



Cycle and circuit designs: odd case

Jean-Claude Bermond, Dominique Sotteau

► To cite this version:

Jean-Claude Bermond, Dominique Sotteau. Cycle and circuit designs: odd case. Beitrage zur Graphentheorie and deren Anwendungen, Proc. Coll Oberhof, 1977, 1978, Illmenau, Germany. pp.11-32. hal-02361437

HAL Id: hal-02361437

<https://inria.hal.science/hal-02361437>

Submitted on 13 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CYCLE AND CIRCUIT DESIGNS ODD CASE

1. Introduction.

In what follows, n, k, λ will be positive integers with $n \geq k$ and G will always denote a graph with k vertices. Definitions not given here can be found in Berge's book /1/.

Let us denote by

λK_n : the complete multigraph with n vertices and any two distinct vertices joined by exactly λ edges.

λK_n^* : the complete symmetric directed multigraph with n vertices and any two distinct vertices x and y joined by λ arcs (x,y) and λ arcs (y,x) .

$\lambda K_{n_1, n_2, \dots, n_h}$: the complete multipartite graph with vertex set $X = \bigcup X_i$, $1 \leq i \leq h$, where the X_i are disjoint sets with $|X_i| = n_i$ and where two vertices, which belong to different sets X_i and X_j are joined by λ edges.

$\lambda K_{n_1, n_2, \dots, n_h}^*$: the complete directed multipartite graph.

C_k : the elementary circuit of length k .

\vec{C}_k : the elementary directed circuit of length k .

We shall say that a graph (respectively directed graph) H can be decomposed into subgraphs isomorphic to G , where G is a given graph (respectively directed graph) if we can partition the edges (resp. arcs) of H into subgraphs isomorphic to G .

1.1. An (n, k, λ) G -design is an edge disjoint decomposition of λK_n into subgraphs isomorphic to G . The G -designs have been introduced by Hell et Rosa /12/. In the particular case where G is the complete graph K_k , an (n, k, λ) K_k -design is nothing else than an (n, k, λ) B.I.B.D. (balanced incomplete block design: see Hall /9/).

1.2. A similar definition holds for directed graphs. If G is a directed graph, with k vertices, an (n, k, λ) G -design is a partition of the arcs of λK_n^* into subgraphs isomorphic to G .

1.3. For results concerning the existence of G-designs, the reader can see, for example, the survey we have done in /6/.

Here we are interested in the particular case where G is equal to C_k or \vec{C}_k . An (n, k, λ) C_k -design is also called a B.C.D. (n, k, λ) (balanced circuit design /13/).

1.4. Proposition. Necessary conditions for the existence of an (n, k, λ) C_k -design are

$$\lambda n(n-1) \equiv 0 \pmod{2k}$$

$$\lambda(n-1) \equiv 0 \pmod{2}.$$

1.5. Proposition. A necessary condition for the existence of an (n, k, λ) \vec{C}_k -design is

$$\lambda n(n-1) \equiv 0 \pmod{k}.$$

1.6. In this paper we will restrict ourselves to the case k odd. The case k even will be considered in /7/.

For k odd, it has been proved that the necessary conditions of the proposition 1.4 are sufficient for

k = 3 (case of Steiner Triple System) see Hanani /10/.

k = 5 by Huang and Rosa /13/.

k odd, $n = 2qk + 1$ or $n = 2qk + k$, $\lambda = 1$ by Rosa /16/.

The necessary condition 1.5 of the existence of a \vec{C}_k -design is sufficient for k = 3, $\lambda = 1$ except $n = 6$ in /2/ and also by Bruck /8/ Mendelsohn /14/.

A short proof can be found in /3/ or 4/.

k = 5, $\lambda = 1$ in /3/ and /17/ and also by Merriell /15/.

k = p^α , p an odd prime and $\lambda = 1$ in /17/ and also for k = 7 by Merriell /15/.

It has also been shown that an $(n, k, 1)$ \vec{C}_k -design exists if n is a power of a prime and $n(n-1) \equiv 0 \pmod{k}$ by Hartnell and Milgram /11/ and if $n \equiv 0$ or $1 \pmod{k}$, k odd in /17/.

Finally we note that the asymptotic results obtained by Wilson /19/ prove that the conditions 1.4 and 1.5 are sufficient, for a given k, for n large enough.

Here we shall prove the following

1.7. Theorem. The necessary conditions of propositions 1.4 and 1.5 are sufficient for $k = 3, 5, 7$ except for $n = 6, k = 3, \lambda = 1$.

2. General Lemmas.

The 3 following lemmas are simple consequences of the definitions.

2.1. Lemma. If there exists an (n, k, λ_1) G-design and an (n, k, λ_2) G-design, then there exists an $(n, k, p\lambda_1 + q\lambda_2)$ G-design (p and q are positive integers).

2.2. Lemma. Let H^* be the directed graph obtained from H by associating to each edge of H two opposite arcs. If H can be decomposed into C_k , then H^* can be decomposed into \vec{C}_k .

Remark. The contrary is false: for example we shall see that there exists a $(4, 3, 1) \vec{C}_3$ -design, but by 1.4 there exists no $(4, 3, 1) C_3$ -design.

2.3. Lemma. If there exists an $(n, k, \lambda) \vec{C}_k$ -design, then there exists an $(n, k, 2\lambda) C_k$ -design.

Remark. The contrary is false: there exists a $(6, 3, 2) C_3$ -design (see Hanani /10/), but there exists no $(6, 3, 1) \vec{C}_3$ -design /2/.

2.4. Lemma. If there exists an (n_i, k, λ) G-design for $1 \leq i \leq h$ and if $\lambda K_{n_1, \dots, n_h}$ (or $\lambda K_{n_1, \dots, n_h}^*$ if G is a directed graph) can be decomposed into subgraphs isomorphic to G , then there exists a $(\sum n_i, k, \lambda)$ G-design.

2.5. Lemma. If there exists an $(n_i + 1, k, \lambda)$ G-design for $1 \leq i \leq h$ and if $\lambda K_{n_1, \dots, n_h}$ (or $\lambda K_{n_1, \dots, n_h}^*$ if G is a directed graph) can be decomposed into subgraphs isomorphic to G then there exists a $((\sum n_i) + 1, k, \lambda)$ G-design.

Proofs. We shall give the proof of lemma 2.5 in the undirected case. (The proof of lemma 2.4 and of the directed case are similar). Let the vertices of $\lambda K_{(\sum n_i) + 1}$ be: $X_1 \cup X_2 \cup \dots \cup X_h \cup \{\infty\}$, with $|X_i| = n_i$, for $1 \leq i \leq h$.

The graph $\lambda K_{(\sum n_i) + 1}$ is the edge disjoint union of:

- the h complete multigraphs, constructed on the sets $X_i \cup \{\infty\}$ which are isomorphic to λK_{n_i+1} and thus can be decomposed (by hypothesis) into subgraphs isomorphic to G .
- the complete multipartite multigraph with vertex sets $\bigcup X_i$ which is isomorphic to $\lambda K_{n_1, \dots, n_h}$ decomposable into subgraphs isomorphic to G by hypothesis.

2.6. Remark. The interest of lemmas 2.4 and 2.5 is that it gives a method of composition which enables us to construct G -designs by knowing the existence of smaller ones. (In the case $G = K_k$, these lemmas are used with different notations by Hanani /10/ and Wilson /19/: indeed a decomposition of $\lambda K_{n_1, \dots, n_h}$ into K_k is equivalent to the existence of group divisible designs).

We shall apply this method with tripartite graphs ($h=3$). The following lemma will enable us to construct decompositions of tripartite multigraphs from decomposition of smaller ones.

2.7. Lemma. If $K_{a,b,c}$ and $K_{a,b,c'}$ can be decomposed into subgraphs isomorphic to G , then also $K_{qa, qb, pc+(q-p)c'}$ with p and q positive integers, $0 \leq p \leq q$.

Proof. Let the vertices of $K_{qa, qb, pc+(q-p)c'}$ be $X \cup Y \cup Z$ with

$$X = \bigcup_{i=1}^q X_i, |X_i| = a \quad (1 \leq i \leq q); \quad Y = \bigcup_{i=1}^q Y_i, |Y_i| = b \quad (1 \leq i \leq q)$$

$$\text{and } Z = \bigcup_{i=1}^p Z_i, |Z_i| = c, \quad 1 \leq i \leq p \text{ and } |Z_i| = c' \text{ for } p+1 \leq i \leq q.$$

Then $K_{qa, qb, pc+(q-p)c'}$ is the edge disjoint union of the q^2 tripartite graphs constructed respectively on X_i, Y_{i+j}, Z_j with $1 \leq i \leq q, 1 \leq j \leq q$ and where the indices of Y are to be taken modulo q . These q^2 graphs are isomorphic to $K_{a,b,c}$ or to $K_{a,b,c'}$ which can be decomposed into subgraphs isomorphic to G .

A similar lemma holds in the directed case.

2.8. Lemma. If $K_{a,b,c}^*$ and $K_{a,b,c'}^*$ can be decomposed into subgraphs isomorphic to G then also $K_{qa, qb, pc+(q-p)c'}^*$ (p, q positive integers $0 \leq p \leq q$).

The two following lemmas can be sometimes useful.

2.9. Lemma. Let $h \equiv 1$ or $3 \pmod{6}$; if K_{n_i} (resp. K_{n_i+1}), $1 \leq i \leq h$ and K_{n_i, n_j, n_k} , $1 \leq i < j < k \leq h$, can be decomposed into subgraphs isomorphic to G then also K_n (resp. K_{n+1}) for $n = \sum_{i=1}^h n_i$.

Sketch of proof: These lemmas result from the existence of $(6t+1, 3, 1)$ or $(6t+3, 3, 1)$ Steiner Triple Systems (see /17/ for more details).

3. Existence of $C_3, C_5, C_7, \vec{C}_3, \vec{C}_5, \vec{C}_7$ -Designs.

In this paragraph, k will always denote an odd integer ≥ 3 . We shall prove the theorems 1.7 ($k = 3, 5, 7$): we give first the ideas of the proof and general propositions which are valid for all (odd) $k \geq 5$.

We apply them only to the particular values of $k : 5, 7$ but the methods can be applied to give other results for greater values of k .

3.1. The proofs of theorem 1.7 for $k = 5$ or 7 are by induction and all of the same kind:

Step 1. First we prove the existence of decomposition of some K_n or K_n^* , for the first values of n in order to start the induction.

Step 2. We prove the existence of decomposition of some $K_{a,b,c}$ or $K_{a,b,c}^*$ which will be needed in the verification.

Step 3. Then we use lemma 2.4 or lemma 2.5 with tripartite graphs ($h = 3$) or lemma 2.9. We indicate in each case, in a table, which lemma we use and which values of n_i occur in the lemma. The verification that the hypothesis of these lemmas are satisfied follows

- for the K_{n_i} or $K_{n_i}^*$ from step 1 or induction hypothesis.
- for the K_{n_1, n_2, n_3} or K_{n_1, n_2, n_3}^* from lemma 2.7 (or 2.8):

we indicate in the same table the values of the parameters a, b, c, c', p and q used. The $K_{a,b,c}$ and $K_{a,b,c}'$ used in these lemmas can be decomposed by step 2.

For step 1 the following proposition is useful

3.2. Proposition. Let k be an odd integer ≥ 3 , then there exists

- a) $a(k, k, 1)$ \vec{C}_k -design
- b) $a(k, k, 1) \vec{C}_k$ -design
- c) $a(k+1, k, 1) \vec{C}_k$ -design
- d) $a(2k, k, 1) \vec{C}_k$ -design for $k \geq 5$.

Proof. a) is well known (see Berge /1/).

b) is a consequence of a) and lemma 2.2.

c) has been proved in /5/: Let $Z_k \cup \{\infty\}$ be the set of vertices of K_{k+1}^* (we recall that Z_k is the additive group of residues modulo k). A decomposition of K_{k+1}^* into \vec{C}_k is given by the following k directed circuits, for $0 \leq i \leq k-1$,
 $(\infty, \dots, \alpha+i, k-1-\alpha+i, \dots, \beta+i, k-2-\beta+i, \dots, \infty)$ with $0 \leq \alpha \leq \lfloor k/4 \rfloor - 1$,
and $\lfloor k/4 \rfloor \leq \beta \leq (k-3)/2$, and the circuit $(\dots, \alpha, \alpha + (k+1)/2, \dots)$ with $0 \leq \alpha \leq (k-1)/2$.

d) There does not exist a $(6, 3, 1) \vec{C}_3$ -design (see /2/). For $k \geq 5$, there exists a $(2k, k, 1) \vec{C}_k$ -design (/4/): let $Z_{k-1} \cup A_\infty$, with $A_\infty = \{\infty_0, \dots, \infty_{k-1}\}$ be the set of vertices of K_{2k}^* . As the complete graph on A_∞ is isomorphic to K_{k+1}^* and thus is decomposable into \vec{C}_k , it suffices to prove that the arcs of K_{2k}^* , which do not join two points of A_∞ , can be partitioned into \vec{C}_k ; that is given by the following $3(k-1)$ directed circuits for $0 \leq i \leq k-2$:

$(\infty_0, i, \infty_1, k-2+i, \dots, \alpha+i, k-2+\alpha+i, \dots, (k-3)/2+i, \infty_0)$ with
 $1 \leq \alpha \leq (k-5)/2$

$(i, \dots, \beta+i, \infty_{\beta+1}, \dots, i)$ and $(i, \dots, k-1-\beta+i, \infty_{\beta+(k+1)/2}, \dots, i)$
with

$1 \leq \beta \leq (k-1)/2$.

For step 2 the following propositions are useful:

3.3. Proposition. Let k be an odd integer ≥ 3 , then

- a) $K_{k,k,k}$ can be decomposed into \vec{C}_k .
- b) $K_{k,k,2k}^*$ can be decomposed into \vec{C}_k , for $k \geq 5$.
- c) $K_{k,2k,2k}^*$ can be decomposed into \vec{C}_k , for $k \geq 5$.
- d) $K_{qk,qk,qk}^*$ can be decomposed into \vec{C}_k , for $k \geq 5$, q integer.

A proof of this proposition can be found in /17/.

3.4. Proposition. For all integers $n, p, n \geq p \geq 1$, $(2p+1)K_{n,n,n}$ can be decomposed into C_{2p+1} .

Proof. Let $X \cup Y \cup Z$ be the set of vertices of $(2p+1)K_{n,n,n}$ with $X = \{x_i\}_{0 \leq i \leq n-1}$, $Y = \{y_i\}_{0 \leq i \leq n-1}$, $Z = \{z_i\}_{0 \leq i \leq n-1}$; the $3n^2$ circuits of length $2p+1$ of the decomposition are given by the following circuits $\{C_{i,j}, C'_{i,j}, C''_{i,j}\}$, for $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$:

$$C_{i,j} = (\dots, x_{i+\alpha}, y_{j+\alpha}, \dots, z_{i+j}, x_i)$$

$$C'_{i,j} = (\dots, y_{i+\alpha}, z_{j+\alpha}, \dots, x_{i+j}, y_i) \quad 0 \leq \alpha \leq p-1$$

$$C''_{i,j} = (\dots, z_{i+\alpha}, x_{j+\alpha}, \dots, y_{i+j}, z_i)$$

3.5. Theorem. There exists an $(n, 5, \lambda)$ C_5 -design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{10}$$

$$\lambda(n-1) \equiv 0 \pmod{2}$$

There exists an $(n, 5, \lambda) \vec{C}_5$ design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{5}.$$

Proof. By lemma 2.1 it suffices to verify the theorem for the minimum values of λ , that is to prove the existence

- of C_5 -designs in the following cases:

- a. $\lambda = 1, n \equiv 1, 5 \pmod{10}$
- b. $\lambda = 2, n \equiv 0, 6 \pmod{10}$
- c. $\lambda = 5, n \equiv 3, 7, 9 \pmod{10}$
- d. $\lambda = 10, n \equiv 2, 4, 8 \pmod{10}.$

- of \vec{C}_5 -designs in the following cases:

- e. $\lambda = 1, n \equiv 0, 1 \pmod{5}$
- f. $\lambda = 5, n \equiv 2, 3, 4 \pmod{5}.$

In order to minimize the verifications we shall use lemmas 2.2 and 2.3 and thus give a proof of the different cases a, b, c, d, e, f in a convenient order.

a. Existence of an $(n, 5, 1)$ C_5 -design for $n \equiv 1$ or $5 \pmod{10}$.
We use the method exposed in 3.1.

Step 1. $K_5, K_{11}, K_{21}, K_{25}, K_{85}$ can be decomposed into C_5 . For K_5 see 3.2 a). For the others it can be done by direct construction (see Rosa /16/). For example for K_{11} : if Z_{11} is the set of vertices of K_{11} a decomposition is given by the circuits: $(i, i+2, i+10, i+6, i)$, $0 \leq i \leq 10$.

Step 2. $K_{5,5,5}, K_{10,10,10}, K_{10,10,20}, K_{15,15,15}, K_{15,15,5}, K_{15,15,25}$ can be decomposed into C_5 .

- $K_{5,5,5}, K_{10,10,10}, K_{15,15,15}$: see proposition 3.3 a), d).

- $K_{10,10,20}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{10,10,20}$ with $X = \{x_i\}_{0 \leq i \leq 9}, Y = \{y_i\}_{0 \leq i \leq 9}, Z = \{z_i\}_{0 \leq i \leq 9} \cup \{\infty_i\}_{0 \leq i \leq 9}$.

A decomposition into C_5 is given by the 100 following circuits:

$$(x_{i+j}, y_j, z_{2i+1+j}, y_{j+1}, \infty_i, x_{i+j})_{0 \leq i \leq 4, 0 \leq j \leq 9}$$

$$(x_j, z_{2i+1+j}, x_{j+1}, \infty_{i+5}, y_{i+j+1}, x_j)_{0 \leq i \leq 4, 0 \leq j \leq 9}$$

- $K_{15,15,5}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{15,15,5}$ with

$$X = \{x_i\}_{0 \leq i \leq 14}, Y = \{y_i\}_{0 \leq i \leq 14}, Z = \{\infty_i\}_{0 \leq i \leq 4}$$

A decomposition into C_5 is given by the 75 following circuits:

$$(x_i, y_{i+3j+1}, x_{i+1}, y_{i+3j+3}, \infty_j, x_i)_{0 \leq j \leq 4, 0 \leq i \leq 14}$$

- $K_{15,15,25}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{15,15,25}$ with

$$X = \{x_i\}_{0 \leq i \leq 14}, Y = \{y_i\}_{0 \leq i \leq 14}, Z = \{z_i\}_{0 \leq i \leq 14} \cup \{\infty_i\}_{0 \leq i \leq 9}$$

A decomposition into C_5 is given by the following circuits:

$$(x_i, y_{i+1}, x_{i+1}, y_{i+3}, z_i, x_i)_{0 \leq i \leq 14}, (x_i, z_{i+2}, x_{i+1}, z_{i+4}, y_{i+6}, x_i)_{0 \leq i \leq 14}$$

$$(x_i, y_{i+\alpha+9}, z_{i+2\alpha+9}, y_{i+\alpha+3}, \infty_\alpha, x_i)_{0 \leq i \leq 14, 0 \leq \alpha \leq 5}, (x_i, z_{i+5}, x_{i+1},$$

$$z_{i+7}, y_{i+8}, x_i)_{0 \leq i \leq 14}$$

$$(x_1, z_{\alpha+i+10}, x_{i+3}, \infty_{\alpha+6}, y_{\alpha+3+i}, x_i)_{0 \leq i \leq 14, 0 \leq \alpha \leq 2}, (x_1, z_{i+14}, x_{i+1}, \infty_9, y_{i+7}, x_i)_{0 \leq i \leq 14}.$$

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
$30t+1$	$t \geq 1$	2.5	$10t$	$10t$	5	-	$2t$	0
$30t+11$	$t \geq 1$	2.5	$10(t-1)+10$	$10(t-1)+20$	10	20	t	1
$30t+15$	$t \geq 0$	2.4	$10t+5$	$10t+5$	5	-	$2t+1$	0
$30t+21$	$t \geq 2$	2.5	$10(t-2)+20$	$10(t-2)+40$	10	20	t	2
$90t+25$	$t \geq 1$	2.4	$30(t-1)+5$	$30(t-1)+25$	15	5	$2t+1$	2
$90t+55$	$t \geq 0$	2.4	$30t+15$	$30t+25$	15	25	$2t+1$	1
$90t+85$	$t \geq 1$	2.4	$30(t-1)+5$	$30(t-1)+25$	15	5	$2t+3$	5

The remaining case $n = 30t+5$ is solved for $t \geq 1$ by lemma 2.9 with $h = 6t+1$, $n_i = 5$ for $1 \leq i \leq h$.

e. Existence of an $(n, 5, 1) \vec{C}_5$ -design for $n \equiv 0, 1 \pmod{5}$.

By lemma 2.2 and case a we know the existence of $(n, 5, 1) \vec{C}_5$ -designs for $n \equiv 1, 5 \pmod{10}$. It remains to prove it for $n \equiv 0, 6 \pmod{10}$.

Step 1. K_5^* , K_6^* , K_{10}^* , K_{11}^* , K_{26}^* can be decomposed into \vec{C}_5 .

- K_5^* , K_{11}^* : case a + lemma 2.2.
- K_6^* , K_{10}^* : see proposition 3.2.c.
- K_{26}^* : K_{26}^* is the edge disjoint union of three complete graphs isomorphic to K_6^* , K_{11}^* , K_{11}^* which can be decomposed into \vec{C}_5 and of a complete tripartite graph isomorphic to $K_{5,10,10}^*$ which can be decomposed into \vec{C}_5 by proposition 3.3.c.

Step 2. $K_{5,5,5}^*$, $K_{5,5,10}^*$ can be decomposed into \vec{C}_5 (proposition 3.3.a, lemma 2.2 and proposition 3.3.b).

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
30t	$t \geq 1$	2.4	10t	10t	5	-	2t	0
30t+6	$t \geq 1$	2.5	10(t-1)+10	10(t-1)+15	5	10	2t	1
30t+10	$t \geq 1$	2.4	10(t-1)+10	10(t-1)+20	5	10	2t	2
30t+16	$t \geq 0$	2.5	10t+5	10t+5	5	-	2t+1	0
30t+20	$t \geq 0$	2.4	10t+5	10t+10	5	10	2t+1	1
30t+26	$t \geq 1$	2.5	10(t-1)+15	10(t-1)+25	5	10	2t+1	2

b. Existence of an $(n, 5, 2)$ C_5 -design for $n \equiv 0, 6 \pmod{10}$:

It results from case e by use of lemma 2.3.

c. Existence of $(n, 5, 5)$ C_5 -design for $n \equiv 3, 7, 9 \pmod{10}$.

Step 1. $5K_5$, $5K_7$, $5K_9$, $5K_{11}$ can be decomposed into C_5 :

- $5K_5$, $5K_{11}$: case a + lemma 2.1.

- $5K_7$: Let Z_7 be the set of vertices of $5K_7$; a decomposition into C_5 is given by the following 21 circuits:

$(i, i+1, i+2, i+3, i+4, i), (i, i+1, i+4, i+6, i+1, i), (i, i+1, i+6, i+2, i+5, i),$
 $0 \leq i \leq 6.$

- $5K_9$: Let Z_9 be the set of vertices of $5K_9$. A decomposition into C_5 is given by the following 36 circuits:

$(i, i+1, i+2, i+3, i+4, i), (i, i+2, i+4, i+6, i+8, i), (i, i+3, i+6, i+1, i+4, i),$
 $(i, i+7, i+2, i+8, i+5, i), \quad 0 \leq i \leq 8.$

Step 2. $5K_{2,2,2}$, $5K_{2,2,4}$ can be decomposed into C_5

- $5K_{2,2,2}$: see proposition 3.4.

- $5K_{2,2,4}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{2,2,4}$ with

$$X = \{x_i\}_{0 \leq i \leq 1}, \quad Y = \{y_i\}_{0 \leq i \leq 1}, \quad Z = \{z_i\}_{0 \leq i \leq 3}$$

then a decomposition of $5K_{2,2,4}$ into C_5 is given by the following 20 circuits:

$(y_j, z_i, y_{j+1}, z_{i+1}, x_j, y_j), (z_i, x_j, z_{i+1}, x_{j+1}, y_j, z_i)_{0 \leq i \leq 3, 0 \leq j \leq 1}$
 $(y_0, z_0, y_1, z_1, x_0, y_0)(y_1, z_2, y_0, z_1, x_1, y_1)(z_3, x_0, z_2, x_1, y_0, z_3)$
 $(z_3, x_1, z_0, x_0, y_1, z_3).$

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
$6t+5$	$t \geq 2$	2.5	$2(t-2)+4$	$2(t-2)+8$	2	4	t	2
$6t+7$	$t \geq 1$	2.5	$2(t-1)+4$	$2(t-1)+4$	2	-	t+1	0
$6t+9$	$t \geq 1$	2.5	$2(t-1)+4$	$2(t-1)+6$	2	4	t+1	1

f. Existence of an $(n, 5, 5) \vec{C}_5$ -design for $n \equiv 2, 3, 4 \pmod{5}$.

By lemma 2.2 and case c, we know the existence of $(n, 5, 5) \vec{C}_5$ -designs for $n \equiv 3, 7, 9 \pmod{10}$. It remains to prove it for $n \equiv 2, 4, 8 \pmod{10}$.

Step 1. $5K_6^*, 5K_8^*, 5K_{10}^*, 5K_{12}^*, 5K_{14}^*, 5K_{16}^*$ can be decomposed into \vec{C}_5 :

- $5K_6^*, 5K_{10}^*, 5K_{16}^*$: case a + lemma 2.1.

- $5K_8^*$: Let $Z_7 \cup \{\infty\}$ be the set of vertices of $5K_8^*$. A decomposition into \vec{C}_5 is given by the following 56 directed circuits:

$(i, i+1, i+3, i+6, i+2, i)(i, i+1, i+3, i+6, i+5, i)(i, i+1, i+4, i+2, i+3, i)$
 $(i, i+4, i+1, i+5, \infty, i) (i, i+1, i+3, i+6, \infty, i) (i, i+2, i+6, i+4, \infty, i)$
 $(i, i+5, i+3, i+2, \infty, i) (i, i+6, i+5, i+4, \infty, i), 0 \leq i \leq 6.$

- $5K_{12}^*$: Let $Z_{11} \cup \{\infty\}$ be the set of vertices of $5K_{12}^*$. A decomposition is given by the following 11 x 12 directed circuits:
 $0 \leq i \leq 10$

$(i, i+1, i+2, i+3, i+4, i)(i, i+2, i+4, i+6, i+8, i)(i, i+3, i+6, i+9, i+1, i)$
 $(i, i+4, i+8, i+1, i+5, i)(i, i+5, i+10, i+4, i+9, i)(i, i+6, i+1, i+7, i+2, i)$
 $(i, i+7, i+3, i+10, i+6, i)(i, i+8, i+5, i+2, \infty, i)(i, i+9, i+7, i+5, \infty, i)$
 $(i, i+8, i+5, i+3, \infty, i)(i, i+10, i+9, i+8, \infty, i)(i, i+10, i+3, i+4, \infty, i).$

- $5K_{14}^*$: Let $Z_{13} \cup \{\infty\}$ be the set of vertices of $5K_{14}^*$. A decomposition is given by the following 14 x 13 directed circuits:
 $0 \leq i \leq 12$

$(i, i+1, i+2, i+3, i+4, i)(i, i+2, i+4, i+6, i+8, i)(i, i+3, i+6, i+9, i+12, i)$
 $(i, i+4, i+8, i+12, i+3, i)(i, i+5, i+10, i+2, i+7, i)(i, i+6, i+12, i+5, i+11, i)$
 $(i, i+7, i+1, i+8, i+2, i)(i, i+8, i+3, i+11, i+6, i)(i, i+9, i+5, i+3, i+1, i)$
 $(i, i+10, i+7, i+4, \infty, i)(i, i+11, i+9, i+7, \infty, i)(i, i+12, i+11, i+10, \infty, i)$

$(1, 1+12, 1+9, 1+7, \infty, 1)(1, 1+8, 1+12, 1+2, \infty, 1)$.

Step 2. $5K_{2,2,2}^*$, $5K_{2,2,4}^*$ can be decomposed into \vec{C}_5 (see step 2 of case c + Lemma 2.2).

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
6t+6	$t \geq 2$	2.4	$2(t-2)+6$	$2(t-2)+6$	2	-	t+1	0
6t+8	$t \geq 2$	2.4	$2(t-2)+6$	$2(t-2)+8$	2	4	t+1	1
6t+10	$t \geq 2$	2.4	$2(t-2)+6$	$2(t-2)+10$	2	4	t+1	2

d. Existence of an $(n, 5, 10)C_5$ -design for $n \equiv 2, 4, 8 \pmod{10}$.
It results from case f by use of lemma 2.3.

3.6. Theorem. There exists an $(n, 7, \lambda)C_7$ -design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{14}$$

$$\lambda(n-1) \equiv 0 \pmod{2}.$$

There exists an $(n, 7, \lambda)\vec{C}_7$ -design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{7}.$$

By lemma 2.1, it suffices to prove the theorem in the following cases:

- for C_7 -designs:

- a. $\lambda = 1, \quad n \equiv 1, 7 \pmod{14}$
- b. $\lambda = 2, \quad n \equiv 0, 8 \pmod{14}$
- c. $\lambda = 7, \quad n \equiv 3, 5, 9, 11, 13 \pmod{14}$
- d. $\lambda = 14, \quad n \equiv 2, 4, 6, 10, 12 \pmod{14}$.

- for \vec{C}_7 -designs:

- e. $\lambda = 1, \quad n \equiv 0, 1 \pmod{7}$
- f. $\lambda = 7, \quad n \equiv 2, 3, 4, 5, 6 \pmod{7}$.

We shall use lemmas 2.2 and 2.3 and give a proof of all these cases in a convenient order.

a. Existence of an $(n, 7, 1)C_7$ -design for $n \equiv 1, 7 \pmod{14}$.

We still use the method exposed in 3.1:

Step 1. $K_7, K_{15}, K_{29}, K_{71}$ can be decomposed into C_7 .

- K_7 : see proposition 3.2.a.

- K_{15} : Let Z_{15} be the set of vertices of K_{15} . A decomposition into C_7 is given by the 15 following circuits:

$$(i, i+1, i+3, i+8, i+14, i+7, i+4, i), \quad 0 \leq i \leq 14.$$

- K_{29} : Let Z_{29} be the set of vertices of K_{29} . A decomposition into C_7 is given by the 58 following circuits:

$$(i, i+14, i+27, i+10, i+21, i+2, i+3, i), (i, i+2, i+6, i+11, i+5, i+12, i+20, i), \\ 0 \leq i \leq 28.$$

- K_{71} : Let Z_{71} be the set of vertices of K_{71} . A decomposition into C_7 is given by the following 5 x 71 circuits:

$$(i, i+4, i+9, i+15, i+47, i+7, i+50, i), (i, i+7, i+15, i+24, i+51, i+6, i+52, i) \\ (i, i+10, i+21, i+33, i+57, i+9, i+58, i), (i, i+14, i+29, i+45, i+62, i+9, i+51, i), \\ (i, i+1, i+3, i+6, i+39, i+5, i+41, i), \quad 0 \leq i \leq 70.$$

Step 2. $K_{7,7,7}, K_{14,14,14}, K_{14,14,28}, K_{7,7,21}$ can be decomposed into C_7 :

- $K_{7,7,7}, K_{14,14,14}$ (see proposition 3.3. a, d.).

- $K_{14,14,28}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{14,14,28}$ with

$$X = \{x_i\}_{0 \leq i \leq 13}, \quad Y = \{y_i\}_{0 \leq i \leq 13}, \quad Z = \{\infty_i\}_{0 \leq i \leq 13};$$

then a decomposition into C_7 is given by the following 10 x 14 circuits:

$$(x_i, y_{i+\alpha}, \infty_{\alpha+4}, x_{i+1}, \infty_{\alpha+12}, y_{i+\alpha+1}, \infty_{\alpha+20}, x_i), \quad 0 \leq i \leq 13, \quad 0 \leq \alpha \leq 7 \\ (x_{9+i}, y_{6+i}, \infty_0, x_{1+i}, y_1, \infty_{1+\varepsilon_1}, y_{1+7}, x_{1+9}), \quad 0 \leq i \leq 13; \quad \varepsilon_1 = 1 \text{ if } 7 \leq i \leq 13 \\ (x_{8+i}, y_{2+i}, \infty_3, x_i, \infty_{1+\varepsilon_1}, x_{7+i}, y_{3+i}, x_{8+i}), \quad 0 \leq i \leq 13; \quad \varepsilon_1 = 1 \text{ if } 7 \leq i \leq 13.$$

- $K_{7,7,21}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{7,7,21}$ with

$$X = \{x_i\}_{0 \leq i \leq 6}, \quad Y = \{y_i\}_{0 \leq i \leq 6}, \quad Z = \{\infty_i\}_{0 \leq i \leq 20}.$$

Then a decomposition into C_7 is given by the following 7 x 7 circuits

$$(x_i, y_{i+\alpha}, x_{i+1}, y_{\alpha+7}, y_{\alpha+1+1}, x_{\alpha+14}, x_i) 0 \leq i \leq 6, 0 \leq \alpha \leq 6.$$

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
42t+1	$t \geq 1$	2.5	$14(t-1)+14$	$14(t-1)+14$	7	-	2t	0
42t+15	$t \geq 1$	2.5	$14(t-1)+14$	$14(t-1)+28$	14	28	t	1
42t+21	$t \geq 0$	2.4	$14t+7$	$14t+7$	7	-	2t+1	0
42t+29	$t \geq 2$	2.5	$14(t-2)+28$	$14(t-2)+56$	14	28	t	2
42t+35	$t \geq 0$	2.4	$14t+7$	$14t+21$	7	21	t	1

The remaining case $n = 42t+7$ is solved for $t \geq 1$ by lemma 2.9 with $h = 6t+1$, $n_i = 7$ for $1 \leq i \leq h$.

e. Existence of an $(n, 7, 1) \vec{C}_7$ -design for $n \equiv 0, 1 \pmod{7}$.

By lemma 2.2 and case a, we know the existence for $n \equiv 1, 7 \pmod{14}$. It remains to prove it for $n \equiv 0, 8 \pmod{14}$:

Step 1. $K_7^*, K_8^*, K_{14}^*, K_{15}^*, K_{36}^*$ can be decomposed into \vec{C}_7 :

- K_7^*, K_{15}^* : case a + lemma 2.2.
- K_8^*, K_{14}^* : proposition 3.2.c.
- K_{36}^* is the edge disjoint union of three complete directed graphs isomorphic to $K_8^*, K_{15}^*, K_{15}^*$ and of a tripartite directed graph isomorphic to $K_{7,14,14}^*$ which can be decomposed into \vec{C}_7 by proposition 3.3.c.

Step 2. $K_{7,7,7}^*, K_{7,7,14}^*$ can be decomposed into \vec{C}_7 : see proposition 3.3.a + lemma 2.2 and proposition 3.3.b.

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
42t	$t \geq 1$	2.4	$14(t-1)+14$	$14(t-1)+14$	7	-	2t	0
42t+8	$t \geq 1$	2.5	$14(t-1)+14$	$14(t-1)+21$	7	14	2t	1
42t+14	$t \geq 1$	2.4	$14(t-1)+14$	$14(t-1)+28$	7	14	2t	2
42t+22	$t \geq 0$	2.5	$14t+7$	$14t+7$	7	-	2t+1	0
42t+28	$t \geq 0$	2.4	$14t+7$	$14t+14$	7	14	2t+1	1
42t+36	$t \geq 1$	2.5	$14(t-1)+21$	$14(t-1)+35$	7	14	2t+1	2

b. Existence of an $(n, 7, 2) \vec{C}_7$ -design for $n \equiv 0, 8 \pmod{14}$.

It's a consequence of case e by use of lemma 2.3.

c. Existence of an $(n, 7, 7)C_7$ -design for $n \equiv 3, 5, 9, 11, 13 \pmod{14}$.

Step 1. $7K_7, 7K_9, 7K_{11}, 7K_{13}, 7K_{15}, 7K_{17}, 7K_{21}, 7K_{27}, 7K_{29}, 7K_{39}$ can be decomposed into C_7 :

- $7K_7, 7K_{15}, 7K_{21}, 7K_{29}$: case a + lemma 2.1.
- $7K_{19}$ is the edge disjoint union of three complete graphs isomorphic to $7K_7$ and of a tripartite graph isomorphic to $7K_{7,7,7}$ which can be decomposed into C_7 by proposition 3.4.
- $7K_{27}$ is the edge disjoint union of three complete graphs isomorphic to $7K_9$ and of a tripartite complete graph isomorphic to $7K_{9,9,9}$ which can be decomposed into C_7 by proposition 3.4.
- $7K_{39}$ is the edge disjoint union of three complete graphs isomorphic to $7K_{13}$ and of a tripartite complete graph isomorphic to $7K_{13,13,13}$ which can be decomposed into C_7 by proposition 3.4.
- $7K_9$: Let Z_9 be the set of vertices of $7K_9$. A decomposition is given by the following 4×9 circuits of length 7:

$$(1, i+1, i+2, i+3, i+4, i+5, i+6, i), (i, i+2, i+4, i+6, i+8, i+1, i+5, i) \\ (i, i+3, i+6, i+1, i+4, i+7, i+2, i), (i, i+4, i+8, i+5, i+1, i+7, i+2, i), \\ 0 \leq i \leq 8.$$

- $7K_{11}$: Let Z_{11} be the set of vertices of $7K_{11}$. A decomposition is given by the following 5×11 circuits of length 7, $\{C_{\beta, i}\}$ for $1 \leq \beta \leq 5, 0 \leq i \leq 10$:

$$C_{\beta, i} = (\dots, \alpha\beta+1, \dots, i)_{0 \leq \alpha \leq 6}.$$

- $7K_{13}$: Let Z_{13} be the set of vertices of $7K_{13}$. A decomposition is given by the following 6×13 circuits $\{C_{\beta, i}\}$ for $1 \leq \beta \leq 6, 0 \leq i \leq 12$,

$$C_{\beta, i} = (\dots, \alpha\beta+1, \dots, i)_{0 \leq \alpha \leq 6}.$$

- $7K_{17}$: Let Z_{17} be the set of vertices of $7K_{17}$. A decomposition is given by the following 8×17 circuits of length 7, $\{C_{\beta, i}\}$ for $1 \leq \beta \leq 8$ and $0 \leq i \leq 16$,

$$C_{\beta, i} = (\dots, \alpha\beta+1, \dots, i)_{0 \leq \alpha \leq 6}.$$

Step 2. $7K_{4,4,4}$, $7K_{4,4,2}$ can be decomposed into C_7 .

- $7K_{4,4,4}$: see proposition 3.4.

- $7K_{4,4,2}$: Let $X \cup Y \cup Z$ be the set of vertices of $7K_{4,4,2}$ with
 $X = \{x_i\}_{0 \leq i \leq 3}$, $Y = \{y_i\}_{0 \leq i \leq 3}$, $Z = \{z_i\}_{0 \leq i \leq 1}$.

A decomposition of $7K_{4,4,2}$ into C_7 is given by the following circuits:

$$\left. \begin{aligned} &(z_i, x_i, z_{i+1}, x_{i+1}, y_{i+3}, x_{i+3}, y_i, z_i) \\ &(z_i, y_i, z_{i+1}, y_{i+1}, x_{i+3}, y_{i+2}, x_i, z_i) \quad 0 \leq i \leq 3 \\ &(z_i, x_i, y_{i+1}, x_{i+1}, y_{i+2}, x_{i+2}, y_i, z_i) \\ &(z_i, x_{i+1}, z_{i+1}, x_{i+2}, y_{i+1}, x_i, y_{i+3}, z_i) \\ &(z_i, y_{i+1}, z_{i+1}, y_{i+2}, x_{i+3}, y_{i+3}, x_{i+1}, z_i) \end{aligned} \right\} \text{ taken twice } 0 \leq i \leq 3.$$

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
12t+1	$t \geq 2$	2.5	$4(t-2)+8$	$4(t-2)+8$	4	-	k	0
12t+3	$t \geq 4$	2.5	$4(t-4)+10$	$4(t-4)+10$	4	2	k+1	5
12t+5	$t \geq 3$	2.5	$4(t-3)+16$	$4(t-3)+8$	4	2	k+1	4
12t+7	$t \geq 2$	2.5	$4(t-2)+12$	$4(t-2)+6$	4	2	k+1	3
12t+9	$t \geq 2$	2.5	$4(t-2)+12$	$4(t-2)+8$	4	2	k+1	2
12t+11	$t \geq 1$	2.5	$4(t-1)+8$	$4(t-1)+6$	4	2	k+1	1

f. Existence of an $(n, 7, 7, \vec{C}_7)$ -design for $n \equiv 2, 3, 4, 5, 6 \pmod{7}$.

By lemma 2.2 and case c, we know the existence for $n \equiv 3, 5, 9, 11, 13 \pmod{14}$. It remains to prove it for $n \equiv 2, 4, 6, 10, 12 \pmod{14}$.

Step 1. $7K_8^*$, $7K_{10}^*$, $7K_{12}^*$, $7K_{14}^*$, $7K_{16}^*$, $7K_{18}^*$, $7K_{20}^*$, $7K_{22}^*$, $7K_{26}^*$, $7K_{28}^*$, $7K_{30}^*$, $7K_{38}^*$ can be decomposed:

- $7K_8^*$, $7K_{14}^*$, $7K_{22}^*$, $7K_{28}^*$: case a + lemma 2.1.

- $7K_{30}^*$ is the edge disjoint union of three complete directed graph isomorphic to $7K_{10}^*$ and of a tripartite complete directed graph isomorphic to $7K_{10,10,10}^*$ which can be decomposed by proposition 3.4.

- $7K_{10}^*$: Let $Z_9 \cup \{\infty\}$ be the set of vertices of $7K_{10}^*$. Then a decomposition is given by the following 9×10 directed circuits:

$$(\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5, 0 \leq i \leq 8, \beta = 2, 5, 7}$$

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6, 0 \leq i \leq 8, \beta = 1, 8, 4.}$$

$$(i, i+3, i+6, i+8, i+2, i+5, \infty, i), (i, i+6, i+3, i+4, i+2, i+8, \infty, i)$$

$$(i, i+7, i+5, i+2, i+8, i+3, \infty, i), (i, i+8, i+2, i+7, i+3, i+5, \infty, i), 0 \leq i \leq 8.$$

- $7K_{12}^*$: Let $Z_{11} \cup \{\infty\}$ be the set of vertices of $7K_{12}^*$. A decomposition of $7K_{12}^*$ into \vec{C}_7 is given by the following 12×11 directed circuits:

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } 0 \leq i \leq 10, \beta = 1, 2, 3, 4, 10}$$

$$(\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5 \text{ for } 0 \leq i \leq 10, \beta = 5, 6, 7, 8, 9.}$$

$$(i, 10+i, 8+i, 5+i, 1+i, 7+i, \infty, i), (i, 2+i, 5+i, 10+i, 6+i, 3+i, \infty, i), 0 \leq i \leq 10.$$

- $7K_{16}^*$: Let $Z_{15} \cup \{\infty\}$ be the set of vertices of $7K_{16}^*$. A decomposition is given by the following 16×15 directed circuits: for $0 \leq i \leq 14$

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } \beta = 1, 2, 4, 7, 8, 11, 13, 14,}$$

$$(i, i+3, i+8, i+14, i+9, i+6, \infty, i) \text{ taken 5 times,}$$

$$(i, i+1, i+3, i+7, i+14, i+13, i+6, i+9, i), (i, i+5, i+10, i+4, i+14, i+8, \infty, i),$$

$$(i, i+10, i+4, i+2, i+11, i+6, \infty, i).$$

- $7K_{18}^*$: Let $Z_{17} \cup \{\infty\}$ be the set of vertices of $7K_{18}^*$. A decomposition is given by the following 18×17 directed circuits: for $0 \leq i \leq 16$

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } 0 \leq \beta \leq 11; (\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5 \text{ for } 12 \leq \beta \leq 16}$$

$$(i, i+1, i+7, i+14, i+9, i+5, \infty, i), (i, i+16, i+14, i+11, i+7, i+2, \infty, i).$$

- $7K_{20}^*$: Let $Z_{19} \cup \{\infty\}$ be the set of vertices of $7K_{20}^*$. A decomposition is given by the following 20×19 directed circuits: for $0 \leq i \leq 18$,

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } 1 \leq \beta \leq 13, (\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5 \text{ for } 14 \leq \beta \leq 18,}$$

$$(i, i+5, i+11, i+3, i+15, i+14, \infty, i), (i, i+4, i+10, i+7, i+5, i+4, \infty, i).$$

- $7K_{26}^*$: Let $Z_{25} \cup \{\infty\}$ be the set of vertices of $7K_{26}^*$. A decomposition is given by the following 26×25 directed circuits: for $0 \leq i \leq 24$

$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6}$, for $1 \leq \beta \leq 24$, $\beta \neq 5, 10, 15, 20, 21$
 $(i, i+5, i+15, i+14, i+4, i+24, \infty, i)$: taken 7 times.

- $7K_{38}^*$: Let $Z_{37} \cup \{\infty\}$ be the set of vertices of $7K_{38}^*$. A decomposition of $7K_{38}^*$ is given by the following 38×37 directed circuits: for $0 \leq i \leq 36$

$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6}$ for $1 \leq \beta \leq 31$, $(\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5}$ for $32 \leq \beta \leq 36$,

$(i, i+36, i+34, i+31, i+27, i+22, \infty, i)$

$(i, i+6, i+18, i+36, i+23, i+16, \infty, i)$.

Step 2. $7K_{4,4,4}^*$, $7K_{4,4,2}^*$ can be decomposed into \vec{C}_7 : see step 2 of case c + lemma 2.2.

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	q-p
12t	$t \geq 2$	2.4	$4(t-2)+8$	$4(t-2)+8$	4	-	k	0
12t+2	$t \geq 4$	2.4	$4(t-4)+20$	$4(t-4)+10$	4	2	k+1	5
12t+4	$t \geq 3$	2.4	$4(t-3)+16$	$4(t-3)+8$	4	2	k+1	4
12t+6	$t \geq 3$	2.4	$4(t-3)+16$	$4(t-3)+10$	4	2	k+1	3
12t+8	$t \geq 2$	2.4	$4(t-2)+12$	$4(t-2)+8$	4	2	k+1	2
12t+10	$t \geq 2$	2.4	$4(t-2)+12$	$4(t-2)+10$	4	2	k+1	1

Case $k = 3$.

It is well known that an $(n, 3, \lambda)C_3$ -design (or $(n, 3, \lambda)B.I.B.D.$) exists if and only if $\lambda n(n-1) \equiv 0 \pmod{6}$ and $\lambda(n-1) \equiv 0 \pmod{2}$ (see /10/).

It has been proved in /2, 3, 8, 14/ that:

3.7. Theorem. An $(n, 3, 1)C_3$ -design exists if and only if $n \neq 6$, and $n \equiv 0$ or $1 \pmod{3}$.

For $\lambda \geq 2$ we have the theorem:

3.8. Theorem. For $\lambda \geq 2$, an $(n, 3, \lambda)C_3$ -design exists if and only if $\lambda n(n-1) \equiv 0 \pmod{3}$.

Proof. By theorem 3.7 and lemma 2.1 it suffices to verify the theorem for $\lambda = 3$, $n \equiv 2 \pmod{3}$ and for $\lambda = 2, 3$ and $n = 6$. Lemma 5.3 of [10] implies that for $n \geq 3$ the edges of K_n can be partitioned into K_3 , K_4 , K_5 , K_6 and K_8 . Thus it suffices to prove the existence of a decomposition for $3K_n^*$ $n = 5, 6, 8$ and $2K_6^*$. For $n = 5$, it follows from the existence of an $(5, 3, 1)C_3$ -design.

For $n = 6$ let the vertices be $Z_5 \cup \{\infty\}$: a decomposition of $2K_6^*$ is given by: $(\infty, i, i+1)$; $(\infty, i+1, i)$; $(i, i+2, i+4)$; $(i+4, i+2, i)$ with $0 \leq i \leq 4$ and for $3K_6^*$ by $(\infty, i, i+1)$ twice and once $(i, i+2, i+4)$, $(i+4, i+2, i)$, $(i+2, i+1, i)$ with $0 \leq i \leq 4$. For $n = 8$ let the vertices be $Z_7 \cup \{\infty\}$ then a decomposition is given by: 3 times the circuits $(i, i+1, i+3)$ twice $(i, i+3, i+1)$ and once $(\infty, i, i+3)$; $(\infty, i+3, i+1)$; $(\infty, i+1, i)$ where $0 \leq i \leq 6$.

References

- /1/ C. Berge
Graphs and Hypergraphs
North Holland, Amsterdam, 1973
- /2/ J. C. Bermond
An application of the solution of Kirkman's schoolgirl problem: the decomposition of the symmetric oriented complete graph into 3-circuits, Discrete Math. 8 (1974), 301 - 304
- /3/ J. C. Bermond
Decomposition of K_n^* into k-circuits and balanced G-designs, in "Recent advances in graph theory" (ed. M. Fiedler), Proc. Symp. Prague Akademiei Prague, (1975), 57 - 68
- /4/ J. C. Bermond
Thesis
University of Paris XI (Orsay) (1975)
- /5/ J. C. Bermond and V. Faber
Decomposition of the complete directed graph into k-circuits
J. Combinatorial Theory (B), 21, 1976, 146 - 155
- /6/ J. C. Bermond and D. Sotteau
Graph decompositions and G-designs.
Proc. 5th British Combinatorial Conference, Aberdeen July 1975, Congressus Numerantium 15, Utilitas Math. Publ., 53 - 72
- /7/ J. C. Bermond, C. Huang and D. Sotteau
Cycle and circuit designs: even case, to appear.
- /8/ R. H. Bruck
Existence of the A-maps of a A.K. Dewney and N. Robertson, unpublished manuscript.
- /9/ M. Hall Jr.
"Combinatorial Theory"
Blaisdell Waltham, Mass. (1967)

- /10/ H. Hanani
Balanced incomplete block designs and related designs
Discrete Math. 11 (1975), 255 - 369
- /11/ B. Hartnell and M. Milgram
Decomposition of K_p , for p a prime into k -circuits
"Proc. Journées franco-belges sur les graphes et les hypergraphes", Paris, Mai 1974, Cahiers du C.E.R.O, 17, (Bruxelles) 1975, 221 - 222
- /12/ P. Hell and A. Rosa
Graph decompositions, handcuffed prisoners and balanced P -designs, Discrete Math. 2 (1972), 229 - 252
- /13/ C. Huang and A. Rosa
Another class of balanced graph designs, balanced circuit designs,
Discrete Mathematics, 12 (1975), 269 - 293
- /14/ N. S. Mendelsohn
A natural generalization of Steiner triple systems, in
"Computers in number Theory" (eds. A.O.L. Atkin and B.J. Birch)
Academic Press, New York (1971), 323 - 338
- /15/ D. Merriel
Partitioning the directed graph into 5-cycles, unpublished manuscript
- /16/ A. Rosa
On the cyclic decomposition of the complete graph into polygons with odd number of edges
Casopis Pest. Math. 91 (1966), 53 - 63
- /17/ D. Sotteau
Decomposition of K_n^* into circuits of odd length,
Discrete Math. 15 (1976), 185 - 191
- /18/ R. M. Wilson
Construction and uses of pairwise balanced designs, in
"Combinatorics, Part I" (eds. M. Hall and J. M. Van Lint),
Math. Centre Tracts, 55, Amsterdam (1974), 18 - 41

/19/ R. M. Wilson

Decompositions of complete graphs into subgraphs isomorphic to a given graph, in Proc. 5th British Combinatorial Conference, Aberdeen, July 1975, Congressus Numerantium 15, Utilitas Math., 647 - 659