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ON NONHAMILTONIAN HOMOGENEOUSLY TRACEABLE DIGRAPHS

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and

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Abstract. A digraph is called homogeneously traceable iff, for each vertex v , there is a hamiltonian path starting at v . We prove that any nonhamiltonian homogeneously traceable digraph with n vertices has at least $2n$ arcs and we construct a family of nonhamiltonian homogeneously traceable oriented graphs (i. e., digraphs without 2-cycles) with n vertices and $2n$ arcs for all $n \geq 7$. For $n < 7$ such oriented graphs do not exist. We also show that nonhamiltonian homogeneously traceable digraphs of order n exist iff $n \geq 5$.

Homogeneously traceable graphs have been recently introduced in [4]. We define the same concept for digraphs as follows:

Definition. A digraph D is *homogeneously traceable* iff, for each vertex v of D , there is a hamiltonian path starting at v .

For brevity we write NHHT instead of "nonhamiltonian homogeneously traceable". It is proved in [2] that the only values of n for which NHHT graphs of order n do not exist are $3 \leq n \leq 8$. From each NHHT graph of order n , $n \geq 9$, given in [2], we cannot obtain NHHT digraphs without 2-cycles. Chartrand, Gould and Kapoor [private communication] raised the question whether NHHT oriented graphs (i.e., digraphs without 2-cycles) exist. We answer this question in the affirmative and furthermore we prove that NHHT digraphs exist not only for $n \geq 9$ but also for $5 \leq n \leq 8$, and do not exist for $n \leq 4$.

We state now:

Lemma 1. *If D is a NHHT digraph with n vertices, then D has at least $2n$ arcs.*

Proof. Suppose $\text{od}(v) = 1$ for some vertex v in D . The hamiltonian path starting at the vertex adjacent from v must end at v , hence there is a hamiltonian cycle in D , a contradiction. Hence $\text{od}(v) \geq 2$ for each v , which proves the lemma.

Denote by D_7 the digraph pictured in Figure 1.

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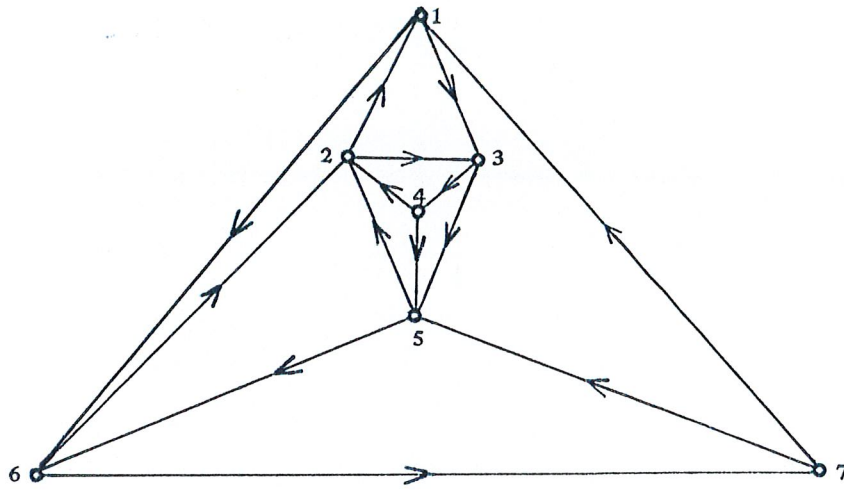


Figure 1.

Lemma 2. D_7 is an arc-minimal NHHT oriented graph.

Proof. D_7 is nonhamiltonian. In fact, a hamiltonian cycle should contain $(6, 7)$ and $(3, 4)$, hence it contains neither $(6, 2)$ nor $(3, 5)$. Therefore it contains either $(4, 2)$ and $(7, 5)$ or $(5, 2)$, $(4, 5)$ and $(7, 1)$. Both cases lead to an impossibility.

D_7 is homogeneously traceable, as can be easily verified, and, by Lemma 1, arc-minimal.

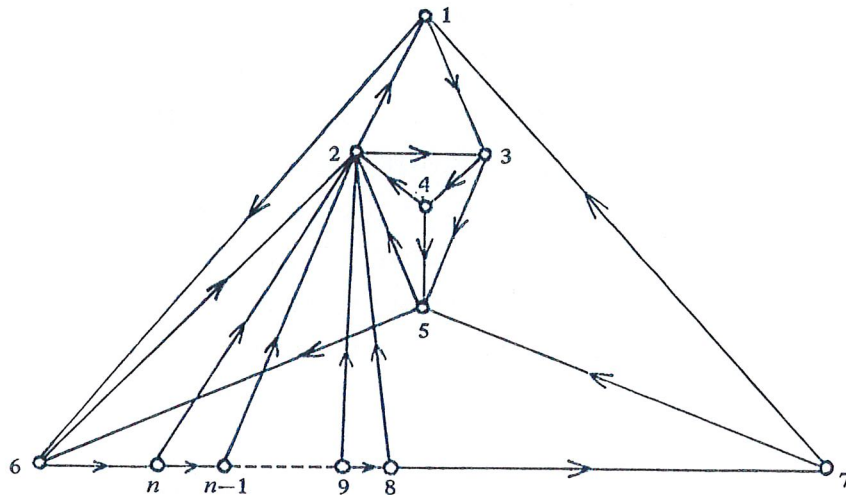


Figure 2.

Now insert in D_7 , between vertices 6 and 7, the new vertices $n, n-1, \dots, 9, 8$, replace the arc $(6, 7)$ by the directed path $(6, n), (n, n-1), \dots, (8, 7)$ and add the arcs $(n, 2), (n-1, 2), \dots, (8, 2)$. Denote the oriented graph so obtained by D_n (Figure 2). We will prove that D_n is NHHT. In fact, if D_n has a hamiltonian cycle Z , then Z contains the vertex sequence $6, n, \dots, 8, 7$ which implies the existence of a hamiltonian cycle in D_7 . Hence D_n is nonhamiltonian. On the other hand hamiltonian paths of D_n starting at each vertex i in D_n may be obtained as follows: For

$i \notin \{7, 8, \dots, n\}$ replace, in the hamiltonian path of D_7 starting at i , the arc $(6, 7)$, which must be used, by the path $(6, n), (n, n-1), \dots, (8, 7)$. For $i \in \{7, 8, \dots, n\}$ such a hamiltonian path is $(i, i-1), \dots, (8, 7), (7, 5), (5, 6), (6, n), (n, n-1), \dots, (i+1, 2), (2, 1), (1, 3), (3, 4)$. Since D_n is obtained from D_7 by adding $n-7$ vertices and $2(n-7)$ arcs, we have thus proved:

Theorem 1. *The graphs D_n form a family of arc-minimal NHHT oriented graphs of all orders $n \geq 7$.*

In the proof of Lemma 1 we have shown that the outdegree of any vertex in a NHHT digraph is at least 2. Using this fact, it is easy to check that NHHT oriented graphs of order n do not exist for $n \leq 5$. M. Syslo [5] has shown, using a computer, that no NHHT oriented graph of order $n = 6$ exists. It follows then from our theorem:

Corollary. *NHHT oriented graphs of order n exist if and only if $n \geq 7$.*

Using the same fact as above it can also be hand checked that no NHHT digraph of order n exists for $n \leq 4$. We give in Figure 3 a NHHT digraph of order $n = 5$. It is easy to see that if D is a NHHT digraph (oriented graph) of order n and D' is obtained from D by selecting a vertex v in D and adding a new vertex u and the arcs $\{(v, u)\} \cup \{(u, w) \mid (v, w) \in D\}$, then D' is a NHHT digraph (oriented graph) of order $n+1$; D' is however not arc-minimal. We therefore obtain:

Theorem 2. *NHHT digraphs of order n exist if and only if $n \geq 5$.*

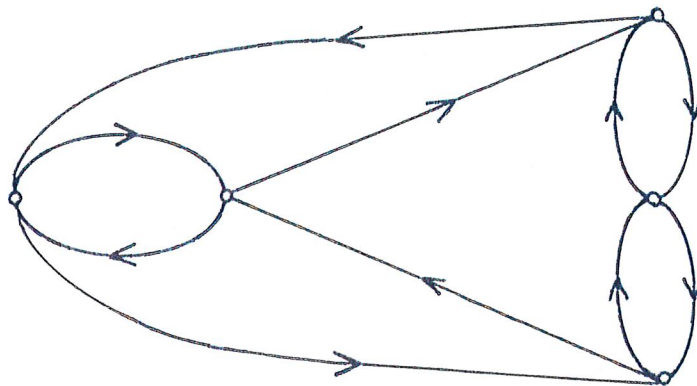


Figure 3.

We would like to mention that hypohamiltonian digraphs are NHHT digraphs (A hypohamiltonian digraph is a nonhamiltonian digraph such that any subdigraph obtained from it by deleting one vertex is hamiltonian). Hypohamiltonian digraphs are known to exist for orders $n \geq 6$ ([3], [6]) and examples of hypohamiltonian oriented graphs are also known [6].

T. Zamfirescu [private communication] has exhibited an NHHT oriented graph of order 8, which is not a member of the family we exhibit in our paper.

Since converses of homogeneously traceable digraphs may or may not be homogene-

ously traceable, it remains an open question for which orders n does there exist an arc-minimal NHHT digraph (oriented graph) whose converse is also an NHHT digraph (oriented graph).

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