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Jean-Claude Bermond, Jan Novak

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ON MAXIMAL SYSTEMS OF QUADRUPLES

J-C.BERMOND and J.NOVAK.

ABSTRACT : We determine the maximum number of quadruples of a set  $X$  of  $n$  elements, such that every pair of elements of  $X$  belong to at most one quadruple (problem of packing of pairs with quadruples) for some values of  $n$  ; in particular for :  $n = 12 st + 2$  and  $n = 12 st + 2s$ , with  $6 + 1$  a power of a prime and  $s \equiv 1$  or  $4 \pmod{12}$  ; and for  $n = 48 t + i$  with  $i = -6, -5, -3, 6, 7, 9$ .

1. INTRODUCTION.

1.1. Let  $X$  be a set of  $n$  elements. We are interested in the following "packing problem" : find the maximum number  $q(n)$  of quadruples of  $X$ , such that every pair of elements of  $X$  belong to at most one quadruple (or equivalently two quadruples meet in at most one point). We will denote such a maximal system by  $K(n, 4, 1)$  and thus  $q(n) = |K(n, 4, 1)|$ . In a graph theoretical language we want to find the maximum number of edge-disjoint  $K_4$ , contained in a  $K_n$ .

1.2. H.Hanani [ 6] has proved that, if  $n \equiv 1$  or  $4 \pmod{12}$ , then there exist maximal systems of quadruples  $K(n, 4, 1)$  such that every pair of elements belong to exactly one quadruple, and in this case  $q(n) = n(n-1) / 12$ .

1.3. R.M.Wilson [13] has proved that  $q(n) \leq f(n)$  where

$$f(n) = \left[ \frac{n}{4} \left[ \frac{n-1}{3} \right] \right] \text{ if } n \not\equiv 7 \text{ or } 10 \pmod{12}$$

$$f(n) = \left[ \frac{n}{4} \left[ \frac{n-1}{3} \right] \right] - 1 \text{ if } n \equiv 7 \text{ or } 10 \pmod{12}.$$

(  $[x]$  denotes the greatest integer less than or equal to  $x$ ).

In [13] R.M.Wilson has also proved that if  $n$  is large enough then  $q(n) = f(n)$ .

1.4. Let  $n \equiv 1$  or  $4 \pmod{12}$  ; if we delete all the quadruples of a  $K(n,4,1)$  containing a given element, we obtain  $f(n-1)$  quadruples, thus forming a  $K(n-1, 4,1)$ . Thus  $q(n) = f(n)$  for  $n \equiv 0$  or  $3 \pmod{12}$ .

1.5. In [3] R.K.Guy has given the values of  $q(n)$  for  $n \leq 16$  ; for these values  $q(n) = f(n)$  except for  $n = 8,9,10,11$  where :

$$q(8) = 2, f(8) = 4 ; q(9) = 3, f(9) = 4 ; q(10) = 5, f(10) = 6 ; \\ q(11) = 6, f(11) = 8.$$

1.6. The aim of this paper is to construct systems  $K(n,4,1)$  with  $q(n) = f(n)$  quadruples in particular for  $n = 12st + 2$  or  $n = 12st + 2s$ , with  $6t + 1$  a power of a prime and  $s \equiv 1$  or  $4 \pmod{12}$  and also for  $n = 48t - 6, 48t - 5, 48t - 3, 48t + 6, 48t + 7, 48t + 9$ .

1.7. Remarks : the same packing problems with triples has been solved in [7, 11, 12] (see also [I, I.6. 16]). The problem of coverings of pairs by quadruples, such that every pair of elements belong to at least one quadruple has been solved by W.H. Mills [8,9].

## 2. CASE $n = 12t + 2$ .

2.1. Theorem : Let  $n = 12t + 2$  and  $6t + 1 = p^r$ ,  $p$  a prime.  
Then  $q(n) = f(n) = 12t^2 + 2t$ .

Proof : Let  $x$  be a primitive root of the Galois Field  $GF(p^r)$ .

We will use the following theorem proved in M.Hall [4, th. 15.3.4.] : the triples  $\{x^i, x^{2t+i}, x^{4t+i}\}$   $i = 0, 1, \dots, t-1$  are base blocks, with respect to the additive group  $A$  of  $GF(p^r)$ , of a steiner triple system on  $6t + 1$  elements. Recall that a steiner triple system is a family of triples, such that every pair of elements belong to exactly one triple : here the triples are  $\{x^i + a, x^{2t+i} + a, x^{4t+i} + a\}$   $i = 0, 1, \dots, t-1$  and  $a \in A$ .

Now let us associate to each triple of this steiner triple system on  $A$ , a quadruple of elements of  $A \times Z_2$  in the following manner : to the base triples  $\{x^i, x^{2t+i}, x^{4t+i}\}$   $i = 0, 1, \dots, t-1$  we associate the quadruple  $\{(x^i, 0), (x^{2t+i}, 0), (x^{4t+i}, 0), (0, 1)\}$   $i = 0, 1, \dots, t-1$ . We claim that these quadruples are base quadruples, with respect to  $A \times Z_2$ , of a  $K(n,4,1)$  ; that is that we have a  $K(n,4,1)$  consisting of the  $f(n) = 12t^2 + 2t$  quadruples  $\{(x^i + a, \alpha), (x^{2t+i} + a, \alpha), (x^{4t+i} + a, \alpha), (a, \alpha + 1)\}$  where  $i = 0, 1, \dots, t-1, a \in A$  and  $\alpha \in Z_2$ .

It suffices to prove that any pair of elements of  $A \times Z_2$  belong to at most one of these quadruples. If the pair of elements is of the form  $\{(y, \omega), (z, \omega)\}$  this follows from the fact, that the pair  $\{y, z\}$  belong to exactly one of the triples  $\{x^i + a, x^{2t+i} + a, x^{4t+i} + a\}$ . Thus suppose that the pair is of the form  $\{(y, 0), (z, 1)\}$ . Note that we have  $x^{6t} = 1$  and  $x^{3t} = -1$  and therefore the non zero elements of  $A$  are  $x^i, x^{2t+i}, x^{4t+i}$  ( $i=0, 1, \dots, t-1$ ) and their opposite  $-x^i = x^{3t+i}, -x^{2t+i} = x^{5t+i}$  and  $-x^{4t+i} = x^{t+i}$ . Thus we are in one of the three following cases

- i)  $y = z$  and the pair  $\{(y, 0), (z, 1)\}$  belong to no quadruple.
- ii)  $y - z = x^{2pt+i}$  with  $p \in \{0, 1, 2\}$  and  $i \in \{0, 1, \dots, t-1\}$  then the pair  $\{(y, 0), (z, 1)\}$  belong to the unique quadruple  $\{(x^i + z, 0), (x^{2t+i} + z, 0), (x^{4t+i} + z, 0), (z, 1)\}$  where  $i$  is defined by  $y - z = x^{2pt+i}$ .
- iii)  $z - y = x^{2pt+i}$  with  $p \in \{0, 1, 2\}$  and  $i \in \{0, 1, \dots, t-1\}$  then the pair  $\{(y, 0), (z, 1)\}$  belong to the unique quadruple  $\{(x^i + y, 1), (x^{2t+i} + y, 1), (x^{4t+i} + y, 1), (y, 0)\}$ . □

### 3. OTHER CASES (COMPOSITION METHODS).

The two following lemmas are known in design's constructions. We give them in a graph theoretical language. Let us denote by  $K_{n \times s}$  the complete  $s$ -partite graph with vertex set the union of  $s$  disjoint sets  $X_i$  ( $i=1, \dots, s$ ) of cardinality  $n$ , two vertices being joined if and only if they belong to two different sets  $X_i$  and  $X_j$ .

3.1. Lemma : If  $n \neq 2, 6$ , the edges of  $K_{n \times 4}$  can be partitioned into  $K_4$ .

Proof : The existence of a partition of the edges of  $K_{n \times 4}$  into  $K_4$  is equivalent to the existence of a transversal design  $T(n, 4, 1)$  [6, p.266] or [14] (the  $K_4$  of the decomposition playing the role of the blocks of the transversal designs), which is equivalent to the existence of an orthogonal array  $OA(4, n)$  [4, chap.13] (the  $K_4$  being associated to the columns of the orthogonal array) and such an  $OA(4, n)$  exists if and only if there exist 2 orthogonal latin squares that is iff  $n \neq 2, 6$  [4, th.13.4.1].

(For more details see [1, I.4] or [2, p.59]). □

3.2. Lemma : If  $n \neq 2, 6$  and if  $s \equiv 1$  or  $4 \pmod{12}$  then the edges of  $K_{nxs}$  can be partitioned into  $K_4$ .

Proof : As  $s \equiv 1$  or  $4 \pmod{12}$ , the edges of  $K_s$  can be partitioned into  $K_4$  ([5]) and thus the edges of  $K_{nxs}$  can be partitioned into edge disjoint  $K_{nx4}$ , and thus in  $K_4$  by lemma 3.1.  $\square$

3.3. Lemma : Let  $n \neq 2, 6$  and let  $s \equiv 1$  or  $4 \pmod{12}$ . Then

- a)  $q(sn) \geq s(s-1)n^2/12 + sq(n)$ .
- b)  $q(sn+1) \geq s(s-1)n^2/12 + sq(n+1)$ .
- c) If  $n+2 \not\equiv 1$  or  $4 \pmod{12}$  :  $q(sn+2) \geq s(s-1)n^2/12 + sq(n+2)$ .
- d)  $q(sn+3) \geq s(s-1)n^2/12 + sq(n+3) - (s-1)$ .

Proof : By lemma 3.2. we know that  $K_{nxs}$  is the edge disjoint union of  $s(s-1)n^2/12 K_4$ .

a) Let the set of vertices of  $K_{sn}$  be  $UX_i$  ( $i = 1, 2, \dots, s$ ) with  $|X_i| = n$  then  $K_{sn}$  is the edge disjoint union of the  $K_{nxs}$  constructed on the  $X_i$  and of the  $s$  complete graphs constructed on each  $X_i$ , isomorphic to  $K_n$ , which contain each  $q(n)$  edge disjoint  $K_4$  and a) follows.

b) Let the set of vertices of  $K_{sn+1}$  be  $\{x\} \cup \{UX_i\}$   $i = 1, 2, \dots, s$  and  $|X_i| = n$ . Then  $K_{sn+1}$  is the edge disjoint union of the  $K_{nxs}$ , constructed on the  $X_i$  and of the  $s$  graphs, isomorphic to  $K_{n+1}$ , constructed on  $X_i \cup \{x\}$  for  $i=1, 2, \dots, s$ , which contain each  $q(n+1) K_4$  and b) follows.

c) Note that if  $n+2 \not\equiv 1$  or  $4 \pmod{12}$ , the quadruples of a  $K(n+2, 4, 1)$  do not contain all the pairs of elements and thus  $K_{n+2}$  minus an edge contains  $q(n+2)$  edge-disjoint  $K_4$ . Let the set of vertices of  $K_{sn+2}$  be  $\{x, y\} \cup \{UX_i\}$  with  $i = 1, 2, \dots, s$  and  $|X_i| = n$ . Then  $K_{sn+2}$  is the edge-disjoint union of the  $K_{nxs}$  constructed on the  $X_i$  and of the  $s$  graphs, constructed on  $X_i \cup \{x, y\}$  which do not contain the edge  $\{x, y\}$ ; these graphs are isomorphic to  $K_{n+2}$  minus an edge and thus contain  $q(n+2)$  edge disjoint  $K_4$  and c) follows.

d) Note that  $K_n$  minus any quadruple and thus  $K_n$  minus any triangle contain at least  $(q(n) - 1)K_4$ . Let the set of vertices of  $K_{sn+3}$  be  $\{x, y, z\} \cup \{UX_i\}$  with  $i = 1, 2, \dots, s$  and  $|X_i| = n$ .

Then  $K_{sn+3}$  is the edge disjoint union of the  $K_{nxs}$  constructed on the  $X_i$ , of the complete graph constructed on  $X_1 \cup \{x, y, z\}$  isomorphic to  $K_{n+3}$ , which contains  $q(n+3) K_4$  and of the  $(s-1)$  graphs constructed on  $X_i \cup \{x, y, z\}$  (for  $i = 2, \dots, s$ ) minus the triangle  $\{x, y, z\}$ , graphs which are isomorphic to  $K_{n+3}$  minus a triangle and thus contain  $(q(n+3) - 1)K_4$  and d) follows  $\square$

Remark : We can obtain an inequality similar to d) for  $q(sn + 4)$ , but it is not useful for what follows.

3.4. Theorem : Let  $t$  be an integer,  $t \geq 1$  . Then  $q(m) = f(m)$  for  $m = 48t-6, 48t-5, 48t-3, 48t+6, 48t+7, 48t+9$  .

Proof : It suffices by 1.3. to prove that  $q(m) \geq f(m)$ . We will prove this by applying lemma 3.3. with  $s = 4$  and using the results (1.2 and 1.4) :  
 $q(12t) = 12t^2 - 3t$  ;  $q(12t+1) = 12t^2 + t$  ;  $q(12t+3) = 12t^2 + 3t$  ;  
 $q(12t+4) = 12t^2 + 7t + 1$  . By lemma

$$3.3.c. \quad q(48t-6) \geq (12t-2)^2 + 4q(12t) = 192t^2 - 60t + 4 = f(48t-6)$$

$$3.3.d. \quad q(48t-5) \geq (12t-2)^2 + 4q(12t+1) - 3 = 192t^2 - 44t + 1 = f(48t-5)$$

$$3.3.b. \quad q(48t-3) \geq (12t-1)^2 + 4q(12t) = 192t^2 - 36t + 1 = f(48t-3)$$

$$3.3.c. \quad q(48t+6) \geq (12t+1)^2 + 4q(12t+3) = 192t^2 + 36t + 1 = f(48t+6)$$

$$3.3.d. \quad q(48t+7) \geq (12t+1)^2 + 4q(12t+4) - 3 = 192t^2 + 52t + 2 = f(48t+7)$$

$$3.3.b. \quad q(48t+9) \geq (12t+2)^2 + 4q(12t+3) = 192t^2 + 60t + 4 = f(48t+9) \quad \square$$

3.5. Theorem : Let  $s \equiv 1$  or  $4 \pmod{12}$  and  $t$  an integer,  $t \geq 1$  .

If  $q(12t+2) = f(12t+2)$  then :

$$a) \quad q(12ts+2s) = f(12ts+2s)$$

$$b) \quad q(48t+5) = f(48t+5)$$

$$c) \quad q(12ts+2) = f(12ts+2).$$

Proof : a) By lemma 3.3.a. with  $n = 12t+2$  we have :

$$\begin{aligned} q(12ts+2s) &\geq s(s-1) (12t+2)^2 / 12 + s q(12t+2) \\ &= 12s^2 t^2 + (4s^2 - 2s) t + (s^2 - 1) / 3 = f(12ts+2s). \end{aligned}$$

b) Lemma 3.3.b. with  $s = 4$  and  $n = 12t+1$  gives :

$$q(48t+5) \geq (12t+1)^2 + 4q(12t+2) = 192t^2 + 32t + 1 = f(48t+5).$$

c) Lemma 3.3.c. with  $n = 12t$  gives :

$$q(12ts + 2) \geq s(s-1) (12t)^2 / 12 + sq(12t+2) = 12s^2 t^2 + 2st = f(12ts+2). \quad \square$$

3.6. Conclusion : This paper was motivated by the study of the  $K(12t+2, 4, 1)$  : see [10]. In particular it seems that  $q(12t+2) = f(12t+2)$  for all  $t$  . Let us call  $t$  regular if  $q(12t+2) = f(12t+2)$ . For example, if we consider  $t \leq 50$  ; by theorem 2.1.  $t$  is regular if  $6t+1$  is a power of a prime, that is  $t \neq 9, 14, 15, 19, 22, 24, 29, 31, 34, 36, 39, 41, 42, 43, 44, 49, 50$  . By theorem 3.5.c.  $t$  is regular if  $t = st_0$  with  $s \equiv 1$  or  $4 \pmod{12}$  and  $t_0$  regular . Thus  $24 = 4.6$  ;  $39 = 13.3$  ;  $44 = 4.11$  ;  $49 = 49.1$  ;  $50 = 25.2$  are regular . By theorem 3.5.a.  $t$  is regular if  $t = st_0 + (2s-2)/12$  with  $s \equiv 1 \pmod{12}$  and  $t_0$  regular . Thus  $15 = 13 + (26-2)/12$  ;  $29 = 25 + (50-2)/12$  ;  $41 = 13.3 + (26-2)/12$  ;  $43 = 37 + (74-2)/12$  are regular.

The values not covered are  $t = 9, 14, 19, 22, 31, 34, 36, 42$ . We can prove that all these values of  $t$  are regular by using partitions of the edges of  $K_{n \times s}$  into  $K_4$  for  $s \not\equiv 1$  or  $4 \pmod{12}$ . For example from a partition of the edges of  $K_{28}$  into  $K_4$ , we obtain by deleting all the quadruples containing a given vertex a partition of the edges of  $K_{3 \times 9}$  into  $K_4$ ; this partition induces a partition of the edges of  $K_{12 \times 9}$  into  $864 K_4$ .

Thus  $q(110) \geq 864 + 9 q(14) = 864 + 126 = 990 = f(110)$ .

Furthermore we can prove that  $t$  is regular if the edges of  $K_{12 \times t}$  can be partitioned into  $12t(t-1) K_4$  and it suffices by [lemma 5.10 of 6] to prove this for  $t = 4, 5, \dots, 12, 14, 15, 18, 19, 23, 27$ .

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