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Decomposition of K_n^* into k -circuits and Balanced G -designs

J. C. Bermond

1. INTRODUCTION

Let K_n^* be the complete directed graph with n vertices, that is, every ordered pair of vertices is joined by exactly one arc. By a k -circuit we mean a directed elementary circuit of length k .

We are interested in the following problem: For which values of n is it possible to partition the arcs of K_n^* into k -circuits ($k \leq n$). A necessary condition for such a partition is that the number of arcs is a multiple of k , that is: $n(n-1) \equiv 0 \pmod{k}$. We conjecture that this condition is sufficient except for $n = 6$, $k = 3$; $n = 4 = k$ and $n = 6 = k$. In [2], the case $k = 3$ is solved completely. In a joint paper with V. Faber [5], two methods are developed: An "oriented difference method" and a method of composition using complete directed bipartite graphs. These methods give many results for k even and, in particular, solve completely the cases $k = 4, 6, 8, 10$. By using refinements of the "difference method" we have proved in [3] that the condition is sufficient for $n = 2k$, except $k = 3$.

In this paper, we develop a "composition method" using lexicographic products. This method gives the complete solution for $k = 3$ (with a simpler proof than in [2]) and for $k = 5$.

Our problem is a special case of the problem of the existence of balanced G -designs, introduced by P. Hell and A. Rosa [10]. Let λK_n (λK_n^*) be the complete (complete directed) multigraph, with n vertices, where two vertices are joined by exactly λ edges (arcs) and let G be a graph (directed graph) with k vertices; an (n, k, λ) G -design consists of a partition of the edges (arcs) of λK_n (λK_n^*) into partial subgraphs isomorphic to G . Furthermore, if every vertex of λK_n (λK_n^*) belongs to the same number of partial subgraphs,

the G -design is said to be balanced. If G is regular, that is, if every vertex of G belongs to the same number of edges (arcs) of G , then every G -design is balanced. In the case $G = K_k$ an $(n, k, \lambda)K_k$ -design is nothing else than a B.I.B.D. (balanced incomplete block design) (see M. Hall [8] or H. J. Ryser . The particular case in which G is a k -circuit \vec{C}_k corresponds to our original problem. Other cases have also been studied: C_k -design, C_k a k -cycle (A. Kotzig [17] and A. Rosa [19,20]), balanced P_k -design, P_k a k -chain (chain with k vertices), which is known under the name of "handcuffed design" (P. Hell and A. Rosa [10], S.H.Y. Hung and N.S. Mendelsohn [13], J.F. Lawless [15,16]), G -design, where G is a bipartite graph (C. Huang and A. Rosa [12]), G -design, where G is the transitive tournament with 3 vertices (S.H.Y. Hung and N.S. Mendelsohn [14]).

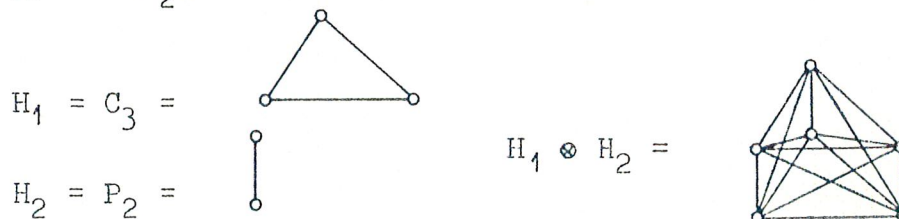
In [3] we have generalized the first two methods to the G -designs, but the composition method using bipartite graphs applies only in some cases (G an even cycle or circuit, G a chain or path). The composition method using the lexicographic product works for all the graphs G . We will first give the general lemmas, but for the applications we will restrict ourselves to the cases $G = K_k, P_k$ and \vec{C}_k , with more emphasis to this last case. (For complements and applications of the methods to packing and covering problems see [3]. For the definitions not given here see C. Berge [1].)

2. LEXICOGRAPHIC PRODUCT AND BASIC LEMMAS

DEFINITION. Let $H_1 = (X_1, U_1)$ and $H_2 = (X_2, U_2)$ be two (directed) graphs; the lexicographic product (called also composition) of H_1 by H_2 , denoted $H_1 \otimes H_2$, is the graph which has the cartesian product $X_1 \times X_2$ as its set of vertices and in which (x_1, x_2) is joined to (y_1, y_2) if and only if $x_1 y_1$ is an edge (arc) of H_1 , or $x_1 = y_1$ and $x_2 y_2$ is an edge (arc) of H_2 .

($H_1 \otimes H_2$ can be considered as formed by replacing each vertex of H_1 by a copy of H_2).

EXAMPLE:



NOTATIONS. We say that the graph H can be decomposed into partial subgraphs isomorphic to G , if we can partition the edges (arcs) of H into partial subgraphs isomorphic to G . S_n will denote a graph formed by n isolated (independent) vertices.

LEMMA 1. If H_1, H_2 and $G \otimes S_n$ (where n is the number of vertices of H_2) can be decomposed into partial subgraphs isomorphic to G , then $H_1 \otimes H_2$ can also be decomposed into partial subgraphs isomorphic to G .

PROOF. Let m be the number of vertices of H_1 and n that of H_2 . Then $H_1 \otimes H_2$ is the edge (arc) disjoint union of $H_1 \otimes S_n$ and $S_m \otimes H_2$. $S_m \otimes H_2$ is the edge (arc) disjoint union of m copies of H_2 and can thus be decomposed into partial subgraphs isomorphic to G if H_2 can be. If H_1 can be decomposed into partial subgraphs isomorphic to G , then $H_1 \otimes S_n$ can be decomposed into partial subgraphs isomorphic to $G \otimes S_n$ and the lemma follows from the hypothesis of the decomposition of $G \otimes S_n$.

PROPOSITION 1. If there exist an (m, k, λ) G -design, an (n, k, λ) G -design and if $G \otimes S_n$ can be decomposed into partial subgraphs isomorphic to G , then there exists an (mn, k, λ) G -design.

PROOF. This result follows from Lemma 1 with $H_1 = \lambda K_m$ or λK_m^* and $H_2 = \lambda K_n$ or λK_n^* .

In order to apply Proposition 1, we will study the decomposition of $G \otimes S_n$ for some particular graphs G .

DEFINITION. (M.Hall [8], chap. 13). An orthogonal array $OA(n, k)$ is a matrix with k rows, n^2 columns, whose elements, denoted $a_{i,j}$ ($i=1, \dots, k; j=1, \dots, n^2$), belong to a set E with n elements and such that:

$$\forall i, i' \{ (a_{i,j}, a_{i',j}) \text{ with } j = 1, \dots, n^2 \} = E \times E.$$

It is known that an orthogonal array $OA(n, k)$ exists if and only if there exist $k - 2$ mutually orthogonal Latin squares of order n .

LEMMA 2. $K_k \otimes S_n$ can be decomposed into partial subgraphs isomorphic to K_k if and only if there exists an orthogonal array $OA(n, k)$.

PROOF. (We will only give a sketch of the proof. The reader can refer to [3] or to the equivalence of transversal designs and orthogonal arrays (R.M.Wilson [21,22] or R.C.Bose and S.S.Shrikharde [6]), the blocks of the transversal design playing the role of the subgraphs of the decomposition). Let us identify the set E of the vertices of $OA(n,k)$ and the set of the vertices of S_n ; thus, the vertices of $K_k \otimes S_n$ are labelled (i,e) with $i = 1, \dots, k$ and $e \in E$. The equivalence of the lemma can be obtained by associating to each column j of $OA(n,k)$ the complete subgraph G_j , isomorphic to K_k , containing the vertices (i,a_{ij},j) and conversely. Orthogonality is equivalent to the fact that each edge of $K_k \otimes S_n$ belongs to exactly one G_j .

PROPOSITION 2. (R.C.Bose and S.S.Shrikharde [7]): If there exist an (m,k,λ) B.I.B.D., an (n,k,λ) B.I.B.D. and an orthogonal array $OA(n,k)$, then there exists an (mn,k,λ) B.I.B.D.

PROOF. Follows from Proposition 1 and Lemma 2.

Let T_k denote any tournament with k vertices; then by taking an orientation of K_k that gives T_k , we obtain the same result as Lemma 2 or Proposition 2 for T_k -designs:

PROPOSITION 2'. If there exists an (m,k,λ) T_k -design, an (n,k,λ) T_k -design and an $OA(n,k)$ then there exists an (mn,k,λ) T_k -design.

LEMMA 3. $\vec{C}_k \otimes S_n$ can be decomposed into k -circuits.

PROOF. This lemma has been proved in [4]. We give here another proof. As the number of arcs of $\vec{C}_k \times S_n$ is kn^2 , it suffices to exhibit n^2 mutually arc-disjoint k -circuits. Let the vertices of \vec{C}_k be the elements of Z_k and those of S_n the elements of Z_n , where Z_p is the additive group of the integers modulo p . The following n^2 k -circuits:

$$(0,j) \dots (i,j+i\alpha) \dots (k-2,j+(k-2)\alpha) (k-1,\alpha) (0,j),$$

where $j \in Z_n$ and $\alpha \in Z_n$, are arc-disjoint. Indeed, suppose for example that:

$$(i,j+i\alpha) (i+1, j+(i+1)\alpha) = (i, j'+i\alpha') (i+1, j'+(i+1)\alpha');$$

then $j+i\alpha = j'+i\alpha'$ and $j+(i+1)\alpha = j'+(i+1)\alpha'$ and by subtraction $\alpha = \alpha'$ and $j = j'$.

From Lemma 1, Proposition 1 and Lemma 3 we deduce:

PROPOSITION 3. If H_1 and H_2 can be decomposed into k -circuits, then also $H_1 \otimes H_2$. In particular, if there exist an $(m, k, \lambda) \vec{C}_k$ -design and an $(n, k, \lambda) \vec{C}_k$ -design then there exists an $(mn, k, \lambda) \vec{C}_k$ -design.

Similar results can be obtained for the k -cycle C_k instead of the k -circuit \vec{C}_k .

LEMMA 4. $P_k \otimes S_n$ can be decomposed into k -chains. (Recall that a k -chain P_k is an elementary chain with k vertices.)

PROOF. Let the vertices of P_k (resp. S_n) be the elements of Z_k (resp. Z_n). Then we have a partition of the edges of $P_k \otimes S_n$ into the following n^2 edge-disjoint k -chains: $(0, j) \dots (i, j + i\alpha) \dots (k-1, j + (k-1)\alpha)$ where $j \in Z_n$ and $\alpha \in Z_n$.

PROPOSITION 4. If there exist an $(m, k, \lambda) P_k$ -design and an $(n, k, \lambda) P_k$ -design, then there exists an $(mn, k, \lambda) P_k$ -design.

REMARK. In the proof of Lemma 4, we notice that the end-vertices of the chains of the decomposition of $P_k \otimes S_n$ are the vertices $(0, j)$ and $(k-1, j)$, $j \in Z_n$ and 0 and $k-1$ are the end-vertices of P_k . Thus we obtain:

PROPOSITION 4'. (P.Hell and A.Rosa [10], J.F.Lawless [16].) If there exist a balanced $(m, k, \lambda) P_k$ -design and a balanced $(n, k, \lambda) P_k$ -design, then there exists a balanced $(mn, k, \lambda) P_k$ -design.

Similar results can also be obtained for k -paths \vec{P}_k .

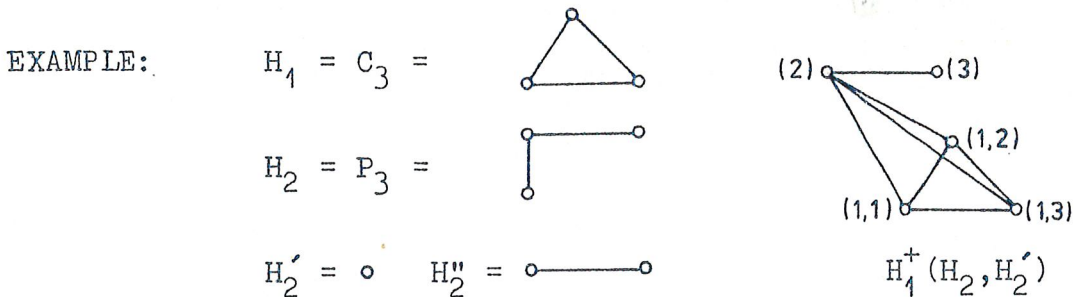
3. GENERALIZATION

We shall now consider a generalization of the lexicographic product which we have introduced in [4]. It is obtained by identifying, in $H_1 \otimes H_2$, h points of each copy of H_2 . The exact definition is:

DEFINITION. Let m, n, h be three integers. Let $H_1 = (X_1, U_1)$ be a graph (directed or not) and $H_2 = (X_2' \cup X_2'', U_2)$ be a graph

(directed or not) with $|X_2'| = n$ and $|X_2''| = h$. The vertices of X_2' are labelled from 1 to n and those of X_2'' from $n+1$ to $n+h$. Let H_2' be the subgraph generated by X_2' and H_2'' the subgraph generated by X_2'' . We define a new graph $H_1^+(H_2, H_2')$ as follows: Its vertex set is $(X_1 \times X_2') \cup X_2''$; the vertices are labelled (i_1, i_2) with $i_1 = 1, \dots, m$; $i_2 = 1, \dots, n$ if they belong to $X_1 \times X_2'$ and (k) with $k = n+1, \dots, n+h$ if they belong to X_2'' . Two vertices are joined if and only if they are of the form:

- (i_1, i_2) and (j_1, j_2) with $i_1 j_1$ an edge (arc) of H_1 or $i_1 = j_1$ and $i_2 j_2$ an edge (arc) of H_2' ;
- (i_1, i_2) and (k) , (i_2, k) being an edge (arc) of H_2 ;
- (k) and (k') , kk' being an edge (arc) of H_2'' .



REMARKS. In the case $h = 0$, we obtain the lexicographic product $H_1 \otimes H_2$. In general, $H_1^+(H_2, H_2')$ is the edge (arc) disjoint union of $H_1 \otimes S_n$, H_2'' and p copies of $H_2 - H_2''$ (graph obtained from H_2 by deleting the edges (arcs) of H_2''). In the particular case $h = 1$, that is H_2'' being reduced to one vertex, $H_1^+(H_2, H_2')$ is the edge (arc)-disjoint union of $H_1 \otimes S_n$ and p copies of H_2 (because there is no arc in H_2''). From this remark, we obtain exactly as in Lemma 1 the following lemma:

LEMMA 5. If $H_1, H_2'', G \otimes S_n$ (where n is the number of vertices of H_2') and $H_2 - H_2''$ (the graph obtained from H_2 by deleting the arcs of H_2'') can be decomposed into partial subgraphs isomorphic to G , then $H_1^+(H_2, H_2')$ can also be decomposed into partial subgraphs isomorphic to G .

LEMMA 5'. If $H_1, H_2, G \otimes S_n$ ($n+1$ being the number of vertices of H_2) can be decomposed into partial subgraphs isomorphic to G , then the same is true for $H_1^+(H_2, \{x\})$.

By taking $H_1 = K_m$ or K_m^* , $H_2 = K_{n+h}$ or K_{n+h}^* , $H_2' = K_n$ or K_n^* and thus $H_2'' = K_n$ or K_n^* , we obtain:

PROPOSITION 5. If there exists an (m, k, λ) G-design, an (h, k, λ) G-design and if $G \otimes S_n$ and $K_{n+h} - K_n$ or $K_{n+h}^* - K_n^*$ can be decomposed into partial subgraphs isomorphic to G, then there exists an $(mn + h, k, \lambda)$ G-design.

PROPOSITION 5'. If there exists an (m, k, λ) G-design, an $(n+1, k, \lambda)$ G-design and if $G \otimes S_n$ can be decomposed into partial subgraphs isomorphic to G, then there exists an $(mn+1, k, \lambda)$ G-design.

In the case $G = K_k$, by using Lemma 2 we obtain:

PROPOSITION 6. If there exists an (m, k, λ) B.I.B.D., an (h, k, λ) B.I.B.D., an orthogonal array $OA(n, k)$ and if $K_{n+h} - K_h$ can be decomposed into partial subgraphs isomorphic to K_k , then there exists an $(mn+h, k, \lambda)$ B.I.B.D.

PROPOSITION 6'. (R.C.Bose and S.S.Shrikhande [7].) If there exist an (m, k, λ) B.I.B.D., an $(n+1, k, \lambda)$ B.I.B.D. and an orthogonal array $OA(n, k)$, then there exists an $(mn+1, k, \lambda)$ B.I.B.D.

Proposition 6 seems to be new and may yield new B.I.B.D.'s. An interesting case of application is the case $h = k$; there always exists a (k, k, λ) B.I.B.D. and the existence of a decomposition of $K_{n+k} - K_k$ into K_k is equivalent to the existence of an $(n+k, k, \lambda)$ B.I.B.D. (it suffices to take as K_k one of the blocks of the B.I.B.D.). Thus we have

COROLLARY. If there exists an (m, k, λ) B.I.B.D., an $(n+k, k, \lambda)$ B.I.B.D., an orthogonal array $OA(n, k)$, then there exists an $(mn+k, k, \lambda)$ B.I.B.D.

As an example we have the following B.I.B.D.'s that cannot be obtained in [7] (they are known from the work of H.Hanani [9]):

(285, 5, 1) B.I.B.D. by taking $m = 5$, $k = 5$, $n = 56$, $n + k = 61$,

(232, 4, 1) B.I.B.D. by taking $m = 4$, $k = 4$, $n = 57$, $n + k = 61$.

In the case $G = \vec{C}_k$, by using Lemma 3 we obtain:

PROPOSITION 7. If K_m^* , K_h^* and $K_{n+h}^* - K_h^*$ can be decomposed into k-circuits, then K_{mn+h}^* can be decomposed into k-circuits.

PROPOSITION 7'. If K_m^* and K_{n+1}^* can be decomposed into k-circuits, then K_{mn+1}^* can be decomposed into k-circuits.

We can also obtain similar theorems with $G = C_k, P_k, \vec{P}_k$ instead of \vec{C}_k . We shall apply Propositions 3, 7 and 7' to solve completely the case $k=3$ and 5 of the decomposition of K_n^* into k -circuits.

4. DECOMPOSITION OF K_n^* INTO 3-CIRCUITS

We shall prove the following theorem which solves completely the problem of the existence of a decomposition of K_n^* into 3-circuits (case $k=3$ of the conjecture of the introduction). This problem was solved in [2] as well as in N.S.Mendelsohn [18], but the proof given here is simpler.

THEOREM 1. Let n be an integer, $n \equiv 0$ or $1 \pmod{3}$, $n \neq 4, 6$; then there exists a decomposition of K_n^* into 3-circuits, containing two opposite 3-circuits (that is a partial subgraph isomorphic to K_3^*).

PROOF. We prove the theorem by induction on n . The theorem is true for $n=3$. For $n=4$ there exists a decomposition of K_4^* into 3-circuits, but no decomposition containing a K_3^* . For $n=6$ there exists no decomposition of K_6^* into 3-circuits (see [2]). For $n=13, 16, 18$ we can find a decomposition containing a K_3^* : For $n=13$ by using the $(13, 3, 1)$ B.I.B.D.; for $n=16$ and 18 by applying the method of [2] with a solution of Kirkman's schoolgirl problem for only $n=15$. Suppose that n_0 is an integer, $n_0 \equiv 0$ or $1 \pmod{3}$ and that the theorem is true for all $n < n_0$ ($n \neq 4, 6$) with $n \equiv 0$ or $1 \pmod{3}$. We break the proof in four cases depending on the congruences modulo 9.

Case 1: $n_0 = 3t$ with $t \equiv 0$ or $1 \pmod{3}$.

$K_{n_0}^*$ can be decomposed by Proposition 3, with $m=3$, $n=t$ and by noticing that K_{mn}^* contains a K_3^* if at least K_m^* (or K_n^*) contains a K_3^* . The case $t=6$, $n_0=18$ is solved above.

Case 2: $n_0 = 3t + 1$ with $t \equiv 0$ or $1 \pmod{3}$.

$K_{n_0}^*$ can be decomposed into 3-circuits by Proposition 7' with $m=t$, $n=3$ ($n+1=4$). This works except if $t=4$ or 6 . If $t=4$, $n_0=13$ has been solved above. If $t=6$, $n_0=19$ and the theorem follows from Proposition 7' with $m=9$, $n=2$.

Case 3: $n_0 = 9k + 7 = 3(3k + 2) + 1$.

The theorem results from Proposition 7' with $m=3$, $n=3k+2$,

$n + 1 = 3k + 3$; except if $k = 1$, but in this case $n_0 = 16$ (solved above).

Case 4: $n_0 = 9k + 6 = 3(3k + 1) + 3$ with $k \neq 0$.

The theorem follows from Proposition 7 with $m=3$, $h=3$, $n=3k+1$, $n + h = 3k + 4$ and by the induction hypothesis that $K_{3k+4}^* - K_3^*$ can be decomposed into 3-circuits.

REMARKS. The theorem gives a new proof of the existence of an $(n,3,2)$ B.I.B.D. If we want to prove the existence of an $(n,3,2)$ B.I.B.D. or an $(n,3,1)$ TT_3 -design (called directed triple systems by S.H.Y.Hung and N.S.Mendelsohn [14]), the proof is quite the same as the proof of Theorem 1 but simpler because there exists a solution for $n = 6$. Indeed, we do not need as induction hypothesis that the decomposition contains a K_3^* : this was used only in Case 4. In this case if $k = 2p$, that is $n_0 = 18p + 6 = 6(3p + 1)$, the result follows from Propositions 2' or 3 with $m = 6$ and $n = 3p + 1$. If $k = 2p + 1$, that is $n_0 = 18p + 15 = 2(9p + 7) + 1$, the result follows from Proposition 5' with $m = 9p + 7$, $n = 2$, $n + 1 = 3$. And we need only to start the induction to have the decomposition for $n = 3, 4, 6$. Thus we have:

THEOREM 2. If $n \equiv 0$ or $1 \pmod{3}$ there exists an $(n,3,2)$ B.I.B.D. and an $(n,3,1)$ TT_3 -design [14].

The same method has been used by A.J.W.Hilton [11] to obtain a proof of the existence of an $(n,3,1)$ B.I.B.D. or Steiner triple systems. The method can be used also to have the existence of an $(n,3,\lambda)$ B.I.B.D. for any λ , or other G -designs where G has 3 vertices.

5. DECOMPOSITION OF K_n^* INTO 5-CIRCUITS

We shall prove the following theorem:

THEOREM 3. K_n^* can be decomposed into 5-circuits if and only if $n \equiv 0$ or $1 \pmod{5}$.

PROOF. To prove the theorem we shall use the Propositions 3, 7', 7 called here respectively A, B, C and some lemmas D_i , whose proofs use the difference method, but are long (the reader can find all these proofs in [3]).

A: If K_m^* and K_n^* can be decomposed into 5-circuits, then K_{mn}^* can be decomposed as well.

B: If K_m^* and K_{n+1}^* can be decomposed into 5-circuits, then K_{mn+1}^* can be decomposed as well.

C: If K_m^* , K_n^* and $K_{n+h}^* - K_h^*$ can be decomposed into 5-circuits, then K_{mn+h}^* can be decomposed as well.

D: There exists a decomposition into 5-circuits of:

$D_1: K_5^*$; $D_2: K_6^*$; $D_3: K_{10}^* - K_5^*$; $D_4: K_{10}^* - K_6^*$.
 $D_5: K_{11}^* - K_5^*$; $D_6: K_{11}^* - K_6^*$; $D_7: K_{15}^*$; $D_8: K_{16}^* - K_{10}^*$.
 $D_9: K_{16}^* - K_6^*$; $D_{10}: K_{20}^* - K_{10}^*$; $D_{11}: K_{40}^* - K_{20}^*$; $D_{12}: K_{45}^* - K_{15}^*$.
 (Recall that $K_{n+h}^* - K_h^*$ means the graph obtained from K_{n+h}^* by deleting the arcs of a K_h^*).

With all these lemmas we can prove the theorem by induction. We shall indicate which propositions or lemmas are used, with the corresponding values.

The theorem is true for

$n = 5$ (D_1); $n = 6$ (D_2); $n = 10$ (D_3 or D_4); $n = 11$ (D_5 or D_6);
 $n = 15$ (D_7); $n = 16$ (D_8 or D_9) and $n = 20$ (D_{10}).

Suppose that $n_0 \equiv 0$ or $1 \pmod{5}$ and that the theorem is true for all $n < n_0$ with $n \equiv 0$ or $1 \pmod{5}$. We break the proof in many cases depending on the congruences modulo 300.

Case 1: $n_0 = 10t + 1$, $t \geq 2$.

1.a $t=2p$ ($p \geq 1$), $n_0=20p+1$, B: $m=5p$, $n=4$.
 1.b $t=3p$ ($p \geq 1$), $n_0=30p+1$, B: $m=6$, $n=5p$.
 1.c $t=3p+1$ ($p \geq 1$), $n_0=30p+11$, C, D_5 : $m=5p+1$, $n=6$, $h=5$.
 1.d it remains $t \equiv 5 \pmod{6}$
 $t=30k+5$, $n_0=300k+51$, B: $m=60k+10$, $n=5$.
 $t=30k+11$, $n_0=300k+111$, B: $m=30k+11$, $n=10$.
 $t=30k+17$, $n_0=300k+171$, B: $m=5$, $n=60k+34$.
 $t=30k+23$, $n_0=300k+231$, B: $m=60k+46$, $n=5$.
 $t=30k+29$, $n_0=300k+291$, B: $m=10$, $n=30k+29$.
 . .

Case 2: $n_0 = 10t + 5$, $t \geq 2$.

2.a $t=2p$ ($p \geq 1$), $n_0=20p+5$, B: $m=5p+1$, $n=4$.
 2.b $t=3p$ ($p \geq 1$), $n_0=30p+5$, C, D_5 : $m=5p$, $n=6$, $h=5$.
 2.c $t=3p+2$, $n_0=30p+25$, B: $m=6$, $n=5p+4$.
 2.d it remains $t \equiv 1 \pmod{6}$

$t=30k+4,$	$n_0=300k+15,$	A:	$m=15,$	$n=20k+4.$
$t=30k+7,$	$n_0=300k+75,$	A:	$m=15,$	$n=20k+5.$
$t=30k+13.$	$n_0=300k+135,$	$C, D_3:$	$m=60k+26,$	$n=5,$ $h=5.$
$t=30k+19,$	$n_0=300k+195,$	$C, D_{12}:$	$m=10k+6,$	$n=30,$ $h=15.$
$t=30k+25,$	$n_0=300k+255,$	A:	$m=5,$	$n=60k+51.$

Case 3: $n_0 = 10t + 6, \quad t \geq 2.$

3.a	$t=2p \ (p \geq 1),$	$n_0=20p+6,$	$C, D_4:$	$m=5p,$	$n=4,$	$h=6.$
3.b	$t=3p \ (p \geq 1),$	$n_0=30p+6,$	A:	$m=6,$	$n=5p+1.$	
3.c	$t=3p+1 \ (p \geq 1),$	$n_0=30p+16,$	$C, D_8:$	$m=5p+1,$	$n=6,$	$h=10.$
3.d	it remains $t \equiv 5 \pmod{6}$					
	$t=30k+5,$	$n_0=300k+56,$	B:	$m=60k+11,$	$n=5.$	
	$t=30k+11,$	$n_0=300k+116,$	$C, D_9:$	$m=30k+11,$	$n=10,$	$h=6.$
	$t=30k+17,$	$n_0=300k+176,$	B:	$m=60k+35,$	$n=5.$	
	$t=30k+23,$	$n_0=300k+236,$	$C, D_6:$	$m=60k+46,$	$n=5,$	$h=6.$
	$t=30k+29,$	$n_0=300k+296,$	B:	$m=5,$	$n=60k+59.$	

Case 4: $n_0 = 10t + 10, \quad t \geq 2.$

4.a	$t=2p \ (p \geq 1),$	$n_0=20p+10,$	$C, D_4:$	$m=5p+1,$	$n=4,$	$h=6.$
4.b	$t=3p \ (p \geq 1),$	$n_0=30p+10,$	$C, D_8:$	$m=5p,$	$n=6,$	$h=10.$
4.c	$t=3p+2,$	$n_0=30p+30,$	A:	$m=6,$	$n=5p+5.$	
4.d	it remains $t \equiv 1 \pmod{6}$					
	$t=30k+1,$	$n_0=300k+20,$	A:	$m=20,$	$n=15k+4.$	
	$t=30k+7,$	$n_0=300k+80,$	A:	$m=5,$	$n=60k+16.$	
	$t=30k+13,$	$n_0=300k+140,$	$C, D_{11}:$	$m=15k+6,$	$n=20,$	$h=20.$
	$t=30k+19,$	$n_0=300k+200,$	A:	$m=5,$	$n=60k+40.$	
	$t=30k+25,$	$n_0=300k+260,$	A:	$m=10,$	$n=30k+26.$	

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