + \gamma_\phi\gamma_2 2D^2\alpha \hspace{10em} \text{(by Lemma 21, (11), (12), and Lemma 3)} \\
 & \quad + \frac{4dD\beta}{T} \hspace{10em} \text{(because } \phi(y) \in [0, 1], \text{ Proposition 4, and } \alpha \leq \beta) \\
 & = (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \hspace{10em} (56)
 \end{aligned}$$

We now write

$$(X'_B)_{i,j} = (\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau))_{i,j}.$$

Similarly to (48), we see that

$$\begin{aligned}
 (X'_B)_{i,j} & = \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \\
 & \quad \cdot \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 & = \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \\
 & \quad \cdot \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \Bigg)_{i,j} \\
 & = (Y_B)_{i,j} \\
 & \quad + \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right)_{i,j} \\
 & \quad + \left(\begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \cdot \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
 & \quad + \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \right. \\
 & \quad \left. \cdot \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j},
 \end{aligned}$$

where we used the notation

$$(\text{grad}(f_B^i) \cdot \text{grad}(f_B^j))_{i,j} = (Y_B)_{i,j}.$$

We now see by (56), (15), Cauchy-Schwarz and the triangle inequality that

$$\begin{aligned}
 |(X'_B)_{i,j} - (Y_B)_{i,j}| & \leq 2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & \quad + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2.
 \end{aligned}$$

This also means that for the operator norm of the matrix we find

$$\|(X'_B - Y_B)_{i,j}\|_\infty \leq (n + 1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)$$

$$1176 \quad + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (57)$$

1177 Similarly to (35) and (36), we also see that (56) gives

$$1178 \quad |X'_{i,j}| \leq \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2, \quad (58)$$

1179 and thus,

$$1180 \quad \|X'_{i,j}\|_\infty \leq (n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (59)$$

1181 Friedland's bound (37) now gives

$$1182 \quad \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x,\tau))_{i,j}$$

$$1183 \quad \geq \gamma_0^B - (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n$$

$$1184 \quad \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right.$$

$$1185 \quad \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)$$

1186 where we used (57), and (59). ◀

1187 **Proof of Lemma 34.** As we have seen in the calculation in (56), the τ component of $\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}^i)$
 1188 is bounded in absolute value by $\left| ((f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x))^i \right| \leq 2D^2\beta$. On the other
 1189 hand the x component is lower bounded by

$$1190 \quad \sqrt{\gamma_0^B} / (\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T},$$

1191 as a consequence of Lemma 7 and the calculation in (56). This gives that

$$1192 \quad \tan \angle(\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}), \Xi) \leq \frac{2D^2\beta}{\sqrt{\gamma_0^B} / (\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T}}.$$

1193 ◀

1194 **► Lemma 36.** Let v be a vertex in \mathcal{T} , $x_1, x_2 \in \text{star}(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that
 1195 $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1)$ and $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)$ are well defined, then

$$1196 \quad |\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)| \leq g_8(D),$$

1197 with $g_8(D) = \mathcal{O}(D)$. In fact,

$$1198 \quad g_8(D) = 2\sqrt{dd}\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)$$

1199 **Proof of Lemma 36.** The proof follows the same steps as the proof of Lemma 27. By
 1200 expansion we see that

$$1201 \quad |\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)|$$

$$\begin{aligned}
 &= \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} \right. \\
 &\quad \left. + \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2) \right| \\
 &\leq \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} \right| + \left| \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} \right| \\
 &\quad + \left| \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2) \right| \quad (\text{by the triangle inequality}) \\
 &\leq 2\sqrt{dd}\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 &\hspace{15em} (\text{by (39), (41) and (56) twice})
 \end{aligned}$$

1207

1208 ► **Lemma 38.** *Under the same conditions as in Corollary 28,*

$$\begin{aligned}
 &\det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right)_{i,j} \\
 &\geq \gamma_0^B - g_7(D) - g_9(D),
 \end{aligned}$$

1211 where $g_9(D) = \mathcal{O}(D)$. In fact,

$$\begin{aligned}
 g_7(D) &= (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 &\quad \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 &\quad \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \\
 g_9(D) &= (n+1)^{n+1} \left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
 &\quad \cdot \left(2 \left(2\sqrt{dd}\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right) \\
 &\quad \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 &\quad + \left(2\sqrt{dd}\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2
 \end{aligned}$$

1219 **Proof of Lemma 38.** The proof is more or less the same as the proofs of Lemmas 16 and
 1220 28, but with even more complicated bounds. Let $x_0 \in \text{star}(v)$ and $\tau_0 \in [0, 1]$, be such that
 1221 $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior
 1222 of a d -simplex in \mathcal{T} . Lemma 23 gives that

$$\begin{aligned}
 &|\det(\tilde{X}'_B)_{i,j}| = |\det(\text{grad}_{(x,\tau)}(F_{PL,2,\epsilon}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)}(F_{PL,2,\epsilon}^j(x_0, \tau_0)))_{i,j}| \\
 &\quad > \gamma_0^B - (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n
 \end{aligned}$$

1224

$$\begin{aligned}
 & \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 & \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right).
 \end{aligned}$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give a bound on the determinant of

$$(Z'_B)_{i,j} = \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right).$$

As before we also need a bound on the operator norm $\|(Z'_B)_{i,j}\|_\infty$. This bound for this we notice that the bound (58) also holds for the convex hull, that is

$$\begin{aligned}
 & \left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right| \\
 & \leq \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2.
 \end{aligned}$$

And thus, similarly to (52), we have that

$$\|(Z'_B)_{i,j}\|_\infty \leq (n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (60)$$

We note that

$$\begin{aligned}
 \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \right| & \leq \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \\
 & \quad \text{(because } \operatorname{grad}(f_B^i) \leq \gamma_1^B, \text{ and (56))}
 \end{aligned} \quad (61)$$

Similarly to (45), we find that

$$\begin{aligned}
 & |(Z'_B)_{i,j} - (\tilde{X}'_B)_{i,j}| \\
 & = \left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right. \\
 & \quad \left. - \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) \right| \\
 & \leq \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right| \\
 & \quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) \right| \\
 & \quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right| \\
 & \quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \right| \\
 & \quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right|
 \end{aligned}$$

$$\begin{aligned}
 & \cdot \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right| \\
 & \leq 2 \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \quad (\text{by Corollary 37}) \\
 & \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \quad (\text{by (61)}) \\
 & + \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \quad (\text{by Corollary 37})
 \end{aligned}$$

Similarly to (34), we have that

$$\begin{aligned}
 \| (Z'_B - \tilde{X}'_B)_{i,j} \|_\infty & \leq 2(n+1) \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \\
 & \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & + (n+1) \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \quad (62)
 \end{aligned}$$

Indeed, Friedland's bound now gives,

$$\begin{aligned}
 |\det(Z'_B)_{i,j}| & = \left| \det \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right| \\
 & \geq \gamma_0^B - (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^n \\
 & \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
 & \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \quad (\text{by Lemma 32}) \\
 & - (n+1)^{n+1} \left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \quad (\text{by (60)}) \\
 & \cdot \left(2 \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right. \\
 & \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
 & \left. + \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \right) \\
 & \quad (\text{by (62) and factoring out } (n+1))
 \end{aligned}$$

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