



**HAL**  
open science

## Some Ramsey numbers for directed graphs

Jean-Claude Bermond

► **To cite this version:**

Jean-Claude Bermond. Some Ramsey numbers for directed graphs. *Discrete Mathematics*, 1974, 9 (4), pp.313-321. 10.1016/0012-365X(74)90077-6 . hal-02366537

**HAL Id: hal-02366537**

**<https://inria.hal.science/hal-02366537>**

Submitted on 16 Nov 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# SOME RAMSEY NUMBERS FOR DIRECTED GRAPHS

J.C. BERMOND

*Centre de Mathématique Sociale, 54 Bd Raspail, 75.270 Paris Cedex 06, France*

**Abstract.** Given  $k$  directed graphs  $G_1, \dots, G_k$ , the Ramsey number  $R(G_1, \dots, G_k)$  is the smallest integer  $n$  such that for any partition  $(U_1, \dots, U_k)$  of the arcs of the complete symmetric directed graph  $K_n^*$ , there exists an integer  $i$  such that the partial graph generated by  $U_i$  contains  $G_i$  as a subgraph. In this article we give a necessary and sufficient condition for the existence of Ramsey numbers, and, when they exist, an upper bound function. We also give exact values for some classes of graphs. Our main result is:  $R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, G) = n_1 \dots n_{k-1} (p-1) + 1$ , where  $G$  is a hamiltonian directed graph with  $p$  vertices and  $\vec{P}_{n_i}$  denotes the directed path of length  $n_i$ .

## 1. Definitions

Let  $G = (X, U)$  denote a finite directed graph with no loops and no multiple arcs. We associate with  $G$  an undirected simple graph, called the underlying graph of  $G$ , which has the same vertex set  $X$  and which contains an edge incident with two vertices  $x$  and  $y$ , provided at least one of the arcs  $xy$  or  $yx$  belongs to  $V$ .

We will use the following notations (for definitions on tournaments see Moon [11], for other definitions see Berge [1]):

- $K_p$  : complete undirected graph with  $p$  vertices.
- $K_p^*$  : complete symmetric directed graph with  $p$  vertices.
- $T_p$  : a tournament with  $p$  vertices.
- $TT_p$  : the transitive tournament with  $p$  vertices.
- $\vec{P}_n$  : directed simple path of length  $n$  ( $\vec{P}_n$  has  $n+1$  vertices).
- $\vec{C}_p$  : directed simple circuit of length  $p$ .

The Ramsey number  $R(G_1, \dots, G_k)$  of the directed graphs  $G_1, \dots, G_k$  is the smallest integer  $n$  such that for any partition  $(U_1, \dots, U_k)$  of the arcs of  $K_n^*$  into  $k$  sets, some of which may be empty, there exists an integer  $i$ ,  $1 \leq i \leq k$ , such that  $G_i$  is a subgraph of the partial graph of  $K_n^*$  generated by  $U_i$  (which we denote also by  $U_i$ , when no confusion is possible).

The partition  $(U_1, \dots, U_k)$  is called a  $k$ -coloring of the arcs of  $K_n^*$ . If we do not allow the sets  $U_i$  to be empty, our results remain true, if we suppose that no  $G_i$  is reduced to one arc.

The Ramsey number of undirected graphs is defined analogously. We also denote by the same symbol the set of edges  $E_i$  and the partial graph of  $K_n$  generated by  $E_i$ .

From the definitions it follows easily that if  $G_i$  is a directed graph and  $H_i$  is the underlying graph of  $G_i$ , then

$$(1) \quad R(G_1, \dots, G_k) \geq R(H_1, \dots, H_k).$$

The purpose of this article is to provide a criterion for the existence of the Ramsey number of directed graphs and then to give an upper bound for them. Note that a lower bound is given by (1). Furthermore, we give exact values for some classes of directed graphs.

There is an extensive literature concerned with Ramsey numbers of undirected graphs. For directed graphs we have the existence of the Ramsey numbers for two graphs and  $R(TT_m, TT_n) = R(K_m, K_n)$  by Harary and Hell (see [10]), and the value of  $R(\vec{P}_m, \vec{P}_n)$  by Williamson [16].

## 2. Existence and upper bounds for $R(G_1, \dots, G_k)$

Let  $\nu(n)$  denote the smallest integer  $\nu$ , such that every tournament  $T_\nu$  contains a transitive subtournament  $TT_n$ .

**Lemma 2.1** (Erdős and Moser [5], Stearns [15]).  *$\nu(n)$  is finite and  $\nu(n) \leq 2^{n-1}$ .*

Only few values of  $\nu$  are known:  $\nu(2) = 2$ ;  $\nu(3) = 4$ ;  $\nu(4) = 8$ ;  $\nu(5) = 14$ ;  $\nu(6) = 28$ . The last two values are due to Reid and Parker [13].

We denote by  $r(n_1, \dots, n_{k-1})$  the Ramsey number  $R(K_{n_1}, \dots, K_{n_{k-1}})$  and note that  $r(n) = n$ .

**Theorem 2.2.**  $R(TT_{n_1}, \dots, TT_{n_{k-1}}, K_p^*) \leq r(\nu[r(n_1, \dots, n_{k-1})], p)$ .

**Proof.** Let  $(U_1, \dots, U_k)$  be a  $k$ -coloring of the arcs of  $K_n^*$ ; then

$(U_1 \cup \dots \cup U_{k-1}, U_k)$  is a 2-coloring. Let  $(E_1, E_2)$  be the following 2-coloring of the edges of  $K_n$ : the graph  $E_1$  is the underlying graph of  $U_1 \cup \dots \cup U_{k-1}$  and  $E_2$  is the complement of  $E_1$ . Note that  $E_2$  is not the underlying graph of  $U_k$ ; and edge  $\{x, y\}$  belongs to  $E_2$  if and only if  $xy$  and  $yx$  belongs to  $U_k$ . Then, by definition, if  $n \geq r(v, p)$ , either  $E_1$  contains a  $K_p$  or  $E_2$  contains a  $K_p$ . Thus, by the construction, we have that either  $U_1 \cup \dots \cup U_{k-1}$  contains a  $T_p$  or  $U_k$  contains a  $K_p^*$ . Suppose that  $U_1 \cup \dots \cup U_{k-1}$  contains a  $T_p$ . If  $v \geq v[r(n_1, \dots, n_{k-1})]$ , then  $T_p$  contains a transitive subtournament  $TT_{r(n_1, \dots, n_{k-1})}$ . Let  $(E_1, \dots, E_{k-1})$  be the following  $k-1$  coloring of the edges of  $K_{r(n_1, \dots, n_{k-1})}$ : a given edge belongs to  $E_i$  if the unique corresponding arc in  $TT_{r(n_1, \dots, n_{k-1})}$  belongs to  $U_i$ . Then, by definition of  $r(n_1, \dots, n_{k-1})$ , there exists an integer  $i$ ,  $1 \leq i \leq k-1$ , such that  $E_i$  contains a  $K_{n_i}$ . Thus there exists an integer  $i$ ,  $1 \leq i \leq k-1$ , such that  $TT_{r(n_1, \dots, n_{k-1})}$  and consequently  $T_p$  contains a  $TT_{n_i}$  included in  $U_i$ . Summarizing: if  $n \geq r(v[r(n_1, \dots, n_{k-1})], p)$ , then for any  $k$ -coloring  $(U_1, \dots, U_k)$  of  $K_n^*$ , either there exists an integer  $i$ ,  $1 \leq i \leq k-1$ , such that  $U_i$  contains a  $TT_{n_i}$ , or  $U_k$  contains a  $K_p^*$ .

**Theorem 2.3.** *Let  $G_1, \dots, G_k$  be directed graphs,  $R(G_1, \dots, G_k)$  exists if and only if at most one of the  $G_i$  contains a circuit.*

**Proof.** Suppose that no  $G_i$ ,  $1 \leq i \leq k-1$ , contains a circuit. Let  $n_i$  be the number of vertices of  $G_i$ . Then, for  $1 \leq i \leq k-1$ ,  $G_i$  is contained in a  $TT_{n_i}$  and  $R(G_1, \dots, G_{k-1}, G_k) \leq R(TT_{n_1}, \dots, TT_{n_{k-1}}, K_{n_k}^*)$ . Thus the sufficiency results from Theorem 2.2 and the classic Ramsey's theorem.

Conversely, suppose that two graphs  $G_{i_1}$  and  $G_{i_2}$  contain circuits. For each  $n$ , choose the following  $k$ -coloring of  $K_n^*$ :  $E_{i_1}$  is the transitive tournament  $TT_n$ ,  $E_{i_2}$  is the transitive tournament complement of  $E_{i_1}$  and  $E_i$  is empty for  $i \neq i_1, i_2$ . For this  $k$ -coloring,  $K_n^*$  does not contain any  $G_i$  as a subgraph of  $E_i$ , and the necessity follows.

The following propositions will show that in some cases the upper bound in Theorem 2.2 is attained.

**Proposition 2.4.**  $R(TT_n, K_2^*) = v(n)$ .

**Proof.** By Theorem 2.2, we have  $R(TT_n, K_2^*) \leq r(v(n), 2) = v(n)$ . By

definition of  $\nu(n)$ , there exists a tournament  $T_{\nu(n)-1}$  that does not contain any  $TT_n$ . Let us consider the 2-coloring  $(U_1, U_2)$  of  $K_{\nu(n)-1}^*$ , where  $U_1$  is the tournament  $T_{\nu(n)-1}$  defined above and  $U_2$  is the complement of  $U_1$ . Then  $U_1$  does not contain any  $TT_n$  and  $U_2$  does not contain a  $K_2^*$ . Thus  $R(TT_n, K_2^*) \geq \nu(n)$ .

**Proposition 2.5.**  $R(TT_3, K_3^*) = 9$ .

**Proof.** By Theorem 2.2,  $R(TT_3, K_3^*) \leq r(\nu(3), 3) = r(4, 3) = 9$ . Consider the following 2-coloring  $(U_1, U_2)$  of  $K_8^*$ , whose vertices are labelled from 1 to 8:  $ij$  is an arc of  $U_1$  if and only if  $j-i \equiv 2$  or  $3 \pmod{8}$ . It can be shown that  $U_1$  does not contain any  $TT_3$ . Let  $U_2$  be the complement of  $U_1$  and let  $(E_1, E_2)$  be the 2-coloring of  $K_8$ , where  $E_1$  is the underlying graph of  $U_1$ . By a result of Graver and Yackel [8, p. 148],  $E_2$  does not contain a  $K_3$  and so  $U_2$  does not contain a  $K_3^*$ .

It may be that the upper bound of Theorem 2.2 is the exact value of  $R(TT_n, K_m)$ , but in general it is not the exact value as shown in the following propositions concerning the case where  $G_p = K_2^*$ .

**Proposition 2.6.** Let  $f(n_1, \dots, n_{k-1}) = R(TT_{n_1}, \dots, TT_{n_{k-1}}, K_2^*) - 1$ . Then

- (i)  $f(n_1, \dots, n_{k-2}, 2) = f(n_1, \dots, n_{k-2})$ , and
- (ii)  $f(n_1, \dots, n_{k-1}) \leq 1 + 2 \sum_{i=1}^{k-1} f(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_{k-1})$ .

**Proof.** (i) is evident. To prove (ii), let  $(U_1, \dots, U_k)$  be a  $k$ -coloring of  $K_n^*$ . If  $U_k$  does not contain a  $K_2^*$ , then  $U_k$  is included in a tournament and  $U_1 \cup U_2 \cup \dots \cup U_{k-1}$  contains a tournament  $T_n$ . Let  $x_0$  be a vertex of  $T_n$ ; if  $n \geq 2m + 2$ , then either the outdegree or the indegree of  $x_0$  is greater than or equal to  $m + 1$ . Suppose that it is the outdegree. Thus we have  $m + 1$  arcs starting from  $x_0$ . If

$$m \geq \sum_{i=1}^{k-1} f(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_{k-1}),$$

then there exists an integer  $i$ ,  $1 \leq i \leq k-1$ , such that  $U_i$  contains at least  $f(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_{k-1}) + 1$  arcs issuing from  $x_0$ . Consider the subgraph of  $K_n^*$  generated by the terminal vertices  $x$  of the arcs  $x_0x$  belonging to  $U_i$ . This subgraph has at least  $f(n_1, \dots, n_{i-1}, n_i - 1, \dots, n_{k-1}) + 1$

vertices and as it does not contain a  $K_2^*$  included in  $U_k$ , by definition of  $f$ , either it contains a  $TT_{n_j}$  for  $j \neq i$ , or it contains a  $TT_{n_{i-1}}$ . But in the latter case this  $TT_{n_{i-1}}$  and the arcs  $x_0x$  belonging to  $U_i$  induce a  $TT_{n_i}$  included in  $U_i$ .

**Proposition 2.7.**  $R(TT_3, TT_3, K_2^*) = 14$ .

**Proof.** By Proposition 2.6,  $f(3, 3) \leq 1 + 4f(3, 2) = 1 + 4f(3)$  and by Proposition 2.4,  $f(3) = \nu(3) - 1 = 3$ . Thus  $f(3, 3) \leq 13$  and  $R(TT_3, TT_3, K_2^*) \leq 14$ . Consider the following 3-coloring  $(U_1, U_2, U_3)$  of  $K_{13}^*$ :  $U_1$  is the set of arcs  $ij$  with  $j-i \equiv 1, 3$  or  $9 \pmod{13}$ ,  $U_2$  is the set of arcs  $ij$  with  $j-i \equiv 2, 5$  or  $6 \pmod{13}$ . The graph generated by  $U_1 \cup U_2$  is a tournament. Its complement  $U_3$  is also a tournament and does not contain a  $K_2^*$ . It can be easily checked that neither  $U_1$  nor  $U_2$  contain a  $TT_3$ .

The upper bound of Theorem 2.2 for  $R(TT_3, TT_3, K_2^*)$  is  $r(\nu(r(3, 3)), 2) = \nu(r(3, 3)) = \nu(6) = 28$ , which is far from the upper bound of Proposition 2.6 if  $k \geq 3$  and  $n_i \geq 3$ .

### 3. Some path Ramsey numbers

The determination of Ramsey numbers for directed graphs is more difficult than in the undirected case. It is evident for example that  $R(K_n, K_2) = n$ , but  $R(TT_n, K_2^*) = \nu(n)$  is known only if  $n \leq 6$ . We shall see, however, that it is possible to give the exact values in some cases for which the same problem for the underlying graphs is not solved. We shall deal now with the case where  $G_i, 1 \leq i \leq k-1$ , is a path.

Our first aim was to determine  $R(\vec{P}_m, G)$ , where  $G$  was a circuit. The case where  $G$  is a path has been solved by Williamson [16] where it is proved that  $R(\vec{P}_1, \vec{P}_n) = n + 1$  and  $R(\vec{P}_m, \vec{P}_n) = m + n - 1$ , for  $2 \leq m \leq n$ . But the determination of  $R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, G)$ , where  $G$  contains a circuit, uses the same ideas as for one path. Thus we give results directly for the general case.

We shall use the following theorem due to Chvátal [4], which is a generalisation of a theorem found independently by Gallai [6] and Roy [14].

**Lemma 3.1** (Chvátal [4]). *Let  $G = (X, U)$  be a directed graph and suppose that  $(U_1, \dots, U_{k-1})$  is a partition of  $U$  into  $k-1$  sets. If the length of the maximal directed (simple) path contained in  $U_i$  is less than or equal to  $n_i-1$ , then  $\chi(G) \leq n_1 n_2 \dots n_{k-1}$ .*

**Proof.** We give here only a sketch of a proof (very similar to Chvátal's elegant proof). The argument will explain the origin of Lemma 3.4.

Let  $V$  be a minimal set of arcs of  $G$  such that the graph  $G-V$ , obtained from  $G$  by deleting the arcs of  $V$ , is without circuits (for properties of sets  $V$  and related problems see [2, 3]). Consider the sets of vertices  $C_{i_1, \dots, i_{k-1}}$ ,  $0 \leq i_j \leq n_j-1$ , of  $G$  defined as follows:  $x$  belongs to  $C_{i_1, \dots, i_{k-1}}$  if and only if  $i_j$  is the maximal length of a directed simple path, with initial vertex  $x$ , belonging to  $(U-V) \cap U_j$ , for  $j = 1, 2, \dots, k-1$ . As in Gallai's proof [6] in the case  $k = 2$ , it can be shown that the  $C_{i_1, \dots, i_{k-1}}$  are independent sets of vertices in  $G-V$  and also in  $G$ . Since the number of these sets is  $n_1 n_2 \dots n_{k-1}$ , the lemma is proved.

**Lemma 3.2.**  $R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, K_p^*) \leq n_1 \dots n_{k-1} (p-1) + 1$ .

**Proof.** Let  $(U_1, \dots, U_{k-1}, U_k)$  be a  $k$ -coloring of  $K_n^*$ , where  $n = n_1 \dots n_{k-1} (p-1) + 1$ . Let us suppose that  $U_k$  does not contain a  $K_p^*$ . Then the stability number (maximum number of independent vertices) of the graph  $U_1 \cup U_2 \cup \dots \cup U_{k-1}$  is less than or equal to  $p-1$ . Thus  $\chi(G) > n_1 n_2 \dots n_{k-1}$  and, by Lemma 3.1, there exists an integer  $i$ ,  $1 \leq i \leq k-1$  such that  $U_i$  contains a  $\vec{P}_{n_i}$ .

**Remark 3.3.** For  $k = 2$ , Lemma 3.2 is also a consequence of the Gallai–Milgram theorem [7]: the arcs of a directed graph  $G$  can be partitioned into  $\alpha(G)$  vertex-disjoint paths ( $\alpha(G)$  denotes the stability number of  $G$ ). If the number of vertices of  $G$  is greater than or equal to  $n_1 (p-1) + 1$ , then either  $\alpha(G) \geq p$  or there exists a path of length  $n_1$ . Lemma 3.2 follows by applying this result to  $U_1$ .

**Lemma 3.4.**  $R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, \vec{C}_p) > n_1 \dots n_{k-1} (p-1)$ .

**Proof.** Let  $C_{i_1, \dots, i_{k-1}}$ ,  $0 \leq i_j \leq n_j-1$ , be disjoint sets of  $p-1$  elements. Let  $X$  be the union of all the  $C_{i_1, \dots, i_{k-1}}$ , so  $n = |X| = n_1 \dots n_{k-1} (p-1)$ . Let  $(U_1, \dots, U_k)$  be the following  $k$ -coloring of  $K_n^*$  (with vertex set  $X$ ):

for  $1 \leq j \leq k-1$ ,  $U_j$  contains the arcs  $xy$ , where  $x$  belongs to  $C_{i_1, \dots, i_j, \dots, i_{k-1}}$  and  $y$  to  $C_{i_1, \dots, i'_j, \dots, i'_{k-1}}$ , with  $i_h = i'_h$  for  $1 \leq h \leq j-1$  and  $i_j > i'_j$ . (There is no condition on the  $i_h$  and  $i'_h$  for  $h > j$ .) Now  $U_k$  is the complement of  $U_1 \cup U_2 \cup \dots \cup U_{k-1}$  and  $U_j$  does not contain a  $\vec{P}_{n_j}$  because  $0 \leq i_j \leq n_j - 1$ . A vertex  $x$  of  $C_{i_1, \dots, i_{k-1}}$  is joined by an arc of  $U_k$  to a point of  $C_{i'_1, \dots, i'_{k-1}}$  if and only if the  $(k-1)$ -tuple  $(i_1, \dots, i_{k-1})$  is less than or equal to  $(i'_1, \dots, i'_{k-1})$  in the lexicographic ordering. Thus the circuits of  $U_k$  are contained in the subgraphs generated by  $C_{i_1 \dots i_{k-1}}$  and their lengths are at most  $p-1$ .

**Theorem 3.5.** *Let  $G$  be a directed hamiltonian graph with  $p$  vertices. Then  $R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, G) = n_1 \dots n_{k-1} (p-1) + 1$ .*

**Proof.** Since  $G$  has  $p$  vertices and contains a hamiltonian circuit  $\vec{C}_p$ ,

$$\begin{aligned} R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, \vec{C}_p) &\leq R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, G) \\ &\leq R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, K_p^*) \end{aligned}$$

and the theorem follows from Lemmas 3.2 and 3.4.

Theorem 3.5 applies, in particular, to the cases in which  $G$  is taken to be  $\vec{C}_p$ ,  $K_p^*$ , any strong tournament  $TT_p$ , and for  $p$  even, the complete symmetric directed bipartite graph  $K_{p/2, p/2}^*$ .

Theorem 3.5 gives also an upper bound for  $R(\vec{P}_{n_1}, \dots, \vec{P}_{n_{k-1}}, TT_p)$  and  $R(P_{n_1}, \dots, P_{n_{k-1}}, K_p)$ , where  $P_{n_i}$  denotes the undirected path of length  $n_i$ . In the case  $k = 2$ , this upper bound is the exact value, but this is not true for  $k > 2$  as shown by the following results.

**Theorem 3.6** (Parsons [12]).  $R(P_n, K_p) = n(p-1) + 1$ .

**Proof.** By Theorem 3.5,  $R(P_n, K_p) \leq R(\vec{P}_n, K_p^*) = n(p-1) + 1$ .

$R(P_n, K_p) > n(p-1)$  is obtained by considering the following 2-coloring  $(E_1, E_2)$  of  $K_{n(p-1)}$ :  $E_1$  is a set of  $p-1$  disjoint  $K_n$  and  $E_2$  is the complement.

**Corollary 3.7.**  $R(\vec{P}_n, TT_p) = n(p-1) + 1$ .

**Proposition 3.8.**  $R(P_2, P_2, K_3) = R(\vec{P}_2, \vec{P}_2, TT_3) = 5$ .



The proof is left to the reader, the lower bound 5 follows from Proposition 3.9.

**Proposition 3.9.**  $R(\vec{P}_m, \vec{P}_n, TT_p) > (m+n-2)(p-1)$  for  $2 \leq m \leq n$ , and  $R(P_m, P_n, K_p) > (n + \lfloor \frac{1}{2}(m+1) \rfloor - 1)(p-1)$  for  $2 \leq m \leq n$ .

**Proof.** Let  $2 \leq m \leq n$  (the case  $m = 1$  is trivial, indeed we have  $R(\vec{P}_m, \vec{P}_n, TT_p) = R(\vec{P}_n, TT_p) = n(p-1) + 1$ ). To prove the proposition consider the following 3-coloring  $(U_1, U_2, U_3)$  of  $K_n^*$ , with  $n = (m+n-2)(p-1)$ .  $U_1 \cup U_2$  is formed by  $p-1$  disjoint copies of  $K_{m+n-2}^*$ , such that there is no  $\vec{P}_m$  in  $U_1$  and no  $\vec{P}_n$  in  $U_2$ , which is possible by the result of Williamson [16]:  $R(\vec{P}_m, \vec{P}_n) = m+n-1$ ;  $U_3$  is the complement of  $U_1 \cup U_2$ . One can show analogously, using the result of Gyárfás and Gerencsér [9]:  $R(P_m, P_n) = n + \lfloor \frac{1}{2}(m+1) \rfloor$ , that  $R(P_m, P_n, K_p) > (n + \lfloor \frac{1}{2}(m+1) \rfloor - 1)(p-1)$ .

We conjecture that  $R(\vec{P}_m, \vec{P}_n, TT_p) = (m+n-2)(p-1) + 1$  and  $R(P_m, P_n, K_p) = (n + \lfloor \frac{1}{2}(m+1) \rfloor - 1)(p-1) + 1$  for  $2 \leq m \leq n$ .

**Added in proof.** Some of the results of Section 2 (2.2, 2.3, 2.5) are also in an article of F. Harary and P. Hell (Generalised Ramsey theory for graphs V, The Ramsey number of a digraph, to appear).

## References

- [1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1972).
- [2] J.C. Bermond, Ordres à distance minimum d'un tournoi et graphes partiels sans circuits maximaux, Math. Sci. Humaines 37 (1972) 5–25.
- [3] J.C. Bermond, The circuit hypergraph of a tournament, Proceedings of Kesthely (1973) to appear.
- [4] V. Chvátal, Monochromatic paths in edge-colored graphs, J. Combin. Theory 13 (B) (1972) 69–70.
- [5] P. Erdős and L. Moser, On the representation of a directed graph as unions of orderings, Publ. Math. Inst. Hungar. Acad. Sci. 9 (1964) 125–132.
- [6] T. Gallai, On directed paths and circuits, in: P. Erdős and G. Katona, eds., Theory of graphs (Academic Press, New York, 1968) 115–118.
- [7] T. Gallai and A.N. Milgram, Verallgemeinerung eines Graphen theoretischen Satzes von Redei, Acta Sci. Math. 21 (1960) 181–186.
- [8] J. Graver and J. Yackel, Some graph theoretic results associated with Ramsey's theorem, J. Combin. Theory 4 (1968) 125–175.

- [9] A. Gyárfás and L. Gerencsér, On Ramsey type problems, *Ann. Univ. Sci. Budapest. Eötvös* 10 (1967) 167–170.
- [10] F. Harary and P. Hell, Generalized Ramsey theory for graphs IV, Ramsey numbers for digraphs, in: F. Harary, ed., *Recent Results on Generalized Ramsey Theory for Graphs*, *Lectures Notes in Math.* 303 (Springer, Berlin, 1973) 125–138.
- [11] J.W. Moon, *Topics on Tournaments* (Holt, New York, 1968).
- [12] T.D. Parsons, The Ramsey numbers  $r(P_m, K_n)$ , *Discrete Math.* 6 (1973) 159–162.
- [13] K.B. Reid and E.T. Parker, Disproof of a conjecture of Erdős and Moser on tournaments, *J. Combin. Theory* 9 (1970) 225–238.
- [14] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, *R. Inf. Res. Oper.* 1 (1967) 129–132.
- [15] R. Stearns, The voting problem, *Am. Math. Monthly* 66 (1959) 761–763.
- [16] J.E. Williamson, A Ramsey-type problem for paths in digraphs, *Math. Ann.* 203 (1973) 117–118.