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ON THE MAXIMUM NUMBER OF EDGES IN A HYPER-GRAPH WHOSE LINEGRAPH CONTAINS NO CYCLE

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Soit $H = (X, \mathcal{E})$ un hypergraphe h -uniforme avec $|X| = n$ et soit $L_{h-1}(H)$ le graphe dont les sommets représentent les arêtes de H , deux sommets étant reliés si et seulement si les arêtes qu'ils représentent intersectent en $h-1$ sommets. Nous montrons que si $L_{h-1}(H)$ ne contient pas de cycle, alors $|\mathcal{E}| < \binom{n}{h-1}/h - 1$, la borne étant exacte pour $h=2$ et pour des valeurs de n pour $h=3$. Ce problème mène à une conjecture sur les "presque systèmes de Steiner".

Let $H = (X, \mathcal{E})$ be a h -uniform hypergraph, with $|X| = n$ and let $L_{h-1}(H)$ be the graph, whose vertices are the edges of H , two vertices being joined if and only if the edges they represent intersect in $h-1$ vertices. We prove that, if $L_{h-1}(H)$ contains no cycle, then $|\mathcal{E}| < \binom{n}{h-1}/h - 1$; moreover the bound is exact for $h=2$ and with some values of n for $h=3$. This problem leads to a conjecture on "almost Steiner systems".

1. Introduction

Let X be a set of n elements and let $H = (X, \mathcal{E})$ be a h -uniform hypergraph, that is \mathcal{E} is a family of h -subsets of X .

We denote by $L_{h-1}(H)$ the graph, whose vertices are the edges of H and where $E_1, E_2 \in \mathcal{E}$ are joined by an edge iff $|E_1 \cap E_2| = h-1$. These graphs were previously studied (see for references [2]).

In this note we answer the following problem, which is a version of a problem due to Kachelsky and Kleitman, presented at the Problem Session of the 7th British Combinatorial Conference (Cambridge, August 1979).

Theorem 1. Suppose $L_{h-1}(H)$ contains no cycle, then

$$|\mathcal{E}| < \frac{\binom{n}{h-1}}{h-1}. \quad (1)$$

Moreover the bound is exact for $h=2$ and for some values of n for $h=3$.

2. Proof of inequality (1)

Let $F \subset X$, $|F| = h-1$. The F is contained in at most 2 different members of \mathcal{E} , as otherwise for $E_1, E_2, E_3 \in \mathcal{E}$ and containing F , we would have a triangle in $L_{h-1}(H)$.

Hence, for $F \subset E \in \mathcal{E}$, $|F| = h-1$ there are only two possibilities, either F is contained only in E or there is exactly one more edge E' , with $F \subset E'$.

Let $a(E)$ denote the number of $(h-1)$ -subsets of $E \in \mathcal{E}$, which are contained in some $E' \in \mathcal{E}$, $E' \neq E$. Then the vertex E is joined to exactly $a(E)$ other vertices of $L_{h-1}(H)$.

Let m be the number of edges of $L_{h-1}(H)$, then we obtain

$$2m = \sum_{E \in \mathcal{E}} a(E). \quad (2)$$

On the other hand, $L_{h-1}(H)$ contains no cycle, hence we also have

$$m \leq |\mathcal{E}| - 1. \quad (3)$$

Combining (2) and (3) we obtain

$$|\mathcal{E}| - 1 \geq \frac{1}{2} \sum_{E \in \mathcal{E}} a(E). \quad (4)$$

Let $c(H)$ be the number of $(h-1)$ -subsets of X , which are contained in some edge E of H . Obviously $c(H) \leq \binom{n}{h-1}$. We obtain

$$c(H) = \sum_{E \in \mathcal{E}} (h - a(E)) + \frac{1}{2} \sum_{E \in \mathcal{E}} a(E) \leq \binom{n}{h-1}$$

or equivalently

$$c(H) = h |\mathcal{E}| - \frac{1}{2} \sum_{E \in \mathcal{E}} a(E) \leq \binom{n}{h-1}. \quad (5)$$

From (5) using (4) we deduce

$$h |\mathcal{E}| - (|\mathcal{E}| - 1) = (h-1) |\mathcal{E}| + 1 \leq \binom{n}{h-1}$$

or equivalently $|\mathcal{E}| < \binom{n}{h-1} / (h-1)$ proving (1).

Remark. Let us define $L_{\geq i}(H)$ to be the graph whose vertices are the edges of H and where $E_1 \neq E_2 \in \mathcal{E}$ are joined iff $|E_1 \cap E_2| \geq i$ ($1 \leq i \leq h-1$). Suppose that

$L_{\geq i}(H)$ contains no cycle. Then the above proof yields

$$|\mathcal{E}| < \binom{n}{i} / \left(\binom{h}{i} - 1 \right).$$

3. Constructions

For $h=2$, we can take any tree to show that the bound (1) is exact.

Let now $h=3$.

Suppose $n=8k+5$ is a prime and 2 is a primitive root modulo n , that is the numbers $\pm 2^\alpha$, $\alpha=0, 1, \dots, \frac{1}{2}(n-3)$ are all different. (It is conjectured in number theory that there exists an infinite number of such n 's.)

Let H be a hypergraph with vertex set $\{0, 1, \dots, n-1\}$ and edge set:

$$\begin{aligned} \{E_i^\alpha = \{i, i+2^{2\alpha}, i+2^{2\alpha+1}\}, i=1, \dots, n-1; \alpha=0, 1, \dots, \frac{1}{4}(n-5)\} \\ \cup \{E_0^\alpha = \{0, 2^{2\alpha+1}, 2^{2\alpha+2}\}, \alpha=0, 1, \dots, \frac{1}{4}(n-9)\} \end{aligned}$$

(all the numbers are taken modulo n).

We assert that $L_{h-1}(H)$ contains no cycle. To see this, observe that the subgraph generated by $X^\alpha = \{E_i^\alpha, i=1, \dots, n-1\}$ is a Hamiltonian path on X^α .

Moreover $|E_i^\alpha \cap E_j^{\alpha'}| \leq 1$, for $1 \leq i, j \leq n-1$, $\alpha \neq \alpha'$ as 2 is a primitive root. Therefore there are no edges in $L_{h-1}(H)$ joining X^α to $X^{\alpha'}$.

Finally the vertex E_0^α is joined in $L_{h-1}(H)$ only to $E_{2^{2\alpha+1}}^\alpha$ and to $E_{n-2^{2\alpha+2}}^{\alpha+1}$. Hence $L_{h-1}(H)$ is a tree (see Fig. 1 for an example, $n=13$).

We have $|\mathcal{E}| = \frac{1}{4}(n-1)(n-1) + \frac{1}{4}(n-5) = \frac{1}{4}n(n-1) - 1$ which proves that the bound (1) is optimal for the mentioned values of n .

For the other values of n , we can prove:

Theorem 2. For every n , there exists a 3-uniform hypergraph H of order n such that $L_2(H)$ has no cycle and $\frac{1}{2}\binom{n}{2} - |H| \leq \frac{1}{4}n$.

We use the following lemma:

Lemma 1. There exists a sequence of matchings (graphs with maximum degree 1) C_j such that

- (a) $|C_{2i-1}| = |C_{2i}| = i$,
- (b) the vertex-set of $G_j = C_1 \cup \dots \cup C_j$ is $\{x_1, \dots, x_{j+1}\}$
- (c) $C_j \cap C_k = \emptyset$ ($j \neq k$),
- (d) the degree-sequence of G_{2i} is dominated by $\{1, 2, 3, \dots, 2i-3, 2i-2, 2i-1, 2i-1\}$, for $i \geq 3$.

The construction of H in Theorem 2 is the following:

$$H = \{\{x_j\} \cup E \mid E \in C_{j-2}, j=3, \dots, n\}.$$

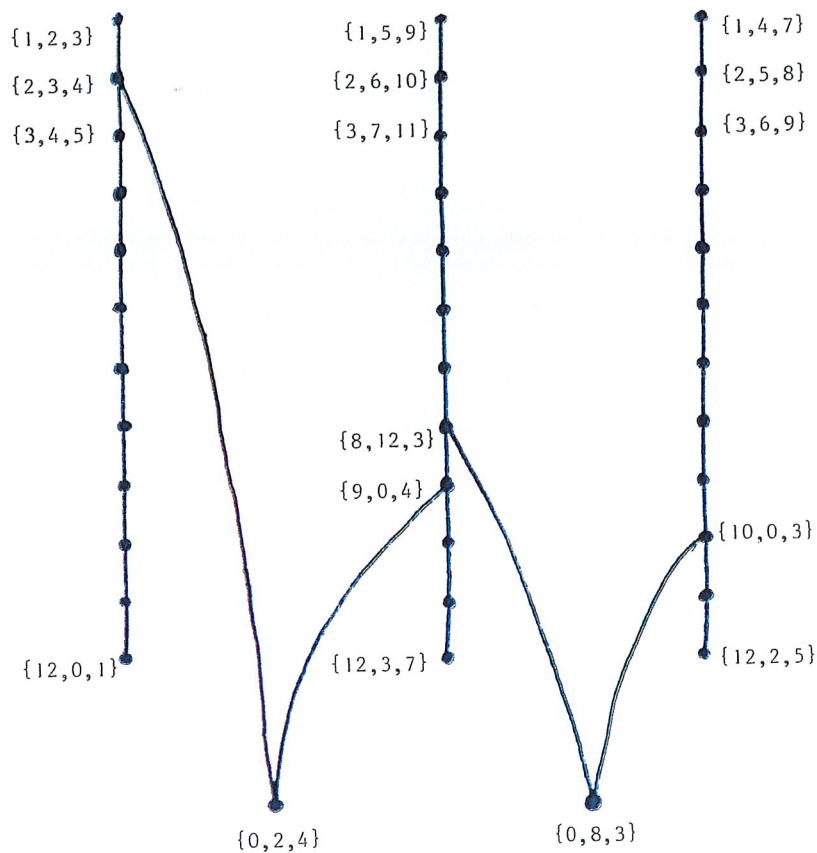


Fig. 1.

Lemma 1 is proved by induction, with the aid of the Tutte–Berge matching theorem (see for instance [1]). This method is also valid for $h > 3$. However, in this case, Lemma 1 should be replaced by results not yet proved such as the following one:

Conjecture. There exists a sequence of $(h - 1)$ -uniform hypergraphs S_j such that:

- (a) the vertex-set of $S_1 \cup \dots \cup S_j$ is $\{x_1, \dots, x_{j+h-2}\}$,
- (b) $S_j \cap S_k = \emptyset$ ($j \neq k$),
- (c) every two edges of S_i meet in at most $h - 3$ vertices,
- (d) $|S_j| = \binom{j+h-2}{h-2} / (h-1) - O(j^{h-3})$.

The last two conditions mean that S_j is an “almost Steiner system”. (Note $O(j^{h-3}) \leq c_h j^{h-3}$ for some constant c_h).

References

- [1] C. Berge, *Graphs and hypergraphs* (North-Holland, Amsterdam, 1976).
- [2] J.-C. Bermond, A. Germa and M.-C. Heydemann, *Graphes representatifs d'hypergraphes*, in: *Coll. Math. Disretes, Bruxelles* (1978), *Cahiers du C.E.R.O.* 20 (1978) 325–329.