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Jean-Claude Bermond

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On Hamiltonian Walks

J.C. BERMOND

C.M.S., 54 Bd. Raspail, 75006-Paris.

Abstract: If G is a connected graph, of order n , a hamiltonian walk of G is a closed walk of minimum length which contains every vertex of G . The length of a hamiltonian walk is greater than or equal to n , with equality if and only if G is hamiltonian. Following J.L. Jolivet, we define $s(G)$ to be the difference of the length of a hamiltonian walk in G and n . Thus $s(G) \geq 0$ and $s(G) = 0$ if and only if G is hamiltonian. We prove the following generalizations of Ore's and Pósa's theorems (thus answering conjectures of J.L. Jolivet):

Theorem: Let $c \leq n$; if for every pair of non-adjacent vertices x and y , $d(x) + d(y) \geq c$, then $s(G) \leq n - c$.

Theorem: Let $c \leq n$; if:

$$|\{x : d(x) \leq j\}| \leq j - 1 \quad \text{for } 0 \leq j \leq (c-1)/2$$

$$|\{x : d(x) \leq (c-1)/2\}| \leq (c-1)/2 \quad (\text{if } c \text{ is odd})$$

then $s(G) \leq n - c$.

In order to prove these theorems we need results concerning the existence of elementary cycles of length at least c in connected and 2-connected graphs.

1. INTRODUCTION

We shall follow the terminology of Berge's book [1], in which the reader can find all the definitions not given here.

G will always denote a *connected graph of order n* .

1.1. Recall that a hamiltonian cycle (path) of G is an elementary cycle (path) of length n , that is, a cycle (path), which contains every vertex of G exactly once. Not all graphs are hamiltonian, and several weaker conditions can be considered.

1.2. One of the simplest conditions concerns the existence of a cycle of length c or of length at least c . We shall return to this problem in §3.

Another possible generalization of a hamiltonian path is a tree with at most h end vertices: existence conditions have been obtained in this case by Las Vergnas [12] and the author [2].

1.3. Here we are interested in another generalization introduced by Goodman and Hedetniemi [8] and Jolivet [10,11]. We define a spanning closed walk of a connected graph G to be a closed walk (called sometimes cycle), which contains every vertex of G . (This is called in [10] a pseudo-hamiltonian cycle.) The length of a spanning closed walk is greater than or equal to n ; if the length is n it is a hamiltonian cycle.

Exactly we are interested in spanning closed walks of minimum length $h(G)$, which we call hamiltonian walks such as in [8]. Following Jolivet [10,11] we define for a connected graph G , $s(G)$ to be the least integer k such that there exists a spanning closed walk of length $n+k$, that is $s(G) = h(G) - n$. Thus $s(G) \geq 0$ and $s(G) = 0$ if and only if G is hamiltonian. For example it is proved in [10], that for a tree T , $s(T) = n-2$, and thus $s(G) \leq n-2$ for every graph G .

1.4. Remark: The parameter $s(G)$ gives a measure of the hamiltonicity of a graph. Another parameter considered in [4,9,10,14,15] is the minimum number of edges, which must be added to G in order to make it hamiltonian. If G is not hamiltonian this number is equal to the minimum number of vertex-disjoint paths covering the vertices of G .

1.5. The following lemmas of [10,11] give a relation between the existence of elementary cycles and the parameter $s(G)$ and will be very useful.

1.5.1. If G contains an elementary cycle of length c , then $s(G) \leq n-c$.

1.5.2. Lemma: If G contains two elementary cycles of length c and c' , having at most one vertex in common, then $s(G) \leq n+2-(c+c')$.

Lemma 1.5.2. was proved in [11] for two disjoint cycles, but a similar proof shows that if the two cycles have one vertex in common, then $s(G) \leq n+1-(c+c')$.

1.6. The aim of this paper is to prove for hamiltonian walks generalizations of Ore's and Posa's theorems, thus answering conjectures of Jolivet [10,11]. We first recall some results on the existence of hamiltonian cycles; then we give sufficient conditions for the existence of elementary cycles of length at least c for a connected and a 2-connected graph. These are used to prove the theorems.

The results given here are part of the author's thesis [3] and we shall generally indicate only sketches of the proofs.

1.7. Notations: In what follows cycle will always mean elementary cycle; E will denote the set of edges of G ; $d(x)$ means the degree of the vertex x ; $\{x_1, x_2, \dots, x_n\}$ is an arbitrary ordering of the vertices of G ; the sequence of the degrees of the vertices in a non-decreasing order will be denoted by d_1, d_2, \dots, d_n .

2. HAMILTONIAN CYCLES

The following results are known. The reader can find proofs, references and more results in Bondy and Chvátal [6].

G has an hamiltonian cycle if:

2.1. Dirac: $d(x) \geq n/2$ for every vertex x .

2.2. Ore: $d(x) + d(y) \geq n$ if $\{x, y\} \notin E$.

2.3. Posa: $|\{x : d(x) \leq j\}| < j$ for $j < (n-1)/2$ and

$$|\{x : d(x) \leq (n-1)/2\}| \leq (n-1)/2 \quad \text{for } n \text{ odd.}$$

2.4. Bondy: $j < k$, $d_j \leq j$, $d_k \leq k-1 \Rightarrow d_j + d_k \geq n$.

2.5. Chvátal: $d_j \leq j < n/2 \Rightarrow d_{n-j} \geq n-j$.

2.6. Las Vergnas: $j < k$, $k \geq n-j$, $\{x_j, x_k\} \notin E$

$$\left. \begin{array}{l} d(x_j) \leq j, \quad d(x_k) \leq k-1 \\ \Rightarrow d(x_j) + d(x_k) \geq n. \end{array} \right\}$$

3. ELEMENTARY CYCLES OF LENGTH AT LEAST C.

Case 1 : G connected.

3.1. Theorem : Let $\{x_1, x_2, \dots, x_n\}$ be any ordering of the vertices of G and let $c \leq n$. Suppose that :

$$j < k, \{x_j, x_k\} \notin E \quad \left. \vphantom{j < k} \right\} \Rightarrow \max \{d(x_j); d(x_k)\} \geq c.$$

$$d(x_j) \leq j, d(x_k) \leq k-1$$

Then G contains a cycle of length at least $c+1$.

Proof : Suppose the theorem false for some value of n and let $G = (X, E)$ be a graph with n vertices, which satisfies the hypothesis, but not the conclusion, and has the maximum possible number of edges. Clearly G is not complete. Let x_j and x_k be two non adjacent vertices of G with $j < k$, and such that : i) there exists between x_j and x_k a path having the maximum possible length p.

ii) $j+k$ is as large as possible subject to i).

Then the graph obtained by adding to G the edge $\{x_j, x_k\}$ satisfies the hypothesis of the theorem and by the maximality of E, contains a cycle of length at least $c+1$. Thus the length p of the path between x_j and x_k satisfies $p \geq c$. Let $P(x_j, x_k) = \{x_j, y_1, y_2, \dots, y_{p-1}, x_k\}$ be the path between x_j and x_k and let :

$$J = \{y_1 : y_1 \in P(x_j, x_k) \text{ and } \{x_j, y_{i+1}\} \in E\}$$

$$K = \{y_i : y_i \in P(x_j, x_k) \text{ and } \{y_{i-1}, x_k\} \in E\}$$

We have $|J| = d(x_j)$; indeed, x_j cannot be joined to a vertex z not in $P(x_j, x_k)$; (otherwise we would have between z and x_k a path of length greater than p. Also $\{z, x_k\} \notin E$, otherwise we would have a cycle of length greater than $p+1 \geq c+1$, contradicting i)). Furthermore if $y_i \in J$, $\{y_{i-1}, y_i, \dots, y_{i+1}, \dots, y_{p-1}, x_k\}$ is a path of length p; thus $\{y_i, x_k\} \notin E$ (otherwise we would have a cycle of length $p+1 \geq c+1$), and therefore by the maximality of $j+k$ (condition ii)) the label of y_i is at most j. Then we have $|J|$ points with labels at most j. Thus $|J| = d(x_j) \leq j$. Similarly $|K| = d(x_k)$ and the points of K and x_j have labels at most k. Hence $|K| + 1 = d(x_k) + 1 \leq k$. Thus the conditions of the hypothesis are satisfied and either $d(x_j)$ or $d(x_k)$ is at least c. Suppose, without loss of generality, that $d(x_j) \geq c$; since x_j is only joined to vertices in $P(x_j, x_k)$, we obtain a cycle of length at least $c+1$, contrary to our assumption. \square

3.2. Corollaries : Let $c \leq n$. Then G contains a cycle of length at least $c+1$ if any one of the following conditions is satisfied :

- Posa [13] : $|\{x : d(x) \leq j\}| \leq j$ for $0 \leq j \leq c-1$ ($c \geq 2$).
- $j < k, \{x_j, x_k\} \notin E$ $\left. \vphantom{j < k} \right\} \Rightarrow d(x_j) + d(x_k) \geq \min \{2c-1; n\}$.
 $d(x_j) \leq j, d(x_k) \leq k-1$
- Woodall [16] : $d(x) + d(y) \geq \min \{2c-1; n\}$ if $\{x, y\} \notin E$.

Proofs : In order to prove b) and c), we need the results on hamiltonian cycles (2.6. and 2.2.), when $2c-1 \geq n$.

3.3. Conjectures : We conjecture that each of the following conditions is sufficient to ensure the existence of a cycle of length at least $c+1$.

- $j < k, k \geq \min \{2c-1; n\} - j$ $\left. \vphantom{j < k} \right\} \Rightarrow \max \{d(x_j); d(x_k)\} \geq c$.
 $\{x_j, x_k\} \notin E, d(x_j) \leq j, d(x_k) \leq k-1$
- $j < k, \{x_j, x_k\} \notin E$ $\left. \vphantom{j < k} \right\} \Rightarrow d(x_j) + d(x_k) \geq \min \{2c-1; n\}$.
 $d(x_j) \leq j-1, d(x_k) \leq k-2$

3.4. Remark . Other generalizations are possible ; some of them are false : for example we cannot replace in a) the condition

$$k \geq \min \{2c-1; n\} - j \text{ by } k \geq n-j \text{ or the conditions } d(x_j) \leq j \text{ and } d(x_k) \leq k-1 \text{ by } d(x_j) \leq j-n+c+1 \text{ or } d(x_k) \leq k-n+c. \text{ (see [3])}$$

Case 2 : G 2-connected.

3.5. Theorem : Let G be a 2-connected graph and $c \leq n$. Suppose that :

$$j < k, \{x_j, x_k\} \notin E \quad \left. \vphantom{j < k} \right\} \Rightarrow d(x_j) + d(x_k) \geq c.$$

$$d(x_j) \leq j, d(x_k) \leq k-1$$

Then G contains a cycle of length at least c.

Proof : The proof is very similar to Bondy's proof in [5] of the result given below (2.6.a) and thus we omit it. (For a complete proof see [3]).

3.6. Corollaries : Let G be 2-connected and $c \leq n$. Then G contains a cycle of length at least c if any one of the following conditions is satisfied :

- Bondy [5] : $d_j \leq j, d_k \leq k$ ($j \neq k$) $\Rightarrow d_j + d_k \geq c$.
- Posa [13] : $|\{x : d(x) \leq j\}| \leq j-1$ for $1 \leq j \leq \lfloor (c-1)/2 \rfloor$
- $d(x) + d(y) \geq c$ if $\{x, y\} \notin E$.
- Dirac [7] : $d(x) \geq \lfloor (c+1)/2 \rfloor$ for every vertex x.

3.7. Remark : In contrast with the hamiltonian case, when $c \leq n$, there exist graphs satisfying the hypothesis of 2.6.c) but not 2.6.b) : for example consider the graph containing a complete graph on j vertices , a complete graph on $n-j$ vertices and a matching of cardinality j between these two subgraphs , with $j \leq [(c-1)/2]$.



Example : $j = 2, c = 5, n = 6$

3.8. Conjecture : Let G be a 2-connected graph and let $c \leq n$. We conjecture that the following condition is sufficient for the existence of a cycle of length at least c :

$$\left. \begin{array}{l} j < k, k \geq c-j, \{x_j, x_k\} \notin E \\ d(x_j) \leq j, d(x_k) \leq k-1 \end{array} \right\} \Rightarrow d(x_j) + d(x_k) \geq c.$$

3.9. Remark : The reader can find other results and conjectures in Woodall [16,17] and in [3].

Case 3 : We shall give a refinement of theorem 3.1., which will be very useful in the proofs of theorems 4.3. and 4.4.

3.10. Theorem : Let G' denote a 2-connected graph of order $n+1$ and let G be the graph obtained by deleting a vertex of G' and let $c \leq n$. The degrees being the degrees in the graph G ; G' contains a cycle of length at least $c+2$ if any one of the following conditions is satisfied :

- a) $j < k, \{x_j, x_k\} \notin E$
- b) $|\{x : d(x) \leq j\}| \leq j$ for $0 \leq j \leq c+1$.
- c) $d(x) + d(y) \geq 2c-1$ if $\{x, y\} \notin E$.

Proof : Cases b and c are corollaries of case a). The beginning of the proof of the case a) is the same as the proof of theorem 3.1. Suppose the theorem false for some value of n and let $G' = (X \cup a, E')$ be a 2-connected graph of order $n+1$, which satisfies the condition a) , but has no cycle of length at least $c+2$ and has the maximum possible number of edges. Let G be the subgraph obtained by deleting the vertex a . G is not complete ; otherwise as G' is 2-connected, a is joined to at least two vertices of G and then G' would contain an hamiltonian cycle of length $n+1 \geq c+2$.

Let x_j and x_k be two non adjacent vertices defined as in the proof of theorem 3.1. Similarly as in the proof of this theorem we can show that $\max\{d(x_j), d(x_k)\} \geq c$. If $\max\{d(x_j), d(x_k)\} \geq c+1$, we obtain a cycle of length at least $c+2$. Thus we can suppose without loss of generality that $d(x_j) = c$. Furthermore x_j must be joined to the vertices y_i of the path $P(x_j, x_k)$ such that $2 \leq i \leq c+1$, otherwise we would have a cycle of length at least $c+2$. As G' is 2-connected, there exist in G' two vertex-disjoint paths joining the two set of vertices :

$\{y_i : 1 \leq i \leq c\}$ and $\{y_i : c+2 \leq i \leq p\}$. (Remark that , as $y_p = x_k$ is non adjacent to $y_1 = x_j, y_p$ is different from x_{c+1} and thus the two sets are non empty) . One of these paths does not contain the vertex y_{c+1} . Let $P'(y_r, y_s)$ be the part of this path, where the end vertices y_r and y_s satisfy $1 \leq r \leq c$ and $c+2 \leq s \leq p$ and where the internal vertices do not belong to $P(x_j, x_k)$. Then $\{y_1, y_2, \dots, y_{r-1}, P'(y_r, y_s), y_{s-1}, \dots, y_{r+1}, y_1\}$ is a cycle of length greater than or equal to $c+2$, contrary to our assumption. □

4. EXISTENCE OF HAMILTONIAN WALKS.

We shall prove the generalizations of Ore's and Posa's theorems (2.2. and 2.3.). The generalization of Dirac's theorem has been obtained by Jolivet [10,11] and is a corollary of our results.

When G is 2-connected, lemma 1.5.1. and theorem 3.5. give :

$$4.1. Theorem : \left. \begin{array}{l} j < k, \{x_j, x_k\} \notin E \\ d(x_j) \leq j, d(x_k) \leq k-1 \end{array} \right\} \Rightarrow d(x_j) + d(x_k) \geq c ;$$

then $s(G) \leq n - c$.

4.2. Definitions : A block of a graph G is a maximal 2-connected subgraph of G . A cut-vertex in a connected graph is a vertex whose deletion disconnect G . An end-block is a block containing exactly one cut-vertex . It is known that, if G is not 2-connected, then G contains at least two end-blocks.

4.3. Theorem : Let $c \leq n$. If :

$$\begin{aligned} |\{x : d(x) \leq j\}| &\leq j-1 \text{ for } 0 \leq j \leq (c-1)/2 \text{ and} \\ |\{x : d(x) \leq (c-1)/2\}| &\leq (c-1)/2 \text{ (for } c \text{ odd)} ; \\ \text{then } s(G) &\leq n - c . \end{aligned}$$

Proof : If G is 2-connected the theorem is a corollary of theorem 4.1. .

Thus we suppose that G is not 2-connected and we distinguish 2 cases.

Case 1 : $c = 2p$.

i) Every end-block of G contains at least $p+1$ vertices : Indeed let B be an end-block and let b be the only cut-vertex of G in B . Suppose that $|B| \leq p$; $|B| = j+1$ with $j \leq p-1$. If $x \in B-b$, then x is only joined to the vertices of B (B is an end-block) ; thus $d(x) = d_B(x) \leq j$ and as $|B-b| = j$ we have $|\{x : d(x) \leq j\}| \geq j$ for $j \leq p-1 \leftarrow (c-1)/2$ contrary to the hypothesis.

ii) Every end-block contains a cycle of length at least $p+1$: indeed if $x \in B-b$, $d_{B-b}(x) \geq d(x) - 1$. Therefore the subgraph generated by $B-b$ satisfies by hypothesis the condition : $|\{x : d_{B-b}(x) \leq j\}| \leq j$ for $0 \leq j \leq p-2$; then theorem 3.10.b) with $G' = B$, $G = B-b$, and $c = p-1$, implies the existence in B of a cycle of length at least $p+1$.

As G contains two end-blocks with at most one vertex in common, G contains two cycles of length at least $p+1$ having at most one vertex in common and therefore by lemma 1.5.2. $s(G) \leq n+2-2(p+1) = n-c$.

Case 2 : $c = 2p - 1$.

Similarly as in case 1, it can be shown that each end-block contains p vertices. As G satisfies the hypothesis of corollary 3.2.a) with $c = p$, G contains a cycle C of length at least $p+1$. C is included in a block. Let B be an end-block which do not contain C . As $d_B(x) = d(x)$ for every vertex of $B-b$ we have : $|\{x : d(x) \leq j\}| \leq j$ for $0 \leq j \leq p-2$ and thus by 3.2.a) B contains a cycle of length at least p . Therefore G contains two cycles having at most one vertex in common of length $d \geq p+1$ and $d' \geq p$ and by lemma 1.5.2. $s(G) \leq n + 1 - 2p = n - c$. \square

4.4. Theorem : Let $c \leq n$. If $d(x) + d(y) \geq c$ for every pair of non adjacent vertices x and y , then $s(G) \leq n - c$.

Proof : If G is 2-connected the theorem is a corollary of theorem 4.1. . Thus suppose that G is not 2-connected and that the theorem is false and

let c be the smallest value for which the theorem is false . Among the graphs , which do not verify the theorem for c , let G be a graph with the minimum possible number of vertices n . As the theorem is true for $n = c$ (2.2.) we have $n > c$.

Every end-block of G is non hamiltonian : otherwise suppose that G contains a hamiltonian end-block B and let b be the cut vertex of B and let $P = B-b$. First $|B| < c$, otherwise G would contain a cycle of length at least c and thus $s(G) \leq n-c$. We claim that $s(G) \leq s(G-P) + 1$; indeed let C be a hamiltonian walk of $G-P$ of length $|G-P| + s(G-P)$ and let C' be a hamiltonian cycle of B ; then $C \cup C'$ is a closed spanning walk of G of length $|G| + 1 + s(G-P)$ and therefore $s(G) \leq s(G-P) + 1$. Now let $|P| = k$ and thus $|B| = k+1 < c$. We have $d(x) = d_B(x) \leq k$ for every vertex x of P . As B is an end-block, if $x \in P$ and $y \in G-B$, x and y are not adjacent and then, by hypothesis, $d(x) + d(y) \geq c$. Thus :

$$d_{G-P}(y) = d(y) \geq c-k . \text{ We have also } d_{G-P}(b) \geq 1 . \text{ As } c > k+1 ,$$

$2(c-k) > c+1-k$ and therefore $d_{G-P}(y) + d_{G-P}(z) \geq c+1-k$ for every pair of vertices y and z in $G-P$. The subgraph generated by $G-P$ satisfies the hypothesis of the theorem for $c' = c+1-k < c$ and $n' = |G-P| = n-k < n$. By the choice of c and n (minimality), we have $s(G-P) \leq n'-c' \leq n-c-1$. From $s(G) \leq s(G-P) + 1$ we deduce $s(G) \leq n-c$ contrary to our assumption .

Let B be an end-block ; $P = B-b$ is not complete, otherwise B would be hamiltonian ; thus there exist two non adjacent vertices x and y , which satisfy (by hypothesis) $d(x) + d(y) \geq c$. If $x \in P$, $d_P(x) \geq d(x) - 1$ and therefore $d_P(x) + d_P(y) \geq c-2$. Let $p = \lfloor (c+1)/2 \rfloor$. We have $|P| > p+1$ otherwise , if $|P| \leq p$, $d_P(x) \leq p-2$ and $d_P(y) \leq p-2$ (x and y are not adjacent) and thus $d_P(x) + d_P(y) \leq 2p-4 < c-2$. Therefore the subgraph generated by P satisfies the hypothesis of theorem 3.10.c) with $G = P$, $G' = B$, and $c = p-1$ and thus contains a cycle of length at least $p+1$. As G contains two end-blocks, G contains two cycles of length at least $p+1$, having at most one vertex in common and therefore $s(G) \leq n-c$. (Remark that the minimality of c and n is used only to deal with the case where B is an end block such that $P = B-b$ is a complete subgraph with $|P| < p$.) \square

4.5. *Remark* : The conditions of theorems 4.3. and 4.4. are independent. As we have seen in remark 3.7, there exist graphs satisfying the hypothesis of 4.4. and not of 4.3. . It is known that for $c = n$ there exist graphs, which satisfy the hypothesis of 4.3. and not of 4.4. ; it is possible to exhibit such graphs for $5 < c < n$: for example take a complete graph of order $n-2$ and add two non adjacent vertices of degree 2 and 3 respectively .

Thus it will be interesting to prove the following conjecture (a proof similar to that of theorem 3.3. shows that this conjecture is a consequence of conjecture 3.3.b) .)

4.6. *Conjecture* : Let $c \leq n$ and suppose that

$$\left. \begin{array}{l} j < k, \{x_j, x_k\} \notin E \\ d(x_j) \leq j, d(x_k) \leq k-1 \end{array} \right\} \Rightarrow d(x_j) + d(x_k) \geq c .$$

Then $s(G) \leq n - c$.

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