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J-C Bermond

Graceful graphs, radio antennae and French windmills

A graph G with e edges can be labelled by assigning values to its vertices and assigning to each edge the absolute value of the difference of the numbers at its endpoints. G is said to be graceful if its vertex-labels are distinct members of the set $\{0, 1, \dots, e\}$, and its edge-labels form exactly the set $\{1, 2, \dots, e\}$.

In this paper we survey the theory of graceful graphs and apply it to a problem on the configurations of antennae in radio-astronomy; this problem is related to graceful numberings of some windmills.

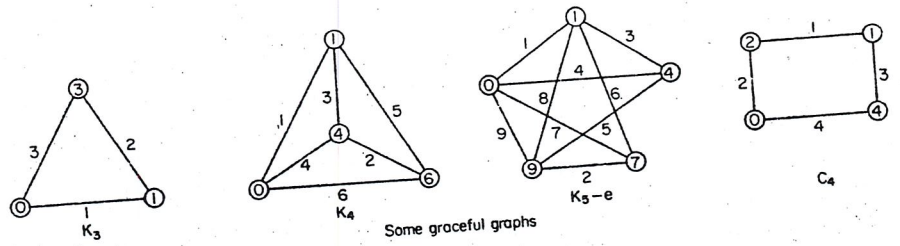
1. INTRODUCTION

A graph $G = (X, E)$ is numbered if each vertex v is assigned a non-negative integer $\phi(v)$, and each edge $\{v, w\}$ is assigned the absolute value of the difference of the numbers at its endpoints - that is, $|\phi(v) - \phi(w)|$.

The numbering is called graceful if, furthermore, we have:

- (a) the vertices are labelled with distinct integers (that is, ϕ is an injection);
- (b) the largest value of the vertex-labels is equal to the number of edges (that is, $\max_v \phi(v) = |E| = e$);
- (c) all the edges of G have distinct labels chosen from the set $\{1, 2, \dots, e\}$.

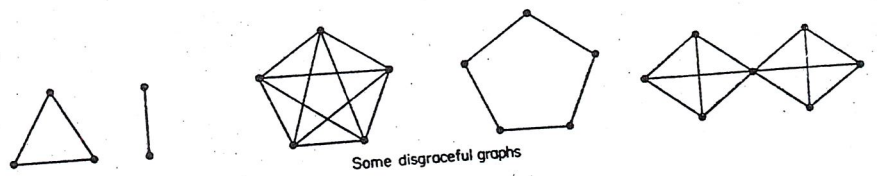
A graph which admits a graceful numbering is said to be a graceful graph. For example, let K_n denote the complete graph on n vertices and C_n denote the cycle of length n . Then Figure 1 gives graceful numberings of $K_3, K_4, K_5 - e$, and C_4 .



Some graceful graphs

Figure 1

Not all graphs are graceful. We leave it to the reader to check that the graphs of Figure 2 (that is, $K_3 \cup K_2, K_5, C_5$, and the graph consisting of two K_4 's with a vertex in common) are not graceful; see later for a proof that K_5 and C_5 are not graceful, and [2] for the last graph.



Some disgraceful graphs

Figure 2

The term 'graceful graph' (or 'graceful numbering') is recent and due to S. W. Golomb [15], and has been popularised by the articles of M. Gardner [12] and S. W. Golomb [16]. But the study of graceful graphs began much earlier. In graph theory the interest in this topic arose from a conjecture of G. Ringel [33] and an article of A. Rosa [34]. (A graceful numbering of a graph G was called a β -valuation by Rosa; note that in Rosa's article other numberings and valuations are also considered.)

This topic has also been considered in other parts of mathematics, in particular in additive number theory under the name of 'restricted difference basis'. More precisely, the problem is as follows: a restricted difference basis (with respect to an integer e) is a set of integers $0 = a_1 < a_2 < \dots < a_{n-1} < a_n = e$, such that every positive integer m with $1 \leq m \leq e$ can be represented in the form $m = a_j - a_i$. The problem is then to find, for a given e , the minimum value $n(e)$ of n , the number of integers in such a restricted difference basis. A more tangible way of posing this question is in terms of the ruler problem: to construct a ruler measuring all integral distances from 1 to the length e of the ruler by marking it at the minimum number of places. Rulers for $e = 6$ and 9 are shown in Figure 3.

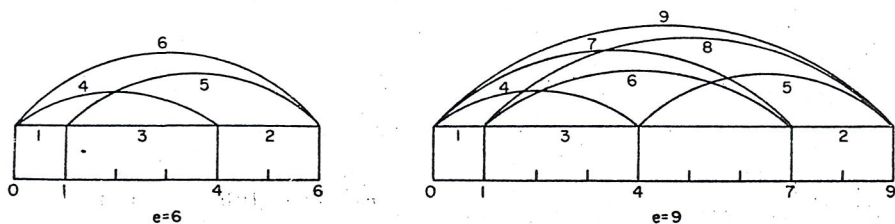


Figure 3

These correspond respectively to the graceful numberings of K_4 and $K_5 - e$, shown in Figure 1, the distances marked off being the vertex-labels. An equivalent problem is to determine the minimum number of vertices in a graceful graph, where the number of edges is prescribed. The first papers on this problem seem to be those of A. Brauer (1945), P. Erdős and I. S. Gál (1948) and L. Rédei and A. Rényi (1948) (see the article of J. Leech [29]). A survey on this problem has been written by J. C. P. Miller [30]. We shall return to this problem in Section 4.

Finally, this topic has also arisen in many applications (for example, coding theory, X-ray crystallography, radar, communication networks and astronomy). We omit further details of these applications, since they are discussed, together with numerous references, in an excellent survey paper by G. S. Bloom and S. W. Golomb [6] (see also [7]). The main unsolved problem is to determine which graphs are graceful. This appears to be very difficult, one reason being that a subgraph of a graceful graph need not be graceful (for example, C_5 is not graceful, although it is a subgraph of K_5). On the other hand, the answer is known for certain classes of graphs. Our object here is to describe the current state of knowledge; since some results have been rediscovered many times, we will mention only those authors we believe to have originated the various results.

2. SOME CLASSES OF GRACEFUL GRAPHS

We begin with a necessary condition due to Rosa [34].

Lemma 2.1. If G is a graceful Eulerian graph with e edges, then $e \equiv 0$ or $3 \pmod{4}$.

Proof Since G is Eulerian, $\sum |\phi(v) - \phi(w)|$ ($vw \in E$) is an even number. But since G is graceful, this number is equal to the sum of the valuations of the edges, that is

$$\sum_{i=1}^e i = \frac{1}{2}e(e+1).$$

Thus $e \equiv 0$ or $3 \pmod{4}$. \square

For example, K_5 and C_5 are Eulerian, but they have 10 and 5 edges, and thus by Lemma 2.1 they are not graceful. For cycles the necessary condition in Lemma 2.1 has been proved (by Rosa [34]) to be sufficient:

Theorem 2.2. The cycle C_n is graceful if and only if $n \equiv 0$ or $3 \pmod{4}$. \square

Rosa also proved the following result:

Theorem 2.3. All complete bipartite graphs are graceful. \square

In Figure 4 we give a graceful numbering of $K_{m,n}$.

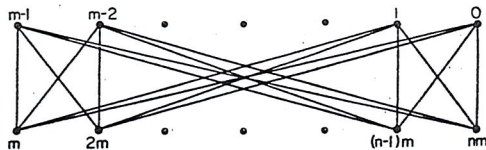


Figure 4

C. Hoede and H. Kuiper [22], and R. Frucht have proved:

Theorem 2.4. All wheels are graceful. \square

Many problems remain open, however - in particular, the following conjecture due to H. Bodendiek, H. Schumacher and H. Wegner, who have proved it in certain special cases [9]:

Conjecture. The graphs consisting of a cycle plus one edge joining two non-adjacent vertices are graceful.

proved by C. Delsime, H. Maheo, K.-K. Koh, Teo, Thullier

In [9] they proved also that the prism P_k consisting of two cycles v_1, v_2, \dots, v_k and v'_1, v'_2, \dots, v'_k plus the edges $v_i v'_i$ for $i = 1, 2, \dots, k$ (this graph has $2k$ vertices and $3k$ edges) is graceful if $k \equiv 0 \pmod{4}$. Recently R. Frucht has proved:

Theorem 2.5. All prisms are graceful. \square

(Figure 5 shows a graceful numbering of the prism P_5 .)

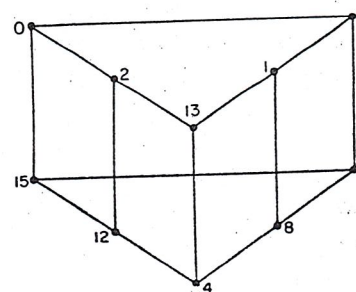


Figure 5

R. Frucht also proved that 'crowns' (cycles plus edges pendant at each vertex) are graceful. T. Gangopadyay and S. P. Rao Hebbare proved some related classes of graphs to be graceful, in particular that the prisms P_k are graceful for k even. They also conjectured that the k -cube is graceful: with A. Germa and M. Maheo we have proved that this is true. To conclude this section we note the following result of D. A. Sheppard [36]:

Theorem 2.6. There are exactly $e!$ gracefully number graphs with e edges. \square

(Two graceful graphs are considered as different if either they are not isomorphic, or if they are isomorphic but have distinct graceful numberings.)

3. GRACEFUL TREES

Trees are certainly the most fascinating graphs in this area. A. Kotzig (see Rosa's article [34]) has made the following conjecture:

Conjecture. All trees are graceful.

Despite many efforts, this conjecture has not been proved; however, many classes of trees have been shown to be graceful, and a wide literature exists on the subject. A survey on this conjecture and on 'the continuing quest to call all trees graceful' has been given by G. S. Bloom [5].

The conjecture is often attributed to G. Ringel [33], but he gave only the following weaker conjecture:

Conjecture. Let T be a given tree with n vertices and $n - 1$ edges; then the edges of K_{2n-1} can be partitioned into $2n - 1$ trees isomorphic to T .

The fact that the second conjecture will imply the first one has been shown by Rosa [34]. To see this, let the vertices of T be $v_1, \dots, v_i, \dots, v_n$, and let ϕ be a graceful numbering; then if the vertices of K_{2n-1} are labelled with the integers $0, 1, \dots, 2n-2$, the $2n-1$ trees T_j of the decomposition ($0 \leq j \leq 2n - 2$) are isomorphic to T and have respectively as vertex v_i the vertex of K_{2n-1} labelled $\phi(v_i) + j$ (where this number is to be taken modulo $2n - 1$). In this case, the decomposition is said to be cyclical. More generally, Rosa [34] has shown, exactly as above, the following result:

Theorem 3.1. If G is a graceful graph with e edges, then the edges of K_{2e+1} can be partitioned into $2n + 1$ graphs isomorphic to G . \square

Note that the converse result is not true; for example, there exists a cyclical decomposition of K_{11} into C_5 's, although C_5 is not graceful. There exists a wide literature on decompositions of complete graphs (see, for example, [4]). In fact it is sufficient (although again not necessary) to suppose in Theorem 3.1 that G admits a less restrictive numbering (called a ρ -valuation by Rosa [34]). The above conjectures for trees have been popularised by R. A. Duke [11], and a great number of results and references concerning these conjectures and more general graceful graphs appear in the American Mathematical Monthly Research Problems section of R. K. Guy (see [17], [18], [19] and [20]).

It is also worth mentioning that one can consider graceful numberings with additional properties. More precisely, Rosa [34] says that a graph admits an α -valuation if the graph is graceful and if there exists an integer r such that if ϕ is a graceful numbering for every edge vw of G , then we have either $\phi(v) \leq x < \phi(w)$ or $\phi(w) \leq x < \phi(v)$. Such a graceful graph is called 'balanced' in [36]. Rosa [34] has proved that not all trees admit an α -evaluation, and Kotzig [27] has conjectured that only a finite number of trees homeomorphic to any given tree do not admit an α -valuation. (The truth of this conjecture would show that almost all trees are graceful and balanced.) Kotzig also proved the following interesting result:

Theorem 3.2. Let T be a given tree. Then the infinite family of trees obtained from T by replacing a given edge by a path of arbitrary length contains only a finite number of trees with no α -valuation. \square

We now describe some classes of trees which have been proved to be graceful. The first result is due to Rosa [34]:

Theorem 3.3. All paths are graceful. \square

Let us now call a caterpillar a tree such that if we delete the end vertices (vertices of degree 1) then the graph obtained is a path. (In [34] a caterpillar is called a snake.) Rosa proved the following result:

Theorem 3.4. All caterpillars are graceful. \square

(Figure 6 shows a caterpillar with a graceful numbering.)

We now use the name lobster for a tree such that if we delete the end-vertices, the graph obtained is a caterpillar (see Figure 7). The first step towards a proof of the first conjecture for trees might be the following natural conjecture:

Conjecture. All lobsters are graceful.

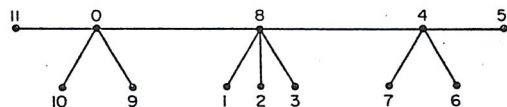


Figure 6

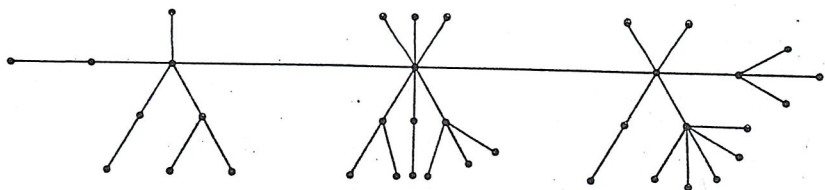


Figure 7

Constructive methods generating graceful trees from smaller ones have been given by many authors; for example, R. G. Stanton and C. R. Zarnke [38], H. Gabow [13], A. Gyárfás and J. Lehel [14], and K. M. Koh, D. G. Rogers and T. Tan [24], [25], [26]. We do not enter in the technical details here; the reader can refer to the survey of Bloom [5], or to any of the articles referenced above. By these methods further classes of trees have been proved to be graceful.

Let us call a tree symmetrical if it is a rooted tree in which every level contains vertices of the same degree. It is proved (for example, in [14]) that:

Theorem 3.5. Symmetrical trees are graceful. \square

Let us call a complete k-ary tree a special symmetrical tree in which the degree of the root is k , the degrees of the vertices on other levels are all $k + 1$, except those of the vertices of the last level, which are all 1. (In Figure 8, we show a binary tree which is gracefully labelled.) As a Corollary of Theorem 3.5, we obtain the following result, first proved in [38]:

Corollary 3.6. Complete k -ary trees are graceful. \square

Note that this result has also been rediscovered by many authors, partly because in 1976 Cahit [10], unaware of Stanton and Zernke's result, conjectured that binary trees are graceful. Other methods, using the adjacency matrix of the tree, have been developed by Haggard and McWha [21], and by Bloom, Haggard and Taylor [8].

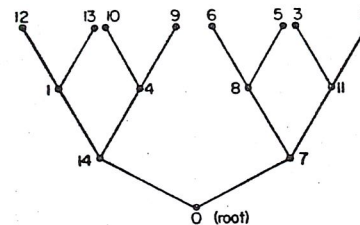


Figure 8

The methods developed can be used to give a graceful labelling to trees having a minimum 'symmetry'. A. M. Pastel and H. Raynaud [31], [32] have conversely shown that some 'asymmetrical' trees are graceful. Let us call an olive-tree a rooted tree consisting of k branches, where the i th branch is a path of length i . It has been proved in [31] that:

Theorem 3.7. All olive-trees are graceful. \square

This theorem answers a conjecture of Gyárfás and Lehel (given in a lecture at the Orsay C.N.R.S. Colloquium in 1976). To prove this, Pastel and Raynaud proved that there exists a graceful numbering of the path of length n , such that any value k ($0 \leq k \leq n$) can be taken as the label of one of the end-vertices of the path. The case $k = 0$ of this last result was first proved by A. Rosa [35]. (Theorem 3.7 has also been proved in a different manner by Koh, Rogers and Tan [24].) In [32] Pastel and Raynaud also exhibited many other families of graceful trees.

To finish the case of graceful trees we mention the following problem due to Kotzig (presented at the Ninth Southeastern Conference on Combinatorics, Boca Raton 1978). To the set of gracefully numbered trees, Kotzig associated the following graph, called G.A.G. in [32]. The vertices of this graph are the graceful trees, gracefully numbered with e edges (two isomorphic trees having different graceful numberings are considered as different). Two vertices are then joined by an edge if the corresponding trees are obtained from each other by deleting one edge and adding another edge with the same label. (See Figure 9 for the case $e = 3$.) Kotzig conjectured that this graph is 1-factorisable (that is, that it has a perfect matching): this conjecture has recently been proved by Pastel (private communication). Kotzig has conjectured also that the graph G.A.G. is connected.

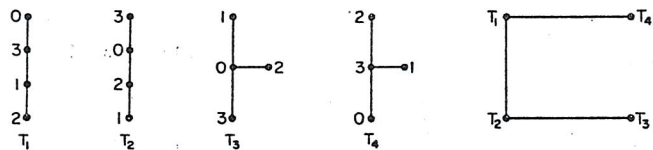


Figure 9

4. COMPLETE GRAPHS

In Figure 1 it was shown that K_3 and K_4 are graceful and it is trivial to see that K_2 can be gracefully labelled. The following result is well known; for a short proof see G. J. Simmons [37, p.630] (see also [7]):

Theorem 4.1. K_n is graceful if and only if $n \leq 4$. \square

In view of Theorem 4.1, it is natural to search for the maximum number of edges $e(n)$ of a graceful graph on n vertices. This question was posed by S. W. Golomb [15], [16], but was in fact considered a long time earlier, in terms of the restricted difference basis we met in the Introduction. More precisely, in these terms one wants to know, for a given e , the minimum number of vertices $n(e)$ of a graceful graph with e edges. In the following table we summarize the known results (see the survey of Miller [30] for examples which attain these bounds).

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$e(n)$	3	6	9	13	17	23	29	36	43	50	59	60	79	90	101	112	123	138

It has also been proved (see [30], for references) that $\lim_{n \rightarrow \infty} e(n)/n^2$ exists, and that $\frac{1}{3} < \lim_{n \rightarrow \infty} e(n)/n^2 < 0.411$.

A related problem is that of finding a set of integers $a_1 < a_2 < \dots < a_n$ such that every positive integer m , $1 \leq m \leq e$, can be represented in the form $m = a_j - a_i$. Such a set is called an unrestricted difference basis with respect to e . The difference between this concept and a restricted difference basis is that here $a_n - a_1$ is not necessarily equal to e . For example, the set $\{0, 8, 18, 19, 22, 24, 31, 39\}$ is an unrestricted difference basis for $e = 24$; indeed the differences $\{a_j - a_i, j > i\}$ are all the integers from 1 to 24, together with 31 and 39 (8 and 31 appearing twice). Note that by the above table, there does not exist a restricted difference basis with $e = 24$ and with only 8 terms. The 'unrestricted difference basis problem' consists of finding, for a given e , the minimum value of the number of terms of an unrestricted difference basis with respect to e . This problem is related to a different kind of labelling of a graph.

Let us call a numbering of a graph \hat{b} -graceful, if it satisfies conditions (a) and (c) of Section 1. A graph which admits a \hat{b} -graceful numbering

will be called a \hat{b} -graceful graph. The unrestricted difference basis problem is thus equivalent to that of finding, for a given e , the minimum number of vertices of a \hat{b} -graceful graph with e edges.

The problem is also equivalent to that of finding for a given n , the maximum number e'_n of edges of a \hat{b} -graceful graph of order n . The following table gives the values known. Here also it has been proved that

$$\lim_{n \rightarrow \infty} e'_n/n^2$$

exists, and that

$$\frac{3}{8} < \lim_{n \rightarrow \infty} e'_n/n^2 < 0.411$$

(see [30] for details).

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
e'_n	3	6	9	13	18	24	29	37	45	51	61	70	79	93	101	113	127

The problem of determining e'_n may be more difficult than that of determining $e(n)$, although in some cases it is easier to characterise some classes of \hat{b} -graceful graphs. For example, it is easy to show by induction on the number of edges that all trees are \hat{b} -graceful.

Another related question, motivated by the fact that K_n is not graceful for $n \geq 5$, is the following problem (see S. W. Golomb [15]). One wants to label the vertices of a graph with distinct non-negative integers in such a way that all of the edges are distinctly labelled. (This numbering satisfies property (a) of Section 1, and a property weaker than (c), since we do not require that all integers between 1 and e appear as labels of edges, but only that they are distinct integers.) So the problem is to find the minimum value $f(G)$ of $\phi(v)$ over all vertices of the graph G with such a numbering. For example, for K_5 we know that this minimum is at least 11 (since K_5 is not graceful); in fact $f(K_5) = 11$, since it suffices to label the vertices respectively 0, 1, 4, 9, 11. All ten edges have different labels - namely, the integers from 1 to 11 except 6. The determination of $f(K_n)$ seems to be a very difficult problem, and the values are known only for $n \leq 11$; the known results are given below. For more details and

references, see Simmons [37] (the problem being exactly that of finding $S(n,1)$ 'synch sets').

n	3	4	5	6	7	8	9	10	11
$f(K_n)$	3	6	11	17	25	34	44	55	72

In view of this table and the known values of $e(n)$, it seems natural to ask whether we always have $e(n) + f(K_n) \geq n(n-1)$.

We conclude this section by mentioning the following extension of Theorem 4.1, due to Kotzig and Turgeon [28]:

Theorem 4.2. The graph consisting of m vertex-disjoint copies of K_n is graceful if and only if $m = 1$ and $n \leq 4$. \square

5. ANTENNAE AND WINDMILLS

To conclude, we mention here how a problem arising in radio-astronomy is related to graceful numbers; for more details see [1] and [2]. The problem is as follows: A few moveable antennae are used in several successive configurations to measure various spatial frequencies relative to some area of sky. Can one obtain an optimal array, where all the distances between antennae are successive multiples of a given increment a without any gaps (otherwise a frequency is missing) and without repetitions (useless)? In practice, a is of the order of the radius of the antennae, and a and $2a$ cannot themselves be obtained (due to overlapping phenomena).

The combinatorial version can be stated as follows: Let m , n and d be positive integers. Is it possible to find m sets A_i each consisting of n integers a_i^j , $1 \leq j \leq n$, such that if

$$D_i = \{ |a_i^j - a_i^{j'}| : 1 \leq j < j' \leq n \},$$

then

$$\sum_{i=1}^m D_i = \{ k : d \leq k \leq \frac{1}{2}mn(n-1) + d - 1 \}.$$

The parameter n corresponds to the number of antennae, m corresponds to the number of days of observation, and d is 1, 2 or 3 (due to the fact that the numbers a and $2a$ are not obtained).

Example 1. If $n = 4, m = 1, d = 1$, we have $A_1 = \{0, 1, 4, 6\}$.

Example 2. If $n = 4, m = 4, d = 1$, we have $A_1 = \{0, 1, 7, 23\}$,
 $A_2 = \{0, 2, 14, 19\}$, $A_3 = \{0, 3, 13, 21\}$, $A_4 = \{0, 4, 15, 24\}$,
 $D_1 = \{1, 6, 7, 16, 22, 23\}$, $D_2 = \{2, 5, 12, 14, 17, 19\}$
 $D_3 = \{3, 8, 10, 13, 18, 21\}$, $D_4 = \{4, 9, 11, 15, 20, 24\}$.

Example 3. If $n = 3, m = 3, d = 2$, we have $A_1 = \{0, 5, 9\}$, $A_2 = \{0, 6, 8\}$,
 $A_3 = \{0, 7, 10\}$.

If $d = 1$, the problem is exactly to determine whether the graph consisting of $m K_n$'s with exactly one vertex in common is graceful. In Figure 10, we show the graceful numbering of 4 K_4 's with one vertex in common (as given in Example 2 above).

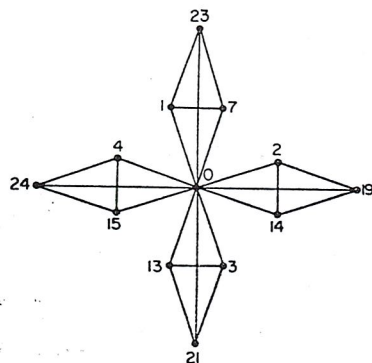


Figure 10

The case $d = 1, n = 3$ was a problem raised by C. Hoede [23], and we shall therefore call a Dutch m-windmill the graph consisting of $m K_3$'s with one vertex in common. In fact, in [1] and [2] we solved the problem just

before Hoede raised his question, for the solution follows immediately from a solution to Skolem's problem: for which values of m is it possible to partition the integers $\{1, 2, \dots, 2m\}$ into m pairs $\{a_i, b_i\}$, $i = 1, 2, \dots, m$, such that $b_i - a_i = i$. The result is:

Theorem 5.1. The Dutch m -windmill is graceful if and only if $m \equiv 0$ or $1 \pmod{4}$. \square

For the general case of the problem stated in the Introduction, we proved in [2] the following result which contains Theorem 4.1 as a special case ($m = d = 1$):

Theorem 5.2. If there exists a solution to the problem, then either $n = 5, m$ is even and $m \geq 4d - 2$;
or $n = 4$ and $m \geq 2d - 1$;
or $n = 3, m \geq 2d - 1$ and $m \equiv 0$ or $1 \pmod{4}$ if d is odd,
 $m \equiv 0$ or $3 \pmod{4}$ if d is even. \square

If $m = d = 1$, the cases $n = 3$ and $n = 4$ are known (the graceful numberings of K_3 and K_4). If $n = 3$ and $d = 1$, the condition is also sufficient by Theorem 5.1. In [2] we solved many cases for $n = 3$; in particular, the cases $d = 2, 3, 4$ and m large enough.

For $n = 4$, only a few results are known. For $d = 1$, as we said above, the problem reduces to that of finding a graceful numbering of the graph consisting of $m K_4$'s with exactly one vertex in common; during the Keszthely Colloquium of 1976, it was proposed to call these graphs French m-windmills. The following conjecture appears in [3]:

Conjecture. French m -windmills are graceful if $m \geq 4$.

Figure 10 shows a graceful French 4-windmill. The French 1-windmill is K_4 , and is thus graceful. The French 2-windmill is not graceful (see the last graph of Figure 2); M. Gardner [12] has called it a 'particularly ungraceful graph'. The French 3-windmill is also not graceful (see [2] for a proof when $m = 2$ and 3).

Other results concerning the case $d \geq 2$ and $n = 5$ are also given in [2]. Note that when the associated graphs are not graceful, one can (exactly as

in Section 4) ask for the maximum number of edges of a graceful subgraph of the graph consisting of $m K_n$'s with exactly one vertex in common.

Let us call a numbering of a graph \hat{a} -graceful if this numbering is defined as in Section 1, except that it does not necessarily satisfy condition (a). A graph which admits an \hat{a} -graceful numbering will be called an \hat{a} -graceful graph. The problem above (in the case $d = 1$) can be considered as the existence of an \hat{a} -graceful numbering of the graph consisting of m vertex-disjoint copies of K_n . The examples above (for example, four copies of K_4) show that there exist \hat{a} -graceful graphs which are not graceful (compare this with Theorem 4.2). If we consider K_8 minus 4 edges, then it can be shown that it is \hat{b} -graceful, but not graceful. It also cannot be \hat{a} -graceful, because if two vertices of K_8 have the same label, then we can only obtain $2l$ values for the edges.

There certainly exist \hat{a} -graceful graphs which are not \hat{b} -graceful, but I am not aware of such an example. As remarked by J. Lehel (private communication), all trees are \hat{a} -graceful; he has also noted that every \hat{a} -graceful graph of diameter 2 is graceful, and that Lemma 2.1 is also valid for \hat{a} -graceful and \hat{b} -graceful graphs.

A related problem, which may be of interest in radio-astronomy (if one wants to move only a restricted number of antennae) is the following:

Problem: For which values of m , n and p is the graph consisting of $m K_n$'s with exactly K_p in common graceful?

The case $p = 1$ is the above problem (for $d = 1$). The case $n = 4$, $p = 3$ was completely solved by H. Meyniel (private communication), who showed that the graph consisting of $m K_4$'s with a K_3 in common is graceful for every m (see Figure 11 for the corresponding graceful numbering).

Solved for $n=4$ $p=2$ by C. Delorme

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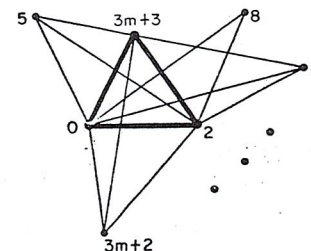


Figure 11

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