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# THE CIRCUIT – HYPERGRAPH OF A TOURNAMENT

J-C. BERMOND\*

Let  $\tau(T_n)$  be the minimal number of arcs which must be deleted (or reversed) in a tournament  $T_n$  in order to obtain a graph without circuits. In this article we give results on the function  $f(n) = \max_{T_n} \tau(T_n)$  (maximum taken over all the tournaments with  $n$  vertices), answering a conjecture of Kotzig [7]. Other known results and references can be found in [2].

## 1. DEFINITIONS AND PRELIMINARY RESULTS

Throughout this paper  $G = (X, U)$  will denote a finite oriented graph, where  $X$  is the set of vertices and  $U$  the set of arcs.  $T_n$  will denote a tournament (graph  $(X, U)$  in which each pair of vertices  $x, y$  is joined by one and only one of the arcs  $\overrightarrow{xy}$  and  $\overrightarrow{yx}$ ), with  $|X| = n$ , and so  $|U| = \frac{n(n-1)}{2}$ . A circuit of  $G$  is a sequence of *different* vertices  $x_1 x_2 \dots x_i x_{i+1} \dots x_p$  such that  $\overrightarrow{x_i x_{i+1}}$  ( $i = 1, 2, \dots, p-1$ ) and  $\overrightarrow{x_p x_1}$  are arcs of  $G$ .

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**Definition 1.** To a graph  $G = (X, U)$  we associate the hypergraph  $C(G) = (U, C)$ , called the *circuit-hypergraph of  $G$* , whose set of vertices is  $U$  (set of arcs of  $G$ ) and whose edges are the circuits of  $G$ . This hypergraph has been studied by Chaty [4].

**Definition 2** (see Berge [1]). A *transversal set* of a hypergraph  $H$  is a set of vertices of  $H$  that meet every edge of the hypergraph and the *transversal number*  $\tau(H)$  is the minimum number of elements of a transversal set.

Let  $\tau(G)$  denote the transversal number of the circuit-hypergraph  $C(G)$  associated to  $G$ , that is, the minimum number of arcs that we must delete from  $G$  in order to obtain a graph without circuits. It has been proved by Grindberg and Dambit, Remage and Thompson (see [2], p. 9) that  $\tau(G)$  is also the minimum number of arcs whose orientation must be reversed in order to obtain a graph without circuits.

**Example.** For the tournament  $T_5$  of Fig. 1 14, 15, 25 is a transversal set, so  $\tau(T_5) \geq 3$ . We shall see later that  $\tau(T_5) = 3$ .

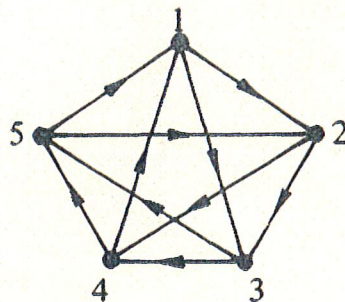


Fig. 1

Instead of  $\tau(G)$ , we can consider the number  $w(G)$ , the stability number of the circuit-hypergraph of  $G$ , which is the maximum cardinality of a set of arcs such that the partial graph generated by these arcs is without circuits (such a set of arcs is called a set of consistent arcs by Erdős and Moon [5]). We have the relation  $\tau(G) + w(G) = |U|$ .  $\tau(G)$  has been introduced by Slater [9] and  $w(G)$  by Erdős and Moon [5]. The number  $\tau(G)$  has importance in "paired comparisons" (see [2])

for further details and references). In this case  $G$  is a tournament and one of the problems is to determine the function  $f(n) = \max_{T_n} \tau(T_n)$ , where the maximum is taken over all the tournaments with  $n$  vertices; as has remarked Kotzig [7],  $f(n)$  is also the maximum of  $\tau(G)$  taken over all the antisymmetric graphs  $G$  with  $n$  vertices.

Various results have been obtained about the function  $f(n)$  (see [2]), essentially upper bounds and that  $\frac{f(n)}{\binom{n}{2}} \rightarrow \frac{1}{2}$  if  $n \rightarrow \infty$  by Erdős and Moon [15]. Here we are interested in lower bounds of  $f(n)$ .

**Definition 3.**  $H$  being a hypergraph, the *matching number* of  $H$ :  $\nu(H)$  is the maximum number of disjoint edges of  $H$ .

Let  $\nu(G)$  denote the matching number of the circuit-hypergraph  $C(G)$ , that is the maximum number of arc-disjoint circuits of  $G$ .

By a theorem on hypergraph ([1], p. 405) we have:

**Proposition 1.** For every graph  $G$ :  $\tau(G) \geq \nu(G)$ .

**Example.** In the tournament  $T_5$  of Fig. 1 we have 3 arc-disjoint circuits: 124, 135, 2345. Thus  $\nu(T_5) \geq 3$  and by Proposition 1  $\tau(T_5) = \nu(T_5) = 3$ .

**Definition 4.** The maximum number of edge-disjoint cycles of the non-oriented complete graph  $K_n$  will be denoted by  $\pi(n)$ . It has been proved by Chartrand, Geller and Hedetniemi [3] that  $\pi(n) = \left\lceil \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rceil$  (where  $\lfloor x \rfloor$  is the greatest integer not exceeding  $x$ ).

**Proposition 2.**  $f(n) \geq \pi(n) = \left\lceil \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rceil$ : (For a proof see [2]).

By using the upper bound of Jung [6] (see also [2]) we have:

**Proposition 3.** For  $n \leq 9$ :  $f(n) = \pi(n)$ .

In [2] p. 21 and 22 we have determined the values of  $f(10)$  and  $f(11)$ .

**Proposition 4.**  $f(10) = 15$  ( $\pi(10) = 13$ );  $f(11) = 20$  ( $\pi(11) = 18$ ).

Kotzig [7] has also found independently Propositions 1, 2, 3 and the value of  $f(10)$ . He has also conjectured that:

**Conjecture 1** (Kotzig [7]). For  $n \geq 10$   $f(n) > \pi(n)$ .

In his article Kotzig has proved that if  $f(n) > \pi(n)$  then  $f(3n) > \pi(3n)$ . Here we shall prove similar results for  $f(3n + 1)$  and  $f(3n + 2)$  and so to prove the conjecture it would be sufficient to prove it for a small number of values of  $n$ . For this purpose we need results on the transversal number of the circuit-hypergraph associated to the lexicographic product (and generalized product) of two graphs.

## 2. LEXICOGRAPHIC PRODUCT OF GRAPHS

**Definition.** Let  $G = (X, U)$  and  $H = (Y, V)$  be two (oriented) graphs the *lexicographic product of  $G$  by  $H$* , noted  $G \otimes H$ , is the graph which has the Cartesian product  $X \times Y$  as set of vertices and in which  $(x_1, y_1)$  is joined to  $(x_2, y_2)$  if and only if either  $\overrightarrow{x_1 x_2}$  is an arc of  $G$  or  $x_1 = x_2$  and  $\overrightarrow{y_1 y_2}$  is an arc of  $H$ .  $G \otimes H$  is called composition of  $G$  with  $H$  in Moon [8] p. 78. (We can consider this graph as obtained by replacing each vertex of  $G$  by a copy of  $H$ ).

**Example:**  $G = C_3$ ,  $H = K_2$ .

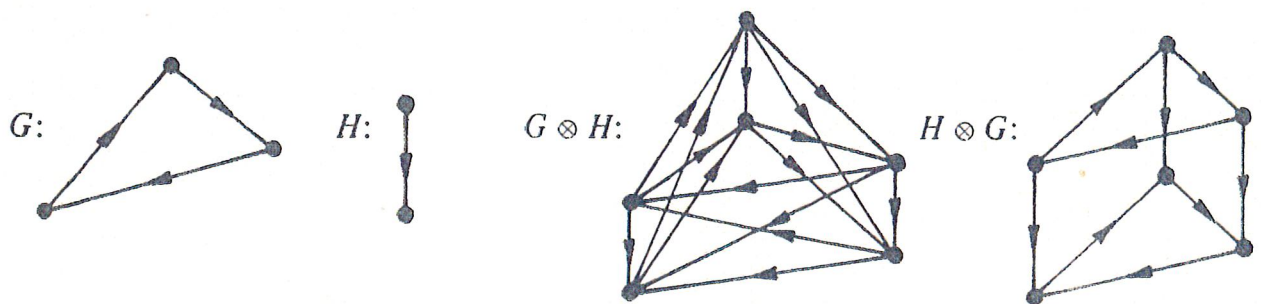


Fig. 2

**Proposition 5.** Let  $G = (X, U)$  and  $H = (Y, V)$  be two graphs with  $|X| = p$  and  $|Y| = q$ , then  $\tau(G \otimes H) \leq p\tau(H) + q^2\tau(G)$ .

**Proof.** First we prove that if  $G$  and  $H$  are two graphs without circuits then  $G \otimes H$  is without circuits. Suppose, on the contrary, that  $G \otimes H$  contains a circuit  $(x_1, y_1)(x_2, y_2) \dots (x_k, y_k)$ . If all the first indices  $x_i$  of the vertices of the circuit were equal, then  $y_1 y_2 \dots y_k$  would be a circuit of  $H$  contrary to the hypothesis. Thus, two of the  $x_i$ 's are different; let us consider the different  $x_i$ 's in the order in which they appear in the circuit:  $x_{i_1}, x_{i_2}, \dots, x_{i_p}$ , then by definition of the lexicographic product  $x_{i_1} x_{i_2} \dots x_{i_p}$  contains a circuit of  $G$ , contrary to the hypothesis.

According to the definition of  $\tau(G)$  and  $\tau(H)$  we can obtain a graph  $G'$  without circuits, by deleting  $\tau(G)$  arcs from  $G$  and a graph  $H'$  without circuits by deleting  $\tau(H)$  arcs from  $H$ . By the first part of the proof  $G' \otimes H'$  is a graph without circuits and is obtained from  $G \otimes H$  by deleting the  $q^2\tau(G)$  arcs of the form  $(x_1, y)(x_2, y')$ ,  $\overrightarrow{x_1 x_2}$  being an arc of  $G - G'$  and the  $p\tau(H)$  arcs of the form  $(x, y_1)(x, y_2)$  where  $\overrightarrow{y_1 y_2}$  is an arc of  $H - H'$  (the  $x$  is the same).

**Conjecture 2.**  $\tau(G \otimes H) = p\tau(H) + q^2\tau(G)$ .

We are able to prove the conjecture, when  $\tau(G) = \nu(G)$ , by considering  $\nu(G \otimes H)$ .

**Proposition 6.** Let  $S_q$  be the graph with  $q$  vertices and no arcs, then  $\nu(G \otimes S_q) \geq q^2\nu(G)$ .

**Proof.** Let the vertices of  $S_p$  be labelled:  $1, 2, \dots, q$ . To the circuit  $x_1 x_2 \dots x_i \dots x_k$  of  $G$ , we associate the  $q^2$  arc-disjoint circuits  $(x_1, j)(x_2, j + \alpha) \dots (x_i, j + (i - 1)\alpha) \dots (x_{k-1}, j + (k - 2)\alpha)(x_k, \alpha)$ , where  $j = 1, 2, \dots, q$ ,  $\alpha = 1, 2, \dots, q$  and where the second indices of the vertices are to be taken modulo  $q$ . Then to the  $\nu(G)$  arc-disjoint circuits of  $G$ , we associate  $q^2\nu(G)$  arc-disjoint circuits in  $G \otimes S_q$ .

**Conjecture 3.**  $\nu(G \otimes S_q) = q^2\nu(G)$ .

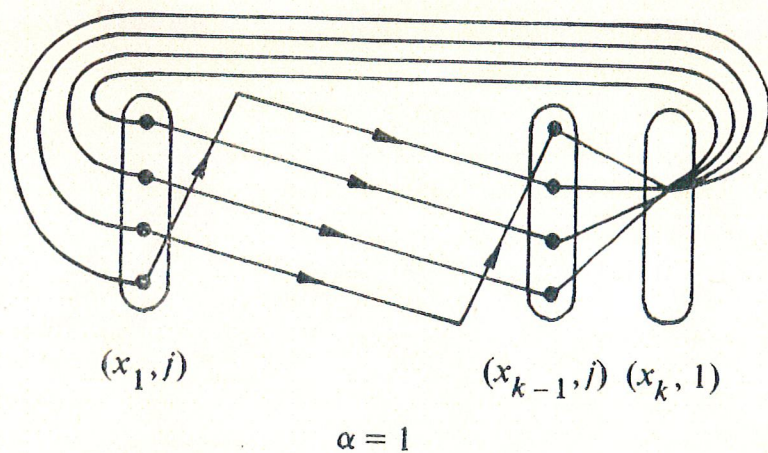
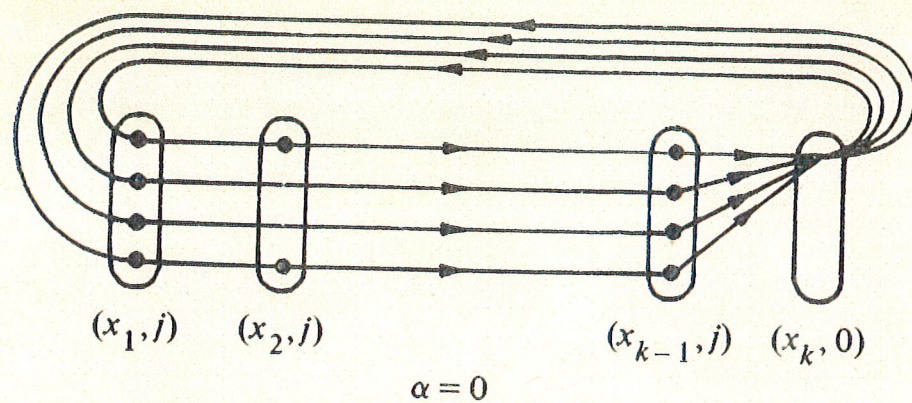


Fig. 3

**Lemma.**  $\tau(S_p \otimes H) = p\tau(H)$  and  $\nu(S_p \otimes H) = p\nu(H)$ .

**Proof.** This results from the decomposition of  $S_p \otimes H$  into the union of  $p$  disjoint copies of  $H$ .

**Proposition 7.**  $\nu(G \otimes H) \geq p\nu(H) + q^2\nu(G)$ .

**Proof.** This results from the decomposition of  $G \otimes H$  into the union of the two arc-disjoint graphs  $S_p \otimes H$  and  $G \otimes S_q$ .

**Proposition 8.**  $\tau(G \otimes H) \geq p\tau(H) + q^2\nu(G)$ .

**Proof.** This results from the decomposition of  $G \otimes H$  into  $S_p \otimes H$

and  $G \otimes S_q$ , the lemma applied to  $\tau(S_p \otimes H)$  and the fact that:  
 $\tau(G \otimes S_q) \geq \nu(G \otimes S_q) \geq q^2 \nu(G)$  (Propositions 1 and 6).

**Corollary.** *When  $\tau(G) = \nu(G)$ , then Conjectures 2 and 3 are true.*

**Theorem 1.**  $f(pq) \geq pf(q) + q^2 \pi(p)$ .

**Proof.** The theorem can be deduced from Proposition 8, by choosing as graph  $G$  a tournament  $T_p$ , with  $\nu(T_p) = \pi(p)$ , and as graph  $H$  a tournament  $T_q$ , with  $\tau(T_q) = f(q)$ .

**Examples.**  $p = 3, q = 10: f(30) \geq 145, (\pi(30) = 140); p = 3, q = 11, f(33) \geq 181, (\pi(33) = 176); p = 7, q = 10, f(70) \geq 805, (\pi(70) = 793), p = 7, q = 11: f(77) \geq 987, (\pi(77) = 975)$ .

In the case  $p = 3$  Theorem 1 leads to  $f(3n) \geq n^2 + 3f(n)$ . This result has been proved by Kotzig ([7]); from this result Kotzig has deduced the following theorem:

**Theorem 2** (Kotzig [7]). *If  $f(n) > \pi(n)$ , then  $f(3n) > \pi(3n) + 1$ .*

**Proof.** From the value of  $\pi(n) = \left\lfloor \frac{n}{3} \left[ \frac{n-1}{2} \right] \right\rfloor$ , we obtain the values of the following table, which proves that  $n^2 + 3\pi(n) \geq \pi(3n) - 1$ .

$n$	$\pi(n)$	$n^2 + 3\pi(n)$	$\pi(3n)$
$6k$	$6k^2 - 2k$	$54k^2 - 6k$	$54k^2 - 6k$
$6k + 1$	$6k^2 + k$	$54k^2 + 15k + 1$	$54k^2 + 15k + 1$
$6k + 2$	$6k^2 + 2k$	$54k^2 + 30k + 4$	$54k^2 + 30k + 4$
$6k + 3$	$6k^2 + 5k + 1$	$54k^2 + 51k + 12$	$54k^2 + 51k + 12$
$6k + 4$	$6k^2 + 6k + 1$	$54k^2 + 66k + 19$	$54k^2 + 66k + 20$
$6k + 5$	$6k^2 + 9k + 3$	$54k^2 + 87k + 34$	$54k^2 + 87k + 35$

If  $f(n) > \pi(n)$ , then  $f(n) \geq \pi(n) + 1$  and so by Theorem 1:

$$f(3n) \geq n^2 + 3f(n) \geq n^2 + 3\pi(n) + 3 \geq \pi(3n) + 2 > \pi(3n) + 1.$$



### 3. GENERALIZATION AND APPLICATIONS TO $f(n)$

**Definition.** Let  $p, q, h$  be three integers; let  $G = (X, U)$  be a graph with  $|X| = p$ ; let  $H = (Y \cup Z, V)$  be a graph with vertex set the union of the disjoint sets  $Y$  and  $Z$ , where  $|Y| = q$  and  $|Z| = h$  (the vertices of  $Y$  are labelled from 1 to  $q$  and those of  $Z$  from  $q + 1$  to  $q + h$ ); and let  $H'$  be the subgraph of  $H$  generated by  $Z$ , and  $H''$  the subgraph of  $H$  generated by  $Y$ . We define a new graph  $G^\pm(H, H')$  as follows: its vertex set is  $(X \times Y) \cup Z$  and the vertices are labelled  $(i, j)$  with  $i = 1, 2, \dots, p$  and  $j = 1, 2, \dots, q$  (if they belong to  $X \times Y$  and  $(k)$  with  $k = q + 1, \dots, q + h$  (if they belong to  $Z$ ); in  $G^\pm(H, H')$  the subgraph generated by  $X \times Y$  is isomorphic to  $G \otimes H''$ , the subgraphs  $H_{i_0}$  ( $p$  in number) generated by the vertices  $(i_0, j)$  with  $j = 1, \dots, q$  and  $(k)$  with  $k = q + 1, \dots, q + h$  are isomorphic to  $H$  for all  $i_0$ . Thus two vertices of  $G^\pm(H, H')$  are joined if and only if they are of the form  $(i_1, j_1)$  and  $(i_2, j_2)$  if either  $\overrightarrow{i_1 i_2}$  is an arc of  $G$  or  $i_1 = i_2$  and  $\overrightarrow{j_1 j_2}$  is an arc of  $H$ ;  $(i, j)$  and  $(k) - \overrightarrow{j k}$  being an arc of  $H$ ;  $(k)$  and  $(k') - \overrightarrow{k k'}$  belong an arc of  $H$ .

In the case  $Z = \phi$ , we obtain the lexicographic product  $G \otimes H$ .

**Example.**

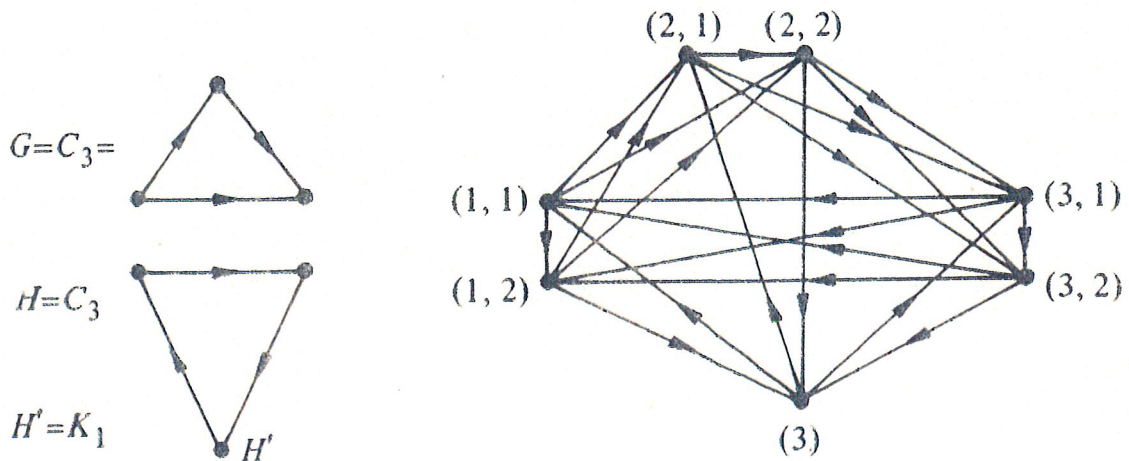


Fig. 4

We want to find  $\tau(G^\pm(H, H'))$  or at least a lower bound; this can be

done by decomposing  $G^\pm(H, H')$  into the union of arc-disjoint graphs. We shall give here a lower bound which is good when  $\frac{h}{q}$  is small.

**Proposition 9.** *Let  $m(H')$  denote the number of arcs of  $H'$ , then:*

$$\tau(G^\pm(H, H')) \geq q^2 \nu(G) + p\tau(H) - (p-1)(m(H') - \tau(H')).$$

**Proof.**  $G^\pm(H, H')$  can be considered as the arc-disjoint union of  $G \otimes S_q$ ,  $H'$ , and the  $p$  copies of  $H - H': H_{i_0} - H'$  (formed by the partial graphs of  $H_{i_0}$  obtained by deleting the arcs of  $H'$ ). Let  $S$  be a transversal set of  $G^\pm(H, H')$ :

$$(1) |S \cap (G \otimes S_q)| \geq \tau(G \otimes S_q) \geq q^2 \nu(G) \text{ by Propositions 1 and 6.}$$

(2)  $|S \cap H'| \leq m(H') - \tau(H')$ . Indeed let us consider the graph  $\bar{H}'$  obtained by reversing the orientation of all the arcs of  $H'$  (in number  $m(H')$ ),  $\tau(\bar{H}') = \tau(H')$  and so if we reverse in  $H'$  strictly more than  $m(H') - \tau(H')$  arcs, the obtained graph can be deduced from  $\bar{H}'$  by reversing the orientation of strictly less than  $\tau(\bar{H}')$  and so it contains a circuit; thus the cardinality of a transversal set is less than or equal to  $m(H') - \tau(H')$ ;  $S \cap H'$  being a transversal set of  $H'$  we have (2).

(3)  $|S \cap H_{i_0}| \geq \tau(H)$  for all  $i_0$  because  $S \cap H_{i_0}$  is a transversal set of  $H_{i_0}$  which is isomorphic to  $H$ . Using the decomposition of  $G^\pm(H, H')$  we have:  $|S \cap G^\pm(H, H')| = |S \cap (G \otimes S_q)| + \sum_{i_0=1}^p |S \cap (H_{i_0} - H')| + |S \cap H'|$ . From  $S \cap (H_{i_0} - H') = S \cap H_{i_0} - S \cap H'$  and from (1), (2), (3) we deduce:  $|S \cap G^\pm(H, H')| \geq q^2 \nu(G) + p\tau(G) - (p-1)(m(H') - \tau(H'))$  and since  $S$  was an arbitrary transversal set, the same lower bound hold for  $\tau(G^\pm(H, H')) = \min |S|$ .

Proposition 9 implies the following theorem for  $f(n)$ :

**Theorem 3.**

$$f(qp + h) \geq q^2 \pi(p) + \max_{H, H'} \{p\tau(H) - (p-1)(m(H') - \tau(H'))\},$$

where the maximum is taken over all graphs  $H$  with  $q + h$  vertices,  $H'$

being any subgraph of  $H$  with  $h$  vertices.

**Remark.** In the case  $h = 0$ , Proposition 9 and Theorem 3 reduce to Proposition 8 and Theorem 1 respectively. We shall give now lower bounds of  $f(pq + h)$  for  $h = 1, 2, 3$  (Theorems 4, 6, 8).

**Theorem 4.**  $f(qp + 1) \geq q^2 \pi(p) + pf(q + 1)$ .

**Proof.** We choose a graph  $H$  with  $\tau(H) = f(q + 1)$ ;  $H'$  being a subgraph with one vertex  $m(H') = 0$  and  $\tau(H') = 0$ .

**Examples.**

$$\begin{array}{llll} p = 3, q = 9 & f(28) \geq 126 & \pi(28) = 121 \\ p = 3, q = 10 & f(31) \geq 160 & \pi(31) = 155 \\ p = 7, q = 9 & f(64) \geq 672 & \pi(64) = 661 \\ p = 7, q = 10 & f(71) \geq 840 & \pi(71) = 828 \end{array}$$

**Theorem 5.** If  $f(n + 1) > \pi(n + 1)$ , then  $f(3n + 1) > \pi(3n + 1) + 1$ .

**Proof.** The following table shows that  $n^2 + 3\pi(n + 1) \geq \pi(3n + 1) - 1$ .

$n$	$\pi(n + 1)$	$n^2 + 3\pi(n + 1)$	$\pi(3n + 1)$
$6k$	$6k^2 + k$	$54k^2 + 3k$	$54k^2 + 3k$
$6k + 1$	$6k^2 + 2k$	$54k^2 + 18k + 1$	$54k^2 + 18k + 1$
$6k + 2$	$6k^2 + 5k + 1$	$54k^2 + 39k + 7$	$54k^2 + 39k + 7$
$6k + 3$	$6k^2 + 6k + 1$	$54k^2 + 54k + 12$	$54k^2 + 54k + 13$
$6k + 4$	$6k^2 + 9k + 3$	$54k^2 + 75k + 25$	$54k^2 + 75k + 26$
$6k + 5$	$6k^2 + 10k + 4$	$54k^2 + 90k + 37$	$54k^2 + 90k + 37$

Thus, if  $f(n + 1) \geq \pi(n + 1) + 1$ , we have by Theorem 4:

$$f(3n + 1) \geq n^2 + 3f(n + 1) \geq n^2 + 3\pi(n + 1) + 3 \geq \pi(3n + 1) + 2.$$

**Theorem 6.**  $f(qp + 2) \geq q^2 \pi(p) + pf(q + 2) - (p - 1)$ .

**Proof.** We choose a graph  $H$  with  $\tau(H) = f(q + 2)$ ;  $H'$  having two vertices  $m(H') = 1$  and  $\tau(H') = 0$ .

### Examples.

$$\begin{array}{lll}
 p = 3, q = 8 & f(26) \geq 107 & \pi(26) = 104 \\
 p = 3, q = 9 & f(29) \geq 139 & \pi(29) = 135 \\
 p = 7, q = 8 & f(58) \geq 547 & \pi(58) = 541 \\
 p = 7, q = 9 & f(65) \geq 701 & \pi(65) = 693
 \end{array}$$

**Theorem 7.** *If  $f(n+2) > \pi(n+2)$  and  $n \not\equiv 2 \pmod{6}$  then  $f(3n+2) > \pi(3n+2)$  and if  $f(n+2) > \pi(n+2) + 1$  and  $n \equiv 2 \pmod{6}$  then  $f(3n+2) > \pi(3n+2) + 2$ .*

**Proof.** The following table shows that if  $n \equiv 2 \pmod{6}$ , we have:  $\pi(3n+2) = n^2 + 3\pi(n+2) + 1$ , and if  $n \not\equiv 2 \pmod{6}$ :  $n^2 + 3\pi(n+2) \geq \pi(3n+2)$ .

$n$	$\pi(n+2)$	$n^2 + 3\pi(n+2)$	$\pi(3n+2)$
$6k$	$6k^2 + 2k$	$54k^2 + 6k$	$54k^2 + 6k$
$6k+1$	$6k^2 + 5k + 1$	$54k^2 + 27k + 4$	$54k^2 + 27k + 3$
$6k+2$	$6k^2 + 6k + 1$	$54k^2 + 42k + 7$	$54k^2 + 42k + 8$
$6k+3$	$6k^2 + 9k + 3$	$54k^2 + 63k + 18$	$54k^2 + 63k + 18$
$6k+4$	$6k^2 + 10k + 4$	$54k^2 + 78k + 28$	$54k^2 + 78k + 28$
$6k+5$	$6k^2 + 13k + 7$	$54k^2 + 99k + 46$	$54k^2 + 99k + 45$

Thus if  $n \not\equiv 2 \pmod{6}$ , by the hypothesis and by Theorem 6  $f(3n+2) \geq n^2 + 3f(n+2) - 2 \geq n^2 + 3\pi(n+2) + 1 \geq \pi(3n+2) + 1$ . If  $n \equiv 2 \pmod{6}$  by the hypothesis  $f(n+2) \geq \pi(n+2) + 2$ , and by Theorem 6  $f(3n+2) \geq n^2 + 3f(n+2) - 2 \geq n^2 + 3\pi(n+2) + 4 \geq \pi(3n+2) + 3$ .

**Proposition 10.** *If, in an interval  $[a, 3a-1]$ ,  $f(n) > \pi(n)$  and for  $n \equiv 4 \pmod{6}$   $f(n) > \pi(n) + 1$ , then in the interval  $[3a, 9a-1]$ ,  $f(n) > \pi(n)$  and for  $n \equiv 4 \pmod{6}$   $f(n) > \pi(n) + 1$ .*

**Proof.** Let  $n$  belong to the interval  $[3a, 9a-1]$ .

*Case 1.*  $n = 3p$ , then  $p$  belongs to  $[a, 3a-1]$ ; thus, by the hypothesis  $f(p) > \pi(p)$  and by Theorem 2  $f(3p) > \pi(3p) + 1$ .

*Case 2.*  $n = 3p + 1$  (in particular this case contains  $n \equiv 4 \pmod{6}$ ). Then  $p + 1$  belongs to  $[a, 3a]$ ; thus by the hypothesis and for  $p = 3a$  by Case 1  $f(p + 1) > \pi(p + 1)$  and by Theorem 5  $f(3p + 1) > \pi(3p + 1) + 1$ .

*Case 3.*  $n = 3p + 2$ , then  $p + 2$  belongs to  $[a, 3a + 1]$ ; more if  $p + 2 \equiv 4 \pmod{6}$ ,  $p \equiv 2 \pmod{6}$ ; thus by the hypothesis and for  $p = 3a$  and  $p = 3a + 1$  by Case 1 and 2,  $f(p + 2) > \pi(p + 2)$  and for  $p \equiv 2 \pmod{6}$ ,  $f(p + 2) > \pi(p + 2) + 1$  and so, by Theorem 7,  $f(3p + 2) > \pi(3p + 2)$ .

**Corollary.** To prove Conjecture 1 of Kotzig:  $f(n) > \pi(n)$  for  $n \geq 10$  it suffices to prove that for  $n$  belonging to  $[10], [29]$ :  $f(n) > \pi(n)$  and for  $n \equiv 4 \pmod{6}$  in this interval  $f(n) > \pi(n) + 1$ .

In Proposition 4 we have proved that  $f(10) > \pi(10) + 1$  and that  $f(11) > \pi(11)$ . In the example of Theorem 4 we have  $f(28) > \pi(28) + 1$  and in the examples of Theorem 6  $f(26) > \pi(26)$  and  $f(29) > \pi(29)$ . We shall decide new values in the following propositions.

**Theorem 8.**  $f(qp + 3) \geq q^2 \pi(p) + pf(q + 3) - 2(p - 1)$ .

**Proof.** Let us choose as graph  $H$  in Theorem 3 a tournament with  $\tau(H) = f(q + 3)$  and, as subgraph  $H'$ , a 3-circuit (this is possible otherwise  $H$  would be a tournament without circuit), then  $m(H') = 3$  and  $\tau(H') = 1$ .

**Corollary.**  $f(24) > \pi(24)$  and  $f(27) > \pi(27)$ .

**Proof.** We apply Theorem 8 with  $p = 3$  and  $q = 7, 8$ ; we deduce  $f(24) \geq 90$   $\pi(24) = 88$ ;  $f(27) \geq 120$   $\pi(27) = 117$ .

**Proposition 11.**  $f(12) \geq 21$  (Conjecture 1 is true for 12).

**Proof.** Let us consider the tournament  $T_{11}$  defined in [2] p. 21, where  $\overrightarrow{ij}$  is an arc of  $T_{11}$ , what we note by  $i > j$ , if and only if  $j - i$  is congruent to a square modulo 11. Let  $T_{12}$  be the tournament obtained by adding to  $T_{11}$  a vertex, denoted by 12, and the arcs  $\overrightarrow{12i}$  for  $i = 1, 2, 4, 5, 6, 10$  and  $\overrightarrow{j12}$  for  $j = 3, 7, 8, 9, 11$ .  $T_{12}$  can be written

$1 > 2, 4, 5, 6, 10,$	$2 > 3, 5, 6, 7, 11,$	$3 > 1, 4, 6, 7, 8, 12,$
$4 > 2, 5, 7, 8, 9,$	$5 > 3, 6, 8, 9, 10,$	$6 > 4, 7, 9, 10, 11,$
$7 > 1, 5, 8, 10, 11, 12,$	$8 > 1, 2, 6, 9, 11, 12,$	$9 > 1, 2, 3, 7, 10, 12,$
$10 > 2, 3, 4, 8, 11,$	$11 > 1, 3, 4, 5, 9, 12,$	$12 > 1, 2, 4, 5, 6, 10.$

To prove Conjecture 1, as  $\pi(12) = 20$ , we have to prove that  $\tau(T_{12}) \geq 21$ . Let  $S$  be a transversal set with  $\tau(T_{12})$  elements.

*Case 1.* Let us suppose that  $S$  contains no arc with end-vertex 12, and let  $T_i$  be the subtournament generated by the vertices 1, 2, 4, 5, 6, 10 and  $T_j$  by the vertices 3, 7, 8, 9, 11; the subtournament generated by 2, 4, 5, 6, 10 and  $T_j$  being isomorphic to the tournament of the example of Fig. 1 we have  $\tau(T_i) = 3$  and  $\tau(T_j) = 3$ .  $S \cap T_i$  must be a transversal set of  $T_i$ , thus  $|S \cap T_i| \geq 3$  and similarly  $|S \cap T_j| \geq 3$ ; further  $S$  must contain all the 15 arcs  $\vec{ij}$  with  $i = 1, 2, 4, 5, 6, 10$ ;  $j = 3, 7, 8, 9, 11$  otherwise  $T_{12} - S$  would contain the circuit  $12ij$ . Thus  $|S| \geq 21$ .

*Case 2.* Let us suppose that  $S$  contains at least an arc with end-vertex 12;  $S \cap T_{11}$  must be a transversal set of  $T_{11}$  and  $\tau(T_{11}) = 20$  (see [2]), thus  $|S| \geq 21$ . In both cases  $|S| \geq 21$  and so  $\tau(T_{12}) \geq 21$ .

**Proposition 12.**  $f(18) > \pi(18)$  and  $f(19) > \pi(19)$ .

**Proof.** We have  $\pi(18) = 48$  and  $\pi(19) = 57$ . Let us consider the tournament  $T_{19}$  in which  $\vec{ij}$  is an arc if and only if  $j - i$  is congruent to a square modulo 19 (quadratic residue). This tournament is vertex transitive and so  $\tau(T_{19}) = 9 + \tau(T_{18})$  where  $T_{18}$  is a tournament obtained by deleting any vertex, for example 19. In  $T_{18}$  all the vertices of score (out-degree) 9 are equivalent and all the vertices of score 8 are equivalent: it suffices to consider the automorphisms associated to the powers of the permutation  $1 \rightarrow 5 \rightarrow 6 \rightarrow 11 \rightarrow 17 \rightarrow 9 \rightarrow 7 \rightarrow 16 \rightarrow 4 \rightarrow 1$  and  $2 \rightarrow 10 \rightarrow 12 \rightarrow 3 \rightarrow 15 \rightarrow 18 \rightarrow 14 \rightarrow 13 \rightarrow 8 \rightarrow 2$ . Thus  $\tau(T_{18}) = 8 + \tau(T_{17})$  where  $T_{17}$  is obtained by deleting any one vertex, for example 18. We shall prove that  $\pi(T_{17}) \geq 41$ , so  $\tau(T_{17}) \geq 41$  and thus  $\tau(T_{18}) \geq 49 > \pi(18)$  and  $\tau(T_{19}) \geq 58 > \pi(19)$ .  $T_{17}$  can be written:

$1 > 2, 5, 6, 7, 8, 10, 12, 17;$	$2 > 3, 6, 7, 8, 9, 11, 13;$
$3 > 1, 4, 7, 8, 9, 10, 12, 14;$	$4 > 1, 2, 5, 8, 9, 10, 11, 13, 15;$
$5 > 2, 3, 6, 9, 10, 11, 12, 14, 16;$	$6 > 3, 4, 7, 10, 11, 12, 13, 15, 17;$
$7 > 4, 5, 8, 11, 12, 13, 14, 16;$	$8 > 5, 6, 9, 12, 13, 14, 15, 17;$
$9 > 1, 6, 7, 10, 13, 14, 15, 16;$	$10 > 2, 7, 8, 11, 14, 15, 16, 17;$
$11 > 1, 3, 8, 9, 12, 15, 16, 17;$	$12 > 2, 4, 9, 10, 13, 16, 17;$
$13 > 1, 3, 5, 10, 11, 14, 17;$	$14 > 1, 2, 4, 6, 11, 12, 15;$
$15 > 1, 2, 3, 5, 7, 12, 13, 16;$	$16 > 1, 2, 3, 4, 6, 8, 13, 14, 17;$
$17 > 2, 3, 4, 5, 7, 9, 14, 15;$	

$T_{17}$  contains the following 41 circuits:

1 2 3; 1 5 16; 1 6 11; 1 7 13; 1 8 15; 1 10 14; 1 12 4; 1 17 9; 2 6 17;  
 2 7 14; 2 8 12; 2 11 15; 2 13 10; 3 4 13; 3 7 5; 3 8 17; 3 9 15; 3 10 11;  
 3 12 16; 3 14 6; 4 8 14; 4 9 6; 4 10 17; 4 11 16; 4 15 7; 5 6 15; 5 9 13;  
 5 10 8; 5 11 17; 6 7 8; 6 10 16; 7 11 9; 7 16 17; 8 9 16; 8 13 11;  
 10 15 12; 12 17 14; 13 17 15; 14 15 16; 7 12 9 10; 11 12 13 14.

By the aid of a computer program Y. Kodratoff (C.N.R.S. Paris) has found that for this tournament  $\tau(T_{17}) = 47$ , and for the subtournament  $T_{16}$ , obtained by deleting the vertex 4,  $\tau(T_{16}) = 40$ ; thus we have:

**Proposition 13.**  $f(19) \geq 64$ ;  $f(18) \geq 55$ ;  $f(17) \geq 47 > 45 = \pi(17)$   
 and  $f(16) \geq 40 > 38 = \pi(16) + 1$ .

The program applied to other tournaments has given:

**Proposition 14.**  $f(13) \geq 27 > 26 = \pi(13)$ .

The program was applied to the following tournament  $T_{13}$ :

$1 > 2, 4, 5, 6, 10, 12;$	$2 > 3, 5, 6, 7, 11, 13;$	$3 > 1, 4, 6, 7, 8, 12;$
$4 > 2, 5, 7, 8, 9, 13;$	$5 > 3, 6, 8, 9, 10, 13;$	$6 > 4, 7, 9, 10, 11, 13;$
$7 > 1, 5, 8, 10, 11, 12;$	$8 > 1, 2, 6, 9, 11, 12;$	$9 > 1, 2, 3, 7, 10, 12;$
$10 > 2, 3, 4, 8, 11, 13;$	$11 > 1, 3, 4, 5, 9, 12;$	$12 > 2, 4, 5, 6, 10, 13;$
$13 > 1, 3, 7, 8, 9, 11.$		

The tournament  $T_{13}$  being regular, we have Proposition 11 as corollary.

**Proposition 15.**  $f(15) \geq 36 > \pi(15)$  and  $f(14) \geq 29 > \pi(14)$ .

Let us consider the tournament  $T_{15}$  isomorphic to the subtournament of the tournament  $T_{31}$  of the quadratic residues module 31, generated by the vertices  $i$  such that  $\overrightarrow{i31}$  is an arc of  $T_{31}$ .  $T_{15}$  can be written:

$$\begin{array}{ll}
 1 > 2, 4, 6, 7, 8, 12, 13; & 2 > 3, 5, 7, 8, 10, 11, 13; \\
 3 > 1, 4, 6, 7, 9, 11, 13; & 4 > 2, 5, 7, 8, 9, 12, 14; \\
 5 > 1, 3, 6, 7, 9, 10, 14; & 6 > 2, 4, 7, 8, 10, 11, 15; \\
 7 > 8, 9, 10, 11, 12, 14, 15; & 8 > 3, 5, 9, 11, 12, 13, 15; \\
 9 > 1, 2, 6, 10, 11, 12, 15; & 10 > 1, 3, 4, 8, 11, 13, 14; \\
 11 > 1, 4, 5, 12, 13, 14, 15; & 12 > 2, 3, 5, 6, 10, 13, 15; \\
 13 > 4, 5, 6, 7, 9, 14, 15; & 14 > 1, 2, 3, 6, 8, 9, 12; \\
 15 > 1, 2, 3, 4, 5, 10, 14.
 \end{array}$$

$T_{15}$  is vertex-transitive; therefore the program was applied to a subtournament  $T_{14}$  and gave  $\tau(T_{14}) = 29$  and therefore  $\tau(T_{15}) = 36$ .

#### Remarks.

1. It is enough to prove Conjecture 1 for the odd values of  $n$ , because if  $f(2p + 1) > \pi(2p + 1)$  then  $f(2p) > \pi(2p)$ : suppose that  $f(2p) \leq \pi(2p)$ , from  $\pi(2p + 1) = \pi(2p) + p$  and  $f(2p + 1) \leq f(2p) + p$  (see [2] p. 11) we deduce  $f(2p + 1) \leq \pi(2p + 1)$  contrary to the hypothesis.

2. The result of Erdős and Moon [5] shows that if  $n \rightarrow \infty$ ,  $f(n) \sim \frac{n(n-1)}{4}$  but  $\pi(n) \sim \frac{n(n-1)}{6}$ ; thus Conjecture 1 is true if  $n$  is sufficiently large.

In summary to prove Conjecture 1 of Kotzig it remains to prove that  $f(n) > \pi(n)$  for  $n = 21, 23, 25$  and that  $f(22) > \pi(22) + 1$ . (We hope to verify the conjecture for these values by computer search).



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