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# Branch-cut-and-price algorithms for the vehicle routing problem with backhauls

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# Motivation 1

Perspectives in the survey by [Koç and Laporte, 2018]:

- ▶ *“Our belief that further studies should focus on developing effective and powerful exact methods, such as branch-and-cut-and-price, to solve all available standard VRPB instances to optimality.”*
- ▶ *“No exact algorithm has yet been proposed for the time windows extension of the VRPB. This type of effective algorithms could be applied to the VRPB with time windows.”*



Koç, Ç. and Laporte, G. (2018).

Vehicle routing with backhauls: Review and research perspectives.

[Computers and Operations Research](#), 91:79 – 91.

## Motivation 2

Generic **Branch-Cut-and-Price (BCP)** state-of-the-art solver for Vehicle Routing Problems (VRPs) [Pessoa et al., 2019].

`vrpsolver.math.u-bordeaux.fr`

**Non-trivial modelling** allows us to solve VRPB more efficiently using the solver.



Pessoa, A., Sadykov, R., Uchoa, E., and Vanderbeck, F. (2019).

A generic exact solver for vehicle routing and related problems.

In Lodi, A. and Nagarajan, V., editors, [Integer Programming and Combinatorial Optimization](#), volume 11480 of [Lecture Notes in Computer Science](#), pages 354–369, Springer International Publishing.

# Plan of the talk

Generic Model for Vehicle Routing Problems

Models for the Vehicle Routing Problem with Backhauls

Results and Conclusions

# Generic Model: Graphs for Resource Constrained Shortest Path (RCSP) generation

Define a set  $R$  of resource indices, partitioned into **main resources**  $R^M$  and **secondary resources**  $R^N$

Define **directed graphs**  $G^k = (V^k, A^k)$ ,  $k \in K$ :

- ▶ **Special vertices**  $v_{\text{source}}^k, v_{\text{sink}}^k$
- ▶ **Arc consumption**  $q_{a,r} \in \mathbb{R}_+$ ,  $a \in A^k$ ,  $r \in R$ 
  - ▶ cycles with zero main resource consumption should not exist
  - ▶ secondary resources may be of a special type that allow negative consumption
- ▶ **Accumulated resource consumption intervals**  $[l_{a,r}, u_{a,r}]$ ,  $a \in A^k$ ,  $r \in R$ 
  - ▶ May also be defined on vertices ( $[l_{v,r}, u_{v,r}]$ ,  $v \in V^k$ ,  $r \in R$ )

Let  $V = \cup_{k \in K} V^k$  and  $A = \cup_{k \in K} A^k$

## Generic Model: Graphs for RCSPs generation (2)

### Resource Constrained Path (disposable resources)

A path  $p = (v_{\text{source}}^k = v_0, a_1, v_1, \dots, a_{n-1}, v_{n-1}, a_n, v_n = v_{\text{sink}}^k)$  over  $G^k$  is resource constrained iff for every  $r \in R$ , the accumulated resource consumption  $t_{j,r}$  at visit  $j$ ,  $0 \leq j \leq n$ , does not exceed  $u_{a_j,r}$ , where

$$t_{j,r} = \begin{cases} 0, & j = 0, \\ \max\{l_{a_j,r}, t_{j-1,r} + q_{a_j,r}\}, & j > 0 \end{cases}$$

# Generic Model: Variables and Mappings

Define continuous and/or integer variables:

1. Mapped  $x$  variables
  - ▶ Each variable  $x_j$ ,  $1 \leq j \leq n_1$ , is mapped into a non-empty set  $M(j) \subseteq A$ .
  - ▶ The inverse mapping of arc  $a$  is  $M^{-1}(a) = \{j | a \in M(j)\}$ .
2. Additional (non-mapped)  $y$  variables
3. Bounds  $[\bar{L}^k, \bar{U}^k]$  for the number of paths in  $P^k$  in a feasible solution.

Define also

- ▶ For each  $k \in K$ ,  $P^k$  is the set of all resource constrained paths in  $G^k$
- ▶  $P = \cup_{k \in K} P^k$
- ▶  $\lambda_p$  = how many times path  $p \in P$  is used in the solution.
- ▶  $h_a^p$  = how many times arc  $a$  is used in path  $p$



## Generic Model: Formulation

$$\text{Min} \quad \sum_{j=1}^{n_1} c_j x_j + \sum_{s=1}^{n_2} f_s y_s \quad (1a)$$

$$\text{S.t.} \quad \sum_{j=1}^{n_1} \alpha_{ij} x_j + \sum_{s=1}^{n_2} \beta_{is} y_s \geq d_i, \quad i = 1, \dots, m, \quad (1b)$$

$$x_j = \sum_{k \in K} \sum_{p \in P^k} \left( \sum_{a \in M(j)} h_a^p \right) \lambda_p, \quad j = 1, \dots, n_1, \quad (1c)$$

$$\bar{L}^k \leq \sum_{p \in P^k} \lambda_p \leq \bar{U}^k, \quad k \in K, \quad (1d)$$

$$\lambda_p \in \mathbb{Z}_+, \quad p \in P, \quad (1e)$$

$$x_j \in \mathbb{N}, y_s \in \mathbb{N} \quad j = 1, \dots, n_1, s = 1, \dots, n_2.$$

(1b) may be separated on demand through callback routines

## Generic Model: Collection of Packing Sets

Define a collection  $\mathcal{S}$  of mutually disjoint **packing sets**, each one being a subset of  $A$ , such that the constraints:

$$\sum_{a \in S} \sum_{p \in P} h_a^p \lambda_p \leq 1, \quad S \in \mathcal{S}, \quad (2)$$

are satisfied by at least one optimal solution  $(x^*, y^*, \lambda^*)$  of Formulation (1).

- ▶ The definition of a proper  $\mathcal{S}$  is **part of the modeling**
- ▶ Packing sets **can be defined on vertices** (each one is a subset of  $V$ )

**Packing sets generalize customers in the classical CVRP**

# Why we need Packing Sets?

Knowledge about packing sets allows the solver to use **state-of-the-art techniques in a generalized form**:

- ▶ **ng-paths** [Baldacci et al., 2011]
  - ▶ **Distance matrix for packing sets** is expected from the user to obtain initial *ng*-neighbourhoods
- ▶ **Limited Memory Rank-1 Cuts** [Pecin et al., 2017]
- ▶ **Elementary path enumeration** [Baldacci et al., 2008]
  - ▶ **Additional condition** to use enumeration:  
*Two partial paths ending in the same vertex and mapped to different columns in (1b) should correspond to different collection of packing sets*

# Rounded Capacity Cuts (RCC) Separators

Interface for separating RCCs [Laporte and Nobert, 1983].  
CVRPSEP code [Lysgaard, 2003] is used by the solver

Each separator is characterized by a triple

- ▶ sub-collection  $\mathcal{S}' \subseteq \mathcal{S}$  of packing sets,
- ▶ a demand  $d_S$  for each  $S \in \mathcal{S}'$ ,
- ▶ capacity  $Q$ .

## Conditions to use

- ▶ Collection of packing sets is defined on vertices.
- ▶ For every  $S \subseteq \mathcal{S}'$ ,  $\sum_{v \in S} \sum_{p \in P} h_v^p \lambda_p = 1$ .
- ▶ For every path  $p \in P$ ,  $\sum_{S \in \mathcal{S}'} d_S \cdot \sum_{v \in S} h_v^p \leq Q$ .

# Vehicle Routing Problem with Backhauls

## Data

- ▶ Depot  $0$
- ▶ Set  $L = \{1, \dots, n\}$  of linehaul vertices
- ▶ Set  $B = \{n + 1, \dots, n + m\}$  of backhaul vertices
- ▶ Graph  $G = (V, A)$ ,  
 $V = \{0\} \cup L \cup B$ ,  
 $A = \{(i, j) : i, j \in V, i \neq j\}$ .
- ▶ Travelling cost  $c_a$ ,  $a \in A$ .
- ▶ Demands  $d_i$ ,  $i \in L \cup B$ .
- ▶  $K$  homogeneous vehicles of capacity  $Q$ .

## Feasible solution

$K$  vehicle routes

- ▶ start and finish at the depot
- ▶ serve at least one linehaul customer
- ▶ serve backhaul customers strictly after linehaul ones
- ▶ total demand of linehaul customers  $\leq Q$
- ▶ total demand of backhaul customers  $\leq Q$

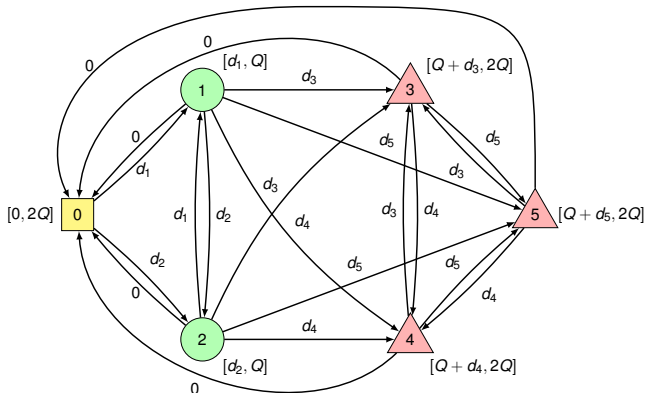
## Objective

Minimize the total travelling cost

## Standard model with one graph

- Graph  $G^1 = (V, A^1)$ ,  $v_{\text{source}}^1 = v_{\text{sink}}^1 = 0$ ,  
 $A^1 = A \setminus \{(0, j) : j \in B\} \setminus \{(j, i) : j \in B, i \in L\}$ ,
- One main capacity resource :

$$q_{(i,j)} = \begin{cases} d_j, & j \in L \cup B, \\ 0, & j = 0. \end{cases} \quad [l_v, u_v] = \begin{cases} [d_v, Q], & v \in L, \\ [Q + d_v, 2Q], & v \in B, \\ [0, 2Q], & j = 0. \end{cases}$$



## Standard model with one graph: formulation

- ▶ Variable  $x_a$  is mapped to arc  $a$ ,  $a \in A^1$ .
- ▶  $[\bar{L}^1, \bar{U}^1] = [K, K]$
- ▶ Formulation:

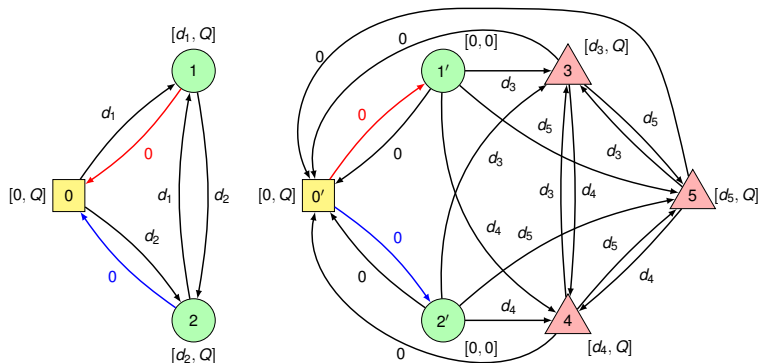
$$\begin{array}{ll} \text{Min} & \sum_{a \in A^1} c_a \cdot x_a \\ \text{S.t.} & \sum_{(i,j) \in A^1} x_a = 1, \quad \forall j \in L \cup B, \\ & x_a \in \{0, 1\}, \quad \forall a \in A^1. \end{array}$$

- ▶ **Packing sets** are defined on vertices:  $\mathcal{S} = \mathcal{S}_L \cup \mathcal{S}_B$ ,  
 $\mathcal{S}_L = \{\{v\} : v \in L\}$ ,  $\mathcal{S}_B = \{\{v\} : v \in B\}$ .
- ▶ **Distance matrix** is based on travelling costs
- ▶ **First RCC separator**:  $(\mathcal{S}_L, \{d_v\}_{v \in L}, Q)$
- ▶ **Second RCC separator**:  $(\mathcal{S}_B, \{d_v\}_{v \in B}, Q)$
- ▶ **Branching** on variables  $x$

## New model with two graphs

- ▶ Graph  $G^1 = (V^1, A^1)$ ,  $V^1 = \{0\} \cup L$ ,  $A^1 = V^1 \times V^1$
- ▶ Graph  $G^2 = (V^2, A^2)$ ,  $V^2 = \{0'\} \cup L' \cup B$ ,  $A^2 = \{(0, j) : j \in L'\} \cup \{(i, j) : i \in L', j \in \{0\} \cup B\} \cup \{(i, j) : i \in B, j \in \{0\} \cup B\}$ .
- ▶ One main capacity resource :

$$q_{(i,j)} = \begin{cases} d_j, & j \in L \cup B, \\ 0, & j \in \{0\} \cup L'. \end{cases} \quad [l_v, u_v] = \begin{cases} [d_v, Q], & v \in L \cup B, \\ [0, 0], & v \in L', \\ [0, Q], & j \in \{0, 0'\}. \end{cases}$$





## New model with two graphs: formulation

- ▶ Variable  $x_a$  is mapped to arc  $a$ ,  $a \in \{(i, j) : j \in L \cup B \cup \{0'\}\}$
- ▶ Variable  $z_j$  is mapped to arc  $(j, 0) \in A^1, j \in L$
- ▶ Variable  $w_j$  is mapped to arc  $(0', j) \in A^2, j \in L'$
- ▶  $[\bar{L}^1, \bar{U}^1] = [\bar{L}^2, \bar{U}^2] = [K, K]$
- ▶ Formulation:

$$\text{Min} \quad \sum_{a \in A^1} c_a \cdot x_a$$

$$\text{S.t.} \quad \sum_{(i,j) \in A^1} x_a = 1, \quad \forall j \in L,$$

$$\sum_{(i,j) \in A^2} x_a = 1, \quad \forall j \in B,$$

$$z_j = w_j, \quad \forall j \in L,$$

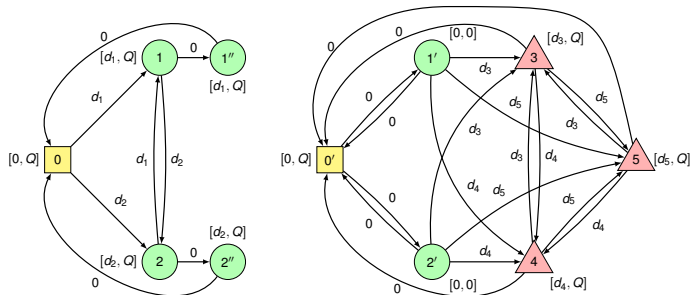
$$x_a \in \{0, 1\}, \quad \forall a \in A^1 \cup A^2,$$

$$z_j, w_j \in \{0, 1\}, \quad \forall j \in L.$$

- ▶ **Branching** on variables  $x$  and  $y$

# New model with two graphs: packing sets

We modify graph  $G^1$  by duplicating vertices in  $L$  to  $L''$ :



- ▶ Packing sets are defined on vertices:  
 $\mathcal{S} = \mathcal{S}_L \cup \mathcal{S}_{L'} \cup \mathcal{S}_{L''} \cup \mathcal{S}_B$ .
- ▶ Packing sets in  $\mathcal{S}_{L'} \cup \mathcal{S}_{L''}$  are artificial and serve to satisfy condition to use enumeration
- ▶ Distance to an artificial packing set is  $\infty$
- ▶ Same two RCC separators as for the first model

# Instances and exact approaches in the literature

## Instances

- ▶ [Goetschalckx and J.-B., 1989]: 25–200 customers
- ▶ [Toth and Vigo, 1997]: 21–100 customers
- ▶ [Uchoa et al., 2017] (modified CVRP): 101–1001 customers

## Exact approaches in the literature

- TV [Toth and Vigo, 1997]: Lagrangian relaxation + branch-and-bound
- MGB [Mingozi et al., 1999]: Heuristic solution of the dual of an LP relaxation of route-based formulation + enumeration of routes with small reduced cost
- GES [Granada-Echeverri and Santa, 2019]: MIP formulation

# Computational results

## Comparison with the literature

Instance set	Solved/Tried instances			
	TV	MGB	GES	Ours
[Goetschalckx and J.-B., 1989]	29/34	30/47	47/62	<b>68/68</b>
[Toth and Vigo, 1997]	23/30	24/33	28/33	<b>33/33</b>

## Comparison of two models

Instances	Size	One graph		Two graphs	
		Time (s)	Nodes	Time (s)	Nodes
Classic	25–200	284	2.0	<b>106</b>	1.9
New	101–167	4814	28	<b>1120</b>	12

## Large instances with 172–1001 customers

- ▶ 77 from 255 instances are solved to optimality in 60 hours
- ▶ The largest solved instance has 655 customers
- ▶ The smallest unsolved instance has 190 customers

# Conclusions

- ▶ Wishes of [Koç and Laporte, 2018] are fulfilled:
  - ▶ All classic instances of the VRPB are solved to optimality
  - ▶ All literature instances of the VRPBTW and HFFVRPB are also solved to optimality (straightforward modifications of the model)
- ▶ Many open instances for future works
- ▶ Model with two graphs is experimentally much better
- ▶ The code of the second model ( $\approx 130$  lines of Julia code) is will be available as a VRPSolver demo at [vrpsolver.math.u-bordeaux.fr](http://vrpsolver.math.u-bordeaux.fr)
- ▶ Demos for several other VRP variants are available (CVRP, VRPTW, HFVRP, PDPTW, TOP, CARP, 2E-CVRP)



Thank you!  
Questions?

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