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ON MAXIMAL SYSTEMS OF QUADRUPLES

J-C.BERMOND and J.NOVAK.

ABSTRACT : We determine the maximum number of quadruples of a set X of n elements, such that every pair of elements of X belong to at most one quadruple (problem of packing of pairs with quadruples) for some values of n ; in particular for : $n = 12 st + 2$ and $n = 12 st + 2s$, with $6 + 1$ a power of a prime and $s \equiv 1$ or $4 \pmod{12}$; and for $n = 48 t + i$ with $i = -6, -5, -3, 6, 7, 9$.

1. INTRODUCTION.

1.1. Let X be a set of n elements. We are interested in the following "packing problem" : find the maximum number $q(n)$ of quadruples of X , such that every pair of elements of X belong to at most one quadruple (or equivalently two quadruples meet in at most one point). We will denote such a maximal system by $K(n, 4, 1)$ and thus $q(n) = |K(n, 4, 1)|$. In a graph theoretical language we want to find the maximum number of edge-disjoint K_4 , contained in a K_n .

1.2. H.Hanani [6] has proved that, if $n \equiv 1$ or $4 \pmod{12}$, then there exist maximal systems of quadruples $K(n, 4, 1)$ such that every pair of elements belong to exactly one quadruple, and in this case $q(n) = n(n-1) / 12$.

1.3. R.M.Wilson [13] has proved that $q(n) \leq f(n)$ where

$$f(n) = \left[\frac{n}{4} \left[\frac{n-1}{3} \right] \right] \text{ if } n \not\equiv 7 \text{ or } 10 \pmod{12}$$

$$f(n) = \left[\frac{n}{4} \left[\frac{n-1}{3} \right] \right] - 1 \text{ if } n \equiv 7 \text{ or } 10 \pmod{12}.$$

($[x]$ denotes the greatest integer less than or equal to x).

In [13] R.M.Wilson has also proved that if n is large enough then $q(n) = f(n)$.

1.4. Let $n \equiv 1$ or $4 \pmod{12}$; if we delete all the quadruples of a $K(n,4,1)$ containing a given element, we obtain $f(n-1)$ quadruples, thus forming a $K(n-1, 4,1)$. Thus $q(n) = f(n)$ for $n \equiv 0$ or $3 \pmod{12}$.

1.5. In [3] R.K.Guy has given the values of $q(n)$ for $n \leq 16$; for these values $q(n) = f(n)$ except for $n = 8,9,10,11$ where :

$$q(8) = 2, f(8) = 4 ; q(9) = 3, f(9) = 4 ; q(10) = 5, f(10) = 6 ; \\ q(11) = 6, f(11) = 8.$$

1.6. The aim of this paper is to construct systems $K(n,4,1)$ with $q(n) = f(n)$ quadruples in particular for $n = 12st + 2$ or $n = 12st + 2s$, with $6t + 1$ a power of a prime and $s \equiv 1$ or $4 \pmod{12}$ and also for $n = 48t - 6, 48t - 5, 48t - 3, 48t + 6, 48t + 7, 48t + 9$.

1.7. Remarks : the same packing problems with triples has been solved in [7, 11, 12] (see also [I, I.6. 16]). The problem of coverings of pairs by quadruples, such that every pair of elements belong to at least one quadruple has been solved by W.H. Mills [8,9].

2. CASE $n = 12t + 2$.

2.1. Theorem : Let $n = 12t + 2$ and $6t + 1 = p^r$, p a prime.
Then $q(n) = f(n) = 12t^2 + 2t$.

Proof : Let x be a primitive root of the Galois Field $GF(p^r)$. We will use the following theorem proved in M.Hall [4, th. 15.3.4.] : the triples $\{x^i, x^{2t+i}, x^{4t+i}\}$ $i = 0, 1, \dots, t-1$ are base blocks, with respect to the additive group A of $GF(p^r)$, of a steiner triple system on $6t + 1$ elements. Recall that a steiner triple system is a family of triples, such that every pair of elements belong to exactly one triple : here the triples are $\{x^i + a, x^{2t+i} + a, x^{4t+i} + a\}$ $i = 0, 1, \dots, t-1$ and $a \in A$.

Now let us associate to each triple of this steiner triple system on A , a quadruple of elements of $A \times Z_2$ in the following manner : to the base triples $\{x^i, x^{2t+i}, x^{4t+i}\}$ $i = 0, 1, \dots, t-1$ we associate the quadruple $\{(x^i, 0), (x^{2t+i}, 0), (x^{4t+i}, 0), (0,1)\}$ $i = 0, 1, \dots, t-1$. We claim that these quadruples are base quadruples, with respect to $A \times Z_2$, of a $K(n,4,1)$; that is that we have a $K(n,4,1)$ consisting of the $f(n) = 12t^2 + 2t$ quadruples $\{(x^i + a, \alpha), (x^{2t+i} + a, \alpha), (x^{4t+i} + a, \alpha), (a, \alpha + 1)\}$ where $i = 0, 1, \dots, t-1, a \in A$ and $\alpha \in Z_2$.

It suffices to prove that any pair of elements of $A \times Z_2$ belong to at most one of these quadruples. If the pair of elements is of the form $\{(y, \alpha), (z, \alpha)\}$ this follows from the fact, that the pair $\{y, z\}$ belong to exactly one of the triples $\{x^i + a, x^{2t+i} + a, x^{4t+i} + a\}$. Thus suppose that the pair is of the form $\{(y, 0), (z, 1)\}$. Note that we have $x^{6t} = 1$ and $x^{3t} = -1$ and therefore the non zero elements of A are x^i, x^{2t+i}, x^{4t+i} ($i=0, 1, \dots, t-1$) and their opposite $-x^i = x^{3t+i}, -x^{2t+i} = x^{5t+i}$ and $-x^{4t+i} = x^{t+i}$. Thus we are in one of the three following cases

- i) $y = z$ and the pair $\{(y, 0), (z, 1)\}$ belong to no quadruple.
- ii) $y - z = x^{2pt+i}$ with $p \in \{0, 1, 2\}$ and $i \in \{0, 1, \dots, t-1\}$ then the pair $\{(y, 0), (z, 1)\}$ belong to the unique quadruple $\{(x^i + z, 0), (x^{2t+i} + z, 0), (x^{4t+i} + z, 0), (z, 1)\}$ where i is defined by $y - z = x^{2pt+i}$.
- iii) $z - y = x^{2pt+i}$ with $p \in \{0, 1, 2\}$ and $i \in \{0, 1, \dots, t-1\}$ then the pair $\{(y, 0), (z, 1)\}$ belong to the unique quadruple $\{(x^i + y, 1), (x^{2t+i} + y, 1), (x^{4t+i} + y, 1), (y, 0)\}$. □

3. OTHER CASES (COMPOSITION METHODS).

The two following lemmas are known in design's constructions. We give them in a graph theoretical language. Let us denote by $K_{n \times s}$ the complete s -partite graph with vertex set the union of s disjoint sets X_i ($i=1, \dots, s$) of cardinality n , two vertices being joined if and only if they belong to two different sets X_i and X_j .

3.1. Lemma : If $n \neq 2, 6$, the edges of $K_{n \times 4}$ can be partitioned into K_4 .

Proof : The existence of a partition of the edges of $K_{n \times 4}$ into K_4 is equivalent to the existence of a transversal design $T(n, 4, 1)$ [6, p.266] or [14] (the K_4 of the decomposition playing the role of the blocks of the transversal designs), which is equivalent to the existence of an orthogonal array $OA(4, n)$ [4, chap.13] (the K_4 being associated to the columns of the orthogonal array) and such an $OA(4, n)$ exists if and only if there exist 2 orthogonal latin squares that is iff $n \neq 2, 6$ [4, th.13.4.1].

(For more details see [1, I.4] or [2, p.59]). □

3.2. Lemma : If $n \neq 2, 6$ and if $s \equiv 1$ or $4 \pmod{12}$ then the edges of K_{nxs} can be partitioned into K_4 .

Proof : As $s \equiv 1$ or $4 \pmod{12}$, the edges of K_s can be partitioned into K_4 ([5]) and thus the edges of K_{nxs} can be partitioned into edge disjoint K_{nx4} , and thus in K_4 by lemma 3.1. \square

3.3. Lemma : Let $n \neq 2, 6$ and let $s \equiv 1$ or $4 \pmod{12}$. Then

- a) $q(sn) \geq s(s-1)n^2/12 + sq(n)$.
- b) $q(sn+1) \geq s(s-1)n^2/12 + sq(n+1)$.
- c) If $n+2 \not\equiv 1$ or $4 \pmod{12}$: $q(sn+2) \geq s(s-1)n^2/12 + sq(n+2)$.
- d) $q(sn+3) \geq s(s-1)n^2/12 + sq(n+3) - (s-1)$.

Proof : By lemma 3.2. we know that K_{nxs} is the edge disjoint union of $s(s-1)n^2/12 K_4$.

a) Let the set of vertices of K_{sn} be UX_i ($i = 1, 2, \dots, s$) with $|X_i| = n$ then K_{sn} is the edge disjoint union of the K_{nxs} constructed on the X_i and of the s complete graphs constructed on each X_i , isomorphic to K_n , which contain each $q(n)$ edge disjoint K_4 and a) follows.

b) Let the set of vertices of K_{sn+1} be $\{x\} \cup \{UX_i\}$ $i = 1, 2, \dots, s$ and $|X_i| = n$. Then K_{sn+1} is the edge disjoint union of the K_{nxs} , constructed on the X_i and of the s graphs, isomorphic to K_{n+1} , constructed on $X_i \cup \{x\}$ for $i=1, 2, \dots, s$, which contain each $q(n+1) K_4$ and b) follows.

c) Note that if $n+2 \not\equiv 1$ or $4 \pmod{12}$, the quadruples of a $K(n+2, 4, 1)$ do not contain all the pairs of elements and thus K_{n+2} minus an edge contains $q(n+2)$ edge-disjoint K_4 . Let the set of vertices of K_{sn+2} be $\{x, y\} \cup \{UX_i\}$ with $i = 1, 2, \dots, s$ and $|X_i| = n$. Then K_{sn+2} is the edge-disjoint union of the K_{nxs} constructed on the X_i and of the s graphs, constructed on $X_i \cup \{x, y\}$ which do not contain the edge $\{x, y\}$; these graphs are isomorphic to K_{n+2} minus an edge and thus contain $q(n+2)$ edge disjoint K_4 and c) follows.

d) Note that K_n minus any quadruple and thus K_n minus any triangle contain at least $(q(n) - 1)K_4$. Let the set of vertices of K_{sn+3} be $\{x, y, z\} \cup \{UX_i\}$ with $i = 1, 2, \dots, s$ and $|X_i| = n$.

Then K_{sn+3} is the edge disjoint union of the K_{nxs} constructed on the X_i , of the complete graph constructed on $X_1 \cup \{x, y, z\}$ isomorphic to K_{n+3} , which contains $q(n+3) K_4$ and of the $(s-1)$ graphs constructed on $X_i \cup \{x, y, z\}$ (for $i = 2, \dots, s$) minus the triangle $\{x, y, z\}$, graphs which are isomorphic to K_{n+3} minus a triangle and thus contain $(q(n+3) - 1)K_4$ and d) follows \square

Remark : We can obtain an inequality similar to d) for $q(sn + 4)$, but it is not useful for what follows.

3.4. Theorem : Let t be an integer, $t \geq 1$. Then $q(m) = f(m)$ for $m = 48t-6, 48t-5, 48t-3, 48t+6, 48t+7, 48t+9$.

Proof : It suffices by 1.3. to prove that $q(m) \geq f(m)$. We will prove this by applying lemma 3.3. with $s = 4$ and using the results (1.2 and 1.4) :
 $q(12t) = 12t^2 - 3t$; $q(12t+1) = 12t^2 + t$; $q(12t+3) = 12t^2 + 3t$;
 $q(12t+4) = 12t^2 + 7t + 1$. By lemma

$$3.3.c. \quad q(48t-6) \geq (12t-2)^2 + 4q(12t) = 192t^2 - 60t + 4 = f(48t-6)$$

$$3.3.d. \quad q(48t-5) \geq (12t-2)^2 + 4q(12t+1) - 3 = 192t^2 - 44t + 1 = f(48t-5)$$

$$3.3.b. \quad q(48t-3) \geq (12t-1)^2 + 4q(12t) = 192t^2 - 36t + 1 = f(48t-3)$$

$$3.3.c. \quad q(48t+6) \geq (12t+1)^2 + 4q(12t+3) = 192t^2 + 36t + 1 = f(48t+6)$$

$$3.3.d. \quad q(48t+7) \geq (12t+1)^2 + 4q(12t+4) - 3 = 192t^2 + 52t + 2 = f(48t+7)$$

$$3.3.b. \quad q(48t+9) \geq (12t+2)^2 + 4q(12t+3) = 192t^2 + 60t + 4 = f(48t+9) \quad \square$$

3.5. Theorem : Let $s \equiv 1$ or $4 \pmod{12}$ and t an integer, $t \geq 1$.

If $q(12t+2) = f(12t+2)$ then :

$$a) \quad q(12ts+2s) = f(12ts+2s)$$

$$b) \quad q(48t+5) = f(48t+5)$$

$$c) \quad q(12ts+2) = f(12ts+2).$$

Proof : a) By lemma 3.3.a. with $n = 12t+2$ we have :

$$q(12ts+2s) \geq s(s-1) (12t+2)^2 / 12 + s q(12t+2) \\ = 12s^2 t^2 + (4s^2 - 2s) t + (s^2 - 1) / 3 = f(12ts+2s).$$

b) Lemma 3.3.b. with $s = 4$ and $n = 12t+1$ gives :

$$q(48t+5) \geq (12t+1)^2 + 4q(12t+2) = 192t^2 + 32t + 1 = f(48t+5).$$

c) Lemma 3.3.c. with $n = 12t$ gives :

$$q(12ts + 2) \geq s(s-1) (12t)^2 / 12 + sq(12t+2) = 12s^2 t^2 + 2st = f(12ts+2). \quad \square$$

3.6. Conclusion : This paper was motivated by the study of the $K(12t+2, 4, 1)$: see [10]. In particular it seems that $q(12t+2) = f(12t+2)$ for all t . Let us call t regular if $q(12t+2) = f(12t+2)$. For example, if we consider $t \leq 50$; by theorem 2.1. t is regular if $6t+1$ is a power of a prime, that is $t \neq 9, 14, 15, 19, 22, 24, 29, 31, 34, 36, 39, 41, 42, 43, 44, 49, 50$. By theorem 3.5.c. t is regular if $t = st_0$ with $s \equiv 1$ or $4 \pmod{12}$ and t_0 regular . Thus $24 = 4.6$; $39 = 13.3$; $44 = 4.11$; $49 = 49.1$; $50 = 25.2$ are regular . By theorem 3.5.a. t is regular if $t = st_0 + (2s-2)/12$ with $s \equiv 1 \pmod{12}$ and t_0 regular . Thus $15 = 13 + (26-2)/12$; $29 = 25 + (50-2)/12$; $41 = 13.3 + (26-2)/12$; $43 = 37 + (74-2)/12$ are regular.

The values not covered are $t = 9, 14, 19, 22, 31, 34, 36, 42$. We can prove that all these values of t are regular by using partitions of the edges of $K_{n \times s}$ into K_4 for $s \not\equiv 1$ or $4 \pmod{12}$. For example from a partition of the edges of K_{28} into K_4 , we obtain by deleting all the quadruples containing a given vertex a partition of the edges of $K_{3 \times 9}$ into K_4 ; this partition induces a partition of the edges of $K_{12 \times 9}$ into $864 K_4$.

Thus $q(110) \geq 864 + 9 q(14) = 864 + 126 = 990 = f(110)$.

Furthermore we can prove that t is regular if the edges of $K_{12 \times t}$ can be partitioned into $12t(t-1) K_4$ and it suffices by [lemma 5.10 of 6] to prove this for $t = 4, 5, \dots, 12, 14, 15, 18, 19, 23, 27$.

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