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# A Branch-Cut-and-Price Algorithm for the Location-Routing Problem

Pedro Liguori<sup>1</sup>    Ridha Mahjoub<sup>1</sup>  
**Ruslan Sadykov**<sup>2</sup>    Eduardo Uchoa<sup>3</sup>

1

LAMSADE,  
Université  
Paris-Dauphine,  
France



2

Inria Bordeaux —  
Sud-Ouest, France



3

Universidade Federal  
Fluminense, Brazil



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# Location-Routing Problem (LRP)

## Data

- ▶  $I$  — set of potential depots with opening costs  $f_i$  and capacities  $w_i$ ,  $i \in I$
- ▶  $J$  — set of customers with demands  $d_j$ ,  $j \in J$
- ▶ Sets of edges:  $E = J \times J$ ,  $F = I \times J$
- ▶  $c_e$  — transportation cost of edge  $e \in E \cup F$
- ▶ An unlimited set of vehicles with capacity  $Q$ .

## The problem

- ▶ Decide which depots to open
- ▶ Assign every client to an open depot subject to depot capacity
- ▶ For every depot, divide assigned clients into routes subject to vehicle capacity
- ▶ Minimize the total depot opening and transportation cost

# LRP: an illustration

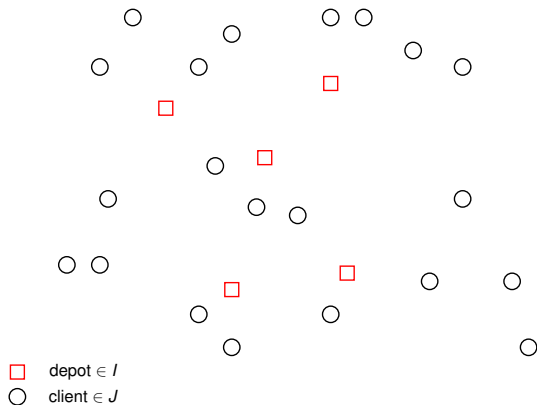
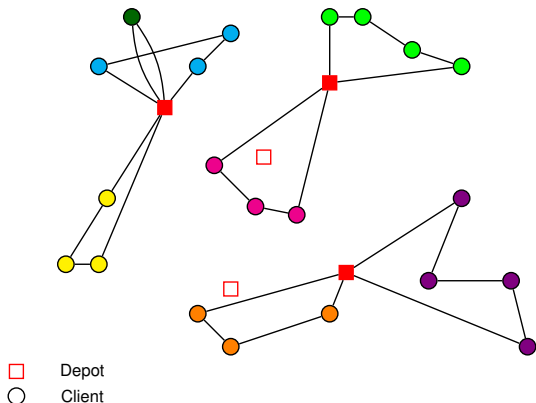


Figure: LRP instance:  $G = (I \cup J, E \cup F)$

## LRP: a solution



**Figure:** Location of depots must be jointly decided with vehicle routing.

# Literature on LRP

- ▶ A combination of **two central OR problems**
- ▶ **≈3000 papers** in Google Scholar with both “location” and “routing” in the title

## Important recent works

- ▶ [Belenguer et al., 2011] — important valid inequalities & Branch-and-Cut;
- ▶ [Baldacci et al., 2011b] — exact “enumeration” & column generation approach
- ▶ [Contardo et al., 2014] — state-of-the-art exact algorithm
- ▶ [Schneider and Löffler, 2019] — state-of-the-art heuristic
- ▶ [Schneider and Drexl, 2017] — the latest survey on LRP

# Our study

- ▶ Recently, **large improvement in exact solution** of classic VRP variants [Pecin et al., 2017b] [Pecin et al., 2017a] [S. et al., 2017] [Pessoa et al., 2018a]
- ▶ A **generic Branch-Cut-and-Price VRP solver** [Pessoa et al., 2019] incorporates all recent advances  
**[vrpsolver.math.u-bordeaux.fr](http://vrpsolver.math.u-bordeaux.fr)**
- ▶ This solver can be applied to the LRP
- ▶ However, **problem-specific cuts are necessary** for obtaining the state-of-the-art performance
- ▶ We review existing families of cuts and propose new ones

## Formulation

- ▶  $\lambda_r^i$ ,  $i \in I$ ,  $r \in R_i$ , equals 1 iff route  $r$  is used for depot  $i$
- ▶  $a_e^r$ ,  $e \in E \cup F$ ,  $r \in \cup_{i \in I} R_i$ , equals 1 iff edge  $e$  is used by  $r$
- ▶  $y_i$ ,  $i \in I$ , equals 1 iff route depot  $i$  is open
- ▶  $z_{ij}$ ,  $i \in I$ ,  $j \in J$ , equals 1 iff client  $j$  is assigned to depot  $i$

$$\min \sum_{i \in I} f_i y_i + \sum_{i \in I} \sum_{r \in R_i} \sum_{e \in E \cup F} c_e a_e^r \lambda_r^i$$

$$\sum_{i \in I} z_{ij} = 1, \quad \forall j \in J,$$

$$\sum_{r \in R_i} \sum_{e \in \delta(j)} a_e^r \lambda_r^i = 2z_{ij}, \quad \forall i \in I, j \in J$$

$$\sum_{j \in J} d_j z_{ij} \leq w_i y_i, \quad \forall i \in I,$$

$$z_{ij} \leq y_i, \quad \forall i \in I, j \in J,$$

$$(z, y, \lambda) \in \{0, 1\}^K$$



## Rounded Capacity Cuts [Laporte and Nobert, 1983]

Given a subset of clients  $C \subset J$ ,

$$\sum_{i \in I} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_e^r \lambda_r^i \geq 2 \cdot \left\lceil \frac{\sum_{i \in C} d_i}{Q} \right\rceil.$$

Separation (embedded in the VRP solver)

CVRPSEP library [Lysgaard et al., 2004]

# Chvátal-Gomory Rank-1 Cuts [Jepsen et al., 2008]

[Pecin et al., 2017c]

Each cut is obtained by a **Chvátal-Gomory rounding of a set  $C \subseteq J$  of set packing constraints** using a vector of multipliers  $\rho$  ( $0 < \rho_j < 1, j \in C$ ):

$$\sum_{i \in I} \sum_{r \in R_i} \left[ \sum_{j \in C} \rho_j \sum_{e \in \delta(j)} \frac{1}{2} a_e^r \right] \lambda_r^i \leq \left[ \sum_{j \in C} \rho_j \right]$$

All best possible vectors  $\rho$  of multipliers for  $|C| \leq 5$  are given in [Pecin et al., 2017c].

**Non-robust** in the terminology of [Pessoa et al., 2008]

Separation (embedded in the VRP solver)

A local search for each vector of multipliers.

## Depot Capacity Cuts [Belenguer et al., 2011]

If a subset of clients  $C \subset J$  cannot be served by a subset of depots  $S \subset I$ ,

$$\sum_{j \in C} d_j > \sum_{i \in S} w_i,$$

then at least one vehicle from a depot  $i \in I \setminus S$  should visit  $C$ :

$$\sum_{i \in I \setminus S} \sum_{r \in R_i} \sum_{e \in \delta(C)} a_e^r \lambda_r^i \geq 2.$$

Separation (in the VRP solver callback)

A heuristic algorithm: combination of GRASP and local search.

## Covering inequalities for depot capacities

Let  $W = \sum_{i \in I} w_i$  and  $D = \sum_{j \in J} d_j$ .

We should have

$$\sum_{i \in I} w_i y_i \geq d(J) \quad \Rightarrow \quad \sum_{i \in I} w_i (1 - y_i) \leq W - D$$

We can generate any valid inequality for this knapsack. For example **covering inequalities**: given a subset of depots  $S \subset I$ ,  $\sum_{i \in S} w_i > W - D$ ,

$$\sum_{i \in S} (1 - y_i) \leq |S| - 1$$

### Separation (in the VRP solver callback)

We optimize an LP which looks for the most violated inequality which is satisfied by all integer solutions of the knapsack.

# Route Load Knapsack Cuts (RLKC)

$x_q^i$  — number of routes with load of exactly  $q \leq Q$  units leaving depot  $i \in I$ . Then:

$$\sum_{q=1}^Q qx_q^i \leq w_i. \quad (1)$$

Any valid inequality for (1) is valid for the LRP.

**Non-robust** in the terminology of [Pessoa et al., 2008]

First separation algorithm

Chvátal-Gomory rounding of (1).

# 1/k-facets of the master knapsack polytope

Theorem ([Aráoz, 1974])

The coefficient vectors  $\xi$  of the knapsack (non-trivial) facets  $\xi x \leq 1$  of  $\sum_{q=1}^n qx_q = n$  with  $\xi_1 = 0$ ,  $\xi_Q = 1$  are the extreme points of the following system of linear constraints

$$\begin{aligned}\xi_1 &= 0, & \xi_Q &= 1, \\ \xi_q + \xi_{Q-q} &= 1 & \forall 1 \leq i \leq n/2, \\ \xi_q + \xi_t &\leq \xi_{q+t} & \text{whenever } q + t < n.\end{aligned}$$

## Definition

A knapsack facet  $\xi x \leq 1$  is called a 1/k-facet if  $k$  is the smallest possible integer such that

$$\xi_q \in \{0/k, 1/k, 2/k, \dots, k/k\} \cup \{1/2\}.$$

## Second separation algorithm

1/6- and 1/8-facets can be efficiently separated using the algorithm by [Chopra et al., 2019]

## Taking into account of RLKCs in the pricing

- ▶ Let  $\bar{\mu}(q)$  be the contribution of RLKCs to the reduced cost of a route variable with load  $q$
- ▶ Pricing problem: **Resource Constrained Shortest Path**
- ▶ It is solved by a labelling algorithm, each label  $L$  is  $(\bar{c}^L + \bar{\mu}(q^L), j^L, q^L)$
- ▶ Dominance relation

$$L \succ L' \quad \text{if } \bar{c}^L \leq \bar{c}^{L'}, j^L = j^{L'}, q^L \leq q^{L'} \quad (2)$$

is valid, as  $\bar{\mu}(q)$  is **non-decreasing**

- ▶ Completion bounds can still be efficiently used as  $\bar{\mu}(q)$  is **super-additive**

## Other components of the Branch-Cut-and-Price

- ▶ Bucket graph-based labelling algorithm for the RCSP pricing [Righini and Salani, 2006] [S. et al., 2017]
- ▶ Partially elementary path (*ng*-path) relaxation [Baldacci et al., 2011a]
- ▶ Automatic dual price smoothing stabilization [Wentges, 1997] [Pessoa et al., 2018b]
- ▶ Reduced cost fixing of (bucket) arcs in the pricing problem [Ibaraki and Nakamura, 1994] [Irnich et al., 2010] [S. et al., 2017]
- ▶ Enumeration of elementary routes [Baldacci et al., 2008]
- ▶ Multi-phase strong branching [Pecin et al., 2017b]
  - ▶ On depot openings (largest priority)
  - ▶ On number of vehicles for each depot
  - ▶ On number of clients per depot
  - ▶ On assignment of clients to depots
  - ▶ On edges of the graph



# Computational results

Open instances solved to optimality. Could not be solved by the state-of-the-art [Contardo et al., 2014] in 5-97 hours

Set	Instance	Optimum	Time
[Prins et al., 2006]	100x5-1b	213568	10m05s
	100x10-1a	287661	1h32m
	100x10-1b	230989	1h38m
	100x10-3a	250882	1h17m
	100x10-3b	203114	11h01m
	200x10-1a	<u>474702</u>	20m42s
	200x10-1b	<u>375177</u>	1h55m
	200x10-2a	<u>448005</u>	4h45m
	200x10-2b	<u>373696</u>	5h53m
[Tuzun and Burke, 1999]	P113112	1238.24	2h29m
	P131112	1892.17	36m52s
	P131212	1960.02	34m59s

Underlined: improved solutions over [Schneider and Löffler, 2019]

## Sensitivity analysis of cuts specific to LRP

26 instances by [Prins et al., 2006] with 5-10 depots and 50-200 clients. Time limit 3 hours

Configuration	Solved	Root gap	Nodes	Time
All but DCCs	22/26	0.87%	27.4	611
All but RLKCs	22/26	0.51%	10.5	480
All but $y$ -knapsack	21/26	0.69%	12.3	578
All cuts	22/26	0.47%	9.9	521

# Conclusions

- ▶ A large **improvement over the state-of-the-art for the LRP by applying the VRP solver** and providing callbacks for problem-specific cuts
- ▶ **Route Load Knapsack Cuts reduce the gap** but not yet worth to include in the VRP solver
- ▶ An extension to the **Two-Echelon Capacitated Vehicle Routing problem** allows us to double the size of instances which can be solved to optimality [Marques et al., 2019]
- ▶ 2E-CVRP **demo is available** on  
**[vrpsolver.math.u-bordeaux.fr](http://vrpsolver.math.u-bordeaux.fr)**

# Perspectives

- ▶ Improve separation of Route Load Knapsack Cuts
- ▶ A polyhedral study is needed for the **Multi Capacitated Depot Vehicle Routing Problem**.
- ▶ You can use the VRP solver to test new families of cuts for vehicle routing problems within state-of-the-art Branch-Cut-and-Price!

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