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The topological correctness of PL-approximations of isomanifolds

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Abstract

Isomanifolds are the generalization of isosurfaces to arbitrary dimension and codimension, i.e. manifolds defined as the zero set of some multivariate multivalued function $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$. A natural (and efficient) way to approximate an isomanifold is to consider its Piecewise-Linear (PL) approximation based on a triangulation \mathcal{T} of the ambient space \mathbb{R}^d . In this paper, we give conditions under which the PL-approximation of an isomanifold is topologically equivalent to the isomanifold. The conditions are easy to satisfy in the sense that they can always be met by taking a sufficiently fine triangulation \mathcal{T} . This contrasts with previous results on the triangulation of manifolds where, in arbitrary dimensions, delicate perturbations are needed to guarantee topological correctness, which leads to strong limitations in practice. We further give a bound on the Fréchet distance between the original isomanifold and its PL-approximation. Finally we show analogous results for the PL-approximation of an isomanifold with boundary.

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1 Introduction

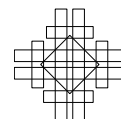
Isosurfacing especially in low dimensions is used in many fields of application, such as CT scans in medicine, biochemistry, biomedicine, deformable modeling, digital sculpting, environmental science, and mechanics and dynamics [45] and the references mentioned there.



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17 The marching cube approach [41] being the most popular approach taken. However the
 18 result of the marching cube algorithm is not necessarily topologically correct.

19 Some results on provable correctness were achieved within the computational geometry
 20 community [10, 47] in three dimensions. In case the isosurfacing is based on simplices instead
 21 of cubes, such as in the marching tetrahedra algorithm [28], some bounds can be given [1, 2],
 22 on for example the one-sided Hausdorff distance.

23 In general homeomorphism proofs in higher dimensions rely on some perturbation scheme
 24 to prove that a triangulation is correct [52, 8, 11, 12, 15]. This is a major difference with
 25 one and two dimensional surfaces where no such requirements exist [30], [49, Section 10.2].
 26 In this paper we shall see that no perturbations are necessary for isomanifolds as well.

27 The techniques used here are also different from many of the standard tools. Manifold
 28 triangulation/reconstruction algorithms use often Delaunay triangulations [50, 24, 19] and
 29 use the closed ball property [32], see for example [3, 18]. Others use Whitney’s lemma [13] or
 30 are based on collapses [4]. While the current paper mainly relies on the non-smooth implicit
 31 function theorem [22] with some Morse theory.

32 We stress that this does not solve the general problem of triangulating manifolds with
 33 boundary, because there exist manifolds that cannot be written as the zero set of a smooth
 34 function, because of the existence of manifolds with a non-trivializable normal bundle (gen-
 35 eralizations of non-orientable manifolds). For an excellent introduction to the topology of
 36 such bundles and manifolds we refer to [44]. We note that locally we can always write an
 37 embedded smooth manifold as the zero set of a smooth function, because it can be para-
 38 metrized as a function from the tangent space to the manifold itself as a consequence of
 39 the implicit function theorem. Non-continuous results have been recently obtained in the
 40 neighbourhood of the manifold [20]. We stress that there are now obstructions to global
 41 closest point projection on manifold [33], and this can even be approximately reconstructed
 42 from a point sample [34].

43 We also emphasize that because we do not use a perturbation scheme, we cannot give
 44 lower bounds on the quality of the linear pieces in the Piecewise-Linear (PL) approximation.
 45 This is a clear difference with previous methods [52, 15, 14, 12] whose output is a thick
 46 triangulation. Although this is an appealing property, it complicates the analysis further
 47 and requires perturbation schemes that work fine in theory, but the constants are miserable
 48 and the methods do not work in practice in high dimension (see a more detailed discussion
 49 in Section 2). Here we ask for less but still provide guarantees on the Fréchet distance and
 50 the approximation of the gradients. Perturbation techniques could be used to improve the
 51 simplex quality (although only to some very limited extent).

52 This rest of the paper is subdivided in two sections. In the first section, we treat closed
 53 isomanifolds, that is compact manifolds without boundary. In the second section, we prove
 54 similar results for isomanifolds with boundary. Extension to general isostratifolds is briefly
 55 discussed in Section 4. Apart from some Delaunay based work on triangulations of stratifolds
 56 in three dimensions [46, 48, 27, 26, 25], we are not aware of similar results on manifolds with
 57 boundary. Significant effort also went in the detection of strata, in this case in arbitrary
 58 dimension, see for example [6, 5].

59 We will often give the simplified statement of results in the main text while complete
 60 versions with all constants will be given in the appendix. All proofs can also be found in
 61 the appendix.

62 This paper part of a collection of closely related papers. The data structure needed to
 63 efficiently propagate along the manifold is presented in [16], while details of the implement-
 64 ation will be discussed in an upcoming paper.

2 Isomanifolds (without boundary)

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ be a smooth function and suppose that 0 is a regular value of f , meaning that at every point x such that $f(x) = 0$, the Jacobian of f is non-degenerate. Then the zero set of f is a manifold as a direct consequence of the implicit function theorem, see for example [29, Section 3.5]. We further assume that $f^{-1}(0)$ is compact. As in [1] we consider a triangulation \mathcal{T} of \mathbb{R}^d . The function f_{PL} is a linear interpolation of the values of f at the vertices if restricted to a single simplex $\sigma \in \mathcal{T}$. For any function $g : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ we write g^i for the components of g .

We prove that under certain conditions that there is an isotopy from the zero set of f to the zero set of f_{PL} . The proof will be using the Piecewise-Linear (PL) map

$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x), \quad (1)$$

which interpolates between f and f_{PL} and is based on the generalized implicit function theorem. The isotopy is in fact stronger than just the existence of a homeomorphism from the zero set of f to that of f_{PL} .

Our result in particular implies that the zero set f_{PL} is a manifold. So this significantly improves on the result of Allgower and Georg [2, Theorem 15.4.1]. The conditions are also weaker, because we do not require that the zero set avoids simplices that have dimension less than the codimension, see [2, Definition 12.2.2] and the text above [2, Theorem 15.4.1]. The idea to avoid these low dimensional simplices originates from Whitney [52], which remarkably was not cited by Allgower and George [2, 1]. Very heavy perturbation schemes for the vertices of the ambient triangulation \mathcal{T} are required for the manifold to be sufficiently far from simplices in the ambient triangulations that have dimension less than the codimension of the manifold [52, 15]. Various techniques have been developed to compute such perturbations with guarantees. They typically consist in perturbing the position of the sample points or in assigning weights to the points. Complexity bounds are then obtained using volume arguments. See, for example [18, 14, 11, 9]. However, these techniques suffer from several drawbacks. The constants in the complexity depend exponentially on the ambient dimension. Moreover the analysis assumes that the probability of the simplices of dimension less than the codimension to intersect the manifold is zero, which is not true when dealing with finite precision. As a result, the actual implementations we are aware of fail to work well in practice except in very simple cases.

We are, by definition, only interested in $f^{-1}(0)$ so we can ignore points that are sufficiently far from this zero set. More precisely, we observe the following: If $f^i(x)$ is positive on a simplex σ then so is f_{PL}^i and thus is F_{PL}^i positive on $\sigma \times [0, 1]$. The same argument holds for negative values. So we see that

► **Remark.** Write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i . Then for all τ , $\{x \mid F_{PL}(x, \tau) = 0\} \subset \Sigma_0$.

The results will be formulated in terms of constants defined in terms of f and the ambient triangulation \mathcal{T} .

104 ► **Definition 1.** We define

$$105 \quad \gamma_0 = \inf_{x \in \Sigma_0} |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| \quad (2)$$

$$106 \quad \gamma_1 = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f^i)| \quad (3)$$

$$107 \quad \alpha = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k \partial_l f^i)_{k,l}\|_2, \quad (4)$$

$$108 \quad D : \text{the longest edge length of a simplex in } \Sigma_0 \quad (5)$$

$$109 \quad T : \text{the smallest thickness of a simplex in } \Sigma_0. \quad (6)$$

110 Here $\text{grad}(f^i) = (\partial_j f^i)_j$ denotes the gradient of component f^i , $\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}$
 111 denotes the determinant of the matrix with entries $\text{grad}(f^i) \cdot \text{grad}(f^j)$, $\|\cdot\|_2$ the operator
 112 2-norm, and $(\partial_k \partial_l f^i)_{k,l}$ the matrix of second order derivatives, that is the Hessian (Hes).
 113 We recall the definition of the operator norm: $\|A\|_p = \max_{x \in \mathbb{R}^n} \frac{|Ax|_p}{|x|_p}$, with $|\cdot|_p$ the p -norm
 114 on \mathbb{R}^n .

115 We will assume that $\gamma_0, \gamma_1, \alpha, D, T \in (0, \infty)$. The thickness is the ratio of the height over
 116 the longest edge length.

117 A good choice for \mathcal{T} is the Coxeter triangulation of type A_d , see [23, 21], or the related
 118 Freudenthal triangulations, see [35, 40, 31, 51], which can be defined for different values of
 119 D while keeping T constant. In this paper, we will thus think of all the above quantities as
 120 well as d and n as constants except D and our results will hold for D small enough.

121 The result

122 We are going to construct an ambient isotopy based on (1). The zero set of $F_{PL}(x, 0)$
 123 (or $f(x)$) gives the smooth isosurface, while the zero set of $F_{PL}(x, 1)$ (or $f_{PL}(x)$) gives
 124 the PL approximation, that is the triangulation of the isosurface after possible barycentric
 125 subdivision. The map $\tau \mapsto \{x \mid F_{PL}(x, \tau) = 0\}$ in fact gives an isotopy. Without too much
 126 extra work we'll also bound the Fréchet distance between $f(x)$ and $f_{PL}(x)$.

130 Proving isotopy consists of two technical steps, which consume most of the space in the
 131 proof below, as well as the use of a standard observation from Morse theory/gradient flow
 132 in the third step. The technical steps are

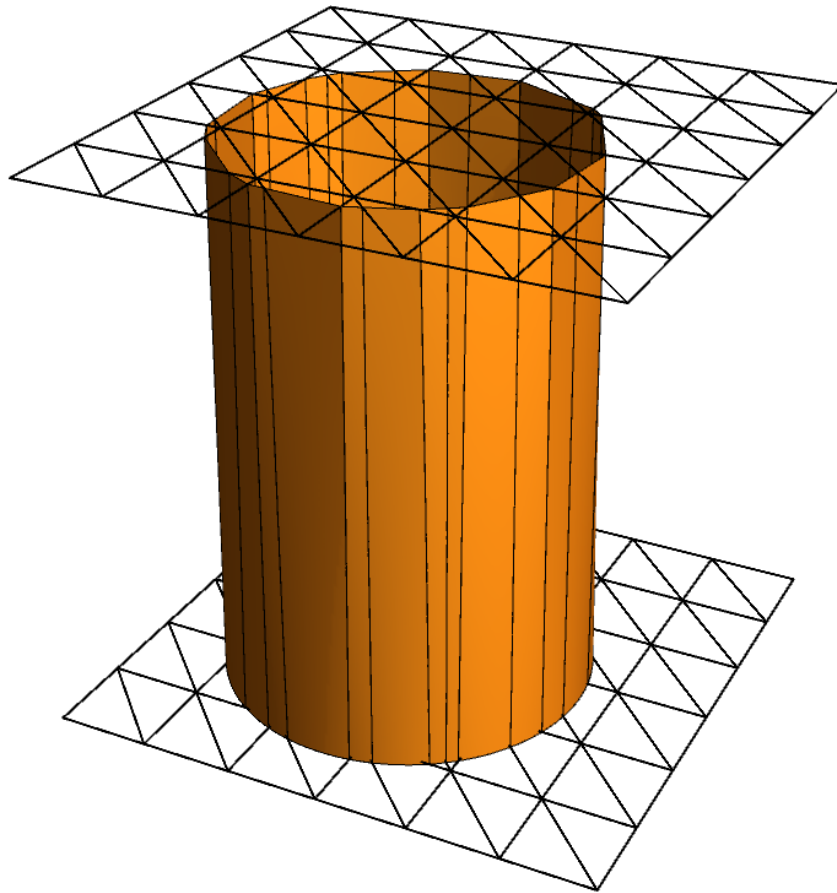
- 133 ■ Let $\sigma \in \mathcal{T}$. We first show that $\{x \mid F_{PL}(x, \tau) = 0\} \cap \sigma \times [0, 1]$ is a smooth manifold,
 134 under certain conditions.
- 135 ■ We prove that $F_{PL}^{-1}(0)$ is a manifold, under certain conditions, using techniques from
 136 nonsmooth analysis.

137 Along the way we shall also see that $F_{PL}^{-1}(0)$ is never tangent to the $\tau = c$ planes, where c
 138 is a constant. This implies that the gradient field of τ restricted to $F_{PL}^{-1}(0)$, is piecewise
 139 smooth and never vanishes.

140 Now we arrive at the third step, which is similar to a standard observation in Morse theory
 141 [42, 43], with the exception that we now consider piecewise-smooth instead of smooth vector
 142 fields. We refer to Milnor [42] for an excellent introduction, see Lemma 2.4 and Theorem
 143 3.1 in particular.

144 ► **Lemma 2** (Gradient flow induced homeomorphisms). *The flow of a non-vanishing piecewise-*
 145 *smooth gradient vector field of a function τ on a compact manifold generates a homeomorph-*
 146 *ism from $\tau = c_1$ to $\tau = c_2$, where c_1 and c_2 are constants.*

147 Bounds on the gradient of τ on the manifold give a bound on the Fréchet distance.



127 **Figure 1** A pictorial overview of the proof. The τ -direction goes upwards. Similarly to Morse
 128 theory we find that $f_{P_L}^{-1}(0)$ (top) and $f^{-1}(0)$ (bottom) are homeomorphic if the function τ restricted
 129 to $F_{P_L}^{-1}(0)$ does not encounter a Morse critical point.

148 2.1 Estimates for a single simplex

149 We now first concentrate on a single simplex σ and write f_L for the linear function whose val-
 150 ues on the vertices of σ coincide with f , that is f_L is the linear extension of the interpolation
 151 of f .

152 2.1.1 Preliminaries and variations of known results

153 We need a simple estimate similar to Proposition 2.1 of Allgower and George [1].

154 ► **Lemma 3.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then $|f_L^i(x) - f^i(x)| \leq 2D^2\alpha$*
 155 *for all $x \in \sigma$.*

156 We will also be using an estimate similar to Proposition 2.2 of Allgower and George [1].

157 ► **Proposition 4.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then*

$$158 \quad |\text{grad} f_L^i - \text{grad} f^i| = \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T},$$

159 *for all $x \in \sigma = \{v_k\}$.*

2.1.2 Estimates on the gradient inside a single simplex

We write

$$F_L(x, \tau) = (1 - \tau)f(x) + \tau f_L(x). \quad (7)$$

We note that F_L extends smoothly outside σ . Here and throughout we restrict ourselves to the setting where $\tau \in [0, 1]$. The function F_L has \mathbb{R}^{d-n} as image.

We can now state the following

► **Lemma 5.** *If we write $\text{grad}_{(x, \tau)}$ for the gradient that includes the τ component, we have*

$$|\det(\text{grad}_{(x, \tau)}(F_L^i) \cdot \text{grad}_{(x, \tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \quad (8)$$

with $g_1(D) = \mathcal{O}(D)$. See (18) in Appendix A for the exact expression of g_1 .

From the previous statement we immediately have that

► **Corollary 6** ($F_L^{-1}(0)$ is a manifold in a neighbourhood of $\sigma \times [0, 1]$). *If $\gamma_0 > g_1(D)$ the implicit function theorem applies to $F_L(x, \tau)$ inside $\sigma \times [0, 1]$. (In fact it applies to an open neighbourhood of this set). In particular, we have proven the first of our two technical steps, $\{x \mid F_{PL}(x, \tau) = 0\} \cap \sigma \times [0, 1]$ is a smooth manifold.*

2.1.3 Transversality with regard to the τ -direction

We will also prove the main result which we need for the third step, that is the gradient of τ restricted to $F_{PL}(x, \tau) = 0$, is piecewise smooth and never vanishes. We now prove inside each $\sigma \times [0, 1]$ the gradient of τ on $F_L = 0$ is smooth and does not vanish.

We first give a simple lower bound on the lengths of vectors v^1, \dots, v^{d-n} , assuming that the norms $|v^i|$ are upper bounded and the determinant of the Gram matrix is lower bounded.

► **Lemma 7.** *Let $v^1, \dots, v^{d-n} \in \mathbb{R}^d$, $|v^i| \leq \gamma_1$, for all i , and assume that $\det(v^i \cdot v^j)_{i,j} > \gamma_0$. Then $|v^i| \geq \sqrt{\gamma_0}/\gamma_1^{d-n-1}$.*

We also need to bound the angle of the vectors $\text{grad}_{(x, \tau)}(F_L^i)$ and the x plane, that is $\mathbb{R}^d \subset \mathbb{R}^{d+1}$. We recall the definition. If $v \in \mathbb{R}^{d+1}$ is a vector and $\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$, is the space spanned by the d basis vectors corresponding to the x -directions, the angle between v and Ξ is $\angle(v, \Xi) = \inf_{w \in \Xi} \angle(v, w)$.

► **Lemma 8.** *Let Ξ be as above. We have*

$$\tan \angle(\text{grad}_{(x, \tau)}(F_L^i), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

In particular the manifold $F_L^{-1}(0)$ inside $\sigma \times [0, 1]$ is never tangent to the $\tau = c$ planes, where c is a constant.

Combining Lemma 8 and Corollary 6 gives:

► **Corollary 9.** *If $\gamma_0 > g_1(D)$, and $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, then inside each $\sigma \times [0, 1]$ the gradient of τ on $F_L^{-1}(0)$ is smooth and does not vanish.*

2.2 Global result

We are now going to prove the global result. For this, we need to recall some definitions and results from non-smooth analysis. We refer to [22] for an extensive introduction.

Definition 10 (Generalized Jacobian, Definition 2.6.1 of [22]). Let $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d-n}$, where F is assumed to be just Lipschitz. The generalized Jacobian of F at x_0 denoted by $J_F(x_0)$, is the convex hull of all $(d-n) \times (d+1)$ -matrices B obtained as the limit of a sequence of the form $J_F(x_i)$, where $x_i \rightarrow x_0$ and F is differentiable at x_i .

Following [22, page 253] we also define:

Definition 11. The generalized Jacobian $J_F(x_0)$ is said to be of maximal rank provided every matrix in $J_F(x_0)$ is of maximal rank.

Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of \mathbb{R}^{d+1} by (x, y) accordingly. Fix a point (a, b) , with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. We now write:

Notation 12 ([22, page 256]). $J_F(x_0, y_0)|_y$ is the set of all $(n+1) \times (n+1)$ -matrices M such that, for some $(n+1) \times (d-n)$ -matrix N , the $(n+1) \times (d+1)$ -matrix $[N, M]$ belongs to $J_F(x_0, y_0)$.

With these definitions and notations we now have:

Theorem 13 (The generalized implicit function theorem [22, page 256]). Suppose that $J_F(a, b)|_y$ is of maximal rank. Then there exists an open set $U \subset \mathbb{R}^{n+1}$ containing a such that there exists a Lipschitz function $g : U \rightarrow \mathbb{R}^{d-n}$, such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.

We recall the definition of F_{PL} ,

$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x). \quad (1)$$

Because of the definition of α , see (4), and Proposition 4, we have that $\text{grad}_{(x, \tau)} F_{PL}(x, \tau)$ and $\text{grad}_{(x, \tau)} F_{PL}(\tilde{x}, \tau)$ are close if x and \tilde{x} are. In particular,

Lemma 14. Let v be a vertex in \mathcal{T} , $x_1, x_2 \in \text{star}(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that $\text{grad}_{(x, \tau)} F_{PL}^i(x_1, \tau_1)$ and $\text{grad}_{(x, \tau)} F_{PL}^i(x_2, \tau_2)$ are well defined, then

$$|\text{grad}_{(x, \tau)} F_{PL}^i(x_1, \tau_1) - \text{grad}_{(x, \tau)} F_{PL}^i(x_2, \tau_2)| \leq \frac{10d^2 D \alpha}{T} + 4\gamma_1 D + 4D^2 \alpha$$

We now immediately have the same bound on points in the convex hull of a number of such vectors:

Corollary 15. Suppose we are in the setting of Lemma 14 and $x_0, x_1, \dots, x_m \in \text{star}(v)$, $\tau_0, \dots, \tau_m \in [0, 1]$, and suppose that μ_1, \dots, μ_m are positive weights such that $\mu_1 + \dots + \mu_m = 1$ then,

$$\left| \text{grad}_{(x, \tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x, \tau)} F_{PL}^i(x_k, \tau_k) \right| \leq \frac{10d^2 D \alpha}{T} + 4\gamma_1 D + 4D^2 \alpha.$$

Using Lemma 5 we see

227 ► **Lemma 16.** *Let v be a vertex in \mathcal{T} , $x_1, \dots, x_m \in \text{star}(v)$, and $\tau_1, \dots, \tau_m \in [0, 1]$, such*
 228 *that $\text{grad}_{(x, \tau)} F_{PL}^i(x_k, \tau_k)$, $k = 0, \dots, m$ are well defined. If we moreover assume $D \leq 1$, and*
 229 *$\frac{6dD\alpha}{T} \leq \gamma_1$ we have that*

$$230 \quad \left| \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x, \tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x, \tau)} F_{PL}(x_k, \tau_k) \right) \right) \right|_{i,j} \geq \gamma_0 - g_2(D),$$

231 *with $g_2(D) = \mathcal{O}(D)$. See (19) in Appendix A for the exact expression of g_2 .*

232 From the previous lemma we immediately have that

233 ► **Corollary 17** ($\{x \mid F_{PL}(x, \tau) = 0\}$ is a manifold). *If $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, and $\gamma_0 > g_2(D)$ the*
 234 *generalized implicit function theorem, Theorem 13, applies to $F_{PL}(x, \tau) = 0$. In particular,*
 235 *$\{x \mid F_{PL}(x, \tau) = 0\}$ is a manifold.*

236 We notice that this bound is stronger than the bound in Corollary 6. This means that
 237 $\{x \mid F_{PL}(x, \tau) = 0\}$ is a Piecewise-Smooth manifold if the conditions of Corollary 17 hold.
 238 The second technical step of the proof is now also completed.

239 The fact that $F_L(x, \tau) = 0$ is a Piecewise-Smooth manifold and Corollary 9 give that
 240 the gradient of τ is a Piecewise-Smooth vector field whose flow we can integrate to give a
 241 homeomorphism from the zero set of f to that of f_{PL} .

242 We summarize in a theorem:

243 ► **Theorem 18.** *If, $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, and $\gamma_0 > g_2(D)$ then the zero*
 244 *set of f is isotopic to the zero set of f_{PL} . We stress that one can satisfy all conditions by*
 245 *choosing D sufficiently small.*

246 2.2.1 Fréchet distance

247 To bound the Fréchet distance (d_F) between the zero sets of $f(x)$ and f_{PL} , it suffices to
 248 bound the angle that the gradient of τ (as restricted to $F_L(x, \tau) = 0$) makes with the
 249 (ambient) τ -direction.

250 For this we will use the angle bound of Lemma 8, together with some estimates that are
 251 similar in spirit to those in [7, Lemma C.13].

252 ► **Lemma 19.** *Let $v^1, \dots, v^{d-n} \in \mathbb{R}^{d+1}$, $|v^i| \leq \tilde{\gamma}_1$, for all i , and assume that $\det(v^i \cdot$
 253 $v^j)_{i,j} > \tilde{\gamma}_0 > 0$. Let e_τ a unit vector. If for all i , $\cos(\angle v^i, e_\tau) \leq \phi_0$, then for any $w \in$
 254 $\text{span}(v^1, \dots, v^{d-n})$*

$$255 \quad \cos \angle(w, e_\tau) \leq \frac{(d-n)d^{d-n-1}\phi_0\tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

256 Let e_τ be the τ direction and let g_τ be the gradient of τ restricted to $F_{PL}^{-1}(0)$, whenever it
 257 exists. We want to bound the angle of g_τ and the τ -direction. Because the homeomorphism
 258 is given by the gradient flow, this bounds the Fréchet distance.

259 There is one subtlety, because the manifold is only Piecewise-Smooth, we need to take
 260 into account the points where g_τ is not uniquely defined. Because for each simplex σ , F_L
 261 extends to a neighbourhood of $\sigma \times [0, 1]$, there exists a limit of $g_\tau(x_i, \tau_i)$ for any sequence
 262 (x_i, τ_i) that lies in $\text{int}(\sigma) \times [0, 1]$, where int denotes the interior. This means that if we bound
 263 g_τ for each simplex we also bound its limits, where the limits are as just described.

264 We are now ready to combine Lemmas 8, 19, and Theorem 18.

► **Corollary 20** (Bound on the Fréchet distance). *Suppose that the conditions of Theorem 18 are satisfied. Then, $d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D)$, with $g_3(D) = \mathcal{O}(D^2)$, where we think of $\gamma_0, \gamma_1, d, n, T$ and α as constants. See (19) in Appendix A for the exact expression of g_2 .*

The most important thing to observe is that $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$, so that we find that $d_F(f^{-1}(0), f_{PL}^{-1}(0)) = \mathcal{O}(D^2)$, where we think of $\gamma_0, \gamma_1, d, n, T$ and α as constants.

3 Isomanifolds with boundary

We'll now consider isomanifolds with boundary. By this we mean that on top of the function $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$, we'll have another function $f_\partial : \mathbb{R}^d \rightarrow \mathbb{R}$ and the set we consider is $M = f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$. This is a manifold with boundary if the gradients of f^i span a $(d-n)$ -dimensional space at each point of $f^{-1}(0)$ and the gradients of f^i and f_∂ span a $(d-n+1)$ -dimensional space at each point of $\partial M = f^{-1}(0) \cap f_\partial^{-1}(0)$, as a consequence of the submersion theorem.

We'll again write f_{PL} for the PL interpolation of f . Similarly we write $f_{\partial, PL}$ for the PL interpolation of f_∂ .

We prove that, under certain conditions, there is an isotopy from $f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$ to $f_{PL}^{-1}(0) \cap f_{\partial, PL}^{-1}([0, \infty))$. The conditions are very similar to the conditions we have before, but of course we need to include bounds on the gradient of $f_{\partial, PL}$.

Overview of the proof

We will again construct an isotopy, but in this case it will consist of two steps.

- In the **first step**, we isotope the part of $f^{-1}(0)$ that is far from $f_\partial^{-1}(0)$ to its piecewise linear approximation, while leaving the part of $f^{-1}(0)$ that is close to $f_\partial^{-1}(0)$ smooth. We will denote the result by $M_1 = (F_{PL,1}(\cdot, 1))^{-1}(0)$.
- In the **second step**, we consider a (small) tubular neighbourhood around $f_\partial^{-1}(0)$ as restricted to M_1 by looking at all $f_\partial^{-1}(\epsilon)$ for $|\epsilon|$ sufficiently small.¹ We then isotope $M_1 \cap f_\partial^{-1}(\epsilon)$ to its piecewise linear approximation. Again the isotopy is chosen in such a way that for ϵ relatively large (for the points such that M_1 is already Piecewise-Linear) it leaves $M_1 \cap f_\partial^{-1}(\epsilon)$ invariant. This gives an isotopy of a tubular neighbourhood of $M_1 \cap f_\partial^{-1}(0)$ to its Piecewise-Linear approximation.

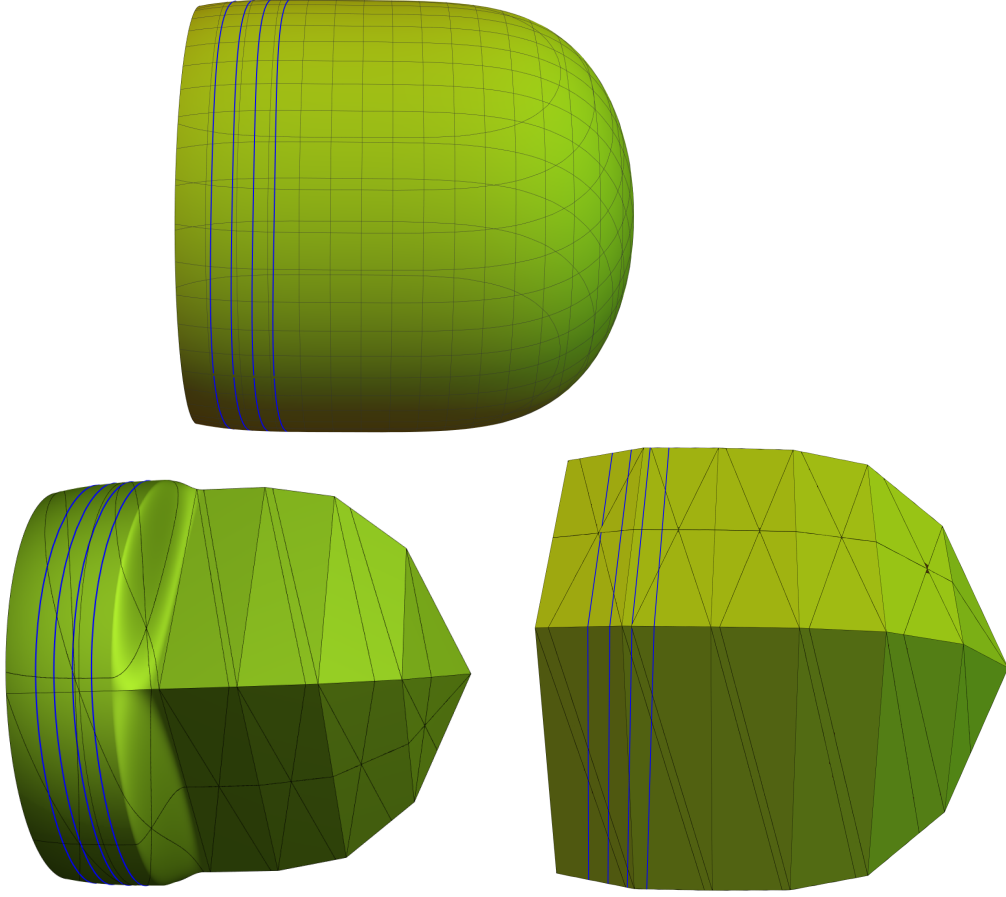
We will first partition the manifold in two parts using a smooth bump function $\phi : \mathbb{R} \rightarrow [0, 1]$ that is zero in a neighbourhood of zero and $\phi(y) = 1$ if $|y| > y_0$, for some $y_0 > 0$. Such bump functions can be easily constructed, see for example [39, Section 2.2]. We will be using the function $\phi(\sum_i (f^i)^2 + f_\partial^2)$.

The first step will be using the zero set of the following function:

$$F_{PL,1}(x, \tau) = \left(1 - \tau \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right)\right) f(x) + \tau \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) f_{PL}(x), \quad (9)$$

on which we'll apply the same gradient flow argument as before.

¹ We stress that ϵ may be negative.



295 **Figure 2** Top: we see the original isosurface with $f_{\partial}^{-1}(-1/10)$, $f_{\partial}^{-1}(0)$, $f_{\partial}^{-1}(1/10)$, and $f_{\partial}^{-1}(2/10)$
 296 indicated in blue. Bottom left: we see that at the end of Step 1 the neighbourhood of the boundary
 297 is intact, while the rest has been isotoped to a Piecewise-Linear approximation. Bottom right:
 298 we have also isotoped the neighbourhood of the boundary to a Piecewise-Linear approximation by
 299 isotoping $f_{\partial}^{-1}(\epsilon)$, to its Piecewise-Linear approximation for all sufficiently small ϵ .

307 The resulting set M_1 is the same zero set of f_{PL} as before if we stay sufficiently far
 308 away from ∂M and the isotopy leaves the manifold invariant close to ∂M . In particular,
 309 $\partial M_1 = \partial M$.

310 In the second step, we define an isotopy that will act only on a small neighbourhood of
 311 ∂M . Consider the sets $B_1(\epsilon) = M_1 \cap f_{\partial}^{-1}(\epsilon)$ and, for each of them, define the function

$$\begin{aligned}
 312 \quad F_{PL,2,\epsilon}(x, \tau) &= \left(1 - \tau \psi \left(\sum_i (f^i)^2 + f_{\partial}^2 \right) \right) (F_{PL,1}(x, 1), f_{\partial}(x) - \epsilon) \\
 313 \quad &\quad + \tau \psi \left(\sum_i (f^i)^2 + f_{\partial}^2 \right) (f_{PL}(x), f_{\partial, PL}(x) - \epsilon), \tag{10}
 \end{aligned}$$

314 where $\psi : \mathbb{R} \rightarrow [0, 1]$ is now a smooth bump function that is 1 in a sufficiently large neighbour-
 315 hood of zero (somewhat larger than y_0) and zero outside some compact set. We stress that
 316 $F_{PL,2,\epsilon}$ is a mapping from $\mathbb{R}^d \times [0, 1]$ to \mathbb{R}^{d-n+1} . Using the result for isomanifolds (with some
 317 modifications), we can prove that each individual set $B_1(\epsilon)$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial, PL}^{-1}(\epsilon)$

for small ϵ while, for sufficiently large ϵ , it leaves the set invariant.

3.1 Step 1

The proof closely follows the proof for the case without boundary in Section 2. The main technical difficulty will be to provide bounds that serve as the counterparts of Lemmas 5 and 16. To be able to do so we first need to discuss bounds on the bump functions ϕ and ψ .

3.1.1 Bump functions

Following [39, Section 2.2] we write,

$$\zeta_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

For $0 < y_1 < y_2$ we write $\zeta_2(x) = \zeta_1(x - y_1)\zeta_1(y_2 - x)$. Then we define $\phi_l : \mathbb{R} \rightarrow [0, 1]$ by $\phi_l(x) = \int_x^{y_2} \zeta_2(x') dx' / \int_{y_1}^{y_2} \zeta_2(x') dx'$. Finally define $\phi_b : \mathbb{R} \rightarrow [0, 1]$ by $\phi_b(x) = \phi_l(|x|)$, and let $\phi(x) = 1 - \phi_b(x)$.

► **Lemma 21.** *We have $\phi_b(x) \in [0, 1]$ and, writing $2y_1 = y_2 = y_0$,*

$$\partial_x(\phi_l(x)) \leq 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} = 4 \frac{e^{\frac{2}{3y_0}}}{y_0} = \gamma_\phi. \quad (11)$$

3.1.2 Inside a single simplex

Similarly to Lemma 5, we now give a condition that ensures that the zero set of $F_{PL,1}^i(x, \tau)$ is smooth inside $\sigma \times [0, 1]$. In fact, similarly to (7), we define

$$\begin{aligned} F_{L,1}^i(x, \tau) &= \left(1 - \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right)\right) f^i(x) + \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) f_L^i(x) \\ &= f(x) + \tau \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)), \end{aligned}$$

where ϕ is as defined above. Observe that $F_{L,1}^i(x, \tau)$ can be extended to a neighbourhood of $\sigma \times [0, 1]$.

► **Remark.** For the constants, it is better if y_0 can be chosen as large as possible, but we need y_1 to be quite a bit larger than y_0 . In turn, we cannot choose y_1 arbitrarily large because this would mean that the gradient field $\text{grad} f_\partial|_{f^{-1}(0)}$ (seen as restricted on $f^{-1}(0)$) would never vanish. The latter is in general impossible thanks to the hairy ball theorem [17].

We introduce the following definition that complements Definition 1:

► **Definition 22.**

$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \text{grad} \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| = 2 \sup_{x \in \Sigma_0} \left| \sum_l f^l \text{grad} f^l + f_\partial \text{grad} f_\partial \right| \quad (12)$$

We have then the analog of Lemma 5:

► **Lemma 23.** *We have :*

$$|\det(\text{grad}_{(x,\tau)} F_{L,1}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x,\tau))_{i,j}| > \gamma_0 - g_4(D),$$

with $g_4(D) = \mathcal{O}(D)$. The exact expression of g_4 is given in (21).

The following corollary is then the analog of Corollary 6:

► **Corollary 24** ($F_{L,1}^{-1}(0)$ is a manifold). *If $\gamma_0 > g_4(D)$, where $g_4(D) = \mathcal{O}(D)$ is as in Lemma 23, then $F_{L,1}^{-1}(0)$ is a smooth manifold inside an ϵ neighbourhood of $\sigma \times [0, 1]$.*

3.1.3 Transversality with regard to the τ -direction

We note that, similarly to Lemma 8, we have

► **Lemma 25.** *Let Ξ be as in Lemma 8 and γ_ϕ as in (11).*

$$\tan \angle(\text{grad}_{(x,\tau)}(F_{L,1}), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \gamma_2\gamma_\phi 2D^2\alpha - \frac{4dD\alpha}{T}}.$$

In particular, if $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$, $F_{L,1}^{-1}(0)$ (if it is a manifold) is never tangent to the $\tau = c$ planes, where c is a constant.

Now, similarly to Corollary 9, we find that

► **Corollary 26** (Transversality with respect to τ for Step 1). *Suppose that $\gamma_0 > g_4(D)$ and that $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$. Then, inside each $\sigma \times [0, 1]$, the gradient of τ on $F_{L,1}^{-1}(0)$ is smooth and does not vanish.*

3.1.4 Global result

We now have to prove that $F_{PL,1}^{-1}(0)$ is a manifold. For this, we shall use a bound similar to the one given in Lemma 16, so that we are able to apply the generalized implicit function theorem if this bound is satisfied. But first of all, we need the following bound, which is similar to Lemma 14.

► **Lemma 27.** *Assuming that the gradients are well defined, we have*

$$|\text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2)| \leq g_5(D),$$

with $g_5(D) = \mathcal{O}(D)$. The exact expression for g_5 is given in (22).

Just as in Corollary 15, we immediately have the same bound on points in the convex hull of a number of such vectors:

► **Corollary 28.** *Suppose we are in the setting of Lemma 27 and $x_0, x_1, \dots, x_m \in \text{star}(v)$, $\tau_0, \dots, \tau_m \in [0, 1]$, such that $\text{grad}_{(x,\tau)} F_{PL,1}^i(x_i, \tau_i)$ is well defined for all i . Further assume that μ_1, \dots, μ_m are positive weights such that $\mu_1 + \dots + \mu_m = 1$. Then,*

$$\left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \leq g_5(D).$$

373 ► **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$374 \quad \det \left(\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$375 \quad \geq \gamma_0 - g_4(D) - g_6(D),$$

376 *with $g_6(D) = \mathcal{O}(D)$. The exact expression of g_6 is given in (23).*

377 Lemma 29 immediately yields that

378 ► **Corollary 30** ($F_{PL,1}^{-1}(0)$ is a manifold). *If, $\gamma_0 > g_4(D) + g_6(D)$ the generalized implicit*
 379 *function theorem, Theorem 13, applies to $F_{PL,1}(x, \tau) = 0$. In particular $F_{PL,1}^{-1}(0)$ is a*
 380 *manifold.*

381 We stress again that inside the set $\{x | \phi(\sum_i (f^i)^2(x) + f_\partial^2(x)) = 1\}$ the zero set of
 382 $F_{PL,1}(x, 1)$ coincides with the zero set of $f_{PL}(x)$.

383 3.2 Step 2

384 Before we can proceed we have to specify the bump function ψ . We suppose that

$$385 \quad \psi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{101}{100}y_0 \\ 0 & \text{if } |x| \geq 2y_0. \end{cases}$$

386 In particular we pick $\psi(x) = \phi_b(x)$, with the choice $y_1 = \frac{101}{100}y_0$ and $y_2 = 2y_0$.

387 First we stress that the zero set of $F_{PL,2,\epsilon}(x, 1)$ coincides with the zero set of $(f_{PL}(x), f_{\partial,PL}(x) -$
 388 $\epsilon)$, provided that $\psi(\sum_i f_i(x)^2 + f_\partial(x)^2) = 1$.

389 Secondly, we now claim the following:

390 ► **Lemma 31.** *The zero set of $F_{PL,2,\epsilon}(x, 1)$ is a subset of the zero set of $f_{PL}(x)$, for each ϵ .*

391 The technical result that remains to be proven is the counterpart of Theorem 18 for
 392 $F_{PL,2,\epsilon}(x, \tau)$ and for each sufficiently small ϵ . To be precise it suffices for $\epsilon \leq 2y_0$. We
 393 remark that it is likely that this bound on ϵ can be improved.

394 We again follow the same path to prove this result. That is we first concentrate on a
 395 single simplex and prove that inside that simplex the zero set of $F_{PL,2,\epsilon}$ is a smooth manifold
 396 on which the gradient of τ as restricted to the manifold does not vanish. We then prove
 397 that is the zero set of $F_{PL,2,\epsilon}$ is globally a manifold.

398 3.2.1 Assumptions and notations

399 Because we are now faced with both $f(x)$ and $f_\partial(x)$ we need to introduce a bound on how
 400 far the gradients of all there are from being colinear. We write

$$401 \quad f_B(x) = (f(x), f_\partial(x)). \quad (13)$$

402 Before we were only interested in the set Σ_0 , similarly here we sometimes concentrate
 403 on a neighbourhood of the zero set of both f_∂ and f . Therefore we write B_ν for all $\sigma \in \mathcal{T}$
 404 such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$.

405 We define γ_0^B in terms of the determinant of the Gram matrix of the gradients, that is

$$406 \quad \gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\operatorname{grad}(f_B^i) \cdot \operatorname{grad}(f_B^j))_{i,j}|. \quad (14)$$

We note that because we take the gradients we can just ignore the ϵ constant. For the lengths of the gradients of f_B we define,

$$\gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f_B^i)|, \quad (15)$$

for all $1 \leq i \leq d - n + 1$. Similarly to α , we define β as the bound on the operator 2-norm of all Hessians of f_B , that is

$$\beta = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2. \quad (16)$$

We stress that we have chosen our definitions such that $\alpha \leq \beta$.

We use the same notation for the ambient triangulation \mathcal{T} , the lower bound on the thickness of the simplices T and upper bound on the longest edge length D . We also need to introduce a bound on the differential of the bump function ψ . Similarly to (11) we define,

$$\gamma_\psi = 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} = 2 \frac{e^{\frac{4}{3(2y_0 - \frac{101}{100}y_0)}}}{2y_0 - \frac{101}{100}y_0} = \frac{200}{99} \frac{e^{\frac{400}{297y_0}}}{y_0}, \quad (17)$$

because we picked $y_1 = \frac{101}{100}y_0$ and $y_2 = 2y_0$, for ψ .

3.2.2 Inside a single simplex

Similarly to Lemma 23 we now give a conditions that ensure that the zero set of $F_{PL,2,\epsilon}(x, \tau)$ is smooth inside $\sigma \times [0, 1]$. In fact similarly to (7) we define

$$\begin{aligned} F_{L,2,\epsilon}(x, \tau) = & \left(1 - \tau\psi \left(\sum_i (f^i)^2 + f_\partial^2\right)\right) (F_{L,1}(x, 1), f_\partial(x) - \epsilon) \\ & + \tau\psi \left(\sum_i (f^i)^2 + f_\partial^2\right) (f_L(x), f_{\partial,L}(x) - \epsilon), \end{aligned}$$

which can be extended to a neighbourhood of $\sigma \times [0, 1]$.

► **Lemma 32.** *For all ϵ ,*

$$\det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau))_{i,j} \geq \gamma_0^B - g_7(D)$$

with $g_7(D) = \mathcal{O}(D)$. The exact expression of g_7 is given in (24).

We again have the following corollary.

► **Corollary 33** ($F_{L,2,\epsilon}^{-1}(0)$ is a manifold). *We have that $F_{L,2,\epsilon}^{-1}(0)$ is a smooth manifold inside an small neighbourhood of $\sigma \times [0, 1]$ provided $\gamma_0^B > g_7(D)$, with $g_7(D)$ as in Lemma 32. As usual this can always be satisfied by choosing the triangulation fine enough, that is D sufficiently small.*

3.2.3 Transversality with regard to the τ -direction

Once more similarly to Lemma 8 we have

► **Lemma 34.** *Let Ξ be as in Lemma 8. We have*

$$\tan \angle(\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}), \Xi) \leq \frac{2D^2\beta}{\sqrt{\gamma_0^B/(\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T}}}.$$

437 In particular the manifold $F_{L,2,\epsilon}^{-1}(0)$ inside $\sigma \times [0, 1]$, if well defined, is never tangent to the
 438 $\tau = c$ planes, where c is a constant, if

$$439 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T}.$$

440 Now similarly to Corollary 9 we find that

441 ► **Corollary 35** (Transversality with respect to τ for Step 2). Suppose that the conditions of
 442 Corollary 33 are satisfied. If moreover

$$443 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T},$$

444 then inside each $\sigma \times [0, 1]$ the gradient of τ on $F_{L,2,\epsilon}^{-1}(0)$ is smooth and does not vanish.

445 3.2.4 Global result

446 We now have to prove that $F_{PL,2,\epsilon}^{-1}(0)$ is a manifold, for all sufficiently small ϵ . For this we
 447 shall use a bound similar to the one given in Lemma 16, so that we are able to apply the
 448 generalized implicit function theorem if this bound is satisfied. For this we first need the
 449 following bound, which is similar to Lemma 27.

450 ► **Lemma 36.** Let v be a vertex in \mathcal{T} , $x_1, x_2 \in \text{star}(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that
 451 $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1)$ and $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)$ are well defined, then

$$452 \quad |\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)| \leq g_8(D),$$

453 with $g_8(D) = \mathcal{O}(D)$. The exact expression of g_8 is given in (25).

454 Just as in Corollary 15, we immediately have the same bound on points in the convex
 455 hull of a number of such vectors:

456 ► **Corollary 37.** Suppose $x_0, x_1, \dots, x_m \in \text{star}(v)$, $\tau_0, \dots, \tau_m \in [0, 1]$, such that $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_i, \tau_i)$
 457 is well defined for all i . Further assume that μ_1, \dots, μ_m are positive weights such that
 458 $\mu_1 + \dots + \mu_m = 1$. Then,

$$459 \quad \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right| \leq g_8(D).$$

460 ► **Lemma 38.** Under the same conditions as in Corollary 37,

$$461 \quad \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$462 \quad \geq \gamma_0^B - g_7(D) - g_9(D),$$

463 where $g_9(D) = \mathcal{O}(D)$. The exact expression of g_9 is given in (26).

464 Lemma 38 immediately yields that

465 ► **Corollary 39** (The generalized implicit function theorem in Step 2). If, $\gamma_0^B > g_7(D) +$
 466 $g_9(D)$ the generalized implicit function theorem, Theorem 13, applies to $F_{PL,1}(x, \tau) = 0$. In
 467 particular $F_{PL,1}^{-1}(0)$ is a manifold.

468 We stress that this condition only needs to be satisfied in a when $\sum_l (f^l)^2 + (f_\partial)^2$ is small,
469 outside this neighbourhood the isotopy leaves the zero set invariant.

470 ► **Theorem 40.** *If,*

$$471 \quad \sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T} \quad (\text{Corollary 26})$$

$$472 \quad \gamma_0 > g_4(D) + g_6(D) \quad (\text{Corollaries 24 and 30})$$

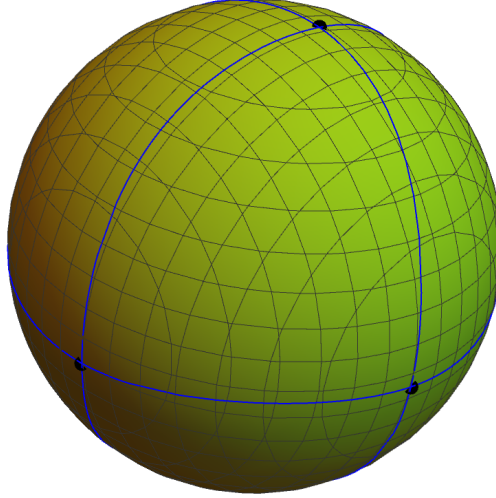
$$473 \quad \sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \quad (\text{Corollary 35})$$

$$474 \quad \gamma_0^B > g_7(D) + g_9(D), \quad (\text{Corollaries 33 and 39})$$

475 *then $f^{-1}(0) \cap f_\partial^{-1}([0, \infty))$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial, PL}^{-1}([0, \infty))$. We stress that one can satisfy*
476 *all conditions by choosing D sufficiently small. We refer to Appendix A for the definitions*
477 *of $g_i(D)$.*

478 4 Isostratifolds

479 There is no obstruction in principle that prevents us from extending the approach above
480 to isostratifolds. By isostratifolds we mean stratifolds that are given by the zero sets of
481 functions. For example suppose that we want to find a PL approximation of the unit sphere
482 centred at 0 in \mathbb{R}^3 including the PL approximations of the intersections of the sphere with
483 slightly deformed $x = 0$, $y = 0$, and $z = 0$ -planes, as depicted in Figure 3. This would also
484 give PL approximations of the respective ‘octants’ of the sphere.



485 ■ **Figure 3** An example of an isostratifold.

486 We could follow the same procedure as for a manifold with boundary to give density con-
487 ditions that ensure that the PL approximation is correct. However this would mean that we
488 have to introduce an extra bump function for each stratum as well as an extra isotopy. Even
489 though that in theory this should be possible the analysis becomes prohibitively lengthy.

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A Overview of constants

We give an overview. We write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i . We write B_ν for all $\sigma \in \mathcal{T}$ such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$. We write

$$f_B(x) = (f(x), f_\partial(x)) \quad (13)$$

$$\gamma_0 = \inf_{x \in \Sigma_0} |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| \quad (2)$$

$$\gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\text{grad}(f_B^i) \cdot \text{grad}(f_B^j))_{i,j}| \quad (14)$$

$$\gamma_1 = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f^i)| \quad (3)$$

$$\gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\text{grad}(f_B^i)| \quad (15)$$

$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \text{grad} \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| = \sup_{x \in \Sigma_0} \left| \sum_l f^l \text{grad} f^l + f_\partial \text{grad} f_\partial \right| \quad (12)$$

$$\alpha = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f^i)\|_2 = \sup_x \max_i \|(\partial_k \partial_l f^i)_{k,l}\|_2 \quad (4)$$

$$\beta = \sup_{x \in \Sigma_0} \max_i \|\text{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2 \quad (16)$$

D : the longest edge length of a simplex in Σ_0

T : the smallest thickness of a simplex in Σ_0 .

$\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$ is the space spanned by the d basis vectors corresponding to the x -directions.

The bump functions give rise to the following:

$$\gamma_\phi = 4 \frac{e^{\frac{2}{3y_0}}}{y_0} \quad (11)$$

$$\gamma_\psi = \frac{200}{99} e^{\frac{400}{297y_0}}. \quad (17)$$

The precise expressions for the $g_i(D)$ are:

$$g_1(D) = n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right) \quad (18)$$

$$g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right) \quad (19)$$

$$g_3(D) = \left(\frac{(d-n)d^{d-n-1} \sqrt{\gamma_0} 2D^2 \alpha \gamma_1^{d-n-1} (\gamma_1 + 2D^2 \alpha)^{d-n}}{\sqrt{\gamma_0 + (2D^2 \alpha \gamma_1^{d-n-1})^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right) \quad (20)$$

$$g_4(D) = n \left(\left(n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \cdot n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right) \quad (21)$$

$$g_5(D) = 2\sqrt{d} d \alpha D + 2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \quad (22)$$

$$g_6(D) = n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1}$$

$$\begin{aligned}
& \cdot \left(2 \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right. \\
& \left. + \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \right). \tag{23}
\end{aligned}$$

$$\begin{aligned}
g_7(D) = & (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
& \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
& \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \tag{24}
\end{aligned}$$

$$g_8(D) = 2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \tag{25}$$

$$\begin{aligned}
g_9(D) = & (n+1)^{n+1} \left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
& \cdot \left(2 \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right. \\
& \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
& \left. + \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \right) \tag{26}
\end{aligned}$$

If $\frac{4dD\alpha}{T} \leq \gamma_1$, $g_1(D)$ can be replaced by the simpler $34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}$.

657 **B** Recap of essential standard results

658 We now quickly recall two essential results, namely the smooth implicit function theorem
659 and Friedland's determinant bound.

660 ► **Theorem 41** (Smooth implicit function theorem). *Let $F : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d-n}$ be a continuously*
661 *differentiable function. Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of \mathbb{R}^{d+1} by*
662 *(x, y) accordingly. Fix a point (a, b) , with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. If the Jacobian $J_{F,y}(a, b) =$*
663 *$(\frac{\partial F^i}{\partial y^j}(a, b))_{i,j}$ is of maximal rank (or equivalently invertible), then there exists an open set*
664 *$U \subset \mathbb{R}^{n+1}$ containing a such that there exists a unique continuously differentiable function*
665 *$g : U \rightarrow \mathbb{R}^{d-n}$ such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.*

666 The following result by Friedland [36] will also be used throughout the proofs:

$$667 \quad |\det(A + E) - \det(A)| \leq n \max\{\|A\|_p, \|A + E\|_p\}^{n-1} \|E\|_p, \quad (27)$$

668 where A and E are $n \times n$ -matrices and $\|\cdot\|_p$ is the p -norm, with $1 \leq p \leq \infty$.

C Proofs

C.1 Proofs of variations of known results in Section 2.1

The following proofs are not exactly the same as the original, but do not differ greatly from known results. We have included the results for completeness.

Proof of Lemma 2. This is a straightforward consequence of the existence and uniqueness of the solution to a differential equation. ◀

The following two proofs are similar to the proofs in Allgower and George [1], however because the statement are a bit different and for completeness we have included these.

Proof of Lemma 3. Taylor's theorem, see for example [29, Theorem 2.8.4], yields that

$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \quad (28)$$

with

$$\begin{aligned} R(v_k) &= 2 \sum_{j \neq l} (v_k - x)^j (v_k - x)^l \int_0^1 (1-t)^2 \partial_j \partial_l f^i(v_k - t(v_k - x)) dt \\ &\quad + 2 \sum_j \frac{((v_k - x)^j)^2}{2} \int_0^1 (1-t)^2 \partial_j^2 f^i(v_k - t(v_k - x)) dt \\ &\leq 2|v_k - x|^2 \alpha \quad (\text{by (4) and Cauchy-Schwarz}) \\ &\leq 2D^2 \alpha \quad (x \in \sigma) \end{aligned}$$

The function f_L at the point $x = \sum_k \lambda_k v_k$, where $\sum_k \lambda_k = 1$, is by construction

$$\begin{aligned} f_L^i(x) &= \sum_k \lambda_k f^i(v_k) \\ &= \sum_k \lambda_k \left(f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k) \right) \quad (\text{by (28)}) \\ &= \sum_k \lambda_k f^i(x) + \sum_j \partial_j f^i(x) \left(\sum_k \lambda_k v_k - x \right)^j + \sum_k \lambda_k R(v_k) \\ &= f^i(x) + 0 + \sum_k \lambda_k R(v_k) \end{aligned}$$

Thanks to the bounds on $R(v_k)$ and Cauchy-Schwarz one has

$$|f_L^i(x) - f^i(x)| \leq 2D^2 \alpha$$

Proof of Proposition 4. We again use that

$$f^i(v_k) = f^i(x) + \sum_j \partial_j f^i(x)(v_k - x)^j + R(v_k), \quad (28)$$

with

$$|R(v_k)| \leq 2D^2 \alpha \quad (29)$$

696 Subtracting $f^i(v_l)$ from $f^i(v_k)$ now yields

$$697 \quad f^i(v_k) - f^i(v_l) = \sum_j \partial_j f^i(x)(v_k - v_l)^j + R(v_k) - R(v_l).$$

698 Because f_L is the linear interpolation of f , we have $f^i(v_k) - f^i(v_l) = \sum_j \partial_j f_L^i(x)(v_k - v_l)$,
699 and thus

$$700 \quad \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(v_k - v_l)^j \right| \leq |R(v_k) - R(v_l)| \leq 4D^2\alpha.$$

701 Let now $u = \mu_k v^k$ and $w = \tilde{\mu}_k v_k$, with $\sum \mu_k = \sum \tilde{\mu}_k = 1$, that is $u, w \in \sigma$. Clearly
702 $\sum \mu_k - \tilde{\mu}_k = 0$, so $u - w$ can be developed in a sum such that the m -th term in the sum is
703 $c_m(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})$, with $c_m > 0$, and $\sum |c_m| = 1$. We stress that $v_{\tilde{k}(m)}$ may be the same as
704 $v_{\tilde{k}(m')}$, with $m \neq m'$. We now see that

$$\begin{aligned} 705 \quad \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(u - w)^j \right| &= \left| \sum_{j,m} (\partial_j f_L^i(x) - \partial_j f^i(x))c_m(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right| \\ 706 \quad &\leq \sum_m c_m \left| \sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))(v_{\tilde{k}(m)} - v_{\tilde{l}(m)})^j \right| \\ &\quad \text{(by the triangle inequality and } c_m > 0) \\ 707 \quad &\leq \sum_m 4c_m D^2\alpha \quad \text{(because } \sum |c_m| = 1) \\ 708 \quad &= 4D^2\alpha. \end{aligned} \tag{30}$$

709 Because the simplex σ contains a ball of radius the smallest altitude over d centred at its
710 barycentre, that is TD/d with T the thickness, the vector $u - w$ can be chosen to be any
711 vector of length less than tD/d . In particular we can choose

$$712 \quad (u - w)^j = \frac{tD}{d} \frac{(\partial_j f_L^i(x) - \partial_j f^i(x))}{\sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2}}.$$

713 Plugging this choice into (30) gives

$$714 \quad \frac{tD}{d} \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq 4D^2\alpha$$

715 So that

$$716 \quad \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T}.$$

717 ◀

718 C.2 Proofs of new results in of Section 2.1

719 ► **Lemma 5.** Let $\text{grad}(f^i) = (\partial_j f_i)_j$ denote the gradients of the components f^i . Suppose
720 that the absolute value of the determinant of the Gram matrix of these gradients is lower
721 bounded, that is

$$722 \quad |\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

and $|\text{grad}(f^i)| \leq \gamma_1$, for all i . Assume further that the conditions of Proposition 4 are satisfied, then

$$|\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| > \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right). \quad (31)$$

If we write $\text{grad}_{(x,\tau)}$ for the gradient that includes the τ component, we find that

$$|\det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \quad (8)$$

where $g_1(D) = \mathcal{O}(D)$, precisely

$$g_1(D) = n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right). \quad (32)$$

Proof of Lemma 5. In this proof we shall be using the following notation:

$$(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j} = (Y)_{i,j},$$

and write $Y_{i,j}$ for the entries of the matrix.

By definition of F_L , we have that

$$\begin{aligned} \det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j} \\ &= \det(\text{grad}((1-\tau)f^i(x) + \tau f_L(x)^i) \cdot \text{grad}((1-\tau)f^j(x) + \tau f_L(x)^j))_{i,j} \\ &= \det(\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x))))_{i,j} \end{aligned}$$

We shall write

$$(X)_{i,j} = (\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}$$

Proposition 4 now yields that

$$\begin{aligned} |X_{i,j} - Y_{i,j}| \\ &= |\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x))) \\ &\quad - \text{grad}(f^i(x)) \cdot \text{grad}(f^j(x))| \\ &= |\text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x)) + \text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x)) \\ &\quad + \text{grad}(\tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(\tau(f_L(x)^j - f^j(x)))| \\ &\leq 2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2. \end{aligned} \quad (33)$$

This estimate is used to bound the $\|E\|_p$ term in Friedland's bound, for $p = \infty$. In particular,

$$\|(X - Y)_{i,j}\|_\infty \leq n \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right). \quad (34)$$

Furthermore,

$$\begin{aligned} |X_{i,j}| &= |\text{grad}(f^i(x) + \tau(f_L(x)^i - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L(x)^j - f^j(x)))| \\ &\leq \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2, \end{aligned} \quad (35)$$

where we used Cauchy-Schwarz. Note that the latter estimate is used to bound the $\max\{\|A\|_p, \|A + E\|_p\}$ term in Friedland's bound, for $p = \infty$. In particular,

$$\| (X)_{i,j} \|_\infty \leq n \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2. \quad (36)$$

754 This means that Friedland's bound now yields

$$|\det(X)_{i,j} - \det(Y)_{i,j}| \leq n(\|(X)_{i,j}\|_\infty)^{n-1} \|(X - Y)_{i,j}\|_\infty. \quad (37)$$

756 Thus,

$$\begin{aligned} & |\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j}| \geq |\det(\text{grad}(f^i(x)) \cdot \text{grad}(f^j(x)))_{i,j}| \\ & \quad - n^{n+1} \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right) \\ & > \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^{2(n-1)} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 \right). \end{aligned}$$

The addition of the τ component gives a small extra contribution to the inner product to the gradients

$$\begin{aligned} \tilde{X}_{i,j} &= \text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j \\ &= \text{grad}(f^i(x) + \tau(f_L^i(x) - f^i(x))) \cdot \text{grad}(f^j(x) + \tau(f_L^j(x) - f^j(x))) \\ &\quad + (f_L^i(x) - f^i(x))(f_L^j(x) - f^j(x)). \end{aligned}$$

765 Combining this with (33) and Lemma 3 yields

$$\begin{aligned} |\tilde{X}_{i,j} - Y_{i,j}| &= |\text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j - \text{grad}(f^i(x)) \cdot \text{grad}(f^j(x))| \\ &\leq 2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \end{aligned}$$

768 Similarly to (35), we also have that

$$|\tilde{X}_{i,j}| = |\text{grad}_{(x,\tau)} F_L^i \cdot \text{grad}_{(x,\tau)} F_L^j| \leq \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2. \quad (38)$$

770 Applying Friedland's bound again gives

$$\begin{aligned} & |\det(\tilde{X})_{i,j}| = \left| \det(\text{grad}_{(x,\tau)}(F_L^i) \cdot \text{grad}_{(x,\tau)}(F_L^j))_{i,j} \right| \\ & > \gamma_0 - n^{n+1} \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \cdot \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right) \\ & \geq \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right). \end{aligned}$$

(because $T \leq 1$)

774

775 ► **Corollary 42.** *If on top of the conditions of Lemma 5, we also assume that $\frac{4dD\alpha}{T} \leq \gamma_1$,*
776 *the bound (31) can be simplified, at the cost of weakening the bound, to*

$$\det(\text{grad}(F_L^i) \cdot \text{grad}(F_L^j))_{i,j} > \gamma_0 - 3 \cdot 2^{2n+1} n^{n+1} (\gamma_1)^{2n-1} \frac{dD\alpha}{T}.$$

778 The bound (8) can be simplified, at the cost of weakening the bound, to

$$779 \quad |\det(\operatorname{grad}_{(x,\tau)}(F_L^i) \cdot \operatorname{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

780 **Proof of Corollary 42.** If we assume $\frac{4dD\alpha}{T} \leq \gamma_1$, a short calculation shows that (31) reduces
781 to

$$782 \quad |\det(\operatorname{grad}(F_L^i) \cdot \operatorname{grad}(F_L^j))_{i,j}| > \gamma_0 - n^{n+1} (2\gamma_1)^{2(n-1)} \left(3\gamma_1 \frac{4dD\alpha}{T}\right) \\ 783 \quad = \gamma_0 - 3 \cdot 2^{2n+1} n^{n+1} (\gamma_1)^{2n-1} \frac{dD\alpha}{T}.$$

784 A straightforward calculation shows (8) reduces to

$$785 \quad \det(\tilde{X})_{i,j} \geq \gamma_0 - n^{n+1} \left(\frac{5}{2}\gamma_1\right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \frac{3}{2}\gamma_1 \frac{6dD\alpha}{T}\right) \\ 786 \quad = \gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

787

788 **Proof of Lemma 7.** Let $A_1 = \operatorname{span}(v^1, \dots, v^{d-n})$. Write v_A^i for the projection of v^i on A_1 .
789 Note that the vector v_A^i is $d-n$ -dimensional. By construction,

$$790 \quad \det(v^i \cdot v^j)_{i,j} = \det(v_A^i \cdot v_A^j)_{i,j} = \det((v_A^i)_i^t (v_A^j)_j) = \det((v_A^i)_i)^2,$$

791 where $(v_A^i)_i$ denotes the $(d-n) \times (d-n)$ -matrix whose columns are v_A^i , and $(v_A^i)_i^t$ its
792 transposed. Moreover, $|v^i| = |v_A^i|$. Hadamard's inequality [38] now gives

$$793 \quad \det((v_A^j)_j) \leq \prod_j |v_A^j| \leq \gamma_1^{d-n-1} |v_A^i|.$$

794 Using that $\det(v^i \cdot v^j)_{i,j} > \gamma_0$ now yields $\sqrt{\gamma_0}/\gamma_1^{d-n-1} \leq |v_A^i|$. ◀

795 **Proof of Lemma 8.** The statement follows fairly straightforwardly from Lemma 7 and Lemma 3.
796 In particular the τ component of $\operatorname{grad}_{(x,\tau)}(F_L^i)$ is bounded in absolute value by $|f_L(x)^i -$
797 $f^i(x)| \leq 2D^2\alpha$. On the other hand the x component is lower bounded by

$$798 \quad \sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T},$$

799 as a consequence of Proposition 4 and Lemma 7. This gives that

$$800 \quad \tan \angle(\operatorname{grad}_{(x,\tau)}(F_L^i), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

801

802 C.3 Proofs of Section 2.2

803 **Proof of Lemma 14.** Using the standard bounds on matrix norm, see for example [37,
804 Equation (2.3.11)], and (4), we have that $\sqrt{d}\alpha \geq |\partial_k \partial_l f^i|$, for all k, l and i . Because the

supremum of the absolute value of a derivative of a function bounds the Lipschitz constant of the function, we have that

$$|\partial_l f^i(x_1) - \partial_l f^i(x_2)| \leq \sqrt{d}\alpha |x_1 - x_2| \quad (39)$$

$$|f^i(x_1) - f^i(x_2)| \leq \gamma_1 |x_1 - x_2|. \quad (40)$$

Note furthermore that

$$|x_1 - x_2| \leq 2D, \quad (41)$$

by the triangle inequality.

Lemma 3 gives that $|f_{PL}^i(x) - f^i(x)| \leq 2D^2\alpha$. Using the triangle inequality we now find that

$$\begin{aligned} |f_{PL}^i(x_1) - f_{PL}^i(x_2)| &\leq |f_{PL}^i(x_1) - f^i(x_1) + f^i(x_1) - f^i(x_2) + f^i(x_2) - f_{PL}^i(x_2)| \\ &\leq 2\gamma_1 D + 4D^2\alpha \end{aligned} \quad (42)$$

Because

$$\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) = \begin{pmatrix} \text{grad}(f^i(x) + \tau_1(f_{PL}^i(x) - f^i(x))) \\ f_{PL}^i(x) - f^i(x) \end{pmatrix},$$

we find that

$$\begin{aligned} &|\text{grad}_{(x,\tau)} F_{PL}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL}^i(x_2, \tau_2)| \\ &\leq \sum_l |\partial_l f^i(x_1) - \partial_l f^i(x_2)| \\ &\quad + \sum_l \tau_1 |\partial_l f_{PL}^i(x_1) - \partial_l f^i(x_1)| + \sum_l \tau_2 |\partial_l f_{PL}^i(x_2) - \partial_l f^i(x_2)| \\ &\quad + |f_{PL}^i(x_1) - f^i(x_1) - f_{PL}^i(x_2) + f^i(x_2)| \\ &\leq 2\sqrt{d}d\alpha D \quad (\text{by (39), and (41)}) \\ &\quad + d \frac{8dD\alpha}{T} \quad (\text{Proposition 4, and } \tau_1, \tau_2 \in [0, 1]) \\ &\quad + 2\gamma_1 D + 4D^2\alpha + 2\gamma_1 D \quad (\text{by (40), (41) and (42)}) \\ &\leq 2\sqrt{d}d\alpha D + d \frac{8dD\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \\ &\leq \frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha, \end{aligned}$$

where we used the triangle inequality several times. ◀

► **Lemma 16.** *Suppose that*

$$|\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

and $|\text{grad}(f^i)| \leq \gamma_1$, for all i . Let now v be a vertex in \mathcal{T} , $x_1, \dots, x_m \in \text{star}(v)$, and $\tau_1, \dots, \tau_m \in [0, 1]$, such that $\text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k)$, $k = 0, \dots, m$ are well defined. Then,

$$\left| \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right) \right|_{i,j}$$

$$\begin{aligned}
&\geq \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right) \\
&- n^{n+1} \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \\
&\cdot \left(2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \right. \\
&\left. + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right)^2 \right).
\end{aligned}$$

If we moreover assume $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$, the expression can be simplified, at the cost of weakening the bound, to

$$\left| \det \left(\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right)_{i,j} \right| \geq \gamma_0 - g_2(D),$$

with $g_2(D) = \mathcal{O}(D)$, in fact

$$g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right).$$

Proof of Lemma 16. Let $x_0 \in \operatorname{star}(v)$ and $\tau_0 \in [0, 1]$, be such that $\operatorname{grad}_{(x,\tau)} F_L^i(x_0, \tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior of a d -simplex in \mathcal{T} . Lemma 5 gives that

$$\begin{aligned}
|\det(\tilde{X})_{i,j}| &= |\det(\operatorname{grad}_{(x,\tau)}(F_L^i(x_0, \tau_0)) \cdot \operatorname{grad}_{(x,\tau)}(F_L^j(x_0, \tau_0)))_{i,j}| \\
&> \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right).
\end{aligned}$$

We want to use this together with Corollary 15 and Friedland's bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (38) also holds for the convex hull, that is

$$\left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right| \leq \left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2.$$

We write

$$Z_{i,j} = \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right).$$

Similarly to (36), we have that

$$\|(Z)_{i,j}\|_\infty \leq n \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right) \quad (43)$$

We note that

$$\left| \operatorname{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right| = \left| \begin{pmatrix} \operatorname{grad}(f^i(x) + \tau(f_{PL}^i(x) - f^i(x))) \\ f_{PL}^i(x) - f^i(x) \end{pmatrix} \right|$$

$$\begin{aligned}
&\leq \gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \\
&\quad (\text{because } \text{grad}(f^i) \leq \gamma_1, \text{ Proposition 4, and Lemma 3}) \\
&\tag{44}
\end{aligned}$$

Corollary 15 now gives

$$\begin{aligned}
|Z_{i,j} - \tilde{X}_{i,j}| &= \left| \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right. \\
&\quad \left. - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
&= \left| \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) + \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right) \right. \\
&\quad \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) - \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) + \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right) \\
&\quad \left. - \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
&\leq \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) \right| \\
&\quad + \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right| \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) \right| \\
&\quad + \left| \text{grad}_{(x,\tau)} F_{PL}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right| \\
&\quad \cdot \left| \text{grad}_{(x,\tau)} F_{PL}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right| \\
&\leq 2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
&\quad (\text{by Corollary 15 and (44)}) \\
&\quad + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \\
&\quad (\text{by Corollary 15}) \\
&\tag{45}
\end{aligned}$$

Similarly to (34), we have that

$$\begin{aligned}
\|(Z - \tilde{X})_{i,j}\|_\infty &\leq n2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \\
&\quad + n \left(\frac{10d^2D\alpha}{T} + 4\gamma_1D + 4D^2\alpha \right)^2 \\
&\tag{46}
\end{aligned}$$

Indeed, Friedland now gives,

$$|\det(Z)_{i,j}| = \left| \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL}^j(x_k, \tau_k) \right) \right)_{i,j} \right|$$

$$\begin{aligned}
&\geq \gamma_0 - n^{n+1} \left(\gamma_1 + \frac{6dD\alpha}{T} \right)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \left(\frac{6dD\alpha}{T} \right)^2 \right) \quad (\text{by Lemma 5}) \\
&\quad - n^{n+1} \left(\left(\gamma_1 + \frac{4dD\alpha}{T} \right)^2 + (2D^2\alpha)^2 \right)^{n-1} \quad (\text{by 43}) \\
&\quad \cdot \left(2 \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right) \left(\gamma_1 + \frac{4dD\alpha}{T} + 2dD^2\alpha \right) \right. \\
&\quad \left. + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right)^2 \right) \\
&\hspace{15em} (\text{by 46, note that } n \text{ was factored out})
\end{aligned}$$

If we further assume that $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$, we can simplify the bound to

$$\begin{aligned}
|\det(Z)_{i,j}| &\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left(\frac{6dD\alpha}{T} \right) \right) \\
&\quad - n^{n+1} \left((2\gamma_1)^2 + (2D\alpha)^2 \right)^{n-1} \\
&\quad \cdot \left(2 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D \right) (2\gamma_1 + 2dD\alpha) + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha \right)^2 \right) \\
&\hspace{15em} (\text{because } D \leq 1, \text{ and } T \leq 1) \\
&\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left(\frac{6dD\alpha}{T} \right) \right) \\
&\quad - n^{n+1} (5\gamma_1^2)^{n-1} \\
&\quad \cdot \left(2 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D \right) (2\gamma_1 + \gamma_1) + \left(\frac{10d^2D\alpha}{T} + 4\gamma_1 D + \gamma_1 D \right)^2 \right) \\
&\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \left(2\gamma_1 \frac{4dD\alpha}{T} + \gamma_1 \left(\frac{6dD\alpha}{T} \right) \right) \\
&\quad - n^{n+1} (5\gamma_1^2)^{n-1} \left(6\gamma_1 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D \right) + \left(\frac{10d^2D\alpha}{T} + 5\gamma_1 D \right)^2 \right) \\
&\geq \gamma_0 - n^{n+1} (2\gamma_1)^{2n-1} \gamma_1 \left(\frac{14dD\alpha}{T} \right) \\
&\quad - n^{n+1} (5\gamma_1^2)^{n-1} \left(6\gamma_1 \left(\frac{14d^2D\alpha}{T} + 4\gamma_1 D \right) + (2d\gamma_1 + 5\gamma_1 D) \left(\frac{10d^2D\alpha}{T} + 5\gamma_1 D \right) \right) \\
&\geq \gamma_0 - n^{n+1} 2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) \\
&\quad - n^{n+1} 5^{n-1} \gamma_1^{2n-2} \left((2d\gamma_1 + 5\gamma_1) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right) \\
&\geq \gamma_0 - n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right)
\end{aligned}$$

► **Corollary 17.** If $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, and

$$\gamma_0 > g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14dD\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2D\alpha}{T} + 9\gamma_1 D \right) \right)$$

the generalized implicit function theorem, Theorem 13, applies to $F_L(x, \tau) = 0$.

► **Theorem 18.** Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^{d-n}$ be a smooth function. Write f_{PL} for the Piecewise-Linear function that is the linear interpolation of the values of f at the vertices, if restricted to a single simplex $\sigma \in \mathcal{T}$. Write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i . Suppose that Σ_0 is compact. Suppose that for in each simplex in Σ_0 we have that

- $|\text{grad}(f^i)| \leq \gamma_1$, for all i .
- $|\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0$, for all i, j .

If moreover, $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, and

$$\gamma_0 > g_2(D) = n^{n+1} \left(2^{2n-1} \gamma_1^{2n} \left(\frac{14d\alpha}{T} \right) + 5^{n-1} \gamma_1^{2n-1} (2d+5) \left(\frac{24d^2\alpha}{T} + 9\gamma_1 \right) \right) D,$$

then the zero set of f is isotopic to the zero set of f_{PL} .

Proof of Lemma 19. Write

$$w = \mu_1 v^1 + \dots + \mu_{d-n} v^{d-n},$$

with $\mu_1, \dots, \mu_{d-n} \in \mathbb{R}$. By definition,

$$\cos \angle(w, e_\tau) = \frac{\sum_i \mu_i v^i \cdot e_\tau}{|w|} = \frac{\sum_i \mu_i v^i \cdot e_\tau}{\sqrt{\sum_{i,j} \mu_i \mu_j v^i \cdot v^j}}.$$

We note that $|\sum_i \mu_i v^i \cdot e_\tau| \leq (d-n)|\mu| \tilde{\gamma}_1 \phi_0$, where μ denotes the vector (μ_i) .

The matrix $(v^i \cdot v^j)_{i,j}$ is a symmetric positive definite matrix, because it is a Gram matrix and its determinant strictly positive, by assumption. This means that the smallest eigenvalue λ_{\min} of $(v^i \cdot v^j)_{i,j}$, gives a lower bound, that is

$$\sum_{i,j} \mu_i \mu_j v^i \cdot v^j \geq \lambda_{\min} |\mu|^2.$$

The determinant of a square matrix is the product of its eigenvalues. The largest eigenvalue is bounded by the operator norm, which in turn is bounded by the Frobenius norm [37, (2.3.7)], denoted by $\|\cdot\|_F$. This gives that the largest eigenvalue λ_{\max} of $(\langle v^i, v^j \rangle)$ is bounded:

$$\lambda_{\max} \leq \| (v^i \cdot v^j) \|_2 \leq \| (v^i)^t \|_2 \| (v^i)_i \|_2 \leq \| (v^i)_i \|_F^2 \leq d^2 \tilde{\gamma}_1^2.$$

For the smallest eigenvalue we now find that

$$\tilde{\gamma}_0 < \det(v^i \cdot v^j)_{i,j} \leq \lambda_{\max}^{d-n-1} \lambda_{\min} \leq d^{2(d-n-1)} \tilde{\gamma}_1^{2(d-n-1)} \lambda_{\min}.$$

Combining these results gives

$$\cos \angle(w, e_\tau) = \frac{\sum_i \mu_i \langle v^i, e_\tau \rangle}{\sqrt{\sum_{i,j} \mu_i \mu_j \langle v^i, v^j \rangle}} \leq \frac{(d-n)|\mu| \tilde{\gamma}_1 \phi_0}{|\mu| \frac{\sqrt{\tilde{\gamma}_0}}{d^{d-n-1} \tilde{\gamma}_1^{d-n-1}}} = \frac{(d-n) d^{d-n-1} \phi_0 \tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}$$

◀

► **Corollary 20.** Suppose that the conditions of Theorem 18 are satisfied. Then,

$$d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D),$$

with $g_3(D) = \mathcal{O}(D^2)$, in fact

$$g_3(D) = \left(\frac{(d-n) d^{d-n-1} \sqrt{\gamma_0} 2D^2 \alpha \gamma_1^{d-n-1} (\gamma_1 + 2D^2 \alpha)^{d-n}}{\sqrt{\gamma_0 + (2D^2 \alpha \gamma_1^{d-n-1})^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2}\right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right).$$

931 **Proof of Corollary 20.** Lemma 8 gives

$$932 \quad \tan \angle(\operatorname{grad}_{(x,\tau)}(F^i), \Xi) \leq \frac{2D^2\alpha\gamma_1^{d-n-1}}{\sqrt{\gamma_0}},$$

933 and thus using the notation of Lemma 19,

$$\begin{aligned} 934 \quad \cos(\angle \operatorname{grad}_{(x,\tau)}(F^i), e_\tau) &\leq \sin \arctan \left(\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}} \right) \\ 935 \quad &= \frac{\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}}{\sqrt{1 + \left(\frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}} \right)^2}} \\ 936 \quad &= \frac{2D^2\alpha}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2}}. \end{aligned}$$

937 By Corollary 42

$$938 \quad \tilde{\gamma}_0 = \gamma_0 - 34 \cdot \left(\frac{5}{2} \right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}.$$

939 Lemma 3 and $|\operatorname{grad}(f^i)| \leq \gamma_1$ give

$$940 \quad \tilde{\gamma}_1 = \gamma_1 + 2D^2\alpha.$$

941 Plugging this into Lemma 19 gives

$$942 \quad \cos \angle(w, e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2} \right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}},$$

943 with $w \in \operatorname{span}_i(\operatorname{grad}_{(x,\tau)}(F^i))$. Because the tangent space to $F_L = 0$ is normal to $\operatorname{span}_i(\operatorname{grad}_{(x,\tau)}(F_L^i))$,
944 we see that

$$945 \quad \sin \angle(g_\tau, e_\tau) \leq \frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2} \right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}}.$$

946 This means that the angle between g_τ and e_τ is very small (we can choose D arbitrarily
947 small). This means that as $\tau \in [0, 1]$ the distance between begin and end point of the
948 gradient flow, and thus the Fréchet distance, by $\tan \angle(g_\tau, e_\tau)$, that is

$$\begin{aligned} 949 \quad &d_F(f^{-1}(0), f_{PL}^{-1}(0)) \\ 950 \quad &\leq \tan \arcsin \left(\frac{(d-n)d^{d-n-1}2D^2\alpha(\gamma_1 + 2D^2\alpha)^{d-n}}{\sqrt{(\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T})^2 + (2D^2\alpha)^2} \sqrt{\gamma_0 - 34 \cdot \left(\frac{5}{2} \right)^{2n-1} n^{n+1} \gamma_1^{2n} \frac{dD\alpha}{T}}} \right) \end{aligned}$$

951 ◀

952 C.4 Proofs of Section 3.1

953 **Proof of Lemma 21.** As mentioned, by construction $\phi_b(x) \in [0, 1]$. Because $\partial_x \beta \left(\frac{y_1 + y_2}{2} \right) =$
954 0 and this is the only zero of the derivative in the open interval (y_1, y_2) , we see that

$$955 \quad \beta(x) \leq \beta \left(\frac{y_1 + y_2}{2} \right) = e^{\frac{4}{y_1 - y_2}}.$$

Hence, because $\beta(x) \geq 0$,

$$\int_{y_1}^{y_2} \beta(x) \leq (y_2 - y_1) e^{\frac{4}{y_1 - y_2}}$$

Because $\beta(x)$ is monotone on $[y_1, \frac{y_1 + y_2}{2}]$ we also have

$$\int_{y_1}^{y_2} \beta(x) \geq \frac{y_2 - y_1}{2} \beta\left(\frac{3}{4}y_1 + \frac{1}{4}y_2\right) = \frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}$$

We now have

$$\begin{aligned} \partial_x(\phi_l(x)) &= \partial_x \left(\int_x^{y_2} \beta(x') dx' \middle/ \int_{y_1}^{y_2} \beta(x') dx' \right) \\ &= \beta(x) / \int_{y_1}^{y_2} \beta(x') dx' \\ &\leq \frac{e^{\frac{4}{y_1 - y_2}}}{\frac{y_2 - y_1}{2} e^{\frac{16}{3(y_1 - y_2)}}} \\ &= 2 \frac{e^{\frac{4}{y_1 - y_2} - \frac{16}{3(y_1 - y_2)}}}{y_2 - y_1} \\ &= 2 \frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} \end{aligned} \tag{DefCphi}$$

◀

► **Lemma 23.** *Suppose that*

$$|\det(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j}| > \gamma_0,$$

and $|\text{grad}(f^i)| \leq \gamma_1$, for all i and that the bound (12) is satisfied. Then,

$$\det(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j} > \gamma_0 - g_4(D),$$

with $g_4(D) = \mathcal{O}(D)$, in fact,

$$\begin{aligned} g_4(D) &= n \left(\left(n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right. \\ &\quad \cdot n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \Big), \end{aligned}$$

Proof of Lemma 23. We start with an estimate on the individual $\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau)$

$$\begin{aligned} &\left| \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \right| \\ &= \left| \begin{pmatrix} \text{grad} f^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} \left(f(x) + \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right) \right| \\ &= \left| -\text{grad}_{(x,\tau)} \left(\tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right) \right| \end{aligned}$$

$$\begin{aligned}
&= \left| \begin{pmatrix} -\tau \operatorname{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) - \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \operatorname{grad}(f_L^i(x) - f^i(x)) \\ \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \end{pmatrix} \right| \\
&= \left(\left| \tau \operatorname{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) + \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \operatorname{grad}(f_L^i(x) - f^i(x)) \right|^2 \right. \\
&\quad \left. + \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right)^2 \right)^{1/2} \\
&\leq \left| \tau \operatorname{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L^i(x) - f^i(x)) \right| + \left| \tau \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \operatorname{grad}(f_L^i(x) - f^i(x)) \right| \\
&\quad + \left| \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^i(x) - f^i(x)) \right| \quad (\text{by the triangle inequality}) \\
&\leq \gamma_2 \gamma_\phi |f_L^i(x) - f^i(x)| + \left| \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \operatorname{grad}(f_L^i(x) - f^i(x)) \right| \\
&\quad \quad \quad (\text{because } \tau \leq 1, (12), (11)) \\
&\quad + |f_L^i(x) - f^i(x)| \quad (\text{because } \phi \in [0, 1]) \\
&\leq \gamma_2 \gamma_\phi |f_L^i(x) - f^i(x)| + |\operatorname{grad}(f_L^i(x) - f^i(x))| \quad (\text{because } \phi \in [0, 1]) \\
&\quad + |f_L^i(x) - f^i(x)| \\
&\leq (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \quad (\text{by Lemma 3 and Proposition 4}) \\
\end{aligned} \tag{47}$$

We now write

$$(X')_{i,j} = (\operatorname{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \operatorname{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j}.$$

Expanding yields

$$\begin{aligned}
(X')_{i,j} &= \left(\left(\operatorname{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \operatorname{grad} f^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \operatorname{grad} f^i(x) \\ 0 \end{pmatrix} \right) \\
&\quad \cdot \left(\operatorname{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \operatorname{grad} f^j(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \operatorname{grad} f^j(x) \\ 0 \end{pmatrix} \right) \Bigg|_{i,j} \\
&= \left(\left(\operatorname{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \operatorname{grad} f^i(x) \\ 0 \end{pmatrix} \right) + \begin{pmatrix} \operatorname{grad} f^i(x) \\ 0 \end{pmatrix} \right) \\
&\quad \cdot \left(\operatorname{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \operatorname{grad} f^j(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \operatorname{grad} f^j(x) \\ 0 \end{pmatrix} \right) \Bigg|_{i,j} \\
&= (Y)_{i,j} \\
&\quad + \left(\left(\operatorname{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \operatorname{grad} f^i(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \operatorname{grad} f^j(x) \\ 0 \end{pmatrix} \right) \Bigg|_{i,j} \\
&\quad + \left(\begin{pmatrix} \operatorname{grad} f^i(x) \\ 0 \end{pmatrix} \cdot \left(\operatorname{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \operatorname{grad} f^j(x) \\ 0 \end{pmatrix} \right) \right) \Bigg|_{i,j} \\
&\quad + \left(\left(\operatorname{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) - \begin{pmatrix} \operatorname{grad} f^i(x) \\ 0 \end{pmatrix} \right) \right. \\
\end{aligned}$$

$$\cdot \left(\text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau) - \begin{pmatrix} \text{grad} f^j(x) \\ 0 \end{pmatrix} \right)_{i,j}, \quad (48)$$

where we again used the notation

$$(\text{grad}(f^i) \cdot \text{grad}(f^j))_{i,j} = (Y)_{i,j}.$$

We now see by Cauchy-Schwarz and the triangle inequality that

$$|X'_{i,j} - Y_{i,j}| \leq 2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2,$$

by (47). This also means that for the operator norm of the matrix we find

$$\|(X' - Y)_{i,j}\|_\infty \leq n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \quad (49)$$

Similarly to (35) and (36), we also see that (47) gives

$$|X'_{i,j}| \leq \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2, \quad (50)$$

and thus,

$$\|X'_{i,j}\|_\infty \leq n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2. \quad (51)$$

Friedland's bound (37) now gives

$$\begin{aligned} & \det(\text{grad}_{(x,\tau)} F_{L,1}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,1}^j(x, \tau))_{i,j} \\ & > \gamma_0 - n \left(\left(n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \right. \\ & \quad \cdot \left. n \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right), \end{aligned}$$

where we used (49), and (51). ◀

Proof of Lemma 25. As we have seen in the calculation in (47), the τ component of $\text{grad}_{(x,\tau)}(F^i)$ is bounded in absolute value by $|f_L(x)^i - f^i(x)| \leq 2D^2 \alpha$. On the other hand the x component is lower bounded by

$$\sqrt{\gamma_0} / \gamma_1^{d-n-1} - \gamma_2 \gamma_\phi 2D^2 \alpha - \frac{4dD\alpha}{T},$$

as a consequence of Proposition 4 and the calculation in (47). This gives that

$$\tan \angle(\text{grad}_{(x,\tau)}(F_{L,1}), \Xi) \leq \frac{2D^2 \alpha}{\sqrt{\gamma_0} / \gamma_1^{d-n-1} - \gamma_2 \gamma_\phi 2D^2 \alpha - \frac{4dD\alpha}{T}}.$$

◀

1023 ► **Lemma 27.** *Under the same conditions as Lemma 14 we have*

$$1024 \quad |\text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2)|$$

$$1025 \quad \leq g_5(D) = 2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)$$

1026 **Proof of Lemma 27.** By expansion we see that

$$1027 \quad |\text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2)|$$

$$1028 \quad = \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} \right.$$

$$1029 \quad \left. + \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right|$$

$$1030 \quad \leq \left| \text{grad}_{(x,\tau)} F_{PL,1}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} \right| + \left| \begin{pmatrix} \text{grad} f^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} \right|$$

$$1031 \quad + \left| \begin{pmatrix} \text{grad} f^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,1}^i(x_2, \tau_2) \right| \quad (\text{by the triangle inequality})$$

$$1032 \quad \leq 2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \quad (\text{by (39), (41) and (47) twice})$$

1033

1034 ► **Lemma 29.** *Under the same conditions as in Lemma 16,*

$$1035 \quad \det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right)_{i,j}$$

$$1036 \quad \geq \gamma_0 - g_4(D) - g_6(D),$$

1037 with $g_6(D) = \mathcal{O}(D)$, where as usual we regard, α , γ_1 , γ_2 , γ_ϕ , d , n and T as constants. In

1038 fact,

$$1039 \quad g_4(D) = n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^{2(n-1)} \right.$$

$$1040 \quad \cdot \left. \left(2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right)$$

$$1041 \quad g_6(D) = n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1}$$

$$1042 \quad \cdot \left(2 \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right.$$

$$1043 \quad \left. + \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \right).$$

1044 **Proof of Lemma 29.** The proof is more or less the same as the proof of Lemma 16, but with

1045 more complicated bounds. Let $x_0 \in \text{star}(v)$ and $\tau_0 \in [0, 1]$, be such that $\text{grad}_{(x,\tau)} F^i(x_0, \tau_0)$

1046 is well defined. Note that it is sufficient for x_0 to lie in the interior of a d -simplex in \mathcal{T} .

1047 Lemma 23 gives that

$$1048 \quad |\det(\tilde{X}')_{i,j}| = |\det(\text{grad}_{(x,\tau)}(F^i(x_0, \tau_0)) \cdot \text{grad}_{(x,\tau)}(F^j(x_0, \tau_0)))_{i,j}|$$

$$\begin{aligned}
&> \gamma_0 - n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^{2(n-1)} \right. \\
&\quad \cdot \left. \left(2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right).
\end{aligned}$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give the bound we search for. However to do so we also need a bound on the operator norm, for this we notice that the bound (50) also holds for the convex hull, that is

$$\begin{aligned}
&\left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right| \\
&\leq \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2.
\end{aligned}$$

We write

$$Z'_{i,j} = \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right).$$

Similarly to (36), we have that

$$\|(Z')_{i,j}\|_\infty \leq n \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right)^2 \quad (52)$$

We note that

$$\begin{aligned}
&\left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \right| \leq \gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \\
&\quad \text{(because } \operatorname{grad}(f^i) \leq \gamma_1, \text{ and (47))} \\
&\quad \quad \quad (53)
\end{aligned}$$

Similarly to (45), we find that

$$\begin{aligned}
|Z'_{i,j} - \tilde{X}'_{i,j}| &= \left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right. \\
&\quad \left. - \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) \right| \\
&\leq \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) \right| \\
&\quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right| \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) \right| \\
&\quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right| \\
&\quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right| \\
&\leq 2 \left(2\sqrt{dd\alpha}D + 2 \left((1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2 \gamma_\phi) 2D^2 \alpha + \frac{4dD\alpha}{T} \right) \\
&\quad \text{(by Corollary 28 and (53))}
\end{aligned}$$

$$+ \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \quad (\text{by Corollary 28})$$

Similarly to (34), we have that

$$\begin{aligned} \|(Z' - \tilde{X}')_{i,j}\|_\infty &\leq 2n \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \\ &\quad + n \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \end{aligned} \quad (54)$$

Indeed, Friedland's bound now gives,

$$\begin{aligned} |\det(Z')_{i,j}| &= \left| \det \left(\left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,1}^j(x_k, \tau_k) \right) \right) \right|_{i,j} \\ &\geq \gamma_0 - n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^{2(n-1)} \right. \\ &\quad \cdot \left. \left(2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \gamma_1 + \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right) \right) \\ &\quad \quad \quad (\text{by Lemma 23}) \\ &\quad - n^{n+1} \left(\left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right)^2 \right)^{n-1} \quad (\text{by 52}) \\ &\quad \cdot \left(2 \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right) \left(\gamma_1 + (1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right. \\ &\quad \quad \left. + \left(2\sqrt{d}d\alpha D + 2 \left((1 + \gamma_2\gamma_\phi)2D^2\alpha + \frac{4dD\alpha}{T} \right) \right)^2 \right) \\ &\quad \quad \quad (\text{by 54, note that } n \text{ was factored out}) \end{aligned}$$

◀

C.5 Proofs of Section 3.2

Proof of Lemma 31. We focus on the first n coordinates of (10). We see that

$$\begin{aligned} &\left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) F_{PL,1}(x, 1) + \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) f_{PL}(x) \\ &= \left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \left(\left(1 - \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) f(x) + \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) f_{PL}(x) \right) \\ &\quad + \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) f_{PL}(x) \\ &= \left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \left(1 - \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) f(x) \\ &\quad + \left(\left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) + \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) f_{PL}(x) \\ &= \left(\left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) + \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) f_{PL}(x), \quad (55) \end{aligned}$$

1091 where we used that

$$1092 \quad 1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) = 0,$$

1093 if

$$1094 \quad \sum_i (f^i)^2 + f_\partial^2 \leq \frac{101}{100} y_0,$$

1095 and

$$1096 \quad \left(1 - \phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) = 0,$$

1097 if

$$1098 \quad \sum_i (f^i)^2 + f_\partial^2 \geq y_0.$$

1099 We can further rewrite (55),

$$\begin{aligned} 1100 & \left(\left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \left(\phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) - 1 + 1 \right) + \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) f_{PL}(x) \\ 1101 & = \left(\left(1 - \psi \left(\sum_i (f^i)^2 + f_\partial^2 \right) \right) \left(\phi \left(\sum_i (f^i)^2 + f_\partial^2 \right) - 1 \right) + 1 \right) f_{PL}(x) \\ 1102 & = f_{PL}(x), \end{aligned}$$

1103 where we used the same argument as before. ◀

1104 ► **Lemma 32.** For all ϵ ,

$$1105 \quad \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x,\tau))_{i,j} \geq \gamma_0^B - g_\tau(D)$$

1106 with $g_\tau(D) = \mathcal{O}(D)$. In fact,

$$\begin{aligned} 1107 \quad g_\tau(D) &= (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\ 1108 & \quad \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\ 1109 & \quad \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \end{aligned}$$

1110 **Proof of Lemma 32.** We start with an estimate on the individual $\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau)$. We
1111 will write $(v, w)^i$ for the i -th coordinate of the composed vector (v, w) . We now see that

$$\begin{aligned} 1112 & \left| \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \right| \\ 1113 & = \left| \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} \left(\left(1 - \tau\psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (F_{L,1}(x, 1), f_\partial(x) - \epsilon)^i \right) \right| \end{aligned}$$

$$\begin{aligned}
& + \tau \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x), f_{\partial,L}(x) - \epsilon)^i \Big| \\
1114 & \\
1115 & = \left| \left(\begin{array}{c} \text{grad} f_B^i(x) \\ 0 \end{array} \right) - \text{grad}_{(x,\tau)} \left(\left(1 - \tau \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) \right. \right. \\
1116 & \cdot \left. \left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) f(x) + \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) f_L(x), f_\partial(x) - \epsilon \right)^i \right. \\
& \quad \left. \left. \right. \right. \quad \text{(by definition of } F_{L,1}) \\
1117 & + \tau \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x), f_{\partial,L}(x) - \epsilon)^i \Big| \\
1118 & = \left| \left(\begin{array}{c} \text{grad} f_B^i(x) \\ 0 \end{array} \right) - \text{grad}_{(x,\tau)} \left(\left(1 - \tau \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) \right. \right. \\
1119 & \cdot \left(f(x) + \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), f_\partial(x) - \epsilon \right)^i \\
1120 & + \tau \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x), f_{\partial,L}(x) - \epsilon)^i \Big| \\
1121 & = \left| \left(\begin{array}{c} \text{grad} f_B^i(x) \\ 0 \end{array} \right) - \text{grad}_{(x,\tau)} \left(\left(f(x) + \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), f_\partial(x) - \epsilon \right)^i \right. \right. \\
1122 & + \tau \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \left(f_L(x) - f(x) - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \Big| \\
1123 & = \left| -\text{grad}_{(x,\tau)} \left(\left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), 0 \right)^i \right. \right. \quad \text{(by definition of } f_B^i(x)) \\
1124 & + \tau \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \Big| \\
1125 & = \left| \left(\begin{array}{c} -\text{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), 0 \right)^i \\ 0 \end{array} \right) \right. \\
1126 & + \left(\begin{array}{c} \tau \text{grad} \left(\psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \left((1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right)) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \right) \\ \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \left((1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right)) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \end{array} \right) \Big| \\
1127 & \leq \max_j \left| \text{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^j(x) - f^j(x)) \right) \right| \quad \text{(by the triangle inequality)} \\
1128 & + \left| \text{grad} \left(\psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \right) \right| \\
& \quad \text{(because } \tau \in [0, 1]) \\
1129 & + \left| ((f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x))^i \right| \quad \text{(because } \phi(y), \psi(y) \in [0, 1], \text{ for all } y) \\
1130 & \leq \max_j \left| \text{grad} \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| |f_L^j(x) - f^j(x)|
\end{aligned}$$

$$\begin{aligned}
& + \max_j \left| \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| |\text{grad}(f_L^j(x) - f^j(x))| \\
& \quad \text{(by the Leibniz rule, Cauchy-Schwarz, and the triangle inequality)} \\
& + \left| \text{grad} \left(\psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) \right| \left| \left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \right| \\
& + \left| \psi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| \left| \text{grad} \left(\left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \right) \right| \\
& \quad \text{(by the Leibniz rule, and the triangle inequality)} \\
& + 2D^2\beta \quad \text{(by Lemma 3)} \\
& \leq \max_j \gamma_\phi \gamma_2 |f_L^j(x) - f^j(x)| + \max_j |\text{grad}(f_L^j(x) - f^j(x))| \\
& \quad \text{(by Lemma 21, (11), (12), and since } \phi(y) \in [0, 1]) \\
& + \gamma_\psi \gamma_2 \left| \left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \right| \\
& \quad \text{(by Lemma 21, (12), (17))} \\
& + \left| \text{grad} \left(\left(\left(1 - \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right) (f_L(x) - f(x)), f_{\partial,L}(x) - f_\partial(x) \right)^i \right) \right| \\
& \quad \text{(because } \psi(y) \in [0, 1]) \\
& + 2D^2\beta \\
& \leq \gamma_\phi \gamma_2 2D^2\alpha + \frac{4dD\alpha}{T} \quad \text{(by Lemma 3 and Proposition 4)} \\
& + \gamma_\psi \gamma_2 2D^2\beta \quad \text{(by Lemma 3 and since } \phi(y) \in [0, 1]) \\
& + \left| \text{grad} (f_L(x) - f(x), f_{\partial,L}(x) - f_\partial(x))^i \right| \\
& + \left| \text{grad} \left(\left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L(x) - f(x)), 0 \right)^i \right) \right| \quad \text{(by the triangle inequality)} \\
& + 2D^2\beta \\
& \leq \gamma_\phi \gamma_2 2D^2\beta + \frac{4dD\beta}{T} \quad \text{(because by definition } \alpha \leq \beta) \\
& + \gamma_\psi \gamma_2 2D^2\beta \\
& + \frac{4dD\beta}{T} \quad \text{(by Proposition 4)} \\
& + \max_j \left| \text{grad} \left(\phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) (f_L^j(x) - f^j(x)) \right) \right| \\
& + 2D^2\beta \\
& \leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1) 2D^2\beta + \frac{8dD\beta}{T} \\
& + \max_j \left| \text{grad} \phi \left(\sum_l (f^l)^2 + f_\partial^2 \right) \right| |(f_L^j(x) - f^j(x))|
\end{aligned}$$

$$\begin{aligned}
& + \max_j \left| \phi \left(\sum_l (f^l)^2 + f_\delta^2 \right) \right| \left| \text{grad}(f_L^j(x) - f^j(x)) \right| \\
& \quad \quad \quad (\text{By the Leibniz rule and the triangle inequality}) \\
& \leq (\gamma_2(\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{8dD\beta}{T} \\
& \quad + \gamma_\phi\gamma_2 2D^2\alpha \quad \quad \quad (\text{by Lemma 21, (11), (12), and Lemma 3}) \\
& \quad + \frac{4dD\beta}{T} \quad \quad \quad (\text{because } \phi(y) \in [0, 1], \text{ Proposition 4, and } \alpha \leq \beta) \\
& = (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \quad (56)
\end{aligned}$$

We now write

$$(X'_B)_{i,j} = (\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau))_{i,j}.$$

Similarly to (48), we see that

$$\begin{aligned}
(X'_B)_{i,j} &= \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \right. \\
&\quad \cdot \left. \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
&= \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \right. \\
&\quad \cdot \left. \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
&= (Y_B)_{i,j} \\
&\quad + \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right)_{i,j} \\
&\quad + \left(\begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \cdot \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j} \\
&\quad + \left(\left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x, \tau) - \begin{pmatrix} \text{grad} f_B^i(x) \\ 0 \end{pmatrix} \right) \right. \\
&\quad \cdot \left. \left(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x, \tau) - \begin{pmatrix} \text{grad} f_B^j(x) \\ 0 \end{pmatrix} \right) \right)_{i,j},
\end{aligned}$$

where we used the notation

$$(\text{grad}(f_B^i) \cdot \text{grad}(f_B^j))_{i,j} = (Y_B)_{i,j}.$$

We now see by (56), (15), Cauchy-Schwarz and the triangle inequality that

$$\begin{aligned}
|(X'_B)_{i,j} - (Y_B)_{i,j}| &\leq 2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
&\quad + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2.
\end{aligned}$$

This also means that for the operator norm of the matrix we find

$$\|(X'_B - Y_B)_{i,j}\|_\infty \leq (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)$$

$$+ \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (57)$$

Similarly to (35) and (36), we also see that (56) gives

$$|X'_{i,j}| \leq \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2, \quad (58)$$

and thus,

$$\|X'_{i,j}\|_\infty \leq (n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (59)$$

Friedland's bound (37) now gives

$$\begin{aligned} & \det(\text{grad}_{(x,\tau)} F_{L,2,\epsilon}^i(x,\tau) \cdot \text{grad}_{(x,\tau)} F_{L,2,\epsilon}^j(x,\tau))_{i,j} \\ & \geq \gamma_0^B - (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\ & \quad \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\ & \quad \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \end{aligned}$$

where we used (57), and (59). ◀

Proof of Lemma 34. As we have seen in the calculation in (56), the τ component of $\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}^i)$ is bounded in absolute value by $\left| ((f_L(x) - f(x)), f_{\partial,L}(x) - f_{\partial}(x))^i \right| \leq 2D^2\beta$. On the other hand the x component is lower bounded by

$$\sqrt{\gamma_0^B / (\gamma_1^B)^{d-n-2}} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T},$$

as a consequence of Lemma 7 and the calculation in (56). This gives that

$$\tan \angle(\text{grad}_{(x,\tau)}(F_{L,2,\epsilon}), \Xi) \leq \frac{2D^2\beta}{\sqrt{\gamma_0^B / (\gamma_1^B)^{d-n-2}} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T}}.$$

◀

► **Lemma 36.** Let v be a vertex in \mathcal{T} , $x_1, x_2 \in \text{star}(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1)$ and $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)$ are well defined, then

$$|\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)| \leq g_8(D),$$

with $g_8(D) = \mathcal{O}(D)$. In fact,

$$g_8(D) = 2\sqrt{d}\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)$$

Proof of Lemma 36. The proof follows the same steps as the proof of Lemma 27. By expansion we see that

$$|\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2)|$$

$$\begin{aligned}
&= \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} + \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} \right. \\
&\quad \left. + \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2) \right| \\
&\leq \left| \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_1, \tau_1) - \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} \right| + \left| \begin{pmatrix} \text{grad} f_B^i(x_1) \\ 0 \end{pmatrix} - \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} \right| \\
&\quad + \left| \begin{pmatrix} \text{grad} f_B^i(x_2) \\ 0 \end{pmatrix} - \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_2, \tau_2) \right| \quad (\text{by the triangle inequality}) \\
&\leq 2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
&\quad (\text{by (39), (41) and (56) twice})
\end{aligned}$$

1206

1207 ► **Lemma 38.** *Under the same conditions as in Corollary 28,*

$$\begin{aligned}
&\det \left(\left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right)_{i,j} \\
&\geq \gamma_0^B - g_7(D) - g_9(D),
\end{aligned}$$

1210 where $g_9(D) = \mathcal{O}(D)$. In fact,

$$\begin{aligned}
g_7(D) &= (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
&\quad \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
&\quad \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \\
g_9(D) &= (n+1)^{n+1} \left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
&\quad \cdot \left(2 \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right) \\
&\quad \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
&\quad + \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2
\end{aligned}$$

1218 **Proof of Lemma 38.** The proof is more or less the same as the proofs of Lemmas 16 and
 1219 28, but with even more complicated bounds. Let $x_0 \in \text{star}(v)$ and $\tau_0 \in [0, 1]$, be such that
 1220 $\text{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0)$ is well defined. Note that it is sufficient for x_0 to lie in the interior
 1221 of a d -simplex in \mathcal{T} . Lemma 23 gives that

$$\begin{aligned}
&|\det(\tilde{X}'_B)_{i,j}| = |\det(\text{grad}_{(x,\tau)}(F_{PL,2,\epsilon}^i(x_0, \tau_0) \cdot \text{grad}_{(x,\tau)}(F_{PL,2,\epsilon}^j(x_0, \tau_0)))_{i,j}| \\
&> \gamma_0^B - (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n
\end{aligned}$$

1223

$$\begin{aligned}
& \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
& \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right).
\end{aligned}$$

We want to use this together with Corollary 28 and Friedland's bound, see (27), to give a bound on the determinant of

$$(Z'_B)_{i,j} = \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right).$$

As before we also need a bound on the operator norm $\|(Z'_B)_{i,j}\|_\infty$. This bound for this we notice that the bound (58) also holds for the convex hull, that is

$$\begin{aligned}
& \left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right| \\
& \leq \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2.
\end{aligned}$$

And thus, similarly to (52), we have that

$$\|(Z'_B)_{i,j}\|_\infty \leq (n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \quad (60)$$

We note that

$$\begin{aligned}
& \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \right| \leq \gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \\
& \quad \text{(because } \operatorname{grad}(f_B^i) \leq \gamma_1^B, \text{ and (56))} \\
& \quad \quad \quad (61)
\end{aligned}$$

Similarly to (45), we find that

$$\begin{aligned}
& |(Z'_B)_{i,j} - (\tilde{X}'_B)_{i,j}| \\
& = \left| \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right. \\
& \quad \left. - \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \cdot \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) \right| \\
& \leq \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right| \\
& \quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) \right| \\
& \quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right| \\
& \quad \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) \right| \\
& \quad + \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right|
\end{aligned}$$

$$\begin{aligned}
& \cdot \left| \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_0, \tau_0) - \sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right| \\
& \leq 2 \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \quad (\text{by Corollary 37}) \\
& \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \quad (\text{by (61)}) \\
& + \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \quad (\text{by Corollary 37})
\end{aligned}$$

Similarly to (34), we have that

$$\begin{aligned}
\|(Z'_B - \tilde{X}'_B)_{i,j}\|_\infty & \leq 2(n+1) \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \\
& \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
& + (n+1) \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \quad (62)
\end{aligned}$$

Indeed, Friedland's bound now gives,

$$\begin{aligned}
|\det(Z'_B)_{i,j}| & = \left| \det \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^i(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^m \mu_k \operatorname{grad}_{(x,\tau)} F_{PL,2,\epsilon}^j(x_k, \tau_k) \right) \right| \\
& \geq \gamma_0^B - (n+1) \left((n+1) \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \\
& \cdot (n+1) \left(2\gamma_1^B \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right. \\
& \left. + \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right) \quad (\text{by Lemma 32}) \\
& - (n+1)^{n+1} \left(\left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right)^2 \right)^n \quad (\text{by (60)}) \\
& \cdot \left(2 \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right) \right. \\
& \cdot \left(\gamma_1^B + (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \\
& \left. + \left(2\sqrt{d}d\beta D + 2 \left((\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \right) \right)^2 \right) \\
& \quad (\text{by (62) and factoring out } (n+1))
\end{aligned}$$