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Analytic combinatorics of chord and hyperchord diagrams with k crossings

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Abstract. Using methods from Analytic Combinatorics, we study the families of perfect matchings, partitions, chord diagrams, and hyperchord diagrams on a disk with a prescribed number of crossings. For each family, we express the generating function of the configurations with exactly k crossings as a rational function of the generating function of crossing-free configurations. Using these expressions, we study the singular behavior of these generating functions and derive asymptotic results on the counting sequences of the configurations with precisely k crossings. Limiting distributions and random generators are also studied.

Keywords: Quasi-Planar Configurations — Chord Diagrams — Analytic Combinatorics — Generating Functions

1 Introduction

Quasi-planar chord diagrams. Let V be a set of n points on the unit circle. A *chord diagram* on V is a set of chords between points of V. We say that two chords *cross* when their relative interior intersect. The *crossing graph* of a chord diagram is the graph with a vertex for each chord and an edge between any two crossing chords. The enumeration properties of crossing-free (or planar) chord diagrams have been widely studied in the literature, see in particular the results of P. Flajolet and M. Noy in [FN99]. From the work of J. Touchard [Tou52] and J. Riordan [Rio75], we know a remarkable explicit formula for the distribution of crossings among all perfect matchings, which was exploited in [FN00] to derive, among other parameters, the limit distribution of the number of crossings for matchings with many chords.

A more recent trend studies chord diagrams with some but restricted crossings. The several ways to restrict their crossings lead to various notions of *quasi-planar chord diagrams*. Among others, it is interesting to study chord diagrams

(i) with at most k crossings, or
(ii) with no $(k+2)$-crossing (meaning $k+2$ pairwise crossing edges), or
(iii) where each chord crosses at most k other chords, or
(iv) which become crossing-free when removing at most k well-chosen chords.

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These conditions are natural restrictions on the crossing graphs of the chord diagrams. Namely, the crossing graphs have respectively (i) at most k edges, (ii) no $(k+2)$-clique, (iii) vertex degree at most k, and (iv) a vertex cover of size k. For $k = 0$, all these conditions coincide and lead to crossing-free chord diagrams.

Families of $(k+2)$-crossing-free chord diagrams have been studied in recent literature. On the one hand, $(k+2)$-crossing-free matchings (as well as their $(k+2)$-nesting-free counterparts) were enumerated in [CDD$^+$07]. On the other hand, maximal $(k+2)$-crossing-free chord diagrams, also called $(k+1)$-triangulations, were enumerated in [Jon05], see also [PS09]. As far as we know, Conditions (i), (iii) and (iv), as well as other natural notions of quasi-planar chord diagrams, still remain to be studied in details. We focus in this paper on the Analytic Combinatorics of chord diagrams under Condition (i).

Rationality of generating functions. In this paper, we study enumeration and asymptotic properties for different families of configurations: perfect matchings, partitions, chord diagrams, and hyperchord diagrams (our results also extend to partitions and hyperchord diagrams with prescribed block sizes, see [PR13]). Examples of these configurations are represented in Figure 1.

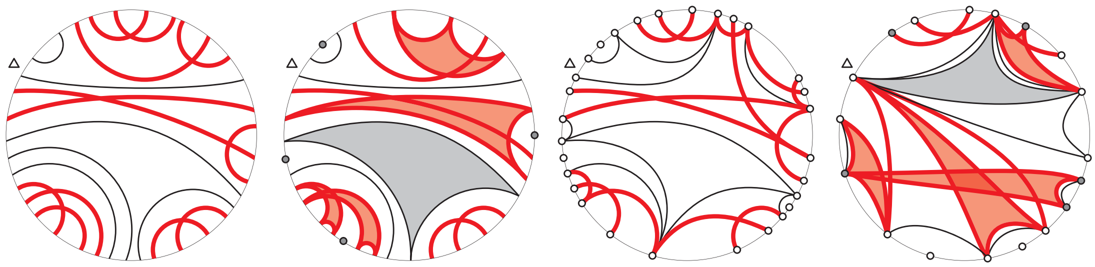

Fig. 1: The four families of (hyper)chord configurations studied in this paper. Their cores are highlighted in bold red.

Let $\mathcal{C}$ denote one of these families of configurations. To avoid handling symmetries, we insert a root in each configuration between two consecutive vertices, and we consider two rooted configurations C and C' of $\mathcal{C}$ as equivalent if there is a continuous bijective automorphism of the circle which sends the root, the vertices, and the (hyper)chords of C to that of C'. We focus on three parameters of the configurations of $\mathcal{C}$: their number n of vertices, their number m of (hyper)chords, and their number k of crossings. Note that for hyperchord diagrams and partitions, we count all crossings involving two chords contained in two distinct hyperchords. We denote by $\mathcal{C}(n, m, k)$ the set of configurations in $\mathcal{C}$ with n vertices, m (hyper)chords and k crossings, and we let $\mathbf{C}_k(x, y) := \sum_{n,m \in \mathbb{N}} |\mathcal{C}(n, m, k)| \, x^n y^m$ denote the generating function of the configurations in $\mathcal{C}$ with precisely k crossings. Our first result concerns the rationality of this function.

Theorem 1 *The generating function $\mathbf{C}_k(x, y)$ of configurations in $\mathcal{C}$ with exactly k crossings is a rational function of the generating function $\mathbf{C}_0(x, y)$ of planar configurations in $\mathcal{C}$ and of the variables x and y.*

The idea behind this result is to confine crossings of the configurations of $\mathcal{C}$ to finite subconfigurations. Namely, we define the *core configuration* $C^\star$ of a configuration $C \in \mathcal{C}$ to be the subconfiguration formed by all (hyper)chords of C involved in at least one crossing. The key observation is that

(i) there are only finitely many core configurations with k crossings, and
(ii) any configuration C of $\mathcal{C}$ with k crossings can be constructed from its core configuration $C^\star$ by inserting crossing-free subconfigurations in the connected components of the complement of $C^\star$ in the disk.

This translates in the language of generating functions to a rational expression of $\mathbf{C}_k(x, y)$ in terms of $\mathbf{C}_0(x, y)$ and its successive derivatives with respect to x, which in turn are rational in $\mathbf{C}_0(x, y)$ and the variables x and y. Similar decomposition ideas were used for example by E. Wright in his study of graphs with fixed excess [Wri77, Wri78], or more recently by G. Chapuy, M. Marcus, G. Schaeffer in their enumeration of unicellular maps on surfaces [CMS09].

Note that Theorem 1 extends a specific result of M. Bóna [Bón99] who proved that the generating function of the partitions with k crossings is a rational function of the generating function of the Catalan numbers. We note that his method was slightly different. The advantage of our decomposition scheme is to be sufficiently elementary and general to apply to the various families of configurations mentioned above.

Asymptotic analysis and random generation. From the expression of the generating function $\mathbf{C}_k(x, y)$ in terms of $\mathbf{C}_0(x, y)$, we can extract the asymptotic behavior of configurations in $\mathcal{C}$ with k crossings.

Theorem 2 *For $k \geq 1$, the number of configurations in $\mathcal{C}$ with k crossings and n vertices is*

$$[x^n]\,\mathbf{C}_k(x, 1) \underset{n\to\infty}{=} \Lambda\, n^\alpha\, \rho^{-n}\,(1 + o(1)),$$

for certain constants $\Lambda, \alpha, \rho \in \mathbb{R}$ depending on the family $\mathcal{C}$ and on the parameter k as follows.

family	*constant* Λ	*exponent* α	*singularity* ρ^{-1}	*Prop.*
matchings(i)	$\dfrac{\sqrt{2}\,(2k-3)!!}{4^{k-1}\,k!\,\Gamma\big(k-\frac{1}{2}\big)}$	$k - \dfrac{3}{2}$	2	*11*
partitions	$\dfrac{(2k-3)!!}{2^{3k-1}\,k!\,\Gamma\big(k-\frac{1}{2}\big)}$	$k - \dfrac{3}{2}$	4	*13*
chord diagrams	$\dfrac{\big(-2+3\sqrt{2}\big)^{3k}\sqrt{-140+99\sqrt{2}}\,(2k-3)!!}{2^{3k+1}\,(3-4\sqrt{2})^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}$	$k - \dfrac{3}{2}$	$6+4\sqrt{2}$	*17*
hyperchord diagrams(ii)	$\simeq \dfrac{1.034^{3k}\;0.003655\,(2k-3)!!}{0.03078^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}$	$k - \dfrac{3}{2}$	$\simeq 64.97$	*21*

From the rational expression of the generating function $\mathbf{C}_k(x, y)$, we also derive random generation schemes for the configurations in $\mathcal{C}$ with precisely k crossings, using the methods developed in [DFLS04].

Overview. Due to space limitation, we have decided to present the detailed analysis only for perfect matchings with k crossings, since we believe that it already gives the flavor of our results and illustrates our method, while remaining technically elementary. For the remaining families, we only report our results and skip the detailed analysis. We refer the interested reader to the long version of this paper [PR13].

(i) The asymptotic estimate for the number of matchings with n vertices is obviously only valid when n is even.

(ii) The expression of ρ^{-1} and Λ for hyperchord diagrams is obtained from approximations of roots of polynomials, and approximate evaluations of analytic functions. Details can be found in Propositions 19 and 21.

2 Perfect matchings

2.1 Perfect matchings and their cores

In this section, we consider the family $\mathcal{M}$ of perfect matchings with endpoints on the unit circle. Each perfect matching M of $\mathcal{M}$ is *rooted*: we mark (with the symbol $\triangle$) an arc of the circle between two endpoints of M. See Figure 1 (left). Let $\mathcal{M}(n,k)$ denote the set of matchings in $\mathcal{M}$ with n vertices and k crossings. We denote by $\mathbf{M}_k(x) := \sum_{n\in\mathbb{N}} |\mathcal{M}(n,k)|\, x^n$ the generating function of perfect matchings with exactly k crossings, where x encodes the number of vertices.

Let M be a perfect matching with some crossings. Our goal is to isolate the contribution of the chords of M involved in crossings from that of the chords of M with no crossings.

Definition 3 *A core matching is a perfect matching where each chord is involved in a crossing. It is a k-core matching if it has exactly k crossings. The core $M^\star$ of a perfect matching M is the submatching of M formed by all its chords involved in at least one crossing. See Figure 1 (left).*

Let K be a core matching. We let $n(K)$ denote its number of vertices and $k(K)$ denote its number of crossings. We call *regions* of K the connected components of the complement of K in the unit disk. A region has i *boundary arcs* if its intersection with the unit circle has i connected arcs. We let $n_i(K)$ denote the number of regions of K with i boundary arcs, and we set $\mathbf{n}(K) := (n_i(K))_{i\in[k(K)]}$ (all regions of K have at most $k(K)$ boundary arcs). Since a crossing only involves 2 chords, a k-core matching can have at most $2k$ chords. This immediately implies that there are only finitely many k-core matchings.

Definition 4 *We encode the finite list of all possible k-core matchings K and their parameters $n(K)$ and $\mathbf{n}(K) := (n_i(K))_{i\in[k]}$ in the k-core matching polynomial*

$$\mathbf{KM}_k(\mathbf{x}) := \mathbf{KM}_k(x_1,\dots,x_k) := \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{\mathbf{x}^{\mathbf{n}(K)}}{n(K)} := \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{1}{n(K)} \prod_{i\in[k]} {x_i}^{n_i(K)}.$$

Example 5 *The 7-core of the matching of Figure 1 (left) contributes to $\mathbf{KM}_7(\mathbf{x})$ as $\frac{1}{24}\, {x_1}^{17}{x_2}^{2}x_3$.*

Remark 6 *There is an efficient enumeration algorithm to generate all connected matchings (i.e. whose crossing graph is connected), from which we can easily compute the k-core matching polynomial $\mathbf{KM}_k(\mathbf{x})$. We refer to [PR13, Sections 2.2 and 2.3] for details on this algorithm.*

2.2 Generating function of matchings with k crossings

We study perfect matchings with k crossings focussing on their k-cores. For this, we consider the following weaker notion of rooting of perfect matchings. We say that a perfect matching with k crossings is *weakly rooted* if we have marked an arc between two consecutive vertices of its k-core. Note that a rooted perfect matching is automatically weakly rooted (the weak root marks the arc of the k-core containing the root of the matching), while a weakly rooted perfect matching corresponds to several rooted perfect matchings. To overcome this technical problem, we use the following rerooting argument.

Lemma 7 *Let K be a k-core with $n(K)$ vertices. The number $M_K(n)$ of rooted matchings on n vertices with core K and the number $\bar{M}_K(n)$ of weakly rooted matchings on n vertices with core K are related by*

$$n(K)M_K(n) = n\bar{M}_K(n).$$

Observe now that we can construct any perfect matching with k crossings by inserting crossing-free submatchings in the regions left by its k-core. From the k-core matching polynomial $\mathbf{KM}_k(\mathbf{x})$, we can therefore derive the following expression of the generating function $\mathbf{M}_k(x)$.

Proposition 8 *For any $k \geq 1$, the generating function $\mathbf{M}_k(x)$ of the rooted perfect matchings with k crossings is given by*

$$\mathbf{M}_k(x) = x\frac{d}{dx}\mathbf{KM}_k\left(x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{M}_0(x)\big)\right).$$

In particular, $\mathbf{M}_k(x)$ is a rational function of $\mathbf{M}_0(x)$ and x.

Proof: Consider a rooted crossing-free perfect matching M. We say that M is *i-marked* if we have placed $i-1$ additional marks between consecutive vertices of M. Note that we can place more than one mark between two consecutive vertices but that the root is always distinguishable from the other marks. Since we have $\binom{n+i-1}{i-1}$ possible ways to place these $(i-1)$ additional marks, the generating function of the i-marked crossing-free perfect matchings is given by

$$\frac{1}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{M}_0(x)\big).$$

Consider now a weakly rooted perfect matching M with $k \geq 1$ crossings. We decompose this matching into several submatchings as follows. On the one hand, the core $M^\star$ contains all crossings of M. This core is rooted by the root of M. On the other hand, each region R of $M^\star$ contains a (possibly empty) crossing-free submatching M_R. We root this submatching M_R as follows:

(i) if the root of M is not the region R, then M_R is just rooted by the root of M;

(ii) otherwise, M_R is rooted on the boundary arc of $M^\star$ just before the root of M in clockwise direction.

Moreover, we place additional marks on the remaining boundary arcs of the complement of R in the unit disk. We thus obtain a rooted i-marked crossing-free submatching M_R in each region R of $M^\star$ with i boundary arcs. Conversly, we can reconstruct the weakly rooted perfect matching M from its rooted core $M^\star$ and its rooted i-marked crossing-free submatchings M_R.

By this bijection, we thus obtain the generating function of weakly rooted perfect matchings with k crossings. From this generating function, and by application of Lemma 7, we derive the generating function $\mathbf{M}_k(x)$ of rooted perfect matchings with k crossings:

$$\begin{aligned}\mathbf{M}_k(x) &= \sum_{\substack{K\ k\text{-core}\\ \text{matching}}} \frac{x}{n(K)}\frac{d}{dx}x^{n(K)}\prod_{i\geq 1}\left(\frac{1}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{M}_0(x)\big)\right)^{n_i(K)} \qquad (1)\\ &= x\frac{d}{dx}\mathbf{KM}_k\left(x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{M}_0(x)\big)\right).\end{aligned}$$

Since $\mathbf{M}_0(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$ satisfies the functional equation $\mathbf{M}_0(x) = 1 + x^2\,\mathbf{M}_0(x)^2$, all its successive derivatives, and thus $\mathbf{M}_k(x)$, are rational in $\mathbf{M}_0(x)$ and x. □

Example 9 *Since $\mathbf{KM}_1(\mathbf{x}) = \frac{1}{4}\,{x_1}^4$, we obtain the coefficients of $\mathbf{M}_1(x)$ (see Seq. A002694 in OIES):*

$$\mathbf{M}_1(x) = \frac{x^4\,\mathbf{M}_0(x)^4}{1-2x^2\,\mathbf{M}_0(x)} = \frac{\big(1-\sqrt{1-4x^2}\big)^4}{16x^4\sqrt{1-4x^2}} = x^4 + 6\,x^6 + 28\,x^8 + 120\,x^{10} + 495\,x^{12} + 2002\,x^{14}\ldots$$

2.3 Asymptotic analysis

We now describe the asymptotic behavior of the number of perfect matchings with $k \geq 1$ crossings. The method consists in studying the asymptotic behavior of $\mathbf{M}_0(x)$ and of all its derivatives around their dominant singularities, and to exploit the rational expression of $\mathbf{M}_k(x)$ in terms of $\mathbf{M}_0(x)$ and x given in Proposition 8. Along the way, we naturally study which k-cores have the main asymptotic contributions. For that, we need the following lemma, whose detailed proof can be found in [PR13, Lemma 2.13].

Lemma 10 *The following assertions are equivalent for an (unrooted) k-core matching K:*
(i) K belongs to the family of the k-core matchings whose first five elements are shown in Figure 2.
(ii) $n_1(K) = 3k$, $n_k(K) = 1$ and $n_i(K) = 0$ for all other values of i (here, $k \geq 2$).
(iii) K maximizes $n_1(K)$ among all possible k-core matchings (here, $k \geq 3$).
(iv) K maximizes the potential $\phi(K) := \sum_{i>1}(2i-3)\, n_i(K)$ among all possible k-core matchings.
We call a k-core matching maximal *if it satisfies these conditions.*

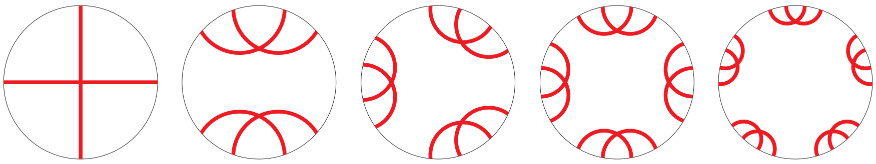

Fig. 2: Maximal core matchings (unrooted) for $k = 1, \ldots, 5$.

Proposition 11 *For any $k \geq 1$, the number of perfect matchings with k crossings and $n = 2m$ vertices is*

$$[x^{2m}]\, \mathbf{M}_k(x) \underset{m\to\infty}{=} \frac{(2k-3)!!}{2^{k-1}\, k!\, \Gamma\big(k-\frac{1}{2}\big)}\, m^{k-\frac{3}{2}}\, 4^m\, (1 + o(1)),$$

where $(2k-3)!! := (2k-3)\cdot(2k-5)\cdots 3\cdot 1$ and $(-1)!! = 1$.

Proof: We assume here that $k \geq 2$ (the case $k = 1$ can be derived from a direct analysis of the expression of $\mathbf{M}_1(x)$ given in Example 9). We first study the asymptotic behavior of $\mathbf{M}_0(x)$ and of all its derivatives around their dominant singularities. The generating function $\mathbf{M}_0(x)$ defines an analytic function around the origin. Its dominant singularities are located at $x = \pm\frac{1}{2}$. Denoting by $X_+ := \sqrt{1-2x}$, the singular expansions of $\mathbf{M}_0(x)$ and its derivative around $x = \frac{1}{2}$ are

$$\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2 - 2\sqrt{2}\, X_+ + O\left({X_+}^2\right), \qquad \frac{d}{dx}\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2\sqrt{2}\, {X_+}^{-1} + O(1),$$

$$\text{and} \qquad \frac{d^i}{dx^i}\mathbf{M}_0(x) \underset{x\sim\frac{1}{2}}{=} 2\sqrt{2}\,(2i-3)!!\, {X_+}^{1-2i} + O\left({X_+}^{2-2i}\right) \qquad \text{for all } i \geq 2,$$

where $(2i-3)!! := (2i-3)\cdot(2i-5)\cdots 3\cdot 1$. These expansions are valid in a dented domain at $x = \frac{1}{2}$, [FS09].

We now exploit the expression of the generating function $\mathbf{M}_k(x)$ given by Equation (1) in the proof of Proposition 8. The dominant singularities of $\mathbf{M}_k(x)$ are located at $x = \pm\frac{1}{2}$. We provide the full analysis around $x = \frac{1}{2}$, the computation for $x = -\frac{1}{2}$ being similar. For conciseness in the following expressions, we set by convention $(-1)!! = 1$. We therefore obtain:

$$\begin{aligned}
\mathbf{M}_k(x) &= x \frac{d}{dx} \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{1}{n(K)} \prod_{i \geq 1} \left(\frac{x^i}{(i-1)!} \frac{d^{i-1}}{dx^{i-1}} \left(x^{i-1} \mathbf{M}_0(x) \right) \right)^{n_i(K)} \\
&\underset{x \sim \frac{1}{2}}{=} \frac{1}{2} \frac{d}{dx} \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{1}{n(K)} \prod_{i > 1} \left(\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!} X_+^{\,3-2i} + O\left(X_+^{\,4-2i} \right) \right)^{n_i(K)} \\
&\underset{x \sim \frac{1}{2}}{=} \frac{1}{2} \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{\phi(K)}{n(K)} \prod_{i > 1} \left(\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!} \right)^{n_i(K)} X_+^{\,-\phi(K)-2} + O\left(X_+^{\,-\phi(K)-1} \right),
\end{aligned}$$

where $\phi(K) := \sum_{i>1} (2i-3)\, n_i(K)$ is the potential function introduced in Lemma 10. Observe that in order to obtain the second equality, we used the fact that $k > 1$, and thus, that there exists k-cores K such that $n_i(K) \neq 0$ when $i > 1$. Combining Lemma 10 and the Transfer Theorem of singularity analysis [FO90], we conclude that the main contribution in the asymptotic of the previous sum arises from maximal k-cores, as they maximize the value $2 + \phi(K)$. By Lemma 10, there are exactly 4 rooted maximal k-cores with $n_1(K) = 3k$, $n_k(K) = 1$, $n(K) = 4k$, and $\phi(K) = 2k - 3$. Hence,

$$\begin{aligned}
[x^n]\, \mathbf{M}_k(x) &\underset{x \sim \frac{1}{2}}{=} [x^n]\, \frac{1}{2} \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \frac{\phi(K)}{n(K)} \prod_{i > 1} \left(\frac{\sqrt{2}\,(2i-5)!!}{4^{i-1}\,(i-1)!} \right)^{n_i(K)} X_+^{\,-\phi(K)-2} + O\left(X_+^{\,-\phi(K)-1} \right) \\
&\underset{x \sim \frac{1}{2}}{=} \frac{2\sqrt{2}\,(2k-3)!!}{4^k\, k!} [x^n]\ \sqrt{1-2x}^{\,1-2k} + O\left((1-2x)^{1-k} \right) \\
&\underset{n \to \infty}{=} \frac{2\sqrt{2}\,(2k-3)!!}{4^k\, k!\, \Gamma\left(k - \frac{1}{2}\right)} n^{k-\frac{3}{2}}\, 2^n\, (1 + o(1)),
\end{aligned}$$

where the last equality is obtained by application of the Transfer Theorem of singularity analysis [FO90].

Finally, we obtain the stated result by adding the expressions obtained when studying $\mathbf{M}_k(x)$ around $x = \frac{1}{2}$ and $x = -\frac{1}{2}$. In fact, one can check that the asymptotic estimate of $[x^n]\, \mathbf{M}_k(x)$ around $x = -\frac{1}{2}$ is the same but with an additional multiplicative constant $(-1)^n$. Consequently, the contribution is equal to 0 when n is odd and to the estimate in the statement when n is even. □

2.4 Random generation

The composition scheme presented in Proposition 8 can also be exploited in order to provide Boltzmann samplers for random generation of perfect matchings with k crossings. Throughout this section we consider a positive real number $\theta < \frac{1}{2}$, which acts as a "control-parameter" for the random sampler (see [DFLS04] for further details). The Boltzmann sampler works in three steps:

(i) We first decide which is the core of our random object.

(ii) Once this core is chosen, we complete the matching by means of non-crossing matchings.

(iii) Finally, we place the root of the resulting perfect matching with k crossings.

We start with the choice of the k-core. For each k-core K, let $\mathbf{M}_K(x)$ denote the generating function of matchings with k crossings and whose k-core is K, where x marks as usual the number of vertices. Note that this generating function is computed as in Proposition 8, using only the contribution of the k-core K. Therefore, we have

$$\mathbf{M}_k(x) = \sum_{\substack{K\ k\text{-core} \\ \text{matching}}} \mathbf{M}_K(x)\,.$$

This sum defines a probability distribution as follows: once fixed the parameter θ, let $p_K = \frac{\mathbf{M}_K(\theta)}{\mathbf{M}_k(\theta)}$. This set of values defines a discrete probability distribution $\{p_K\}_{\substack{K\ k\text{-core} \\ \text{matching}}}$, which can be easily simulated.

Once we have fixed the core of the random matching, we continue in the second step filling in its regions with crossing-free perfect matchings. For this purpose it is necessary to have a procedure to generate crossing-free perfect matchings, namely $\Gamma\mathbf{M}_0(\theta)$. As $\mathbf{M}_0(\theta)$ satisfies the recurrence relation $\mathbf{M}_0(\theta) = 1 + \theta^2\mathbf{M}_0(\theta)^2$, a Boltzmann sampler $\Gamma\mathbf{M}_0(\theta)$ can be defined in the following way. Let $p = \frac{1}{\Gamma\mathbf{M}_0(\theta)}$. Then, using the language of [DFLS04],

$$\Gamma\mathbf{M}_0(\theta) := \mathrm{Bern}(p) \longrightarrow \varnothing \mid (\Gamma\mathbf{M}_0(\theta)\ ,\ \bullet - \bullet\ ,\ \Gamma\mathbf{M}_0(\theta)),$$

where $\bullet - \bullet$ means that the Boltzmann sampler is generating a single chord (or equivalently, two vertices in the border of the circle). This Boltzmann sampler is defined when $\theta < \frac{1}{2}$, in which case the defined branching process is subcritical. In such situation the algorithm stops in finite expected time, see [DFLS04].

Once this random sampler is performed, we can deal with a term of the form $\frac{d^{i-1}}{dx^{i-1}}x^{i-1}\mathbf{M}_0(\theta)$. Indeed, once a random crossing-free perfect matching $\Gamma\mathbf{M}_0(\theta)$ of size $n(\Gamma\mathbf{M}_0(\theta))$ is generated, there exist

$$\binom{n(\Gamma\mathbf{M}_0(\theta)) + i - 1}{i-1}$$

i-marked crossing-free perfect matchings arising from $\Gamma\mathbf{M}_0(\theta)$. Hence, with uniform probability we can choose one of these i-marked crossing-free perfect matchings. As this argument follows for each choice of i, and $\mathbf{KM}_K(\mathbf{x})$ is a polynomial, we can combine the generator of i-marked crossing-free diagrams with the Boltzmann sampler for the cartesian product of combinatorial classes (recall that we need to provide the substitution $x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left(x^{i-1}\mathbf{M}_0(x)\right)$).

Finally, we need to apply the root operator, which can be done by means of similar arguments as in the case of i-marked crossing-free diagrams.

Concerning the statistics of the random variable N corresponding to the size of the element generated by means of the previous random sampler, as it is shown in [DFLS04], the expected value $\mathbb{E}[N]$ and the variance $\mathbb{V}\mathrm{ar}[N]$ of the random variable N satisfy

$$\mathbb{E}[N] = \theta\,\frac{\mathbf{M}_k'(\theta)}{\mathbf{M}_k(\theta)} \qquad \text{and} \qquad \mathbb{V}\mathrm{ar}[N] = \frac{\theta^2(\mathbf{M}_k''(\theta)\,\mathbf{M}_k(\theta) - \theta\,\mathbf{M}_k'(\theta)^2) + \theta\,\mathbf{M}_k'(\theta)}{\mathbf{M}_k(\theta)^2}.$$

Hence, when θ tends to $\frac{1}{2}$, the expected value of the generated element tends to infinity, and the variance for the expected size also diverges.

3 Extension to other families of chord diagrams

3.1 Partitions

We first extend our results to the family $\mathcal{P}$ of partitions of point sets on the unit circle. See Figure 1. As before, the partitions are rooted by a mark on an arc between two vertices. A *crossing* between two blocks U, V of a partition P is a pair of crossing chords u_1u_2 and v_1v_2 where $u_1, u_2 \in U$ and $v_1, v_2 \in V$. We count crossings with multiplicity: two blocks U, V cross as many times as the number of such pairs of crossing chords among U and V. Note that perfect matchings are particular partitions where all blocks have size 2. Applying the same method as in Section 2.2, we obtain an expression of the generating function $\mathbf{P}_k(x, y)$ of partitions with k crossings in terms of the k-core partition polynomial $\mathbf{KP}_k(\mathbf{x}, y)$.

Proposition 12 *For any $k \geq 1$, the generating function $\mathbf{P}_k(x, y)$ of partitions with k crossings is given by*

$$\mathbf{P}_k(x, y) = x\frac{d}{dx}\mathbf{KP}_k\left(x_i \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\big(x^{i-1}\mathbf{P}_0(x, y)\big), y\right).$$

In particular, $\mathbf{P}_k(x, y)$ is a rational function of $\mathbf{P}_0(x, y)$ and x.

From this expression, we can extract as in Section 2.3 asymptotic estimates for the number of partitions with k crossings, and we can as well construct efficient random generators as in Section 2.4.

Proposition 13 *For any $k \geq 1$, the number of partitions with k crossings and n vertices is*

$$[x^n]\,\mathbf{P}_k(x, 1) \underset{n\to\infty}{=} \frac{(2k-3)!!}{2^{3k-1}\,k!\,\Gamma\big(k-\frac{1}{2}\big)}\,n^{k-\frac{3}{2}}\,4^n\,(1 + o(1)).$$

Remark 14 *Our results on matchings and partitions can even be extended to analyze the generating function $\mathbf{P}_k^S(\mathbf{x}, y)$ of partitions with k crossings and whose block sizes all belong to S. In contrast to Proposition 12 which can be directly adapted to this context, the asymptotic analysis of $\mathbf{P}_k^S(\mathbf{x}, y)$ involves more technical tools, including the theory of A. Meir and J. Moon on the singular behavior of generating functions defined by a smooth implicit function schema [MM89]. See [PR13, Section 2.10].*

3.2 Chord diagrams

We now consider the family $\mathcal{D}$ of all chord diagrams on the unit circle. Remember that a chord diagram is given by a set of vertices on the unit circle, and a set of chords between them. In particular, we allow isolated vertices, as well as several chords incident to the same vertex, but not multiple chords with the same two endpoints. We are interested in the generating function $\mathbf{D}_k(x, y)$ of chord diagrams with k crossings. The generating function $\mathbf{D}_0(x, y)$ of crossing-free chord diagrams was studied in [FN99].

Proposition 15 ([FN99, Equation (22)]) *The generating function $\mathbf{D}_0(x, y)$ of crossing-free chord diagrams satisfies the functional equation*

$$y\,\mathbf{D}_0(x, y)^2 + \big(x^2(1+y) - x(1+2y) - 2y\big)\,\mathbf{D}_0(x, y) + x(1+2y) + y = 0.$$

Therefore, all derivatives $\frac{d^i}{dx^i}\mathbf{D}_0(x, y)$ are rational functions in $\mathbf{D}_0(x, y)$ and x. Moreover, we have

$$\mathbf{D}_0(x, 1) \underset{x\sim\rho}{=} \alpha - \beta\sqrt{1-\rho^{-1}x} + O(1-\rho^{-1}x),$$

where $\rho^{-1} = 6 + 4\sqrt{2}$, $\alpha = -1 + 3\frac{\sqrt{2}}{2}$, *and* $\beta = \frac{1}{2}\sqrt{-140 + 99\sqrt{2}}$.

As for matchings, we can construct any chord diagram with k crossings by inserting crossing-free subdiagrams in the regions left by its k-core. We can therefore derive the following expression for the generating function $\mathbf{D}_k(x,y)$ of diagrams with k crossings, in terms of the generating function $\mathbf{D}_0(x,y)$ of crossing-free diagrams, of the k-core diagram polynomial $\mathbf{KD}_k(x,y)$, and of the polynomials

$$\mathbf{D}_0^n(y) := [x^n]\,\mathbf{D}_0(x,y) \qquad \text{and} \qquad \mathbf{D}_0^{\leq p}(x,y) := \sum_{n\leq p} \mathbf{D}_0^n(y)\,x^n = \sum_{\substack{n\leq p\\ m\geq 0}} |\mathcal{D}(n,m,0)|\,x^n y^m.$$

Proposition 16 *For any $k \geq 1$, the generating function $\mathbf{D}_k(x,y)$ of chord diagrams with k crossings is given by*

$$\mathbf{D}_k(x,y) = x\frac{d}{dx}\mathbf{KD}_k\left(x_{0,j} \leftarrow \frac{\mathbf{D}_0^j(y)}{x^j},\ x_{i,j} \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\frac{\mathbf{D}_0(x,y) - \mathbf{D}_0^{\leq i+j}(x,y)}{x^{i+j+1}},\ y\right).$$

In particular, $\mathbf{D}_k(x,y)$ is a rational function of $\mathbf{D}_0(x,y)$ and x.

Similarly to our asymptotic analysis in Section 2.3, we can obtain asymptotic results for the number of chord diagrams with k crossings, in terms of the constants ρ, α, and β defined in Proposition 15. Random generators can as well be constructed, see [PR13, Section 3.6].

Proposition 17 *For any $k \geq 1$, the number of chord diagrams with k crossings and n vertices is*

$$[x^n]\,\mathbf{D}_k(x,1) \underset{n\to\infty}{=} \frac{\alpha^{3k}\,\beta\,(2k-3)!!}{(2\rho)^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}\,n^{k-\frac{3}{2}}\,\rho^{-n}\,(1+o(1)).$$

3.3 Hyperchord diagrams

As from matchings to partitions, we can finally extend our results from chord diagrams to hyperchord diagrams. A *hyperchord* is the convex hull of finitely many points of the unit circle. Given a point set V on the circle, a *hyperchord diagram* on V is a set of hyperchords with vertices in V. Note that we allow isolated vertices in hyperchord diagrams. As for partitions, a *crossing* between two hyperchords U, V is a pair of crossing chords u_1u_2 and v_1v_2, with $u_1, u_2 \in U$ and $v_1, v_2 \in V$. We consider the family $\mathcal{H}$ of hyperchord diagrams, and we want to analyze the generating function $\mathbf{H}_k(x,y)$ of hyperchord diagrams with k crossings, counted with multiplicities. As for chord diagrams, our first step is to study the generating function $\mathbf{H}_0(x,y)$ of crossing-free hyperchord diagrams, extending the results of P. Flajolet and M. Noy for chord diagrams [FN99] that we presented in Proposition 15.

Proposition 18 *The generating function $\mathbf{H}_0(x,y)$ of crossing-free hyperchord diagrams satisfies the functional equation*

$$p_3(x,y)\,\mathbf{H}_0(x,y)^3 + p_2(x,y)\,\mathbf{H}_0(x,y)^2 + p_1(x,y)\,\mathbf{H}_0(x,y) + p_0(x,y) = 0,$$

where

$$\begin{aligned}
p_0(x,y) &:= -2x^2 - x + 2xy^3 + y^2 + x^2y^4 - 7x^2y - 7x^2y^2 - x^2y^3 - 3xy,\\
p_1(x,y) &:= -2x^3 - 2x^3y^4 - 8x^3y + 2x - 3y^2 - 12x^3y^2 - 8x^3y^3\\
&\quad + 6xy - x^2y^4 + x^2 + 4x^2y + 4x^2y^2 - 4xy^3,\\
p_2(x,y) &:= x^2y^3 + x^2 + 3x^2y^2 - x - 3xy + 2xy^3 + 3x^2y + 3y^2,\\
p_3(x,y) &:= -y^2.
\end{aligned}$$

We use this functional equation to derive the asymptotic behavior of $\mathbf{H}_0(x, y)$.

Proposition 19 *The smallest singularity of the generating function $\mathbf{H}_0(x, 1)$ of crossing-free hyperchord diagrams is located at the smallest real root $\rho \simeq 0.015391$ of the polynomial*

$$R(x) := 256\,x^4 - 768\,x^3 + 736\,x^2 - 336\,x + 5.$$

Moreover, when y varies uniformly in a small neighborhood of 1, the singular expansion of $\mathbf{H}_0(x, y)$ is

$$\mathbf{H}_0(x, y) \underset{y\sim 1}{=} h_0(y) - h_1(y)\sqrt{1 - \frac{x}{\rho(y)}} + O\left(1 + \frac{x}{\rho(y)}\right),$$

valid in a domain dented at $\rho(y)$ (for each choice of y), where $h_0(y)$, $h_1(y)$ and $\rho(y)$ are analytic functions around $y = 1$, with $\rho(1) = \rho \simeq 0.015391$, $h_0(1) \simeq 1.034518$ and $h_1(1) \simeq 0.00365515$.

Using a similar method as in Section 2.2, we obtain the following expression of the generating function $\mathbf{H}_k(x, y)$ of hyperchord diagrams with k crossings, in terms of the k-core hyperchord diagram polynomial $\mathbf{KH}_k(\mathbf{x}, y)$, and of the polynomials

$$\mathbf{H}_0^n(y) := [x^n]\,\mathbf{H}_0(x, y) \qquad \text{and} \qquad \mathbf{H}_0^{\leq p}(x, y) := \sum_{n\leq p} \mathbf{H}_0^n(y)\,x^n = \sum_{\substack{n\leq p\\ m\geq 0}} |\mathcal{H}(n, m, 0)|\,x^n y^m.$$

Proposition 20 *For any $k \geq 1$, the generating function $\mathbf{H}_k(x, y)$ of the hyperchord diagrams with k crossings is given by*

$$\mathbf{H}_k(x, y) = x\frac{d}{dx}\mathbf{KH}_k\left(x_{0,j} \leftarrow \frac{\mathbf{H}_0^j(y)}{x^j},\, x_{i,j} \leftarrow \frac{x^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\frac{\mathbf{H}_0(x, y) - \mathbf{H}_0^{\leq i+j}(x, y)}{x^{i+j+1}},\, y\right).$$

In particular, $\mathbf{H}_k(x, y)$ is a rational function of the function $\mathbf{H}_0(x, y)$ and of the variables x and y.

Finally, using the expression of the generating function $\mathbf{H}_k(x, y)$ given by Proposition 20, we can derive the asymptotic behavior of the number of hyperchord diagrams with k crossings. The analysis is identical to that of the proof of Proposition 17.

Proposition 21 *For any $k \geq 1$, the number of hyperchord diagrams with k crossings and n vertices is*

$$[x^n]\,\mathbf{D}_k(x, 1) \underset{n\to\infty}{=} \frac{h_0(1)^{3k}\,h_1(1)\,(2k-3)!!}{(2\rho)^{k-1}\,k!\,\Gamma(k-\frac{1}{2})}\,n^{k-\frac{3}{2}}\,\rho^{-n}\,(1 + o(1)),$$

where $\rho \simeq 0.015391$ is the smallest real root of $R(x) := 256\,x^4 - 768\,x^3 + 736\,x^2 - 336\,x + 5$, and where $h_0(1) \simeq 1.034518$ and $h_1(1) \simeq 0.00365515$.

Remark 22 *Our results on chord and hyperchord diagrams can as well be extended to analyze the generating function $\mathbf{H}_k^S(\mathbf{x}, y)$ of hyperchord diagrams with k crossings and whose hyperchord sizes all belong to S, although the results are more difficult to express. We refer the reader to [PR13, Section 3.8].*

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