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G-DECOMPOSITION OF K_n , WHERE G HAS FOUR VERTICES OR LESS

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The complete graph K_n is said to have a G -decomposition if it is the union of edge disjoint subgraphs each isomorphic to G . The set of values of n for which K_n has a G -decomposition is determined if G has four vertices or less.

1. Introduction

The complete graph K_n is said to have a G -decomposition if it is the union of edge disjoint subgraphs each isomorphic to G . The G -decomposition problem is to determine the set of naturals $N(G)$, such that K_n has a G -decomposition if and only if $n \in N(G)$.

G -decomposition problems, with $G = K_r$ have been raised and solved, for some values of r , long time ago, as existence problems of Block Designs. A review of results may be seen in [7]. Only relatively recently other graphs than K_r have been considered and some results appeared as existence problems for G -designs, generalizing Block Designs [8]. A survey of results in this direction and a description of general methods used may be found in [1] and [2].

As said in the title, this paper accomplishes the solution of the problem for small graphs G , namely having four or less vertices. Observe that from the point of view of the problem, isolated points of G are irrelevant, consequently we will consider only graphs G without isolated vertices.

There are only three such graphs with less than four vertices, namely: the graph consisting in a single edge $G = e$, the path of length two $G = P_3$ and the triangle $G = K_3$.

The solutions of the corresponding G -decomposition problems are as follows: Trivially $N(e)$ is the set of all naturals, it is known and also easy to see that $N(P_3) = \{n \mid n \equiv 0 \text{ or } 1 \pmod{4}\}$; the set $N(K_3)$ has been determined in the classical formulation involving Steiner Triple Systems to be

$$N(K_3) = \{n \mid n \equiv 1 \text{ or } 3 \pmod{6}\}.$$

It can be easily seen that for any graph G having at most n vertices, the following two conditions are necessary for the existence of a G -decomposition of K_n :

- (i) $\frac{1}{2}n(n-1)$ is a multiple of the number of edges of G .
- (ii) $n-1$ is a multiple of the greatest common divisor of the degrees of the vertices of G .

The real problem is to establish sufficiency. Recent Wilson type results [4, 11] show that asymptotically, sufficiency always holds, hence researches like this paper may be regarded as attempts to determine the exceptions in each case.

We will see that when G has four vertices or less there are only three exceptions namely $n=3$ and $n=4$ if G is a star with three edges and $n=5$ if G is a quadrilateral with one diagonal.

A more general problem is to consider G -decompositions of other graphs than K_n . Although this is not the topic of the present paper, we will mention such decompositions for some complete tripartite graphs, since they will be used in the proof of our main result.

2. Graphs G_4 with four vertices and review of known G_4 -decompositions

There are seven nonisomorphic graphs with four nonisolated vertices:

- (i) The complete graph K_4 .
- (ii) The quadrilateral with one diagonal Q .
- (iii) The circuit of length four C_4 .
- (iv) The triangle with attached edge D .
- (v) The path of length three P_4 .
- (vi) The star having three branches S_3 .
- (vii) Two edges having no common vertex E_2 .

The solution of the corresponding problem is well known for K_4 and C_4 , namely Hanani [6] proved

$$N(K_4) = \{n \mid n \equiv 1 \text{ or } 4 \pmod{12}\}$$

and Kotzig [10] proved

$$N(C_4) = \{n \mid n \equiv 1 \pmod{8}\}.$$

It is easy to see that

$$N(E_2) = \{n \mid n \equiv 0 \text{ or } 1 \pmod{4}\}.$$

More recently $N(S_3)$ and $N(P_4)$ have been determined by P. Cain [3] respectively by one of us [1] (see also [2]) to be

$$N(S_3) = \{n \mid n \equiv 0 \text{ or } 1 \pmod{3}; 3 \neq n \neq 4\}$$

and

$$N(P_4) = \{n \mid n \equiv 0 \text{ or } 1 \pmod{3}\}.$$

A more complete list of references is given in [2], in addition mention also [12].
The remaining two cases will be the object of the following two sections.

3. On the set $N(Q)$

Let Q be as in Section 2 a quadrilateral with one diagonal. A quadrilateral with sides ab, bc, cd, da , will be denoted by $(abcd)$ and the same quadrilateral with diagonal ac respectively bd will be denoted $(\overline{abcd}), (\overline{abcd})$. Commas will be used in some cases in order to avoid confusion.

The main result of this paper is to establish the following theorem.

Theorem 3.1. $N(Q) = \{n \mid n \equiv 0 \text{ or } 1 \pmod{5}, n \neq 5\}$. (1)

The following Lemmas 3.2–3.9 describe some direct constructions, while Lemmas 3.10–3.11 and corollary contain composition methods used in the proof.

Notation. Let the vertex set of K_n be Z_n , unless otherwise specified, in particular, all addition of vertex labels are done mod n . Let $\{a, b, c, d\}$ be a subset of Z_n . Denote the set of graphs $(a + j, b + j, c + j, d + j), j \in Z_n$ by $(a, b, c, d)(\text{mod } n)$. Denote further the least integer greater than or equal to $k/2$ by $\{k/2\}$ and the greatest integer less than or equal to $k/2$ by $[k/2]$.

Lemma 3.2. *If $n = 10k + 1$ then K_n has a Q decomposition. For $k \geq 4$ the following $k(10k + 1)$ graphs form a Q decomposition of K_n :*

$$\left. \begin{array}{l} (\overline{1, 1 + 2i, 2k + 2, 4k + i + 1}), i = 3, 4, \dots, [k/2] \\ (\overline{1, 2k + 3 - 2i, 2k + 2, 5k + 2 - i}), i = 1, 2, \dots, \{k/2\} \\ (\overline{1, 3, 4k + 1, 6k}) \\ (\overline{1, 5, 2k + 2, 6k + 1}) \end{array} \right\} (\text{mod } 10k + 1)$$

for $k = 1, 2$ and 3 the decompositions are respectively:

$$\begin{array}{l} (\overline{1, 2, 10, 5}) (\text{mod } 11) \\ \\ \left. \begin{array}{l} (\overline{1, 3, 11, 15}) \\ (\overline{1, 2, 7, 10}) \end{array} \right\} (\text{mod } 21) \\ \\ \left. \begin{array}{l} (\overline{1, 2, 5, 13}) \\ (\overline{1, 10, 27, 11}) \\ (\overline{1, 3, 7, 14}) \end{array} \right\} (\text{mod } 31) \end{array}$$

Lemma 3.3. If $n = 10k + 6$, the following $k(10k + 6) + 5k + 3$ graphs form a Q -decomposition of K_n :

$$\left. \begin{array}{l} (1, \overbrace{2i, 2+2k, 4k+3+i}, i = 1, 2, \dots, \{k/2\}) \\ (1, \overbrace{2i + 2\{k/2\}, 2+2k, 4k+3+\{k/2\}+i}, i = 1, 2, \dots, [k/2]) \\ (1 + j, \overbrace{2+2k+j, 5k+4+j, 7k+5+j}, j = 0, 1, \dots, 5k+2. \end{array} \right\} \pmod{10k+6}$$

Proof of Lemmas 3.2 and 3.3. The above constructions are essentially an application of Bose's difference method. For Lemma 3.2 each edge $\{x, y\}$ of K_n occurs in some graph of the decomposition since one of the differences $x - y$ or $y - x$ occurs in one of the k graphs listed above, the unicity follows from the fact that the total number of edges in the decomposition is precisely the number of edges of K_n . For Lemma 3.3 the proof is similar but a little more complicated, see all details in [2].

Lemma 3.4. K_{10} has a Q -decomposition.

Proof. The decomposition is

$$\begin{array}{ll} \overbrace{1325} & \overbrace{0291} \\ \overbrace{1624} & \overbrace{0493} \\ \overbrace{8574} & \overbrace{0695} \\ \overbrace{7386} & \overbrace{7980} \\ \overbrace{1827} & \end{array}$$

Lemma 3.5. K_{15} has a Q -decomposition.

Proof. The decomposition is:

$$\begin{array}{l} (\overbrace{1\ 2\ 4\ 5}) (\overbrace{1\ 7\ 8\ 9}) (\overbrace{1\ 10\ 11\ 12}) (\overbrace{1\ 13\ 14\ 15}) \\ (\overbrace{2\ 3\ 5\ 6}) (\overbrace{2\ 7\ 10\ 13}) (\overbrace{2\ 8\ 11\ 14}) (\overbrace{2\ 9\ 12\ 15}) \\ (\overbrace{3\ 4\ 6\ 1}) (\overbrace{3\ 7\ 9\ 14}) (\overbrace{3\ 8\ 12\ 10}) (\overbrace{3\ 11\ 15\ 13}) \\ (\overbrace{4\ 7\ 14\ 12}) (\overbrace{4\ 8\ 15\ 10}) (\overbrace{4\ 9\ 13\ 11}) \\ (\overbrace{5\ 8\ 13\ 12}) (\overbrace{5\ 9\ 10\ 14}) (\overbrace{5\ 11\ 7\ 15}) \\ (\overbrace{6\ 10\ 8\ 14}) (\overbrace{6\ 11\ 9\ 15}) (\overbrace{6\ 12\ 7\ 13}) \end{array}$$

Lemma 3.6. K_{20} has a Q -decomposition.

Proof. Let the vertex set of K_{20} be $Z_{10} \times \{0, 1\}$. Then the following 38 graphs form a Q -decomposition of K_{20} .

$(n, 0)(m, 0)(n, 1)(m, 1)$	with	m	1	2	3	4	5	6	7	8	9	0
		n	0	5	9	3	0	0	4	5	0	2

$(c, 0)(a, 0)(c, 1)(b, 1)$		a	1	1	3	4	5	6	7
$(c, 1)(a, 1)(c, 0)(b, 0)$		b	2	5	7	8	6	8	9
$(d, 0)(a, 0)(d, 1)(b, 0)$	with	c	3	7	8	9	3	1	2
$(d, 1)(a, 1)(d, 0)(b, 1)$		d	4	9	0	0	4	2	6

Lemma 3.7. K_{25} has a Q -decomposition.

Proof. Let the vertex set of K_{25} be $\{\infty\} \cup \mathbb{Z}_{12} \times \{0, 1\}$. Then the following 60 graphs are a Q -decomposition of K_{25} .

$$\left. \begin{array}{l} (j, 0)(j, 1)(j+6, 0)(j+6, 1) \\ (j+1, 1)(j, 0)(j+7, 1)(j+6, 0) \end{array} \right\} \text{with } j = 0, 1, \dots, 5$$

$$\left. \begin{array}{l} (i, 0)(i+4, 0)(i+6, 0)(i+1, 0) \\ (i+2, 1)(i, 0)(i+10, 1)(i+5, 1) \\ (i+3, 1)(i, 0)(i+8, 1)(i+4, 1) \\ (\infty)(i, 0)(i+11, 1)(i+9, 1) \end{array} \right\} \text{with } i = 0, 1, \dots, 11$$

Lemma 3.8. The complete tripartite graph $K_{5,5,5}$ has a Q -decomposition.

Proof. The decomposition is

$$\left. \begin{array}{l} (x_{i+1}, \overbrace{y_i, x_{i+2}, z_i}) \\ (y_{i+1}, \overbrace{z_i, y_{i+2}, x_i}) \\ (z_{i+1}, \overbrace{x_i, z_{i+2}, y_i}) \end{array} \right\} i = 0, 1, 2, 3, 4;$$

the vertex set of $K_{5,5,5}$ being $X \cup Y \cup Z$, where $X = \{x_i\}$, $Y = \{y_i\}$, $Z = \{z_i\}$ and $i = 0, 1, \dots, 4$.

Lemma 3.9. The complete tripartite graph $K_{5,5,10}$ has a Q -decomposition.

Proof. The decomposition is

$$\left. \begin{array}{l} (z_i, \overbrace{x_i, z_{i+5}, y_i}) \\ (z_{i+2}, \overbrace{x_i, z_{i+7}, y_{i+1}}) \\ (z_{i+4}, \overbrace{x_i, z_{i+9}, y_{i+2}}) \\ (z_{i+6}, \overbrace{x_i, z_{i+1}, y_{i+3}}) \\ (z_{i+8}, \overbrace{x_i, z_{i+3}, y_{i+4}}) \end{array} \right\} i = 0, 1, 2, 3, 4$$

with X, Y as in Lemma 3.8 and $Z = \{z_i\}, i = 0, 1, \dots, 9$.

Lemma 3.10. *If $K_{q,r,s}$ and $K_{q,r,t}$ have a G -decomposition, the $K_{pq,pr,\alpha s+(p-\alpha)t}$ has a G -decomposition for every $\alpha, 0 \leq \alpha \leq p$.*

Proof. Denote the last tripartite graph by H , the vertex set being $X \cup Y \cup Z$, and partition X into p sets each containing q vertices, Y into p sets each containing r vertices and Z into p sets α of which containing each s vertices and the remaining $p - \alpha$ containing each t vertices.

As well known [5] the graph $K_{p,p,p}$ may be decomposed into triangles and hence the graph H is decomposed into edge disjoint tripartite graphs isomorphic either to $K_{q,r,s}$ or to $K_{q,r,t}$. Each such graph having a G -decomposition by assumption, H has also a G -decomposition.

Corollary. $K_{5p,5p,5\alpha+10(p-\alpha)}$ has a Q -decomposition for $0 \leq \alpha \leq p$. This follows from Lemma 3.8 and 3.9 putting $q = r = s = 5, t = 10$ and $G = Q$ in Lemma 3.10.

Finally we mention the following self-evident Lemma.

Lemma 3.11. *If $K_{p,r,q}, K_p, K_q$ and K_r have a G -decomposition, then also K_{p+q+r} has a G -decomposition.*

Proof of Theorem 3.1. The necessity follows from conditions (i), and (ii) in Section 1 and by observing that K_5 has no Q -decomposition.

For the sufficiency we will consider the two possible cases separately.

Case 1. $n \equiv 1 \pmod{5}$. In this case, K_n has a Q -decomposition by Lemma 1, if $n \equiv 1 \pmod{10}$, and by Lemma 3.3, if $n \equiv 6 \pmod{10}$.

Case 2. $5 \neq n \equiv 0 \pmod{5}$. We will proceed by induction. For $n = 10, 15, 20, 25$ K_n has a Q -decomposition by Lemmas 3.4–3.6 and 3.7 respectively. Suppose this is true for $n < n_0$ and $n_0 \geq 30$. Then $n_0 = 15p + h$, with $p \geq 2$ and $h = 0, 5$ or 10 . By induction hypothesis K_{5p} and K_{5p+h} have a Q -decomposition, hence by the Corollary, with $\alpha = p, p - 1, p - 2$ for $h = 0, 5, 10$ respectively $K_{5p,5p,5p+h}$ has a Q -decomposition and finally by Lemma 3.11 K_{n_0} has a Q -decomposition.

Remark 1. It is possible to use induction also in the proof of the first case, we need only the decompositions of K_6 , K_{11} , K_{26} and the further self evident lemma:

Lemma 3.12. *If K_{q+1} , K_{r+1} , K_{s+1} and $K_{q,r,s}$ have a G -decomposition then also $K_{q+r+s+1}$ has a G -decomposition.*

Remark 2. The idea behind the decomposition of K_{10} and K_{15} can be generalized to decompositions of K_n for $n \equiv 10 \pmod{30}$ and $n \equiv 15 \pmod{210}$.

4. On the set $N(D)$

Let D be as in Section 2 a triangle with attached edge. The triangle abc with attached edge cd will be denoted by $(abc) - d$.

The complete solution of the D -decomposition problem is given in the following theorem. The necessity of the condition in Theorem 4.1 follows from conditions (i) and (ii) of Section 1. The sufficiency when $n \equiv 1 \pmod{8}$ has been proved in [9], we will give a much simpler proof in this case and will also prove the case $n \equiv 0 \pmod{8}$ which has been announced at a conference [4].

Theorem 4.1. $N(D) = \{n \mid n \equiv 0 \text{ or } 1 \pmod{8}\}$ (2)

Proof. For the sufficiency, if $n = 8m + 1$, a D -decomposition of K_n is given by $(1, 2m - 2i, 4m - i + 1) - (6m - 3i + 1)$, $i = 0, 1, \dots, m - 1 \pmod{8m + 1}$ while if $n = 8m$ a D -decomposition is given by

$$\left. \begin{array}{l} (1, 2m - 2i, 4m - i) - (6m - 3i) \quad i = 1, 2, \dots, m - 1 \\ (1, 2m, 4m) - \infty \end{array} \right\} \pmod{8m - 1}$$

where K_{8m} has the vertices $Z_{8m-1} \cup \{\infty\}$.

This last notation means as usual that the $8m^{\text{th}}$ vertex is denoted ∞ and does not change taking the shifts $\pmod{8m - 1}$.

Remark. The proof of Theorem 4.1 may also be done by induction as in Case 2 of Theorem 3.1.

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