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THE CIRCUIT – HYPERGRAPH OF A TOURNAMENT

J-C. BERMOND*

Let $\tau(T_n)$ be the minimal number of arcs which must be deleted (or reversed) in a tournament T_n in order to obtain a graph without circuits. In this article we give results on the function $f(n) = \max_{T_n} \tau(T_n)$ (maximum taken over all the tournaments with n vertices), answering a conjecture of Kotzig [7]. Other known results and references can be found in [2].

1. DEFINITIONS AND PRELIMINARY RESULTS

Throughout this paper $G = (X, U)$ will denote a finite oriented graph, where X is the set of vertices and U the set of arcs. T_n will denote a tournament (graph (X, U) in which each pair of vertices x, y is joined by one and only one of the arcs \overrightarrow{xy} and \overrightarrow{yx}), with $|X| = n$, and so $|U| = \frac{n(n-1)}{2}$. A circuit of G is a sequence of *different* vertices $x_1 x_2 \dots x_i x_{i+1} \dots x_p$ such that $\overrightarrow{x_i x_{i+1}}$ ($i = 1, 2, \dots, p-1$) and $\overrightarrow{x_p x_1}$ are arcs of G .

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Definition 1. To a graph $G = (X, U)$ we associate the hypergraph $C(G) = (U, C)$, called the *circuit-hypergraph of G* , whose set of vertices is U (set of arcs of G) and whose edges are the circuits of G . This hypergraph has been studied by Chaty [4].

Definition 2 (see Berge [1]). A *transversal set* of a hypergraph H is a set of vertices of H that meet every edge of the hypergraph and the *transversal number* $\tau(H)$ is the minimum number of elements of a transversal set.

Let $\tau(G)$ denote the transversal number of the circuit-hypergraph $C(G)$ associated to G , that is, the minimum number of arcs that we must delete from G in order to obtain a graph without circuits. It has been proved by Grindberg and Dambit, Remage and Thompson (see [2], p. 9) that $\tau(G)$ is also the minimum number of arcs whose orientation must be reversed in order to obtain a graph without circuits.

Example. For the tournament T_5 of Fig. 1 14, 15, 25 is a transversal set, so $\tau(T_5) \geq 3$. We shall see later that $\tau(T_5) = 3$.

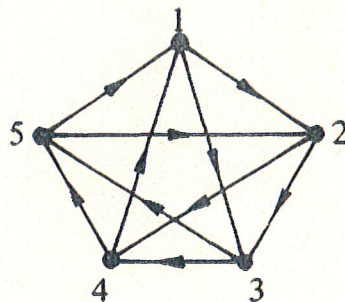


Fig. 1

Instead of $\tau(G)$, we can consider the number $w(G)$, the stability number of the circuit-hypergraph of G , which is the maximum cardinality of a set of arcs such that the partial graph generated by these arcs is without circuits (such a set of arcs is called a set of consistent arcs by Erdős and Moon [5]). We have the relation $\tau(G) + w(G) = |U|$. $\tau(G)$ has been introduced by Slater [9] and $w(G)$ by Erdős and Moon [5]. The number $\tau(G)$ has importance in "paired comparisons" (see [2])

for further details and references). In this case G is a tournament and one of the problems is to determine the function $f(n) = \max_{T_n} \tau(T_n)$, where the maximum is taken over all the tournaments with n vertices; as has remarked Kotzig [7], $f(n)$ is also the maximum of $\tau(G)$ taken over all the antisymmetric graphs G with n vertices.

Various results have been obtained about the function $f(n)$ (see [2]), essentially upper bounds and that $\frac{f(n)}{\binom{n}{2}} \rightarrow \frac{1}{2}$ if $n \rightarrow \infty$ by Erdős and Moon [15]. Here we are interested in lower bounds of $f(n)$.

Definition 3. H being a hypergraph, the *matching number* of H : $\nu(H)$ is the maximum number of disjoint edges of H .

Let $\nu(G)$ denote the matching number of the circuit-hypergraph $C(G)$, that is the maximum number of arc-disjoint circuits of G .

By a theorem on hypergraph ([1], p. 405) we have:

Proposition 1. For every graph G : $\tau(G) \geq \nu(G)$.

Example. In the tournament T_5 of Fig. 1 we have 3 arc-disjoint circuits: 124, 135, 2345. Thus $\nu(T_5) \geq 3$ and by Proposition 1 $\tau(T_5) = \nu(T_5) = 3$.

Definition 4. The maximum number of edge-disjoint cycles of the non-oriented complete graph K_n will be denoted by $\pi(n)$. It has been proved by Chartrand, Geller and Hedetniemi [3] that $\pi(n) = \left\lceil \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rceil$ (where $\lfloor x \rfloor$ is the greatest integer not exceeding x).

Proposition 2. $f(n) \geq \pi(n) = \left\lceil \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rceil$: (For a proof see [2]).

By using the upper bound of Jung [6] (see also [2]) we have:

Proposition 3. For $n \leq 9$: $f(n) = \pi(n)$.

In [2] p. 21 and 22 we have determined the values of $f(10)$ and $f(11)$.

Proposition 4. $f(10) = 15$ ($\pi(10) = 13$); $f(11) = 20$ ($\pi(11) = 18$).

Kotzig [7] has also found independently Propositions 1, 2, 3 and the value of $f(10)$. He has also conjectured that:

Conjecture 1 (Kotzig [7]). For $n \geq 10$ $f(n) > \pi(n)$.

In his article Kotzig has proved that if $f(n) > \pi(n)$ then $f(3n) > \pi(3n)$. Here we shall prove similar results for $f(3n + 1)$ and $f(3n + 2)$ and so to prove the conjecture it would be sufficient to prove it for a small number of values of n . For this purpose we need results on the transversal number of the circuit-hypergraph associated to the lexicographic product (and generalized product) of two graphs.

2. LEXICOGRAPHIC PRODUCT OF GRAPHS

Definition. Let $G = (X, U)$ and $H = (Y, V)$ be two (oriented) graphs the *lexicographic product of G by H* , noted $G \otimes H$, is the graph which has the Cartesian product $X \times Y$ as set of vertices and in which (x_1, y_1) is joined to (x_2, y_2) if and only if either $\overrightarrow{x_1 x_2}$ is an arc of G or $x_1 = x_2$ and $\overrightarrow{y_1 y_2}$ is an arc of H . $G \otimes H$ is called composition of G with H in Moon [8] p. 78. (We can consider this graph as obtained by replacing each vertex of G by a copy of H).

Example: $G = C_3$, $H = K_2$.

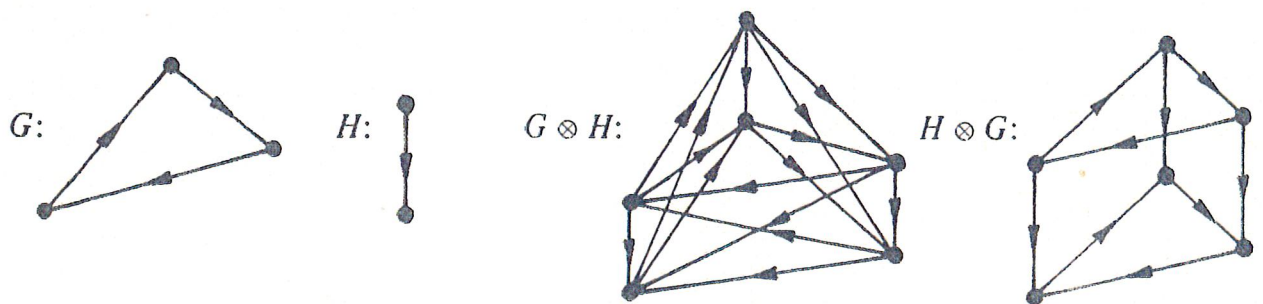


Fig. 2

Proposition 5. Let $G = (X, U)$ and $H = (Y, V)$ be two graphs with $|X| = p$ and $|Y| = q$, then $\tau(G \otimes H) \leq p\tau(H) + q^2\tau(G)$.

Proof. First we prove that if G and H are two graphs without circuits then $G \otimes H$ is without circuits. Suppose, on the contrary, that $G \otimes H$ contains a circuit $(x_1, y_1)(x_2, y_2) \dots (x_k, y_k)$. If all the first indices x_i of the vertices of the circuit were equal, then $y_1 y_2 \dots y_k$ would be a circuit of H contrary to the hypothesis. Thus, two of the x_i 's are different; let us consider the different x_i 's in the order in which they appear in the circuit: $x_{i_1}, x_{i_2}, \dots, x_{i_p}$, then by definition of the lexicographic product $x_{i_1} x_{i_2} \dots x_{i_p}$ contains a circuit of G , contrary to the hypothesis.

According to the definition of $\tau(G)$ and $\tau(H)$ we can obtain a graph G' without circuits, by deleting $\tau(G)$ arcs from G and a graph H' without circuits by deleting $\tau(H)$ arcs from H . By the first part of the proof $G' \otimes H'$ is a graph without circuits and is obtained from $G \otimes H$ by deleting the $q^2\tau(G)$ arcs of the form $(x_1, y)(x_2, y')$, $\overrightarrow{x_1 x_2}$ being an arc of $G - G'$ and the $p\tau(H)$ arcs of the form $(x, y_1)(x, y_2)$ where $\overrightarrow{y_1 y_2}$ is an arc of $H - H'$ (the x is the same).

Conjecture 2. $\tau(G \otimes H) = p\tau(H) + q^2\tau(G)$.

We are able to prove the conjecture, when $\tau(G) = \nu(G)$, by considering $\nu(G \otimes H)$.

Proposition 6. Let S_q be the graph with q vertices and no arcs, then $\nu(G \otimes S_q) \geq q^2\nu(G)$.

Proof. Let the vertices of S_p be labelled: $1, 2, \dots, q$. To the circuit $x_1 x_2 \dots x_i \dots x_k$ of G , we associate the q^2 arc-disjoint circuits $(x_1, j)(x_2, j + \alpha) \dots (x_i, j + (i - 1)\alpha) \dots (x_{k-1}, j + (k - 2)\alpha)(x_k, \alpha)$, where $j = 1, 2, \dots, q$, $\alpha = 1, 2, \dots, q$ and where the second indices of the vertices are to be taken modulo q . Then to the $\nu(G)$ arc-disjoint circuits of G , we associate $q^2\nu(G)$ arc-disjoint circuits in $G \otimes S_q$.

Conjecture 3. $\nu(G \otimes S_q) = q^2\nu(G)$.

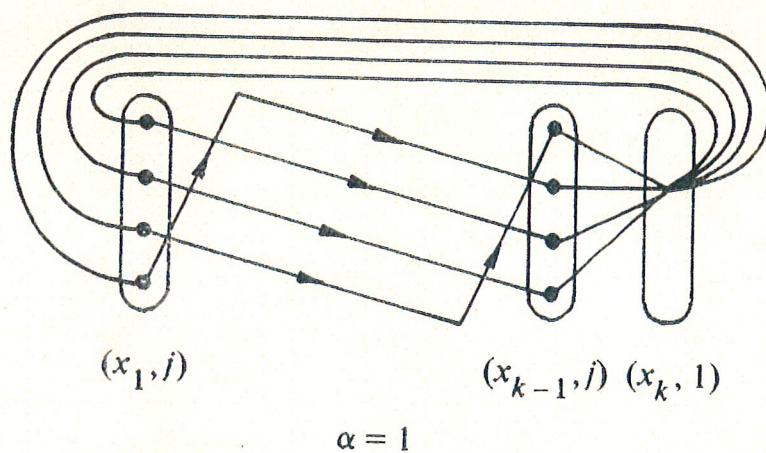
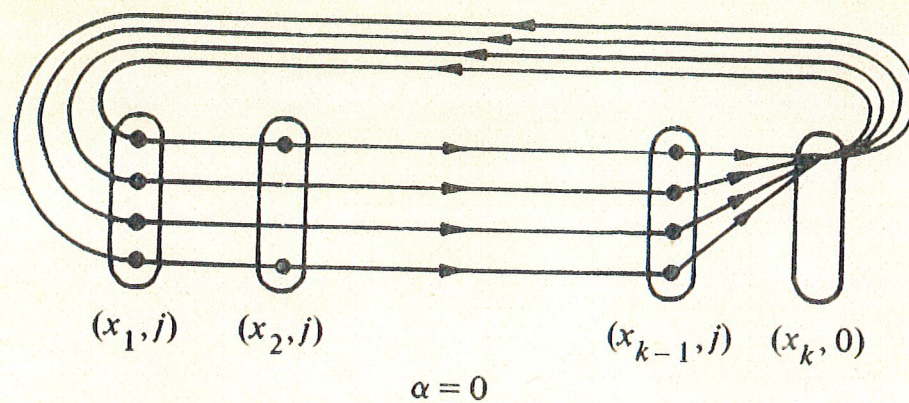


Fig. 3

Lemma. $\tau(S_p \otimes H) = p\tau(H)$ and $\nu(S_p \otimes H) = p\nu(H)$.

Proof. This results from the decomposition of $S_p \otimes H$ into the union of p disjoint copies of H .

Proposition 7. $\nu(G \otimes H) \geq p\nu(H) + q^2\nu(G)$.

Proof. This results from the decomposition of $G \otimes H$ into the union of the two arc-disjoint graphs $S_p \otimes H$ and $G \otimes S_q$.

Proposition 8. $\tau(G \otimes H) \geq p\tau(H) + q^2\nu(G)$.

Proof. This results from the decomposition of $G \otimes H$ into $S_p \otimes H$

and $G \otimes S_q$, the lemma applied to $\tau(S_p \otimes H)$ and the fact that:
 $\tau(G \otimes S_q) \geq \nu(G \otimes S_q) \geq q^2 \nu(G)$ (Propositions 1 and 6).

Corollary. *When $\tau(G) = \nu(G)$, then Conjectures 2 and 3 are true.*

Theorem 1. $f(pq) \geq pf(q) + q^2 \pi(p)$.

Proof. The theorem can be deduced from Proposition 8, by choosing as graph G a tournament T_p , with $\nu(T_p) = \pi(p)$, and as graph H a tournament T_q , with $\tau(T_q) = f(q)$.

Examples. $p = 3, q = 10: f(30) \geq 145, (\pi(30) = 140); p = 3, q = 11, f(33) \geq 181, (\pi(33) = 176); p = 7, q = 10, f(70) \geq 805, (\pi(70) = 793), p = 7, q = 11: f(77) \geq 987, (\pi(77) = 975)$.

In the case $p = 3$ Theorem 1 leads to $f(3n) \geq n^2 + 3f(n)$. This result has been proved by Kotzig ([7]); from this result Kotzig has deduced the following theorem:

Theorem 2 (Kotzig [7]). *If $f(n) > \pi(n)$, then $f(3n) > \pi(3n) + 1$.*

Proof. From the value of $\pi(n) = \left\lfloor \frac{n}{3} \left[\frac{n-1}{2} \right] \right\rfloor$, we obtain the values of the following table, which proves that $n^2 + 3\pi(n) \geq \pi(3n) - 1$.

n	$\pi(n)$	$n^2 + 3\pi(n)$	$\pi(3n)$
$6k$	$6k^2 - 2k$	$54k^2 - 6k$	$54k^2 - 6k$
$6k + 1$	$6k^2 + k$	$54k^2 + 15k + 1$	$54k^2 + 15k + 1$
$6k + 2$	$6k^2 + 2k$	$54k^2 + 30k + 4$	$54k^2 + 30k + 4$
$6k + 3$	$6k^2 + 5k + 1$	$54k^2 + 51k + 12$	$54k^2 + 51k + 12$
$6k + 4$	$6k^2 + 6k + 1$	$54k^2 + 66k + 19$	$54k^2 + 66k + 20$
$6k + 5$	$6k^2 + 9k + 3$	$54k^2 + 87k + 34$	$54k^2 + 87k + 35$

If $f(n) > \pi(n)$, then $f(n) \geq \pi(n) + 1$ and so by Theorem 1:

$$f(3n) \geq n^2 + 3f(n) \geq n^2 + 3\pi(n) + 3 \geq \pi(3n) + 2 > \pi(3n) + 1.$$

3. GENERALIZATION AND APPLICATIONS TO $f(n)$

Definition. Let p, q, h be three integers; let $G = (X, U)$ be a graph with $|X| = p$; let $H = (Y \cup Z, V)$ be a graph with vertex set the union of the disjoint sets Y and Z , where $|Y| = q$ and $|Z| = h$ (the vertices of Y are labelled from 1 to q and those of Z from $q + 1$ to $q + h$); and let H' be the subgraph of H generated by Z , and H'' the subgraph of H generated by Y . We define a new graph $G^\pm(H, H')$ as follows: its vertex set is $(X \times Y) \cup Z$ and the vertices are labelled (i, j) with $i = 1, 2, \dots, p$ and $j = 1, 2, \dots, q$ (if they belong to $X \times Y$ and (k) with $k = q + 1, \dots, q + h$ (if they belong to Z); in $G^\pm(H, H')$ the subgraph generated by $X \times Y$ is isomorphic to $G \otimes H''$, the subgraphs H_{i_0} (p in number) generated by the vertices (i_0, j) with $j = 1, \dots, q$ and (k) with $k = q + 1, \dots, q + h$ are isomorphic to H for all i_0 . Thus two vertices of $G^\pm(H, H')$ are joined if and only if they are of the form (i_1, j_1) and (i_2, j_2) if either $\overrightarrow{i_1 i_2}$ is an arc of G or $i_1 = i_2$ and $\overrightarrow{j_1 j_2}$ is an arc of H ; (i, j) and $(k) - \overrightarrow{j k}$ being an arc of H ; (k) and $(k') - \overrightarrow{k k'}$ belong an arc of H .

In the case $Z = \phi$, we obtain the lexicographic product $G \otimes H$.

Example.

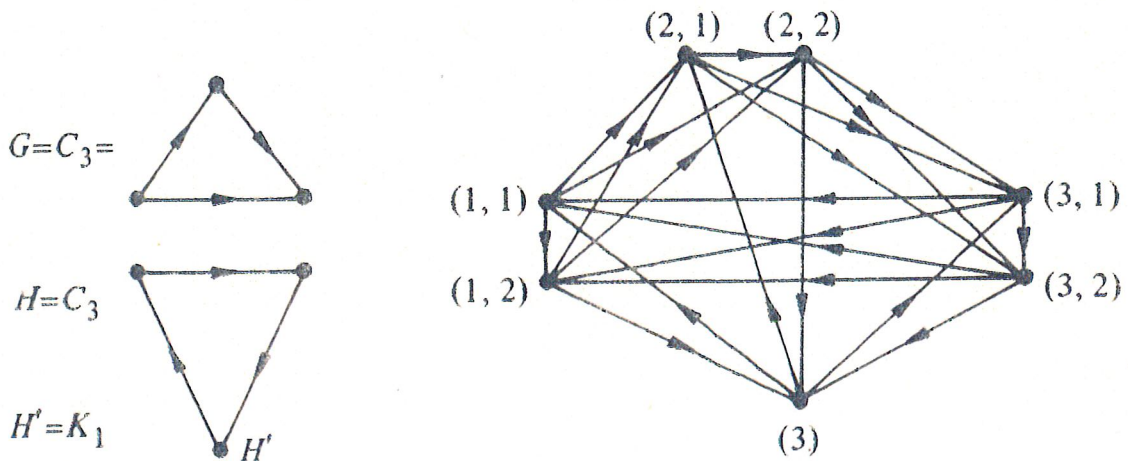


Fig. 4

We want to find $\tau(G^\pm(H, H'))$ or at least a lower bound; this can be

done by decomposing $G^\pm(H, H')$ into the union of arc-disjoint graphs. We shall give here a lower bound which is good when $\frac{h}{q}$ is small.

Proposition 9. *Let $m(H')$ denote the number of arcs of H' , then:*

$$\tau(G^\pm(H, H')) \geq q^2 \nu(G) + p\tau(H) - (p-1)(m(H') - \tau(H')).$$

Proof. $G^\pm(H, H')$ can be considered as the arc-disjoint union of $G \otimes S_q$, H' , and the p copies of $H - H': H_{i_0} - H'$ (formed by the partial graphs of H_{i_0} obtained by deleting the arcs of H'). Let S be a transversal set of $G^\pm(H, H')$:

$$(1) |S \cap (G \otimes S_q)| \geq \tau(G \otimes S_q) \geq q^2 \nu(G) \text{ by Propositions 1 and 6.}$$

(2) $|S \cap H'| \leq m(H') - \tau(H')$. Indeed let us consider the graph \bar{H}' obtained by reversing the orientation of all the arcs of H' (in number $m(H')$), $\tau(\bar{H}') = \tau(H')$ and so if we reverse in H' strictly more than $m(H') - \tau(H')$ arcs, the obtained graph can be deduced from \bar{H}' by reversing the orientation of strictly less than $\tau(\bar{H}')$ and so it contains a circuit; thus the cardinality of a transversal set is less than or equal to $m(H') - \tau(H')$; $S \cap H'$ being a transversal set of H' we have (2).

(3) $|S \cap H_{i_0}| \geq \tau(H)$ for all i_0 because $S \cap H_{i_0}$ is a transversal set of H_{i_0} which is isomorphic to H . Using the decomposition of $G^\pm(H, H')$ we have: $|S \cap G^\pm(H, H')| = |S \cap (G \otimes S_q)| + \sum_{i_0=1}^p |S \cap (H_{i_0} - H')| + |S \cap H'|$. From $S \cap (H_{i_0} - H') = S \cap H_{i_0} - S \cap H'$ and from (1), (2), (3) we deduce: $|S \cap G^\pm(H, H')| \geq q^2 \nu(G) + p\tau(G) - (p-1)(m(H') - \tau(H'))$ and since S was an arbitrary transversal set, the same lower bound hold for $\tau(G^\pm(H, H')) = \min |S|$.

Proposition 9 implies the following theorem for $f(n)$:

Theorem 3.

$$f(qp + h) \geq q^2 \pi(p) + \max_{H, H'} \{p\tau(H) - (p-1)(m(H') - \tau(H'))\},$$

where the maximum is taken over all graphs H with $q + h$ vertices, H'

being any subgraph of H with h vertices.

Remark. In the case $h = 0$, Proposition 9 and Theorem 3 reduce to Proposition 8 and Theorem 1 respectively. We shall give now lower bounds of $f(pq + h)$ for $h = 1, 2, 3$ (Theorems 4, 6, 8).

Theorem 4. $f(qp + 1) \geq q^2 \pi(p) + pf(q + 1)$.

Proof. We choose a graph H with $\tau(H) = f(q + 1)$; H' being a subgraph with one vertex $m(H') = 0$ and $\tau(H') = 0$.

Examples.

$$\begin{array}{llll} p = 3, q = 9 & f(28) \geq 126 & \pi(28) = 121 \\ p = 3, q = 10 & f(31) \geq 160 & \pi(31) = 155 \\ p = 7, q = 9 & f(64) \geq 672 & \pi(64) = 661 \\ p = 7, q = 10 & f(71) \geq 840 & \pi(71) = 828 \end{array}$$

Theorem 5. If $f(n + 1) > \pi(n + 1)$, then $f(3n + 1) > \pi(3n + 1) + 1$.

Proof. The following table shows that $n^2 + 3\pi(n + 1) \geq \pi(3n + 1) - 1$.

n	$\pi(n + 1)$	$n^2 + 3\pi(n + 1)$	$\pi(3n + 1)$
$6k$	$6k^2 + k$	$54k^2 + 3k$	$54k^2 + 3k$
$6k + 1$	$6k^2 + 2k$	$54k^2 + 18k + 1$	$54k^2 + 18k + 1$
$6k + 2$	$6k^2 + 5k + 1$	$54k^2 + 39k + 7$	$54k^2 + 39k + 7$
$6k + 3$	$6k^2 + 6k + 1$	$54k^2 + 54k + 12$	$54k^2 + 54k + 13$
$6k + 4$	$6k^2 + 9k + 3$	$54k^2 + 75k + 25$	$54k^2 + 75k + 26$
$6k + 5$	$6k^2 + 10k + 4$	$54k^2 + 90k + 37$	$54k^2 + 90k + 37$

Thus, if $f(n + 1) \geq \pi(n + 1) + 1$, we have by Theorem 4:

$$f(3n + 1) \geq n^2 + 3f(n + 1) \geq n^2 + 3\pi(n + 1) + 3 \geq \pi(3n + 1) + 2.$$

Theorem 6. $f(qp + 2) \geq q^2 \pi(p) + pf(q + 2) - (p - 1)$.

Proof. We choose a graph H with $\tau(H) = f(q + 2)$; H' having two vertices $m(H') = 1$ and $\tau(H') = 0$.

Examples.

$$\begin{array}{lll}
 p = 3, q = 8 & f(26) \geq 107 & \pi(26) = 104 \\
 p = 3, q = 9 & f(29) \geq 139 & \pi(29) = 135 \\
 p = 7, q = 8 & f(58) \geq 547 & \pi(58) = 541 \\
 p = 7, q = 9 & f(65) \geq 701 & \pi(65) = 693
 \end{array}$$

Theorem 7. *If $f(n+2) > \pi(n+2)$ and $n \not\equiv 2 \pmod{6}$ then $f(3n+2) > \pi(3n+2)$ and if $f(n+2) > \pi(n+2) + 1$ and $n \equiv 2 \pmod{6}$ then $f(3n+2) > \pi(3n+2) + 2$.*

Proof. The following table shows that if $n \equiv 2 \pmod{6}$, we have: $\pi(3n+2) = n^2 + 3\pi(n+2) + 1$, and if $n \not\equiv 2 \pmod{6}$: $n^2 + 3\pi(n+2) \geq \pi(3n+2)$.

n	$\pi(n+2)$	$n^2 + 3\pi(n+2)$	$\pi(3n+2)$
$6k$	$6k^2 + 2k$	$54k^2 + 6k$	$54k^2 + 6k$
$6k + 1$	$6k^2 + 5k + 1$	$54k^2 + 27k + 4$	$54k^2 + 27k + 3$
$6k + 2$	$6k^2 + 6k + 1$	$54k^2 + 42k + 7$	$54k^2 + 42k + 8$
$6k + 3$	$6k^2 + 9k + 3$	$54k^2 + 63k + 18$	$54k^2 + 63k + 18$
$6k + 4$	$6k^2 + 10k + 4$	$54k^2 + 78k + 28$	$54k^2 + 78k + 28$
$6k + 5$	$6k^2 + 13k + 7$	$54k^2 + 99k + 46$	$54k^2 + 99k + 45$

Thus if $n \not\equiv 2 \pmod{6}$, by the hypothesis and by Theorem 6 $f(3n+2) \geq n^2 + 3f(n+2) - 2 \geq n^2 + 3\pi(n+2) + 1 \geq \pi(3n+2) + 1$. If $n \equiv 2 \pmod{6}$ by the hypothesis $f(n+2) \geq \pi(n+2) + 2$, and by Theorem 6 $f(3n+2) \geq n^2 + 3f(n+2) - 2 \geq n^2 + 3\pi(n+2) + 4 \geq \pi(3n+2) + 3$.

Proposition 10. *If, in an interval $[a, 3a - 1]$, $f(n) > \pi(n)$ and for $n \equiv 4 \pmod{6}$ $f(n) > \pi(n) + 1$, then in the interval $[3a, 9a - 1]$, $f(n) > \pi(n)$ and for $n \equiv 4 \pmod{6}$ $f(n) > \pi(n) + 1$.*

Proof. Let n belong to the interval $[3a, 9a - 1]$.

Case 1. $n = 3p$, then p belongs to $[a, 3a - 1]$; thus, by the hypothesis $f(p) > \pi(p)$ and by Theorem 2 $f(3p) > \pi(3p) + 1$.

Case 2. $n = 3p + 1$ (in particular this case contains $n \equiv 4 \pmod{6}$). Then $p + 1$ belongs to $[a, 3a]$; thus by the hypothesis and for $p = 3a$ by Case 1 $f(p + 1) > \pi(p + 1)$ and by Theorem 5 $f(3p + 1) > \pi(3p + 1) + 1$.

Case 3. $n = 3p + 2$, then $p + 2$ belongs to $[a, 3a + 1]$; more if $p + 2 \equiv 4 \pmod{6}$, $p \equiv 2 \pmod{6}$; thus by the hypothesis and for $p = 3a$ and $p = 3a + 1$ by Case 1 and 2, $f(p + 2) > \pi(p + 2)$ and for $p \equiv 2 \pmod{6}$, $f(p + 2) > \pi(p + 2) + 1$ and so, by Theorem 7, $f(3p + 2) > \pi(3p + 2)$.

Corollary. To prove Conjecture 1 of Kotzig: $f(n) > \pi(n)$ for $n \geq 10$ it suffices to prove that for n belonging to $[10], [29]$: $f(n) > \pi(n)$ and for $n \equiv 4 \pmod{6}$ in this interval $f(n) > \pi(n) + 1$.

In Proposition 4 we have proved that $f(10) > \pi(10) + 1$ and that $f(11) > \pi(11)$. In the example of Theorem 4 we have $f(28) > \pi(28) + 1$ and in the examples of Theorem 6 $f(26) > \pi(26)$ and $f(29) > \pi(29)$. We shall decide new values in the following propositions.

Theorem 8. $f(qp + 3) \geq q^2 \pi(p) + pf(q + 3) - 2(p - 1)$.

Proof. Let us choose as graph H in Theorem 3 a tournament with $\tau(H) = f(q + 3)$ and, as subgraph H' , a 3-circuit (this is possible otherwise H would be a tournament without circuit), then $m(H') = 3$ and $\tau(H') = 1$.

Corollary. $f(24) > \pi(24)$ and $f(27) > \pi(27)$.

Proof. We apply Theorem 8 with $p = 3$ and $q = 7, 8$; we deduce $f(24) \geq 90$ $\pi(24) = 88$; $f(27) \geq 120$ $\pi(27) = 117$.

Proposition 11. $f(12) \geq 21$ (Conjecture 1 is true for 12).

Proof. Let us consider the tournament T_{11} defined in [2] p. 21, where \overrightarrow{ij} is an arc of T_{11} , what we note by $i > j$, if and only if $j - i$ is congruent to a square modulo 11. Let T_{12} be the tournament obtained by adding to T_{11} a vertex, denoted by 12, and the arcs $\overrightarrow{12i}$ for $i = 1, 2, 4, 5, 6, 10$ and $\overrightarrow{j12}$ for $j = 3, 7, 8, 9, 11$. T_{12} can be written

$1 > 2, 4, 5, 6, 10,$	$2 > 3, 5, 6, 7, 11,$	$3 > 1, 4, 6, 7, 8, 12,$
$4 > 2, 5, 7, 8, 9,$	$5 > 3, 6, 8, 9, 10,$	$6 > 4, 7, 9, 10, 11,$
$7 > 1, 5, 8, 10, 11, 12,$	$8 > 1, 2, 6, 9, 11, 12,$	$9 > 1, 2, 3, 7, 10, 12,$
$10 > 2, 3, 4, 8, 11,$	$11 > 1, 3, 4, 5, 9, 12,$	$12 > 1, 2, 4, 5, 6, 10.$

To prove Conjecture 1, as $\pi(12) = 20$, we have to prove that $\tau(T_{12}) \geq 21$. Let S be a transversal set with $\tau(T_{12})$ elements.

Case 1. Let us suppose that S contains no arc with end-vertex 12, and let T_i be the subtournament generated by the vertices 1, 2, 4, 5, 6, 10 and T_j by the vertices 3, 7, 8, 9, 11; the subtournament generated by 2, 4, 5, 6, 10 and T_j being isomorphic to the tournament of the example of Fig. 1 we have $\tau(T_i) = 3$ and $\tau(T_j) = 3$. $S \cap T_i$ must be a transversal set of T_i , thus $|S \cap T_i| \geq 3$ and similarly $|S \cap T_j| \geq 3$; further S must contain all the 15 arcs \vec{ij} with $i = 1, 2, 4, 5, 6, 10$; $j = 3, 7, 8, 9, 11$ otherwise $T_{12} - S$ would contain the circuit $12ij$. Thus $|S| \geq 21$.

Case 2. Let us suppose that S contains at least an arc with end-vertex 12; $S \cap T_{11}$ must be a transversal set of T_{11} and $\tau(T_{11}) = 20$ (see [2]), thus $|S| \geq 21$. In both cases $|S| \geq 21$ and so $\tau(T_{12}) \geq 21$.

Proposition 12. $f(18) > \pi(18)$ and $f(19) > \pi(19)$.

Proof. We have $\pi(18) = 48$ and $\pi(19) = 57$. Let us consider the tournament T_{19} in which \vec{ij} is an arc if and only if $j - i$ is congruent to a square modulo 19 (quadratic residue). This tournament is vertex transitive and so $\tau(T_{19}) = 9 + \tau(T_{18})$ where T_{18} is a tournament obtained by deleting any vertex, for example 19. In T_{18} all the vertices of score (out-degree) 9 are equivalent and all the vertices of score 8 are equivalent: it suffices to consider the automorphisms associated to the powers of the permutation $1 \rightarrow 5 \rightarrow 6 \rightarrow 11 \rightarrow 17 \rightarrow 9 \rightarrow 7 \rightarrow 16 \rightarrow 4 \rightarrow 1$ and $2 \rightarrow 10 \rightarrow 12 \rightarrow 3 \rightarrow 15 \rightarrow 18 \rightarrow 14 \rightarrow 13 \rightarrow 8 \rightarrow 2$. Thus $\tau(T_{18}) = 8 + \tau(T_{17})$ where T_{17} is obtained by deleting any one vertex, for example 18. We shall prove that $\pi(T_{17}) \geq 41$, so $\tau(T_{17}) \geq 41$ and thus $\tau(T_{18}) \geq 49 > \pi(18)$ and $\tau(T_{19}) \geq 58 > \pi(19)$. T_{17} can be written:

$1 > 2, 5, 6, 7, 8, 10, 12, 17;$ $2 > 3, 6, 7, 8, 9, 11, 13;$
 $3 > 1, 4, 7, 8, 9, 10, 12, 14;$ $4 > 1, 2, 5, 8, 9, 10, 11, 13, 15;$
 $5 > 2, 3, 6, 9, 10, 11, 12, 14, 16;$ $6 > 3, 4, 7, 10, 11, 12, 13, 15, 17;$
 $7 > 4, 5, 8, 11, 12, 13, 14, 16;$ $8 > 5, 6, 9, 12, 13, 14, 15, 17;$
 $9 > 1, 6, 7, 10, 13, 14, 15, 16;$ $10 > 2, 7, 8, 11, 14, 15, 16, 17;$
 $11 > 1, 3, 8, 9, 12, 15, 16, 17;$ $12 > 2, 4, 9, 10, 13, 16, 17;$
 $13 > 1, 3, 5, 10, 11, 14, 17;$ $14 > 1, 2, 4, 6, 11, 12, 15;$
 $15 > 1, 2, 3, 5, 7, 12, 13, 16;$ $16 > 1, 2, 3, 4, 6, 8, 13, 14, 17;$
 $17 > 2, 3, 4, 5, 7, 9, 14, 15;$

T_{17} contains the following 41 circuits:

$1\ 2\ 3;$ $1\ 5\ 16;$ $1\ 6\ 11;$ $1\ 7\ 13;$ $1\ 8\ 15;$ $1\ 10\ 14;$ $1\ 12\ 4;$ $1\ 17\ 9;$ $2\ 6\ 17;$
 $2\ 7\ 14;$ $2\ 8\ 12;$ $2\ 11\ 15;$ $2\ 13\ 10;$ $3\ 4\ 13;$ $3\ 7\ 5;$ $3\ 8\ 17;$ $3\ 9\ 15;$ $3\ 10\ 11;$
 $3\ 12\ 16;$ $3\ 14\ 6;$ $4\ 8\ 14;$ $4\ 9\ 6;$ $4\ 10\ 17;$ $4\ 11\ 16;$ $4\ 15\ 7;$ $5\ 6\ 15;$ $5\ 9\ 13;$
 $5\ 10\ 8;$ $5\ 11\ 17;$ $6\ 7\ 8;$ $6\ 10\ 16;$ $7\ 11\ 9;$ $7\ 16\ 17;$ $8\ 9\ 16;$ $8\ 13\ 11;$
 $10\ 15\ 12;$ $12\ 17\ 14;$ $13\ 17\ 15;$ $14\ 15\ 16;$ $7\ 12\ 9\ 10;$ $11\ 12\ 13\ 14.$

By the aid of a computer program Y. Kodratoff (C.N.R.S. Paris) has found that for this tournament $\tau(T_{17}) = 47$, and for the subtournament T_{16} , obtained by deleting the vertex 4, $\tau(T_{16}) = 40$; thus we have:

Proposition 13. $f(19) \geq 64$; $f(18) \geq 55$; $f(17) \geq 47 > 45 = \pi(17)$
and $f(16) \geq 40 > 38 = \pi(16) + 1$.

The program applied to other tournaments has given:

Proposition 14. $f(13) \geq 27 > 26 = \pi(13)$.

The program was applied to the following tournament T_{13} :

$1 > 2, 4, 5, 6, 10, 12;$ $2 > 3, 5, 6, 7, 11, 13;$ $3 > 1, 4, 6, 7, 8, 12;$
 $4 > 2, 5, 7, 8, 9, 13;$ $5 > 3, 6, 8, 9, 10, 13;$ $6 > 4, 7, 9, 10, 11, 13;$
 $7 > 1, 5, 8, 10, 11, 12;$ $8 > 1, 2, 6, 9, 11, 12;$ $9 > 1, 2, 3, 7, 10, 12;$
 $10 > 2, 3, 4, 8, 11, 13;$ $11 > 1, 3, 4, 5, 9, 12;$ $12 > 2, 4, 5, 6, 10, 13;$
 $13 > 1, 3, 7, 8, 9, 11.$

The tournament T_{13} being regular, we have Proposition 11 as corollary.

Proposition 15. $f(15) \geq 36 > \pi(15)$ and $f(14) \geq 29 > \pi(14)$.

Let us consider the tournament T_{15} isomorphic to the subtournament of the tournament T_{31} of the quadratic residues module 31, generated by the vertices i such that $\overrightarrow{i31}$ is an arc of T_{31} . T_{15} can be written:

$$\begin{array}{ll}
 1 > 2, 4, 6, 7, 8, 12, 13; & 2 > 3, 5, 7, 8, 10, 11, 13; \\
 3 > 1, 4, 6, 7, 9, 11, 13; & 4 > 2, 5, 7, 8, 9, 12, 14; \\
 5 > 1, 3, 6, 7, 9, 10, 14; & 6 > 2, 4, 7, 8, 10, 11, 15; \\
 7 > 8, 9, 10, 11, 12, 14, 15; & 8 > 3, 5, 9, 11, 12, 13, 15; \\
 9 > 1, 2, 6, 10, 11, 12, 15; & 10 > 1, 3, 4, 8, 11, 13, 14; \\
 11 > 1, 4, 5, 12, 13, 14, 15; & 12 > 2, 3, 5, 6, 10, 13, 15; \\
 13 > 4, 5, 6, 7, 9, 14, 15; & 14 > 1, 2, 3, 6, 8, 9, 12; \\
 15 > 1, 2, 3, 4, 5, 10, 14.
 \end{array}$$

T_{15} is vertex-transitive; therefore the program was applied to a subtournament T_{14} and gave $\tau(T_{14}) = 29$ and therefore $\tau(T_{15}) = 36$.

Remarks.

1. It is enough to prove Conjecture 1 for the odd values of n , because if $f(2p + 1) > \pi(2p + 1)$ then $f(2p) > \pi(2p)$: suppose that $f(2p) \leq \pi(2p)$, from $\pi(2p + 1) = \pi(2p) + p$ and $f(2p + 1) \leq f(2p) + p$ (see [2] p. 11) we deduce $f(2p + 1) \leq \pi(2p + 1)$ contrary to the hypothesis.

2. The result of Erdős and Moon [5] shows that if $n \rightarrow \infty$, $f(n) \sim \frac{n(n-1)}{4}$ but $\pi(n) \sim \frac{n(n-1)}{6}$; thus Conjecture 1 is true if n is sufficiently large.

In summary to prove Conjecture 1 of Kotzig it remains to prove that $f(n) > \pi(n)$ for $n = 21, 23, 25$ and that $f(22) > \pi(22) + 1$. (We hope to verify the conjecture for these values by computer search).

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