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AN APPLICATION OF THE SOLUTION OF KIRKMAN'S SCHOOLGIRL PROBLEM: THE DECOMPOSITION OF THE SYMMETRIC ORIENTED COMPLETE GRAPH INTO 3-CIRCUITS

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Abstract. In this paper, we show that if K_n^* denotes the symmetric oriented complete graph on n vertices, with $n \neq 6$, then we can partition the arcs of K_n^* into oriented 3-circuits if and only if $n(n-1) \equiv 0 \pmod{3}$. We use the solution of Kirkman's schoolgirl problem given by Ray-Chaudhuri and Wilson [1].

An oriented graph G is a pair (X, U) , where X is a finite set of elements called vertices and U is a subset of $X \times X$ whose members are called arcs (loops are not allowed). One writes (x, y) for the arc with initial vertex x and terminal vertex y . One writes (x, y, z) for the (oriented) 3-circuit (circuit of length 3) formed by the arcs (x, y) , (y, z) and (z, x) taken in this order. K_n^* denotes the symmetric oriented complete graph on n vertices (for every two distinct vertices x, y , there exists an arc (x, y) and an arc (y, x)).

We shall use a theorem of Ray-Chaudhuri and Wilson [1] formulated in terms of balanced incomplete block designs.

Let b, n, r, k, λ be positive integers such that $r = \lambda(n-1)/(k-1)$ and $b = \lambda n(n-1)/k(k-1)$. A (n, k, λ) -balanced incomplete block design (BIBD) consists of a finite set X of n elements and b subsets (called blocks) such that:

- (1) every element occurs in exactly r blocks;
- (2) every block contains exactly k elements;
- (3) every pair of elements occurs together in exactly λ blocks.

A $(n, 3, 1)$ -BIBD is called a Steiner triple system; it is known that such a system exists if and only if $n \equiv 1$ or $3 \pmod{6}$. In terms of graphs (not oriented) the existence of a Steiner triple system means that we can

partition the edges of the complete graph K_n into triangles (or 3-cycles).

A (n, k, λ) -BIBD is said to be resolvable if the b blocks can be partitioned into r classes such that each class consists of a set of pairwise disjoint blocks whose union is X . The theorem of Ray-Chaudhuri and Wilson [1] says that a $(n, 3, 1)$ -resolvable BIBD exists if and only if $n \equiv 3 \pmod{6}$. This means that it is possible to partition the edges of the complete graph K_n , where $n = 6t + 3$, into $(3t + 1)$ classes of $(2t + 1)$ pairwise vertex-disjoint triangles (thus each class covers all the vertices of K_n).

Theorem 1. *Let K_n^* be the symmetric oriented complete graph of n vertices with $n \neq 6$. Then it is possible to partition the arcs of K_n^* into oriented 3-circuits if and only if $n(n-1) \equiv 0 \pmod{3}$.*

Proof. A necessary condition for the existence of a partition of the arcs of K_n^* into oriented 3-circuits is that the number of arcs is a multiple of 3; thus $n(n-1) \equiv 0 \pmod{3}$.

To show that this condition is sufficient (except $n = 6$), we break the proof into 3 cases depending upon the congruent class of n modulo 6.

(1) $n \equiv 1$ or $3 \pmod{6}$. In this case the existence of a Steiner triple system gives a partition of the edges of the complete graph into triangles. If we associate with each triangle $\{x, y, z\}$ of the partition the two opposing 3-circuits (x, y, z) and (x, z, y) , we obtain the desired partition of K_n^* . Moreover in the case $n = 6t + 3$, the partition of the edges of K_n given by the theorem of Ray-Chaudhuri and Wilson implies the existence of a partition of the arcs of K_{6t+3}^* into $2(3t + 1)$ classes of $(2t + 1)$ pairwise vertex-disjoint 3-circuits.

(2) $n \equiv 4 \pmod{6}$; $n = 6t + 4$. Let x_0 be an arbitrary vertex of K_n^* . The oriented graph $K_n^* - \{x_0\}$ obtained by deleting x_0 (and the arcs containing x_0) is a symmetric oriented complete graph with $6t + 3$ vertices. Consider a partition of the arcs of $K_n^* - \{x_0\}$ of the type mentioned above and let $C_1, \dots, C_i, \dots, C_{2t+1}$, where $C_i = (x_i, y_i, z_i)$, be a class of this partition consisting of $(2t + 1)$ pairwise vertex-disjoint 3-circuits.

We obtain a partition of K_n^* with the following 3-circuits:

(i) The $3(2t + 1)$ 3-circuits of the form $(x_0, x_i, y_i), (x_0, y_i, z_i), (x_0, z_i, x_i)$ for all $i = 1, \dots, 2t + 1$.

(ii) The $\frac{1}{3}((6t + 3)(6t + 2)) - (2t + 1) = (2t + 1)(6t + 1)$ 3-circuits of the partition of the arcs of $K_n^* - \{x_0\}$ other than the C_i ($i = 1, \dots, 2t + 1$).

(3) $n \equiv 0 \pmod{6}$, $n \neq 6$; $n = 6t + 6$ with $t \geq 1$. Let x_0, y_0, z_0 be 3 distinct vertices of K_n^* . The oriented graph $K_n^* - \{x_0, y_0, z_0\}$ obtained by deleting the vertices x_0, y_0, z_0 is a symmetric oriented complete graph with $6t + 3$ vertices. Consider a partition of $K_n^* - \{x_0, y_0, z_0\}$ obtained by the theorem of Ray-Chaudhuri and Wilson and let $C_1, \dots, C_i, \dots, C_{2t+1}; C'_1, \dots, C'_j, \dots, C'_{2t+1}; C''_1, \dots, C''_k, \dots, C''_{2t+1}$, where $C_i = (x_i, y_i, z_i)$, $C'_j = (x'_j, y'_j, z'_j)$ and $C''_k = (x''_k, y''_k, z''_k)$, be 3 classes of $(2t + 1)$ pairwise vertex-disjoint 3-circuits (these 3 classes exist if $2(3t + 1) \geq 3$, that is, if $t \geq 1$). We obtain a partition of K_n^* with the following 3-circuits:

- (i) The two 3-circuits (x_0, y_0, z_0) and (x_0, z_0, y_0) .
- (ii) The $3(2t + 1)$ 3-circuits of the form (x_0, x_i, y_i) , (x_0, y_i, z_i) , (x_0, z_i, x_i) for all $i = 1, \dots, 2t + 1$.
- (iii) The $3(2t + 1)$ 3-circuits of the form (y_0, x'_j, y'_j) , (y_0, y'_j, z'_j) , (y_0, z'_j, x'_j) for all $j = 1, \dots, 2t + 1$.
- (iv) The $3(2t + 1)$ 3-circuits of the form (z_0, x''_k, y''_k) , (z_0, y''_k, z''_k) , (z_0, z''_k, x''_k) for all $k = 1, \dots, 2t + 1$.
- (v) The $\frac{1}{3}((6t + 3)(6t + 2)) - 3(2t + 1)$ 3-circuits of the partition of $K_n^* - \{x_0, y_0, z_0\}$ other than the C_i, C'_j, C''_k ($i = 1, \dots, 2t + 1; j = 1, \dots, 2t + 1; k = 1, \dots, 2t + 1$).

Remarks. (1) In the case $n \equiv 1 \pmod{6}$; $n = 6t + 7$ ($t \geq 1$), we can obtain a partition of K_n^* using the partition of K_{6t+3}^* , independent of the existence of a $(6t + 7, 3, 1)$ Steiner triple system.

(2) Theorem 1 gives new $(n, 3, 2)$ -BIBDs for $n(n - 1) \equiv 0 \pmod{3}$.

(3) Note that when Ringel and Youngs [2] inbedded K_n on an oriented surface of genus $\frac{1}{2}((n-3)(n-4))$ for $n \equiv 0, 3, 4, 7 \pmod{12}$, each face was a triangle. Thus they gave a proof of the theorem for $n \equiv 0, 3, 4, 7 \pmod{12}$.

In the case $n = 6$ we have the following proposition.

Proposition 2. *There does not exist a partition of the arcs of K_6^* into 3-circuits.*

Proof. Suppose that there exists such a partition. Let 1 be a vertex.

Among the 3-circuits meeting the vertex 1, either

- (i) two opposing 3-circuits do not exist, without loss of generality we

may suppose that the 3-circuits meeting 1 are labeled $(1,2,3)$, $(1,3,4)$, $(1,4,5)$, $(1,5,6)$, $(1,6,2)$; or

(ii) two opposing 3-circuits exist and without loss of generality we may label the 3-circuits $(1,2,3)$, $(1,3,2)$ and $(1,4,5)$, $(1,5,6)$, $(1,6,4)$.

(It is clear that there can not exist two pairs of opposing circuits meeting 1.)

Case (i). Among the arcs containing the vertex 2 which we have not mentioned: $(2,4)$ must appear in the 3-circuit $(2,4,3)$ (since $(4,5)$ and $(6,2)$ have already been used), $(2,5)$ must appear in the 3-circuit $(2,5,4)$ (since $(5,6)$ and $(3,2)$ have already been used) and $(2,6)$ must appear in the 3-circuit $(2,6,5)$. Then the 6 arcs $(3,5)$, $(3,6)$, $(4,6)$, $(5,3)$, $(6,3)$ and $(6,4)$ remain and with these arcs we can not form two 3-circuits.

Case (ii). Among the arcs containing the vertex 2 which we have not mentioned: $(2,4)$ must appear in the 3-circuit $(2,4,6)$ (since $(4,5)$ and $(3,2)$ have already been used), $(2,5)$ in the 3-circuit $(2,5,4)$ (since $(5,6)$ and $(3,2)$ have already been used) and $(2,6)$ in the 3-circuit $(2,6,5)$. Then the 6 arcs $(3,4)$, $(3,5)$, $(3,6)$, $(4,3)$, $(5,3)$ and $(6,3)$ remain and with these arcs we can not form two 3-circuits.

Since submitting this paper, I have been informed by Professor N. Robertson, that R.H. Bruck has also proved the theorem (by a different method) in an unpublished paper untitled "Existence of the A -maps of A.K. Dewney and N. Robertson" (the theorem was conjectured by Dewney and Robertson).

References

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