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CYCLE AND CIRCUIT DESIGNS ODD CASE

1. Introduction.

In what follows, n, k, λ will be positive integers with $n \geq k$ and G will always denote a graph with k vertices. Definitions not given here can be found in Berge's book /1/.

Let us denote by

λK_n : the complete multigraph with n vertices and any two distinct vertices joined by exactly λ edges.

λK_n^* : the complete symmetric directed multigraph with n vertices and any two distinct vertices x and y joined by λ arcs (x,y) and λ arcs (y,x) .

$\lambda K_{n_1, n_2, \dots, n_h}$: the complete multipartite graph with vertex set $X = \cup X_i, 1 \leq i \leq h$, where the X_i are disjoint sets with $|X_i| = n_i$ and where two vertices, which belong to different sets X_i and X_j are joined by λ edges.

$\lambda K_{n_1, n_2, \dots, n_h}^*$: the complete directed multipartite graph.

C_k : the elementary circuit of length k .

\vec{C}_k : the elementary directed circuit of length k .

We shall say that a graph (respectively directed graph) H can be decomposed into subgraphs isomorphic to G , where G is a given graph (respectively directed graph) if we can partition the edges (resp. arcs) of H into subgraphs isomorphic to G .

1.1. An (n, k, λ) G -design is an edge disjoint decomposition of λK_n into subgraphs isomorphic to G . The G -designs have been introduced by Hell et Rosa /12/. In the particular case where G is the complete graph K_k , an (n, k, λ) K_k -design is nothing else than an (n, k, λ) B.I.B.D. (balanced incomplete block design: see Hall /9/).

1.2. A similar definition holds for directed graphs. If G is a directed graph, with k vertices, an (n, k, λ) G -design is a partition of the arcs of λK_n^* into subgraphs isomorphic to G .

1.3. For results concerning the existence of G-designs, the reader can see, for example, the survey we have done in /6/.

Here we are interested in the particular case where G is equal to C_k or \vec{C}_k . An (n, k, λ) C_k -design is also called a B.C.D. (n, k, λ) (balanced circuit design /13/).

1.4. Proposition. Necessary conditions for the existence of an (n, k, λ) C_k -design are

$$\lambda n(n-1) \equiv 0 \pmod{2k}$$

$$\lambda(n-1) \equiv 0 \pmod{2}.$$

1.5. Proposition. A necessary condition for the existence of an (n, k, λ) \vec{C}_k -design is

$$\lambda n(n-1) \equiv 0 \pmod{k}.$$

1.6. In this paper we will restrict ourselves to the case k odd. The case k even will be considered in /7/. For k odd, it has been proved that the necessary conditions of the proposition 1.4 are sufficient for

k = 3 (case of Steiner Triple System) see Hanani /10/.

k = 5 by Huang and Rosa /13/.

k odd, $n = 2qk + 1$ or $n = 2qk + k$, $\lambda = 1$ by Rosa /16/.

The necessary condition 1.5 of the existence of a \vec{C}_k -design is sufficient for $k = 3$, $\lambda = 1$ except $n = 6$ in /2/ and also by Bruck /8/ Mendelsohn /14/.

A short proof can be found in /3/ or 4/.

k = 5, $\lambda = 1$ in /3/ and /17/ and also by Merriell /15/.

k = p^α , p an odd prime and $\lambda = 1$ in /17/ and also for k = 7 by Merriell /15/.

It has also been shown that an $(n, k, 1)$ \vec{C}_k -design exists if n is a power of a prime and $n(n-1) \equiv 0 \pmod{k}$ by Hartnell and Milgram /11/ and if $n \equiv 0$ or $1 \pmod{k}$, k odd in /17/.

Finally we note that the asymptotic results obtained by Wilson /19/ prove that the conditions 1.4 and 1.5 are sufficient, for a given k, for n large enough.

Here we shall prove the following

1.7. Theorem. The necessary conditions of propositions 1.4 and 1.5 are sufficient for $k = 3, 5, 7$ except for $n = 6, k = 3, \lambda = 1$.

2. General Lemmas.

The 3 following lemmas are simple consequences of the definitions.

2.1. Lemma. If there exists an (n, k, λ_1) G-design and an (n, k, λ_2) G-design, then there exists an $(n, k, p\lambda_1 + q\lambda_2)$ G-design (p and q are positive integers).

2.2. Lemma. Let H^* be the directed graph obtained from H by associating to each edge of H two opposite arcs. If H can be decomposed into C_k , then H^* can be decomposed into \vec{C}_k .

Remark. The contrary is false: for example we shall see that there exists a $(4, 3, 1) \vec{C}_3$ -design, but by 1.4 there exists no $(4, 3, 1) C_3$ -design.

2.3. Lemma. If there exists an $(n, k, \lambda) \vec{C}_k$ -design, then there exists an $(n, k, 2\lambda) C_k$ -design.

Remark. The contrary is false: there exists a $(6, 3, 2) C_3$ -design (see Hanani /10/), but there exists no $(6, 3, 1) \vec{C}_3$ -design /2/.

2.4. Lemma. If there exists an (n_i, k, λ) G-design for $1 \leq i \leq h$ and if $\lambda K_{n_1, \dots, n_h}$ (or $\lambda K_{n_1, \dots, n_h}^*$ if G is a directed graph) can be decomposed into h subgraphs isomorphic to G , then there exists a $(\sum n_i, k, \lambda)$ G-design.

2.5. Lemma. If there exists an $(n_i + 1, k, \lambda)$ G-design for $1 \leq i \leq h$ and if $\lambda K_{n_1, \dots, n_h}$ (or $\lambda K_{n_1, \dots, n_h}^*$ if G is a directed graph) can be decomposed into subgraphs isomorphic to G then there exists a $((\sum n_i) + 1, k, \lambda)$ G-design.

Proofs. We shall give the proof of lemma 2.5 in the undirected case. (The proof of lemma 2.4 and of the directed case are similar). Let the vertices of $\lambda K_{(\sum n_i) + 1}$ be: $X_1 \cup X_2 \cup \dots \cup X_h \cup \{\infty\}$, with $|X_i| = n_i$, for $1 \leq i \leq h$.

The graph $\lambda K_{(\sum n_i) + 1}$ is the edge disjoint union of:

- the h complete multigraphs, constructed on the sets $X_i \cup \{\infty\}$ which are isomorphic to λK_{n_i+1} and thus can be decomposed (by hypothesis) into subgraphs isomorphic to G .

- the complete multipartite multigraph with vertex sets $\cup X_i$ which is isomorphic to $\lambda K_{n_1, \dots, n_h}$ decomposable into subgraphs isomorphic to G by hypothesis.

2.6. Remark. The interest of lemmas 2.4 and 2.5 is that it gives a method of composition which enables us to construct G -designs by knowing the existence of smaller ones. (In the case $G = K_k$, these lemmas are used with different notations by Hanani /10/ and Wilson /19/: indeed a decomposition of $\lambda K_{n_1, \dots, n_h}$ into K_k is equivalent to the existence of group divisible designs).

We shall apply this method with tripartite graphs ($h=3$). The following lemma will enable us to construct decompositions of tripartite multigraphs from decomposition of smaller ones.

2.7. Lemma. If $K_{a,b,c}$ and $K_{a,b,c'}$ can be decomposed into subgraphs isomorphic to G , then also $K_{qa, qb, pc+(q-p)c'}$ with p and q positive integers, $0 \leq p \leq q$.

Proof. Let the vertices of $K_{qa, qb, pc+(q-p)c'}$ be $X \cup Y \cup Z$ with

$$X = \bigcup_{i=1}^q X_i, |X_i| = a \quad (1 \leq i \leq q); \quad Y = \bigcup_{i=1}^q Y_i, |Y_i| = b \quad (1 \leq i \leq q)$$

$$\text{and } Z = \bigcup_{i=1}^q Z_i, |Z_i| = c, \quad 1 \leq i \leq p \text{ and } |Z_i| = c' \text{ for } p+1 \leq i \leq q.$$

Then $K_{qa, qb, pc+(q-p)c'}$ is the edge disjoint union of the q^2 tripartite graphs constructed respectively on X_i, Y_{i+j}, Z_j with $1 \leq i \leq q, 1 \leq j \leq q$ and where the indices of Y are to be taken modulo q . These q^2 graphs are isomorphic to $K_{a,b,c}$ or to $K_{a,b,c'}$ which can be decomposed into subgraphs isomorphic to G .

A similar lemma holds in the directed case.

2.8. Lemma. If $K_{a,b,c}^*$ and $K_{a,b,c'}^*$ can be decomposed into subgraphs isomorphic to G then also $K_{qa, qb, pc+(q-p)c'}^*$ (p, q positive integers $0 \leq p \leq q$).

The two following lemmas can be sometimes useful.

2.9. Lemma. Let $h \equiv 1$ or $3 \pmod{6}$; if K_{n_i} (resp. K_{n_i+1}), $1 \leq i \leq h$ and K_{n_i, n_j, n_k} , $1 \leq i < j < k \leq h$, can be decomposed into subgraphs isomorphic to G then also K_n (resp. K_{n+1}) for $n = \sum_{i=1}^h n_i$.

Sketch of proof: These lemmas result from the existence of $(6t + 1, 3, 1)$ or $(6t + 3, 3, 1)$ Steiner Triple Systems (see /17/ for more details).

3. Existence of $C_3, C_5, C_7, \vec{C}_3, \vec{C}_5, \vec{C}_7$ -Designs.

In this paragraph, k will always denote an odd integer ≥ 3 . We shall prove the theorems 1.7 ($k = 3, 5, 7$): we give first the ideas of the proof and general propositions which are valid for all (odd) $k \geq 5$.

We apply them only to the particular values of $k : 5, 7$ but the methods can be applied to give other results for greater values of k .

3.1. The proofs of theorem 1.7 for $k = 5$ or 7 are by induction and all of the same kind:

Step 1. First we prove the existence of decomposition of some K_n or K_n^* , for the first values of n in order to start the induction.

Step 2. We prove the existence of decomposition of some $K_{a,b,c}$ or $K_{a,b,c}^*$ which will be needed in the verification.

Step 3. Then we use lemma 2.4 or lemma 2.5 with tripartite graphs ($h = 3$) or lemma 2.9. We indicate in each case, in a table, which lemma we use and which values of n_i occur in the lemma. The verification that the hypothesis of these lemmas are satisfied follows

- for the K_{n_i} or $K_{n_i}^*$ from step 1 or induction hypothesis.
- for the K_{n_1, n_2, n_3} or K_{n_1, n_2, n_3}^* from lemma 2.7 (or 2.8):

we indicate in the same table the values of the parameters a, b, c, c', p and q used. The $K_{a,b,c}$ and $K_{a,b,c}'$ used in these lemmas can be decomposed by step 2.

For step 1 the following proposition is useful

3.2. Proposition. Let k be an odd integer ≥ 3 , then there exists

- a) $a(k, k, 1) \vec{C}_k$ -design
- b) $a(k, k, 1) \vec{C}_k$ -design
- c) $a(k+1, k, 1) \vec{C}_k$ -design
- d) $a(2k, k, 1) \vec{C}_k$ -design for $k \geq 5$.

Proof. a) is well known (see Berge /1/).

b) is a consequence of a) and lemma 2.2.

c) has been proved in /5/: Let $Z_k \cup \{\infty\}$ be the set of vertices of K_{k+1}^* (we recall that Z_k is the additive group of residues modulo k). A decomposition of K_{k+1}^* into \vec{C}_k is given by the following k directed circuits, for $0 \leq i \leq k-1$,

$(\infty, \dots, \alpha+i, k-1-\alpha+i, \dots, \beta+i, k-2-\beta+i, \dots, \infty)$ with $0 \leq \alpha \leq \lfloor k/4 \rfloor - 1$, and $\lfloor k/4 \rfloor \leq \beta \leq (k-3)/2$, and the circuit $(\dots, \alpha, \alpha + (k+1)/2, \dots)$ with $0 \leq \alpha \leq (k-1)/2$.

d) There does not exist a $(6, 3, 1) \vec{C}_3$ -design (see /2/). For $k \geq 5$, there exists a $(2k, k, 1) \vec{C}_k$ -design (/4/): let $Z_{k-1} \cup A_\infty$, with $A_\infty = \{\infty_0, \dots, \infty_k\}$ be the set of vertices of K_{2k}^* . As the complete graph on A_∞ is isomorphic to K_{k+1}^* and thus is decomposable into \vec{C}_k , it suffices to prove that the arcs of K_{2k}^* , which do not join two points of A_∞ , can be partitioned into \vec{C}_k ; that is given by the following $3(k-1)$ directed circuits for $0 \leq i \leq k-2$:

$(\infty_0, i, \infty_1, k-2+i, \dots, \alpha+i, k-2+\alpha+i, \dots, (k-3)/2+i, \infty_0)$ with $1 \leq \alpha \leq (k-5)/2$

$(i, \dots, \beta+i, \infty_{\beta+1}, \dots, i)$ and $(i, \dots, k-1-\beta+i, \infty_{\beta+(k+1)/2}, \dots, i)$
with

$1 \leq \beta \leq (k-1)/2$.

For step 2 the following propositions are useful:

3.3. Proposition. Let k be an odd integer ≥ 3 , then

- a) $K_{k,k,k}$ can be decomposed into C_k .
- b) $K_{k,k,2k}^*$ can be decomposed into \vec{C}_k , for $k \geq 5$.
- c) $K_{k,2k,2k}^*$ can be decomposed into \vec{C}_k , for $k \geq 5$.
- d) $K_{qk,qk,qk}$ can be decomposed into C_k , for $k \geq 5$, q integer.

A proof of this proposition can be found in /17/.

3.4. Proposition. For all integers $n, p, n \geq p \geq 1$, $(2p+1)K_{n,n,n}$ can be decomposed into C_{2p+1} .

Proof. Let $X \cup Y \cup Z$ be the set of vertices of $(2p+1)K_{n,n,n}$ with $X = \{x_i\}_{0 \leq i \leq n-1}$, $Y = \{y_i\}_{0 \leq i \leq n-1}$, $Z = \{z_i\}_{0 \leq i \leq n-1}$; the $3n^2$ circuits of length $2p+1$ of the decomposition are given by the following circuits $\{C_{i,j}, C'_{i,j}, C''_{i,j}\}$, for $0 \leq i \leq n-1$ and $0 \leq j \leq n-1$:

$$C_{i,j} = (\dots, x_{i+\alpha}, y_{j+\alpha}, \dots, z_{i+j}, x_i)$$

$$C'_{i,j} = (\dots, y_{i+\alpha}, z_{j+\alpha}, \dots, x_{i+j}, y_i) \quad 0 \leq \alpha \leq p-1$$

$$C''_{i,j} = (\dots, z_{i+\alpha}, x_{j+\alpha}, \dots, y_{i+j}, z_i)$$

3.5. Theorem. There exists an $(n, 5, \lambda) C_5$ -design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{10}$$

$$\lambda(n-1) \equiv 0 \pmod{2}$$

There exists an $(n, 5, \lambda) \vec{C}_5$ design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{5}.$$

Proof. By lemma 2.1 it suffices to verify the theorem for the minimum values of λ , that is to prove the existence

- of C_5 -designs in the following cases:

- a. $\lambda = 1, n \equiv 1, 5 \pmod{10}$
- b. $\lambda = 2, n \equiv 0, 6 \pmod{10}$
- c. $\lambda = 5, n \equiv 3, 7, 9 \pmod{10}$
- d. $\lambda = 10, n \equiv 2, 4, 8 \pmod{10}$.

- of \vec{C}_5 -designs in the following cases:

- e. $\lambda = 1, n \equiv 0, 1 \pmod{5}$
- f. $\lambda = 5, n \equiv 2, 3, 4 \pmod{5}$.

In order to minimize the verifications we shall use lemmas 2.2 and 2.3 and thus give a proof of the different cases a, b, c, d, e, f in a convenient order.

a. Existence of an $(n, 5, 1)$ C_5 -design for $n \equiv 1$ or $5 \pmod{10}$.
 We use the method exposed in 3.1.

Step 1. $K_5, K_{11}, K_{21}, K_{25}, K_{85}$ can be decomposed into C_5 . For K_5 see 3.2 a). For the others it can be done by direct construction (see Rosa /16/). For example for K_{11} : if Z_{11} is the set of vertices of K_{11} a decomposition is given by the circuits: $(i, i+2, i+10, i+6, i)$, $0 \leq i \leq 10$.

Step 2. $K_{5,5,5}, K_{10,10,10}, K_{10,10,20}, K_{15,15,15}, K_{15,15,5}, K_{15,15,25}$ can be decomposed into C_5 .

- $K_{5,5,5}, K_{10,10,10}, K_{15,15,15}$: see proposition 3.3 a), d).

- $K_{10,10,20}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{10,10,20}$ with

$$X = \{x_i\}_{0 \leq i \leq 9}, Y = \{y_i\}_{0 \leq i \leq 9}, Z = \{z_i\}_{0 \leq i \leq 9} \cup \{\infty_i\}_{0 \leq i \leq 9}.$$

A decomposition into C_5 is given by the 100 following circuits:

$$(x_{i+j}, y_j, z_{2i+1+j}, y_{j+1}, \infty_i, x_{i+j})_{0 \leq i \leq 4, 0 \leq j \leq 9}$$

$$(x_j, z_{2i+1+j}, x_{j+1}, \infty_{i+5}, y_{i+j+1}, x_j)_{0 \leq i \leq 4, 0 \leq j \leq 9}.$$

- $K_{15,15,5}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{15,15,5}$ with

$$X = \{x_i\}_{0 \leq i \leq 14}, Y = \{y_i\}_{0 \leq i \leq 14}, Z = \{\infty_i\}_{0 \leq i \leq 4}.$$

A decomposition into C_5 is given by the 75 following circuits:

$$(x_i, y_{i+3j+1}, x_{i+1}, y_{i+3j+3}, \infty_j, x_i)_{0 \leq j \leq 4, 0 \leq i \leq 14}.$$

- $K_{15,15,25}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{15,15,25}$ with

$$X = \{x_i\}_{0 \leq i \leq 14}, Y = \{y_i\}_{0 \leq i \leq 14}, Z = \{z_i\}_{0 \leq i \leq 14} \cup \{\infty_i\}_{0 \leq i \leq 9}.$$

A decomposition into C_5 is given by the following circuits:

$$(x_i, y_{i+1}, x_{i+1}, y_{i+3}, z_i, x_i)_{0 \leq i \leq 14}, (x_i, z_{i+2}, x_{i+1}, z_{i+4}, y_{i+6}, x_i)_{0 \leq i \leq 14}$$

$$(x_i, y_{i+\alpha+9}, z_{i+2\alpha+9}, y_{i+\alpha+3}, \infty_\alpha, x_i)_{0 \leq i \leq 14, 0 \leq \alpha \leq 5}, (x_i, z_{i+5}, x_{i+1},$$

$$z_{i+7}, y_{i+8}, x_i)_{0 \leq i \leq 14}$$

$$(x_i, z_{\alpha+i+10}, x_{i+3}, y_{\alpha+6}, y_{\alpha+3+i}, x_i)_{0 \leq i \leq 14, 0 \leq \alpha \leq 2}, (x_i, z_{i+14}, x_{i+1}, y_{i+7}, x_i)_{0 \leq i \leq 14}.$$

Step 3.

n	Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p	
$30t+1$	$t \geq 1$	2.5	$10t$	$10t$	5	-	$2t$	0
$30t+11$	$t \geq 1$	2.5	$10(t-1)+10$	$10(t-1)+20$	10	20	t	1
$30t+15$	$t \geq 0$	2.4	$10t+5$	$10t+5$	5	-	$2t+1$	0
$30t+21$	$t \geq 2$	2.5	$10(t-2)+20$	$10(t-2)+40$	10	20	t	2
$90t+25$	$t \geq 1$	2.4	$30(t-1)+5$	$30(t-1)+25$	15	5	$2t+1$	2
$90t+55$	$t \geq 0$	2.4	$30t+15$	$30t+25$	15	25	$2t+1$	1
$90t+85$	$t \geq 1$	2.4	$30(t-1)+5$	$30(t-1)+25$	15	5	$2t+3$	5

The remaining case $n = 30t+5$ is solved for $t \geq 1$ by lemma 2.9 with $h = 6t+1$, $n_i = 5$ for $1 \leq i \leq h$.

e. Existence of an $(n, 5, 1) \vec{C}_5$ -design for $n \equiv 0, 1 \pmod{5}$.

By lemma 2.2 and case a we know the existence of $(n, 5, 1) \vec{C}_5$ -designs for $n \equiv 1, 5 \pmod{10}$. It remains to prove it for $n \equiv 0, 6 \pmod{10}$.

Step 1. $K_5^*, K_6^*, K_{10}^*, K_{11}^*, K_{26}^*$ can be decomposed into \vec{C}_5 .

- K_5^*, K_{11}^* : case a + lemma 2.2.

- K_6^*, K_{10}^* : see proposition 3.2.c.

- K_{26}^* : K_{26}^* is the edge disjoint union of three complete graphs isomorphic to $K_6^*, K_{11}^*, K_{11}^*$ which can be decomposed into \vec{C}_5 and of a complete tripartite graph isomorphic to $K_{5,10,10}^*$ which can be decomposed into \vec{C}_5 by proposition 3.3.c.

Step 2. $K_{5,5,5}^*, K_{5,5,10}^*$ can be decomposed into \vec{C}_5 (proposition 3.3.a, lemma 2.2 and proposition 3.3.b).

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
30t	$t \geq 1$	2.4	10t	10t	5	-	2t	0
30t+6	$t \geq 1$	2.5	10(t-1)+10	10(t-1)+15	5	10	2t	1
30t+10	$t \geq 1$	2.4	10(t-1)+10	10(t-1)+20	5	10	2t	2
30t+16	$t \geq 0$	2.5	10t+5	10t+5	5	-	2t+1	0
30t+20	$t \geq 0$	2.4	10t+5	10t+10	5	10	2t+1	1
30t+26	$t \geq 1$	2.5	10(t-1)+15	10(t-1)+25	5	10	2t+1	2

b. Existence of an $(n,5,2) C_5$ -design for $n \equiv 0,6 \pmod{10}$:

It results from case e by use of lemma 2.3.

c. Existence of $(n,5,5) C_5$ -design for $n \equiv 3,7,9 \pmod{10}$.

Step 1. $5K_5, 5K_7, 5K_9, 5K_{11}$ can be decomposed into C_5 :

- $5K_5, 5K_{11}$: case a + lemma 2.1.

- $5K_7$: Let Z_7 be the set of vertices of $5K_7$; a decomposition into C_5 is given by the following 21 circuits:

$(i, i+1, i+2, i+3, i+4, i), (i, i+1, i+4, i+6, i+1, i), (i, i+1, i+6, i+2, i+5, i),$

$$0 \leq i \leq 6.$$

- $5K_9$: Let Z_9 be the set of vertices of $5K_9$. A decomposition into C_5 is given by the following 36 circuits:

$(i, i+1, i+2, i+3, i+4, i), (i, i+2, i+4, i+6, i+8, i), (i, i+3, i+6, i+1, i+4, i),$

$(i, i+7, i+2, i+8, i+5, i), 0 \leq i \leq 8.$

Step 2. $5K_{2,2,2}, 5K_{2,2,4}$ can be decomposed into C_5

- $5K_{2,2,2}$: see proposition 3.4.

- $5K_{2,2,4}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{2,2,4}$ with

$$X = \{x_i\}_{0 \leq i \leq 1}, Y = \{y_i\}_{0 \leq i \leq 1}, Z = \{z_i\}_{0 \leq i \leq 3}$$

then a decomposition of $5K_{2,2,4}$ into C_5 is given by the following 20 circuits:

$(y_j, z_i, y_{j+1}, z_{i+1}, x_j, y_j), (z_i, x_j, z_{i+1}, x_{j+1}, y_j, z_1)_{0 \leq i \leq 3, 0 \leq j \leq 1}$

$(y_0, z_0, y_1, z_1, x_0, y_0)(y_1, z_2, y_0, z_1, x_1, y_1)(z_3, x_0, z_2, x_1, y_0, z_3)$

$(z_3, x_1, z_0, x_0, y_1, z_3).$

Step 3.

n		Lemma	$n_1=n_2$	n_3	$a=b=c$	c'	q	$q-p$
$6t+5$	$t \geq 2$	2.5	$2(t-2)+4$	$2(t-2)+8$	2	4	t	2
$6t+7$	$t \geq 1$	2.5	$2(t-1)+4$	$2(t-1)+4$	2	-	t+1	0
$6t+9$	$t \geq 1$	2.5	$2(t-1)+4$	$2(t-1)+6$	2	4	t+1	1

f. Existence of an $(n,5,5) \vec{C}_5$ -design for $n \equiv 2, 3, 4 \pmod{5}$.

By lemma 2.2 and case c, we know the existence of $(n,5,5) \vec{C}_5$ -designs for $n \equiv 3,7,9 \pmod{10}$. It remains to prove it for $n \equiv 2, 4, 8 \pmod{10}$.

Step 1. $5K_6^*$, $5K_8^*$, $5K_{10}^*$, $5K_{12}^*$, $5K_{14}^*$, $5K_{16}^*$ can be decomposed into \vec{C}_5 :

- $5K_6^*$, $5K_{10}^*$, $5K_{16}^*$: case a + lemma 2.1.

- $5K_8^*$: Let $Z_7 \cup \{\infty\}$ be the set of vertices of $5K_8^*$. A decomposition into \vec{C}_5 is given by the following 56 directed circuits:

$(i, i+1, i+3, i+6, i+2, i)(i, i+1, i+3, i+6, i+5, i)(i, i+1, i+4, i+2, i+3, i)$
 $(i, i+4, i+1, i+5, \infty, i) (i, i+1, i+3, i+6, \infty, i) (i, i+2, i+6, i+4, \infty, i)$
 $(i, i+5, i+3, i+2, \infty, i) (i, i+6, i+5, i+4, \infty, i), 0 \leq i \leq 6.$

- $5K_{12}^*$: Let $Z_{11} \cup \{\infty\}$ be the set of vertices of $5K_{12}^*$. A decomposition is given by the following 11 x 12 directed circuits:
 $0 \leq i \leq 10$

$(i, i+1, i+2, i+3, i+4, i)(i, i+2, i+4, i+6, i+8, i)(i, i+3, i+6, i+9, i+1, i)$
 $(i, i+4, i+8, i+1, i+5, i)(i, i+5, i+10, i+4, i+9, i)(i, i+6, i+1, i+7, i+2, i)$
 $(i, i+7, i+3, i+10, i+6, i)(i, i+8, i+5, i+2, \infty, i)(i, i+9, i+7, i+5, \infty, i)$
 $(i, i+8, i+5, i+3, \infty, i)(i, i+10, i+9, i+8, \infty, i)(i, i+10, i+3, i+4, \infty, i).$

- $5K_{14}^*$: Let $Z_{13} \cup \{\infty\}$ be the set of vertices of $5K_{14}^*$. A decomposition is given by the following 14 x 13 directed circuits:
 $0 \leq i \leq 12$

$(i, i+1, i+2, i+3, i+4, i)(i, i+2, i+4, i+6, i+8, i)(i, i+3, i+6, i+9, i+12, i)$
 $(i, i+4, i+8, i+12, i+3, i)(i, i+5, i+10, i+2, i+7, i)(i, i+6, i+12, i+5, i+11, i)$
 $(i, i+7, i+1, i+8, i+2, i)(i, i+8, i+3, i+11, i+6, i)(i, i+9, i+5, i+3, i+1, i)$
 $(i, i+10, i+7, i+4, \infty, i)(i, i+11, i+9, i+7, \infty, i)(i, i+12, i+11, i+10, \infty, i)$

$(i, i+12, i+9, i+7, \infty, i)(i, i+8, i+12, i+2, \infty, i)$.

Step 2. $5K_{2,2,2}^*$, $5K_{2,2,4}^*$ can be decomposed into \vec{C}_5 (see step 2 of case c + Lemma 2.2).

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
6t+6	$t \geq 2$	2.4	$2(t-2)+6$	$2(t-2)+6$	2	-	t+1	0
6t+8	$t \geq 2$	2.4	$2(t-2)+6$	$2(t-2)+8$	2	4	t+1	1
6t+10	$t \geq 2$	2.4	$2(t-2)+6$	$2(t-2)+10$	2	4	t+1	2

d. Existence of an $(n, 5, 10)C_5$ -design for $n \equiv 2, 4, 8 \pmod{10}$.
It results from case f by use of lemma 2.3.

3.6. Theorem. There exists an $(n, 7, \lambda)C_7$ -design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{14}$$

$$\lambda(n-1) \equiv 0 \pmod{2}.$$

There exists an $(n, 7, \lambda)\vec{C}_7$ -design if and only if

$$\lambda n(n-1) \equiv 0 \pmod{7}.$$

By lemma 2.1, it suffices to prove the theorem in the following cases:

- for C_7 -designs:

- a. $\lambda = 1, \quad n \equiv 1, 7 \pmod{14}$
- b. $\lambda = 2, \quad n \equiv 0, 8 \pmod{14}$
- c. $\lambda = 7, \quad n \equiv 3, 5, 9, 11, 13 \pmod{14}$
- d. $\lambda = 14, \quad n \equiv 2, 4, 6, 10, 12 \pmod{14}$.

- for \vec{C}_7 -designs:

- e. $\lambda = 1, \quad n \equiv 0, 1 \pmod{7}$
- f. $\lambda = 7, \quad n \equiv 2, 3, 4, 5, 6 \pmod{7}$.

We shall use lemmas 2.2 and 2.3 and give a proof of all these cases in a convenient order.

a. Existence of an $(n, 7, 1)C_7$ -design for $n \equiv 1, 7 \pmod{14}$.

We still use the method exposed in 3.1:

Step 1. $K_7, K_{15}, K_{29}, K_{71}$ can be decomposed into C_7 .

- K_7 : see proposition 3.2.a.

- K_{15} : Let Z_{15} be the set of vertices of K_{15} . A decomposition into C_7 is given by the 15 following circuits:

$$(i, i+1, i+3, i+8, i+14, i+7, i+4, i), \quad 0 \leq i \leq 14.$$

- K_{29} : Let Z_{29} be the set of vertices of K_{29} . A decomposition into C_7 is given by the 58 following circuits:

$$(i, i+14, i+27, i+10, i+21, i+2, i+3, i), (i, i+2, i+6, i+11, i+5, i+12, i+20, i), \\ 0 \leq i \leq 28.$$

- K_{71} : Let Z_{71} be the set of vertices of K_{71} . A decomposition into C_7 is given by the following 5 x 71 circuits:

$$(i, i+4, i+9, i+15, i+47, i+7, i+50, i), (i, i+7, i+15, i+24, i+51, i+6, i+52, i) \\ (i, i+10, i+21, i+33, i+57, i+9, i+58, i), (i, i+14, i+29, i+45, i+62, i+9, i+51, i), \\ (i, i+1, i+3, i+6, i+39, i+5, i+41, i), \quad 0 \leq i \leq 70.$$

Step 2. $K_{7,7,7}, K_{14,14,14}, K_{14,14,28}, K_{7,7,21}$ can be decomposed into C_7 :

- $K_{7,7,7}, K_{14,14,14}$ (see proposition 3.3. a, d.).

- $K_{14,14,28}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{14,14,28}$ with

$$X = \{x_i\}_{0 \leq i \leq 13}, \quad Y = \{y_i\}_{0 \leq i \leq 13}, \quad Z = \{\infty_i\}_{0 \leq i \leq 13};$$

then a decomposition into C_7 is given by the following 10 x 14 circuits:

$$(x_i, y_{i+\alpha}, \infty_{\alpha+4}, x_{i+1}, \infty_{\alpha+12}, y_{i+\alpha+1}, \infty_{\alpha+20}, x_i), \quad 0 \leq i \leq 13, \quad 0 \leq \alpha \leq 7 \\ (x_{9+i}, y_{6+i}, \infty_0, x_{1+i}, y_1, \infty_{1+\varepsilon_1}, y_{1+7}, x_{1+9}) \quad 0 \leq i \leq 13; \quad \varepsilon_1 = 1 \text{ if } 7 \leq i \leq 13 \\ (x_{8+i}, y_{2+i}, \infty_3, x_i, \infty_{1+\varepsilon_1}, x_{7+i}, y_{3+i}, x_{8+i}), \quad 0 \leq i \leq 13; \quad \varepsilon_1 = 1 \text{ if } 7 \leq i \leq 13.$$

- $K_{7,7,21}$: Let $X \cup Y \cup Z$ be the set of vertices of $K_{7,7,21}$ with

$$X = \{x_i\}_{0 \leq i \leq 6}, \quad Y = \{y_i\}_{0 \leq i \leq 6}, \quad Z = \{\infty_i\}_{0 \leq i \leq 20}.$$

Then a decomposition into C_7 is given by the following 7 x 7 circuits

$$(x_i, y_{i+\alpha}, x_{i+1}, y_{\alpha+7}, y_{\alpha+1+i}, x_{\alpha+14}, x_i) \quad 0 \leq i \leq 6, \quad 0 \leq \alpha \leq 6.$$

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
42t+1	$t \geq 1$	2.5	$14(t-1)+14$	$14(t-1)+14$	7	-	2t	0
42t+15	$t \geq 1$	2.5	$14(t-1)+14$	$14(t-1)+28$	14	28	t	1
42t+21	$t \geq 0$	2.4	$14t+7$	$14t+7$	7	-	2t+1	0
42t+29	$t \geq 2$	2.5	$14(t-2)+28$	$14(t-2)+56$	14	28	t	2
42t+35	$t \geq 0$	2.4	$14t+7$	$14t+21$	7	21	t	1

The remaining case $n = 42t+7$ is solved for $t \geq 1$ by lemma 2.9 with $h = 6t+1$, $n_i = 7$ for $1 \leq i \leq h$.

e. Existence of an $(n, 7, 1) \vec{C}_7$ -design for $n \equiv 0, 1 \pmod{7}$.

By lemma 2.2 and case a, we know the existence for $n \equiv 1, 7 \pmod{14}$. It remains to prove it for $n \equiv 0, 8 \pmod{14}$:

Step 1. $K_7^*, K_8^*, K_{14}^*, K_{15}^*, K_{36}^*$ can be decomposed into \vec{C}_7 :

- K_7^*, K_{15}^* : case a + lemma 2.2.

- K_8^*, K_{14}^* : proposition 3.2.c.

- K_{36}^* is the edge disjoint union of three complete directed graphs isomorphic to $K_8^*, K_{15}^*, K_{15}^*$ and of a tripartite directed graph isomorphic to $K_{7,14,14}^*$ which can be decomposed into \vec{C}_7 by proposition 3.3.c.

Step 2. $K_{7,7,7}^*, K_{7,7,14}^*$ can be decomposed into \vec{C}_7 : see proposition 3.3.a + lemma 2.2 and proposition 3.3.b.

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
42t	$t \geq 1$	2.4	$14(t-1)+14$	$14(t-1)+14$	7	-	2t	0
42t+8	$t \geq 1$	2.5	$14(t-1)+14$	$14(t-1)+21$	7	14	2t	1
42t+14	$t \geq 1$	2.4	$14(t-1)+14$	$14(t-1)+28$	7	14	2t	2
42t+22	$t \geq 0$	2.5	$14t+7$	$14t+7$	7	-	2t+1	0
42t+28	$t \geq 0$	2.4	$14t+7$	$14t+14$	7	14	2t+1	1
42t+36	$t \geq 1$	2.5	$14(t-1)+21$	$14(t-1)+35$	7	14	2t+1	2

b. Existence of an $(n, 7, 2) C_7$ -design for $n \equiv 0, 8 \pmod{14}$.
It's a consequence of case e by use of lemma 2.3.

c. Existence of an $(n,7,7)C_7$ -design for $n \equiv 3,5,9,11,13 \pmod{14}$.

Step 1. $7K_7, 7K_9, 7K_{11}, 7K_{13}, 7K_{15}, 7K_{17}, 7K_{21}, 7K_{27}, 7K_{29}, 7K_{39}$ can be decomposed into C_7 :

- $7K_7, 7K_{15}, 7K_{21}, 7K_{29}$: case a + lemma 2.1.
- $7K_{19}$ is the edge disjoint union of three complete graphs isomorphic to $7K_7$ and of a tripartite graph isomorphic to $7K_{7,7,7}$ which can be decomposed into C_7 by proposition 3.4.
- $7K_{27}$ is the edge disjoint union of three complete graphs isomorphic to $7K_9$ and of a tripartite complete graph isomorphic to $7K_{9,9,9}$ which can be decomposed into C_7 by proposition 3.4.
- $7K_{39}$ is the edge disjoint union of three complete graphs isomorphic to $7K_{13}$ and of a tripartite complete graph isomorphic to $7K_{13,13,13}$ which can be decomposed into C_7 by proposition 3.4.
- $7K_9$: Let Z_9 be the set of vertices of $7K_9$. A decomposition is given by the following 4×9 circuits of length 7:

$$(i, i+1, i+2, i+3, i+4, i+5, i+6, i), (i, i+2, i+4, i+6, i+8, i+1, i+5, i)$$

$$(i, i+3, i+6, i+1, i+4, i+7, i+2, i), (i, i+4, i+8, i+5, i+1, i+7, i+2, i),$$

$$0 \leq i \leq 8.$$

- $7K_{11}$: Let Z_{11} be the set of vertices of $7K_{11}$. A decomposition is given by the following 5×11 circuits of length 7, $\{C_{\beta, i}\}$ for $1 \leq \beta \leq 5, 0 \leq i \leq 10$:

$$C_{\beta, i} = (\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6}$$

- $7K_{13}$: Let Z_{13} be the set of vertices of $7K_{13}$. A decomposition is given by the following 6×13 circuits $\{C_{\beta, i}\}$ for $1 \leq \beta \leq 6, 0 \leq i \leq 12$,

$$C_{\beta, i} = (\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6}$$

- $7K_{17}$: Let Z_{17} be the set of vertices of $7K_{17}$. A decomposition is given by the following 8×17 circuits of length 7, $\{C_{\beta, i}\}$ for $1 \leq \beta \leq 8$ and $0 \leq i \leq 16$,

$$C_{\beta, i} = (\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6}$$

Step 2. $7K_{4,4,4}$, $7K_{4,4,2}$ can be decomposed into C_7 .

- $7K_{4,4,4}$: see proposition 3.4.

- $7K_{4,4,2}$: Let $X \cup Y \cup Z$ be the set of vertices of $7K_{4,4,2}$ with

$$X = \{x_i\}_{0 \leq i \leq 3}, Y = \{y_i\}_{0 \leq i \leq 3}, Z = \{z_i\}_{0 \leq i \leq 1}.$$

A decomposition of $7K_{4,4,2}$ into C_7 is given by the following circuits:

$$\left. \begin{array}{l} (z_i, x_i, z_{i+1}, x_{i+1}, y_{i+3}, x_{i+3}, y_i, z_i) \\ (z_i, y_i, z_{i+1}, y_{i+1}, x_{i+3}, y_{i+2}, x_i, z_i) \quad 0 \leq i \leq 3 \\ (z_i, x_i, y_{i+1}, x_{i+1}, y_{i+2}, x_{i+2}, y_i, z_i) \\ (z_i, x_{i+1}, z_{i+1}, x_{i+2}, y_{i+1}, x_i, y_{i+3}, z_i) \\ (z_i, y_{i+1}, z_{i+1}, y_{i+2}, x_{i+3}, y_{i+3}, x_{i+1}, z_i) \end{array} \right\} \text{ taken twice } 0 \leq i \leq 3.$$

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
12t+1	t ≥ 2	2.5	4(t-2)+8	4(t-2)+8	4	-	k	0
12t+3	t ≥ 4	2.5	4(t-4)+10	4(t-4)+10	4	2	k+1	5
12t+5	t ≥ 3	2.5	4(t-3)+16	4(t-3)+8	4	2	k+1	4
12t+7	t ≥ 2	2.5	4(t-2)+12	4(t-2)+6	4	2	k+1	3
12t+9	t ≥ 2	2.5	4(t-2)+12	4(t-2)+8	4	2	k+1	2
12t+11	t ≥ 1	2.5	4(t-1)+8	4(t-1)+6	4	2	k+1	1

f. Existence of an $(n, 7, 7, \vec{C}_7)$ -design for $n \equiv 2, 3, 4, 5, 6 \pmod{7}$.

By lemma 2.2 and case c, we know the existence for $n \equiv 3, 5, 9, 11, 13 \pmod{14}$. It remains to prove it for $n \equiv 2, 4, 6, 10, 12 \pmod{14}$.

Step 1. $7K_8^*$, $7K_{10}^*$, $7K_{12}^*$, $7K_{14}^*$, $7K_{16}^*$, $7K_{18}^*$, $7K_{20}^*$, $7K_{22}^*$, $7K_{26}^*$, $7K_{28}^*$, $7K_{30}^*$, $7K_{38}^*$ can be decomposed:

- $7K_8^*$, $7K_{14}^*$, $7K_{22}^*$, $7K_{28}^*$: case a + lemma 2.1.

- $7K_{30}^*$ is the edge disjoint union of three complete directed graph isomorphic to $7K_{10}^*$ and of a tripartite complete directed graph isomorphic to $7K_{10,10,10}^*$ which can be decomposed by proposition 3.4.

- $7K_{10}^*$: Let $Z_9 \cup \{\infty\}$ be the set of vertices of $7K_{10}^*$. Then a decomposition is given by the following 9×10 directed circuits:

$$(\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5, 0 \leq i \leq 8, \beta = 2, 5, 7}$$

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6, 0 \leq i \leq 8, \beta = 1, 8, 4.}$$

$$(i, i+3, i+6, i+8, i+2, i+5, \infty, i), (i, i+6, i+3, i+4, i+2, i+8, \infty, i)$$

$$(i, i+7, i+5, i+2, i+8, i+3, \infty, i), (i, i+8, i+2, i+7, i+3, i+5, \infty, i), 0 \leq i \leq 8.$$

- $7K_{12}^*$: Let $Z_{11} \cup \{\infty\}$ be the set of vertices of $7K_{12}^*$. A decomposition of $7K_{12}^*$ into \vec{C}_7 is given by the following 12×11 directed circuits:

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } 0 \leq i \leq 10, \beta = 1, 2, 3, 4, 10}$$

$$(\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5 \text{ for } 0 \leq i \leq 10, \beta = 5, 6, 7, 8, 9.}$$

$$(i, 10+i, 8+i, 5+i, 1+i, 7+i, \infty, i), (i, 2+i, 5+i, 10+i, 6+i, 3+i, \infty, i), 0 \leq i \leq 10.$$

- $7K_{16}^*$: Let $Z_{15} \cup \{\infty\}$ be the set of vertices of $7K_{16}^*$. A decomposition is given by the following 16×15 directed circuits: for $0 \leq i \leq 14$

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } \beta = 1, 2, 4, 7, 8, 11, 13, 14,}$$

$$(i, i+3, i+8, i+14, i+9, i+6, \infty, i) \text{ taken 5 times,}$$

$$(i, i+1, i+3, i+7, i+14, i+13, i+6, i+9, i), (i, i+5, i+10, i+4, i+14, i+8, \infty, i),$$

$$(i, i+10, i+4, i+2, i+11, i+6, \infty, i).$$

- $7K_{18}^*$: Let $Z_{17} \cup \{\infty\}$ be the set of vertices of $7K_{18}^*$. A decomposition is given by the following 18×17 directed circuits: for $0 \leq i \leq 16$

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } 0 \leq \beta \leq 11; (\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5 \text{ for } 12 \leq \beta \leq 16}$$

$$(i, i+1, i+7, i+14, i+9, i+5, \infty, i), (i, i+16, i+14, i+11, i+7, i+2, \infty, i).$$

- $7K_{20}^*$: Let $Z_{19} \cup \{\infty\}$ be the set of vertices of $7K_{20}^*$. A decomposition is given by the following 20×19 directed circuits: for $0 \leq i \leq 18$,

$$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6 \text{ for } 1 \leq \beta \leq 13, (\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5 \text{ for } 14 \leq \beta \leq 18,}$$

$$(i, i+5, i+11, i+3, i+15, i+14, \infty, i), (i, i+4, i+10, i+7, i+5, i+4, \infty, i).$$

- $7K_{26}^*$: Let $Z_{25} \cup \{\infty\}$ be the set of vertices of $7K_{26}^*$. A decomposition is given by the following 26 x 25 directed circuits: for $0 \leq i \leq 24$

$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6}$, for $1 \leq \beta \leq 24$, $\beta \neq 5, 10, 15, 20, 21$

$(i, i+5, i+15, i+14, i+4, i+24, \infty, i)$: taken 7 times.

- $7K_{38}^*$: Let $Z_{37} \cup \{\infty\}$ be the set of vertices of $7K_{38}^*$. A decomposition of $7K_{38}^*$ is given by the following 38 x 37 directed circuits: for $0 \leq i \leq 36$

$(\dots, \alpha\beta+i, \dots, i)_{0 \leq \alpha \leq 6}$ for $1 \leq \beta \leq 31$, $(\dots, \alpha\beta+i, \dots, \infty, i)_{0 \leq \alpha \leq 5}$ for $32 \leq \beta \leq 36$,

$(i, i+36, i+34, i+31, i+27, i+22, \infty, i)$

$(i, i+6, i+18, i+36, i+23, i+16, \infty, i)$.

Step 2. $7K_{4,4,4}^*$, $7K_{4,4,2}^*$ can be decomposed into \vec{C}_7 : see step 2 of case c + lemma 2.2.

Step 3.

n		Lemma	$n_1=n_2$	n_3	a=b=c	c'	q	q-p
12t	$t \geq 2$	2.4	$4(t-2)+8$	$4(t-2)+8$	4	-	k	0
12t+2	$t \geq 4$	2.4	$4(t-4)+20$	$4(t-4)+10$	4	2	k+1	5
12t+4	$t \geq 3$	2.4	$4(t-3)+16$	$4(t-3)+8$	4	2	k+1	4
12t+6	$t \geq 3$	2.4	$4(t-3)+16$	$4(t-3)+10$	4	2	k+1	3
12t+8	$t \geq 2$	2.4	$4(t-2)+12$	$4(t-2)+8$	4	2	k+1	2
12t+10	$t \geq 2$	2.4	$4(t-2)+12$	$4(t-2)+10$	4	2	k+1	1

Case k = 3.

It is well known that an $(n, 3, \lambda)C_3$ -design (or $(n, 3, \lambda)B.I.B.D.$) exists if and only if $\lambda n(n-1) \equiv 0 \pmod{6}$ and $\lambda(n-1) \equiv 0 \pmod{2}$ (see /10/).

It has been proved in /2, 3, 8, 14/ that:

3.7. Theorem. An $(n, 3, 1)\vec{C}_3$ -design exists if and only if $n \neq 6$, and $n \equiv 0$ or $1 \pmod{3}$.

For $\lambda \geq 2$ we have the theorem:

3.8. Theorem. For $\lambda \geq 2$, an $(n, 3, \lambda)C_3$ -design exists if and only if $\lambda n(n-1) \equiv 0 \pmod{3}$.

Proof. By theorem 3.7 and lemma 2.1 it suffices to verify the theorem for $\lambda = 3$, $n \equiv 2 \pmod{3}$ and for $\lambda = 2, 3$ and $n = 6$. Lemma 5.3 of /10/ implies that for $n \geq 3$ the edges of K_n can be partitioned into K_3, K_4, K_5, K_6 and K_8 . Thus it suffices to prove the existence of a decomposition for $3K_n^*$ $n = 5, 6, 8$ and $2K_6^*$. For $n = 5$, it follows from the existence of an $(5, 3, 1)C_3$ -design.

For $n = 6$ let the vertices be $Z_5 \cup \{\infty\}$: a decomposition of $2K_6^*$ is given by: $(\infty, i, i+1); (\infty, i+1, i); (i, i+2, i+4); (i+4, i+2, i)$ with $0 \leq i \leq 4$ and for $3K_6^*$ by $(\infty, i, i+1)$ twice and once $(i, i+2, i+4), (i+4, i+2, i), (i+2, i+1, i)$ with $0 \leq i \leq 4$. For $n = 8$ let the vertices be $Z_7 \cup \{\infty\}$ then a decomposition is given by: 3 times the circuits $(i, i+1, i+3)$ twice $(i, i+3, i+1)$ and once $(\infty, i, i+3); (\infty, i+3, i+1); (\infty, i+1, i)$ where $0 \leq i \leq 6$.

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