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Timed Negotiations

S. Akshay¹, Blaise Genest², Loic Helouet³, and Sharvik Mital¹

¹ IIT Bombay, Mumbai, India {akshayss,sharky}@cse.iitb.ac.in

² Univ Rennes, CNRS, IRISA, Rennes, France blaise.genest@irisa.fr

³ Univ Rennes, INRIA, Rennes, France loic.helouet@inria.fr

Abstract. Negotiations were introduced in [6] as a model for concurrent systems with multiparty decisions. What is very appealing with negotiations is that it is one of the very few non-trivial concurrent models where several interesting problems, such as soundness, i.e. absence of deadlocks, can be solved in PTIME [2]. In this paper, we introduce the model of timed negotiations and consider the problem of computing the minimum and the maximum execution time of a negotiation. The latter can be solved using the algorithm of [10] computing costs in negotiations, but surprisingly minimum execution time cannot.

In this paper, we propose new algorithms to compute both minimum and maximum execution time, that work in much more general classes of negotiations than [10], that only considered sound and deterministic negotiations. Further, we uncover the precise complexities of these questions, ranging from PTIME to Δ_2^P -complete. In particular, we show that computing the minimum execution time is more complex than computing the maximum execution time in most classes of negotiations we consider.

1 Introduction

Distributed systems are notoriously difficult to analyze, mainly due to the explosion of the number of configurations that have to be considered to answer even simple questions. A challenging task is then to propose models on which analysis can be performed with tractable complexities, preferably within polynomial time. Free choice Petri nets are a classical model of distributed systems that allow for efficient verification, in particular when the nets are 1-safe [5, 4].

Recently, [6] introduced a new model called *negotiations* for workflows and business processes. A negotiation describes how processes interact in a distributed system: a subset of processes in a node of the system take a synchronous decisions among several *outcomes*. The effect of this outcome sends contributing processes to a new set of nodes. The execution of a negotiation ends when processes reach a *final configuration*. Negotiations can be deterministic (once an outcome is fixed, each process knows its unique successor node) or not.

Negotiations are an interesting model since several properties can be decided with a reasonable complexity. The question of *soundness*, i.e., deadlock-freedom: whether from every reachable configuration one can reach a final configuration, is PSPACE-complete. However, for deterministic negotiations, it can be decided

41 in PTIME [7]. The decision procedure uses reduction rules. Reduction techniques
42 were originally proposed for Petri nets [1, 8, 12, 17]. The main idea is to define
43 transformations rules that produce a model of smaller size w.r.t. the original
44 model, while preserving the property under analysis. In the context of negotia-
45 tions, [7, 2] proposed a sound and complete set of soundness-preserving reduction
46 rules and algorithms to apply these rules efficiently. The question of soundness
47 for deterministic negotiations was revisited in [9] and showed NLOGSPACE-
48 complete using anti patterns instead of reduction rules. Further, they show that
49 the PTIME result holds even when relaxing determinism [9]. Negotiation games
50 have also been considered to decide whether one particular process can force ter-
51 mination of a negotiation. While this question is EXPTIME complete in general,
52 for sound and deterministic negotiations, it becomes PTIME [13].

53 While it is natural to consider cost or time in negotiations (e.g. think of the
54 Brexit negotiation where time is of the essence, and which we model as running
55 example in this paper), the original model of negotiations proposed by [6] is
56 only qualitative. Recently, [10] has proposed a framework to associate costs to
57 the executions of negotiations, and adapt a static analysis technique based on
58 reduction rules to compute end-to end cost functions that are not sensitive to
59 scheduling of concurrent nodes. For sound *and* deterministic negotiations, the
60 end-to end cost can be computed in $O(n.(C + n))$, where n is the size of the
61 negotiation and C the time needed to compute the cost of an execution. Requir-
62 ing soundness or determinism seem perfectly reasonable, but asking sound *and*
63 deterministic negotiations is too restrictive: it prevents a process from waiting
64 for decisions of other processes to know how to proceed.

65 In this paper, we revisit time in negotiations. We attach time intervals to
66 outcomes of nodes. We want to compute maximal and minimal executions times,
67 for negotiations that are not necessarily sound and deterministic. Since we are
68 interested in minimal and maximal execution time, cycles in negotiations can be
69 either bypassed or lead to infinite maximal time. Hence, we restrict this study to
70 acyclic negotiations. Notice that time can be modeled as a cost, following [10],
71 and the maximal execution time of a sound and deterministic negotiation can
72 be computed in PTIME using the algorithm from [10]. Surprisingly however, we
73 give an example (Example 3) for which the minimal execution time cannot be
74 computed in PTIME by this algorithm.

75 The first contribution of the paper shows that reachability (whether at least
76 one run of a negotiation terminates) is NP-complete, already for (untimed) deter-
77 ministic acyclic negotiations. This implies that computing minimal or maximal
78 execution time for deterministic (but unsound) acyclic negotiations cannot be
79 done in PTIME (unless NP=PTIME). We characterize precisely the complex-
80 ities of different decision variants (threshold, equality, etc.), with complexities
81 ranging from (co-)NP-complete to Δ_2^P .

82 We thus turn to negotiations that are sound but not necessarily determinis-
83 tic. Our second contribution is a new algorithm, not based on reduction rules,
84 to compute the maximal execution time in PTIME for sound negotiations. It is
85 based on computing the maximal execution time of critical paths in the nego-

86 tiations. However, we show that *minimal* execution time cannot be computed
 87 in PTIME for sound negotiations (unless NP=PTIME): deciding whether the
 88 minimal execution time is lower than T is NP-complete, even for T given in
 89 unary, using a reduction from a Bin packing problem. This shows that minimal
 90 execution time is harder to compute than maximal execution time.

91 Our third contribution consists in defining a class in which the minimal exe-
 92 cution time can be computed in (pseudo) PTIME. To do so, we define the class
 93 of k -layered negotiations, for k fixed, that is negotiations where nodes can be or-
 94 ganized into layers of at most k nodes at the same depth. These negotiations can
 95 be executed without remembering more than k nodes at a time. In this case, we
 96 show that computing the maximal execution time is PTIME, even if the negoti-
 97 ation is neither deterministic nor sound. The algorithm, not based on reduction
 98 rules, uses the k -layer restriction in order to navigate in the negotiation while
 99 considering only a polynomial number of configurations. For minimal execution
 100 time, we provide a pseudo PTIME algorithm, that is PTIME if constants are
 101 given in unary. Finally, we show that the size of constants do matter: deciding
 102 whether the minimal execution time of a k -layered negotiation is less than T
 103 is NP-complete, when T is given in binary. We show this by reducing from a
 104 Knapsack problem, yet again emphasizing that the minimal execution time of a
 105 negotiation is harder to compute than its maximal execution time.

106 This paper is organized as follows. Section 2 introduces the key ingredients of
 107 negotiations, determinism and soundness, known results in the untimed setting,
 108 and provides our running example modeling the Brexit negotiation. Section 3
 109 introduces time in negotiations, gives a semantics to this new model, and for-
 110 malizes several decision problems on maximal and minimal durations of runs in
 111 timed negotiations. We recall the main results of the paper in Section 4. Then,
 112 Section 5 considers timed execution problems for deterministic negotiations, Sec-
 113 tion 6 for sound negotiations, and section 7 for layered negotiations. Proof details
 114 for the last three technical sections are given in the Appendices A, B and C.

115 2 Negotiations: Definitions and Brexit example

116 In this section, we recall the definition of negotiations, of some subclasses (acyclic
 117 and deterministic), as well as important problems (soundness and reachability).

118 **Definition 1 (Negotiation [6, 10]).** *A negotiation over a finite set of pro-*
 119 *cesses P is a tuple $\mathcal{N} = (N, n_0, n_f, \mathcal{X})$, where:*

- 120 – N is a finite set of nodes. Each node is a pair $n = (P_n, R_n)$ where $P_n \subseteq P$
 121 is a non empty set of processes participating in node n , and R_n is a finite
 122 set of outcomes of node n (also called results), with $R_{n_f} = \{r_f\}$. We denote
 123 by R the union of all outcomes of nodes in N .
- 124 – n_0 is the first node of the negotiation and n_f is the final node. Every process
 125 in P participates in both n_0 and n_f .
- 126 – For all $n \in N$, $\mathcal{X}_n : P_n \times R_n \rightarrow 2^N$ is a map defining the transition relation
 127 from node n , with $\mathcal{X}_n(p, r) = \emptyset$ iff $n = n_f, r = r_f$. We denote $\mathcal{X} : N \times P \times$

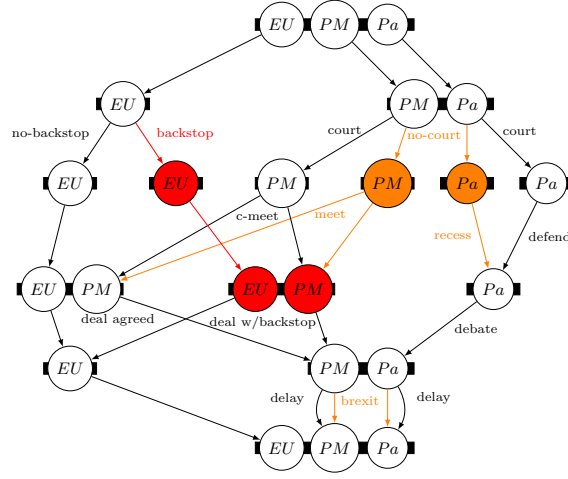


Fig. 1. A (sound but non-deterministic) negotiation modeling Brexit.

128 $R \rightarrow 2^N$ the partial map defined on $\bigcup_{n \in N} (\{n\} \times P_n \times R_n)$, with $\mathcal{X}(n, p, a) =$
 129 $\mathcal{X}_n(p, a)$ for all p, a .

130 Intuitively, at a node $n = (P_n, R_n)$ in a negotiation, all processes of P_n have
 131 to agree on a common outcome r chosen from R_n . Once this outcome r is chosen,
 132 every process $p \in P_n$ is ready to move to any node prescribed by $\mathcal{X}(n, p, r)$. A
 133 new node m can only start when all processes of P_m are ready to move to m .

134 *Example 1.* We illustrate negotiations by considering a simplified model of the
 135 Brexit negotiation, see Figure 1. There are 3 processes, $P = \{EU, PM, Pa\}$. At
 136 first EU decides whether or not to enforce a backstop in any deal (outcome back-
 137 stop) or not (outcome no-backstop). In the meantime, PM decides to prorogue
 138 Pa , and Pa can choose or not to appeal to court (outcome court/no court). If it
 139 goes to court, then PM and Pa will take some time in court (c-meet, defend),
 140 before PM can meet EU to agree on a deal. Otherwise, Pa goes to recess, and
 141 PM can meet EU directly. Once EU and PM agreed on a deal, PM tries to
 142 convince Pa to vote the deal. The final outcome is whether the deal is voted, or
 143 whether Brexit is delayed.

144 **Definition 2 (Deterministic negotiations).** A process $p \in P$ is determinis-
 145 tic iff, for every $n \in N$ and every outcome r of n , $\mathcal{X}(n, p, r)$ is a singleton. A ne-
 146 gotiation is deterministic iff all its processes are deterministic. It is weakly non-
 147 deterministic [9] (called weakly deterministic in [2]) iff, for every node n , one of
 148 the processes in P_n is deterministic. Last, it is very weakly non-deterministic [9]
 149 (called weakly deterministic in [6]) iff, for every n , every $p \in P_n$ and every out-
 150 come r of n , there exists a deterministic process q such that $q \in P_{n'}$ for every
 151 $n' \in \mathcal{X}(n, p, r)$.

152 In deterministic negotiations, once an outcome is chosen, each process knows
 153 the next node it will be involved in. In (very)-weakly non-deterministic nego-
 154 tiations, the next node might depend upon the outcome chosen in other nodes
 155 by other processes. However, once the outcomes have been chosen for all cur-
 156 rent nodes, there is only one next node possible for each process. Observe that
 157 the class of deterministic negotiations is isomorphic to the class of free choice
 158 workflow nets [10]. Coming back to example 1, the Brexit negotiation is non-
 159 deterministic, because process PM is non-deterministic. Indeed, consider out-
 160 comes $c-meet$: it allows two nodes, according to whether the backstop is enforced
 161 or not, which is a decision taken by process EU . However, the Brexit negotiation
 162 is very weakly non-deterministic, as the other processes are deterministic.

163 **Semantics:** A *configuration* [2] of a negotiation is a mapping $M : P \rightarrow 2^N$.
 164 Intuitively, it tells for each process p the set $M(p)$ of nodes p is ready to engage in.
 165 The semantics of a negotiation is defined in terms of moves from a configuration
 166 to the next one. The *initial* M_0 and *final* M_f configurations, are given by $M_0(p) =$
 167 $\{n_0\}$ and $M_f(p) = \emptyset$ respectively for every process $p \in P$. A configuration M
 168 *enables* node n if $n \in M(p)$ for every $p \in P_n$. When n is enabled, a decision
 169 at node n can occur, and the participants at this node choose an outcome $r \in$
 170 R_n . The occurrence of (n, r) produces the configuration M' given by $M'(p) =$
 171 $\mathcal{X}(n, p, r)$ for every $p \in P_n$ and $M'(p) = M(p)$ for remaining processes in $P \setminus P_n$.
 172 Moving from M to M' after choosing (n, r) is called a *step*, denoted $M \xrightarrow{n,r} M'$. A
 173 *run* of \mathcal{N} is a sequence $(n_1, r_1), (n_2, r_2) \dots (n_k, r_k)$ such that there is a sequence of
 174 configurations M_0, M_1, \dots, M_k and every (n_i, r_i) is a step between M_{i-1} and M_i .
 175 A run starting from the initial configuration and ending in the final configuration
 176 is called a *final run*. By definition, its last step is (n_f, r_f) .

177 An important class of negotiations in the context of timed negotiations are
 178 acyclic negotiations, where infinite sequence of steps are impossible:

179 **Definition 3 (Acyclic negotiations).** *The graph of a negotiation \mathcal{N} is the*
 180 *labeled graph $G_{\mathcal{N}} = (V, E)$ where $V = N$, and $E = \{((n, (p, r), n') \mid n' \in$
 181 $\mathcal{X}(n, p, r)\}$, with pairs of the form (p, r) being the labels. A negotiation is acyclic*
 182 *iff its graph is acyclic. We denote by $Paths(G_{\mathcal{N}})$ the set of paths in the graph of a*
 183 *negotiation. These paths are of the form $\pi = (n_0, (p_0, r_0), n_1) \dots (n_{k-1}, (p_k, r_k), n_k)$.*

184 The Brexit negotiation of Fig.1 is an example of acyclic negotiation. Despite
 185 their apparent simplicity, negotiations may express involved behaviors as shown
 186 with the Brexit example. Indeed two important questions in this setting are
 187 whether there is some way to reach a final node in the negotiation from (i) the
 188 initial node and (ii) any reachable node in the negotiation.

189 **Definition 4 (Soundness and Reachability).**

- 190 1. A negotiation is *sound* iff every run from the initial configuration can be
 191 extended to a final run. The problem of soundness is to check if a given
 192 negotiation is sound.
- 193 2. The problem of reachability asks if a given negotiation has a final run.

194 Notice that the Brexit negotiation of Fig.1 is sound (but not deterministic).
 195 It seems hard to preserve the important features of this negotiation while being
 196 both sound *and* deterministic. The problem of soundness has received consider-
 197 able attention. We summarize the results about soundness in the next theorem:

198 **Theorem 1.** *Determining whether a negotiation is sound is PSPACE-Complete.*
 199 *For (very-)weakly non-deterministic negotiations, it is co-NP-complete [9]. For*
 200 *acyclic negotiations, it is in DP and co-NP-Hard [6]. Determining whether an*
 201 *acyclic weakly non-deterministic negotiation is sound is in PTIME [2, 9]. Fi-*
 202 *nally, deciding soundness for deterministic negotiation is NLOGSPACE-complete [9].*

203 Checking reachability is NP-complete, even for deterministic acyclic negoti-
 204 ations (surprisingly, we did not find this result stated before in the literature):

205 **Proposition 1.** *Reachability is NP-complete for acyclic negotiations, even if*
 206 *the negotiation is deterministic.*

Proof (sketch). One can easily guess a run of size $\leq |\mathcal{N}|$ in polynomial time, and
 verify if it reaches n_f , which gives the inclusion in NP. The hardness part comes
 from a reduction from 3-CNF-SAT that can be found in the proof of Theorem 3. \square

207 ***k*-Layered Acyclic Negotiations**

208 We introduce a new class of negotiations which has good algorithmic properties,
 209 namely *k*-layered acyclic negotiations, for *k* fixed. Roughly speaking, nodes of a
 210 *k*-layered acyclic negotiations can be arranged in layers, and these layers contain
 211 at most *k* nodes. Before giving a formal definition, we need to define the depth
 212 of nodes in \mathcal{N} .

213 First, a *path* in a negotiation is a sequence of nodes $n_0 \dots n_\ell$ such that for
 214 all $i \in \{1, \dots, \ell - 1\}$, there exists p_i, r_i with $n_{i+1} \in \mathcal{X}(n_i, p_i, r_i)$. The *length* of a
 215 path n_0, \dots, n_ℓ is ℓ . The *depth* $\text{depth}(n)$ of a node n is the maximal length of a
 216 path from n_0 to n (recall that \mathcal{N} is acyclic, so this number is always finite).

217 **Definition 5.** *An acyclic negotiation is layered if for all node n , every path*
 218 *reaching n has length $\text{depth}(n)$. An acyclic negotiation is *k*-layered if it is layered,*
 219 *and for all $\ell \in \mathbb{N}$, there are at most *k* nodes at depth ℓ .*

220 The Brexit example of Fig.1 is 6-layered. Notice that a layered negotiation
 221 is necessarily *k*-layered for some $k \leq |\mathcal{N}| - 2$. Note also that we can always
 222 transform an acyclic negotiation \mathcal{N} into a layered acyclic negotiation \mathcal{N}' , by
 223 adding dummy nodes: for every node $m \in \mathcal{X}(n, p, r)$ with $\text{depth}(m) > \text{depth}(n) +$
 224 1 , we can add several nodes n_1, \dots, n_ℓ with $\ell = \text{depth}(m) - (\text{depth}(n) + 1)$, and
 225 processes $P_{n_i} = \{p\}$. We compute a new relation \mathcal{X}' such that $\mathcal{X}'(n, p, r) =$
 226 $\{n_1\}$, $\mathcal{X}'(n_\ell, p, r) = \{m\}$ and for every $i \in 1..\ell - 1$, $\mathcal{X}'(n_i, p, r) = n_{i+1}$. This
 227 transformation is polynomial: the resulting negotiation is of size up to $|\mathcal{N}| \times$
 228 $|\mathcal{X}| \times |P|$. The proof of the following Theorem can be found in appendix C.

229 **Theorem 2.** *Let $k \in \mathbb{N}^+$. Checking reachability or soundness for a *k*-layered*
 230 *acyclic negotiation \mathcal{N} can be done in PTIME.*

231 **3 Timed Negotiations**

232 In many negotiations, time is an important feature to take into account. For
 233 instance, in the Brexit example, with an initial node starting at the begining of
 234 September 2019, there are 9 weeks to pass a deal till the 31st October deadline.

235 We extend negotiations by introducing timing constraints on outcomes of
 236 nodes, inspired by time Petri nets [15] and by the notion of negotiations with
 237 costs [10]. We use time intervals to specify lower and upper bounds for the
 238 duration of negotiations. More precisely, we attach time intervals to pairs (n, r)
 239 where n is a node and r an outcome. In the rest of the paper, we denote by
 240 \mathcal{I} the set of intervals with endpoints that are non-negative integers or ∞ . For
 241 convenience we only use closed intervals in this paper (except for ∞), but the
 242 results we show can also be extended to open intervals with some notational
 243 overhead. Intuitively, outcome r can be taken at a node n with associated time
 244 interval $[a, b]$ only after a time units have elapsed from the time all processes
 245 contributing to n are ready to engage in n , and at most b time units later.

246 **Definition 6.** A timed negotiation is a pair (\mathcal{N}, γ) where \mathcal{N} is a negotiation,
 247 and $\gamma : N \times R \rightarrow \mathcal{I}$ associates an interval to each pair (n, r) of node and outcome
 248 such that $r \in R_n$. For a given node n and outcome r , we denote by $\gamma^-(n, r)$ (resp.
 249 $\gamma^+(n, r)$) the lower bound (resp. the upper bound) of $\gamma(n, r)$.

250 *Example 2.* In the Brexit example, we define the following timed constraints γ .
 251 We only specify the outcome names, as the timing only depends upon them.
 252 Backstop and no-backstop both take between 1 and 2 weeks: $\gamma(\text{backstop}) =$
 253 $\gamma(\text{no-backstop}) = [1, 2]$. In case of no-court, recess takes 5 weeks $\gamma(\text{recess}) =$
 254 $[5, 5]$, and PM can meet EU immediatly $\gamma(\text{meet}) = [0, 0]$. In case of court ac-
 255 tion, PM needs to spend 2 weeks in court $\gamma(\text{c-meet}) = [2, 2]$, and depending on
 256 the court delay and decision, Pa needs between 3 (court overrules recess) to 5
 257 (court confirms recess) weeks, $\gamma(\text{defend}) = [3, 5]$. Agreeing on a deal can take
 258 anywhere from 2 weeks to 2 years (104 weeks): $\gamma(\text{deal agreed}) = [2, 104]$ - some
 259 would say infinite time is even possible! It needs more time with the backstop,
 260 $\gamma(\text{deal w/backstop}) = [5, 104]$. All others outcomes are assumed to be immedi-
 261 ate, i.e., associated with $[0, 0]$.

262 **Semantics:** A *timed valuation* is a map $\mu : P \rightarrow \mathbb{R}^{\geq 0}$ that associates a non-
 263 negative real value to every process. A *timed configuration* is a pair (M, μ) where
 264 M is a configuration and μ a timed valuation. There is a *timed step* from (M, μ)
 265 to (M', μ') , denoted $(M, \mu) \xrightarrow{(n,r)} (M', \mu')$, if (i) $M \xrightarrow{(n,r)} M'$, (ii) $p \notin P_n$ implies
 266 $\mu'(p) = \mu(p)$ (iii) $p \in P_n$ implies $(\mu'(p) - \max_{p' \in P_n} \mu(p')) \in \gamma(n, r)$

267 Intuitively a timed step $(M, \mu) \xrightarrow{(n,r)} (M', \mu')$ depicts a decision taken at
 268 node n , and how long each process of P_n waited in that node before taking
 269 decision (n, r) . The last process engaged in n must wait for a duration contained
 270 in $\gamma(n, r)$. However, other processes may spend a time greater than $\gamma^+(n, r)$.

271 A *timed run* is a sequence of steps $\rho = (M_1, \mu_1) \xrightarrow{e_1} (M_2, \mu_2) \dots (M_k, \mu_k)$
 272 where each $(M_i, \mu_i) \xrightarrow{e_i} (M_{i+1}, \mu_{i+1})$ is a timed step. It is *final* if $M_k = M_f$. Its
 273 *execution time* $\delta(\rho)$ is defined as $\delta(\rho) = \max_{p \in P} \mu_k(p)$.

274 Notice that we only attached timing to processes, not to individual steps.
 275 With our definition of runs, timing on steps may not be monotonous (i.e., non-
 276 decreasing) along the run, while timing on processes is. Viewed by the lens of
 277 concurrent systems, the timing is monotonous on the partial orders of the system
 278 rather than the linearization. It is not hard to restrict paths, if necessary, to have
 279 a monotonous timing on steps as well. In this paper, we are only interested in
 280 execution time, which does not depend on the linearization considered.

281 Given a timed negotiation \mathcal{N} , we can now define the minimum and maximum
 282 execution time, which correspond to optimistic or pessimistic views:

283 **Definition 7.** *Let \mathcal{N} be a timed negotiation. Its minimum execution time, de-*
 284 *noted $\text{mintime}(\mathcal{N})$ is the minimal $\delta(\rho)$ over all final timed run ρ of \mathcal{N} . We*
 285 *define the maximal execution time $\text{maxtime}(\mathcal{N})$ of \mathcal{N} similarly.*

286 Given $T \in \mathbb{N}$, the main problems we consider in this paper are the following:

- 287
- 288 – The mintime problem, i.e., do we have $\text{mintime}(\mathcal{N}) \leq T$?
 - 289 In other words, does there exist a final timed run ρ with $\delta(\rho) \leq T$?
 - 290 – The maxtime problem, i.e., do we have $\text{maxtime}(\mathcal{N}) \leq T$?
 - 291 In other words, does $\delta(\rho) \leq T$ for every final timed run ρ ?

292 These questions have a practical interest : in the Brexit example, the question
 293 “is there a way to have a vote on a deal within 9 weeks ?” is indeed a minimum
 294 execution time problem. We also address the equality variant of these decision
 295 problems, i.e., $\text{mintime}(\mathcal{N}) = T$: is there a final run of \mathcal{N} that terminates
 296 in exactly T time units and no other final run takes less than T time units?
 297 Similarly for $\text{maxtime}(\mathcal{N}) = T$.

298 *Example 3.* We use Fig. 1 to show that it is not easy to compute the minimal
 299 execution time, and in particular one cannot use the algorithm from [10] to com-
 300 pute it. Consider the node n with $P_n = \{PM, Pa\}$ and $R_n = \{\text{court}, \text{no_court}\}$.
 301 If the outcome is court, then PM needs 2 weeks before he can talk to EU and Pa
 302 needs at least 3 weeks before he can debate. However, if the outcome is no_court,
 303 then PM need not wait before he can talk to EU , but Pa wastes 5 weeks in re-
 304 cess. This means that one needs to remember different alternatives which could
 305 be faster in the end, depending on the future. On the other hand, the algorithm
 306 from [10] attaches one minimal time to process Pa , and one minimal time to
 307 process PM . No matter the choices (0 or 2 for PM and 3 or 5 for Pa), there
 308 will be futures in which the chosen number will over or underapproximate the
 309 real minimal execution time (this choice is not explicit in [10])⁴. For maximum
 310 execution time, it is not an issue to attach to each node a unique maximal exe-
 311 cution time. The reason for the asymmetry between minimal execution time and
 312 maximal execution time of a negotiation is that the execution time of a path
 313 is $\max_{p \in P} \mu_k(p)$, for μ_k the last timed valuation, hence breaking the symmetry
 314 between min and max.

⁴ the authors of [10] acknowledged the issue with their algorithm for mintime.

315 4 High Level view of the main results

316 In this section, we give a high-level description of our main results. Formal
 317 statements can be found in the sections where they are proved. We gather in
 318 Fig. 2 the precise complexities for the minimal and the maximal execution time
 319 problems for 3 classes of negotiations that we describe in the following. Since we
 320 are interested in minimum and maximum execution time, cycles in negotiations
 321 can be either bypassed or lead to infinite maximal time. Hence, while we define
 322 timed negotiations in general, we always restrict to acyclic negotiations (such as
 323 Brexit) while stating and proving results.

324 In [10], a PTIME algorithm is given to compute different costs for negoti-
 325 ations that are both sound *and* deterministic. One limitation of this result is
 326 that it cannot compute the minimum execution time, as explained in Example
 327 3. A second limitation is that the class of sound and deterministic negotiations
 328 is quite restrictive: it cannot model situations where the next node a process
 329 participates in depends on the outcome from another process, as in the Brexit
 330 example. We thus consider classes where one of these restrictions is dropped.

331 We first consider (Section 5) negotiations that are deterministic, but with-
 332 out the soundness restriction. We show that for this class, no timed problem
 333 we consider can be solved in PTIME (unless NP=PTIME). Further, we show
 334 that the equality problems ($maxtime/mintime(\mathcal{N}) = T$), are complete for the
 335 complexity class DP, i.e., at the second level of the Boolean Hierarchy [16].

336 We then consider (Section 6) the class of negotiations that are sound, but not
 337 necessarily deterministic. We show that maximum execution time can be solved
 338 in PTIME, and propose a new algorithm. However, the minimum execution time
 339 cannot be computed in PTIME (unless NP=PTIME). Again for the mintime
 340 equality problem we have a matching DP-completeness result.

341 Finally, in order to obtain a polytime algorithm to compute the minimum
 342 execution time, we consider the class of k -layered negotiations (see Section 7):
 343 Given $k \in \mathbb{N}$, we can show that $maxtime(\mathcal{N})$ can be computed in PTIME for
 344 k -layered negotiations. We also show that while the $mintime(\mathcal{N}) \leq T?$ problem
 345 is weakly NP-complete for k -layered negotiations, we can compute $mintime(\mathcal{N})$
 346 in pseudo-PTIME, i.e. in PTIME if constants are given in unary.

	Deterministic	Sound	k -layered
Max $\leq T$	co-NP-complete (Thm. 3)	PTIME (Prop. 3)	PTIME (Thm. 6)
Max = T	DP-complete (Prop. 2)		
Min $\leq T$	NP-complete (Thm. 3)	NP-complete* (Thm. 5)	pseudo-PTIME (Thm. 8)
Min = T	DP-complete (Prop. 2)	DP-complete* (Prop. 4)	NP-complete** (Thm. 7)
			pseudo-PTIME (Thm. 8)

Fig. 2. Results for acyclic timed negotiations. *DP* refers to the complexity class, Difference Polynomial time [16], the second level of the Boolean Hierarchy.

* hardness holds even for very weakly non-deterministic negotiations, and T in unary.

** hardness holds even for sound and very weakly non-deterministic negotiations.

347 5 Deterministic Negotiations

348 We start by considering the class of deterministic acyclic negotiations. We show
 349 that both maximal and minimal execution time cannot be computed in PTIME
 350 (unless NP=PTIME), as the threshold problems are (co-)NP-complete.

351 **Theorem 3.** *The $\text{mintime}(\mathcal{N}) \leq T$ decision problem is NP complete, and the*
 352 *$\text{maxtime}(\mathcal{N}) \leq T$ decision problem is co-NP complete for acyclic deterministic*
 353 *timed negotiations.*

354 *Proof.* For $\text{mintime}(\mathcal{N}) \leq T$, containment in NP is easy: we just need to guess a
 355 run ρ (of polynomial size as \mathcal{N} is acyclic), consider the associated timed run ρ^-
 356 where all decisions are taken at their earliest possible dates, and check whether
 357 $\delta(\rho^-) \leq T$, which can be done in time $O(|\mathcal{N}| + \log T)$.

358 For the hardness, we give the proof in two steps. First, we start with a proof
 359 of Proposition 1 that reachability problem is NP-hard using reduction of 3-CNF
 360 SAT, i.e., given a formula ϕ , we build a deterministic negotiation \mathcal{N}_ϕ s.t. ϕ is
 361 satisfiable iff \mathcal{N}_ϕ has a final run. In a second step, we introduce timings on this
 362 negotiation and show that $\text{mintime}(\mathcal{N}_\phi) \leq T$ iff ϕ is satisfiable.

363 Step 1: Reducing 3-CNF-SAT to Reachability problem.

364 Given a boolean formula ϕ with variables $v_i, 1 \leq i \leq n$ and clauses $c_j, 1 \leq j \leq$
 365 m , for each variable v_i we define the sets of clauses $S_{i,t} = \{c_j | v_i \text{ is present in } c_j\}$
 366 and $S_{i,f} = \{c_j | \neg v_i \text{ is present in } c_j\}$. Clauses in $S_{i,t}$ and $S_{i,f}$ are naturally or-
 367 dered: $c_i < c_j$ iff $i < j$. We denote these elements $S_{i,t}(1) < S_{i,t}(2) < \dots$
 368 Similarly for set $S_{i,f}$.

369 Now, we construct a negotiation \mathcal{N}_ϕ (as depicted in Figure 3) with a process
 370 V_i for each variable v_i and a process C_j for each clause c_j :

- 371 – Initial node n_0 has a single outcome r taking each process C_j to node $Lone_{c_j}$,
 372 and each process V_i to node $Lone_{v_i}$.
- 373 – $Lone_{c_j}$ has three outcomes: if literal $v_i \in c_j$, then t_i is an outcome, taking
 374 V_i to $Pair_{c_j, v_i}$, and if literal $\neg v_i \in c_j$, then f_i is an outcome, taking V_i to
 375 $Pair_{c_j, \neg v_i}$.
- 376 – The outcomes of $Lone_{v_i}$ are **true** and **false**. Outcome **true** brings v_i to
 377 node $Tlone_{v_i, 1}$ and outcome **false** brings v_i to node $Flone_{v_i, 1}$.
- 378 – We have a node $Tlone_{v_i, j}$ for each $j \leq |S_{i,t}|$ and $Flone_{v_i, j}$ for each $j \leq |S_{i,f}|$,
 379 with V_i as only process. Let $c_r = S_{i,t}(j)$. Node $Tlone_{v_i, j}$ has two outcomes
 380 $vton$ bringing V_i to $Tlone_{v_i, j+1}$ (or n_f if $j = |S_{i,t}|$), and $vtoc_{i,r}$ bringing V_i
 381 to $Pair_{c_r, v_i}$. The two outcomes from $Flone_{v_i, j}$ are similar.
- 382 – Node $Pair_{c_r, v_i}$ has V_i and C_r as its processes and one outcome $ctof$ which
 383 takes process C_j to final node n_f and process V_i to $Tlone_{v_i, j+1}$ (with $c_r =$
 384 $S_{i,t}(j)$), or to n_f if $j = |S_{i,t}|$. Node $Pair_{c_r, \neg v_i}$ is defined in the same way
 385 from $Flone_{v_i, j}$.

386 With this we claim that \mathcal{N}_ϕ has a final run iff ϕ is satisfiable which completes
 387 the first step of the proof. We give a formal proof of this claim in Appendix A.
 388 Observe that the negotiation \mathcal{N}_ϕ constructed is deterministic and acyclic (but it
 389 is not sound).

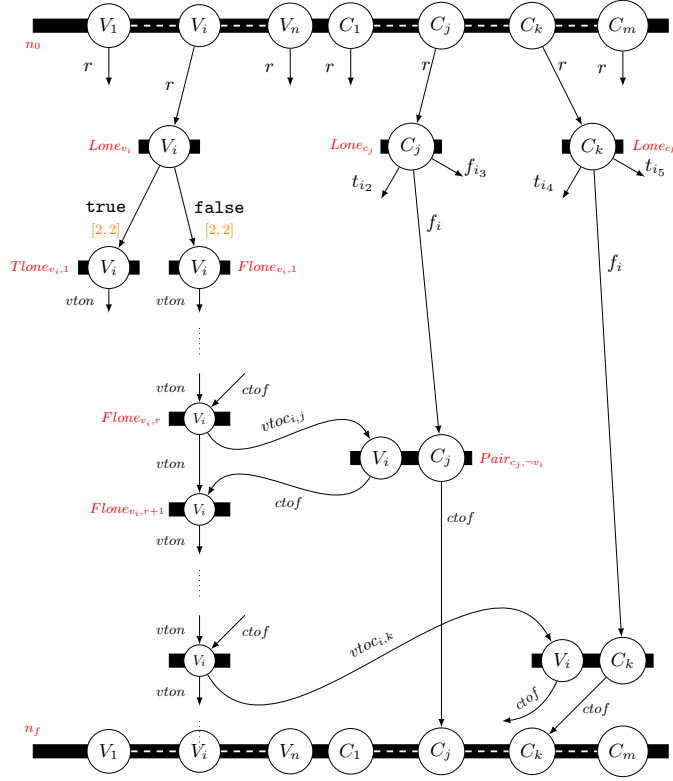


Fig. 3. A part of \mathcal{N}_ϕ where clause c_j is $(i_2 \vee \neg i_1 \vee \neg i_3)$ and clause c_k is $(i_4 \vee \neg i_1 \vee i_5)$. Timing is $[0, 0]$ wherever not mentioned

390 Step 2: Before we introduce timing on \mathcal{N}_ϕ , we introduce a new outcome r'
 391 at n_0 which takes all processes to n_f . Now, the timing function γ associated
 392 with the \mathcal{N}_ϕ is: $\gamma(n_0, r) = [2, 2]$ and $\gamma(n_0, r') = [3, 3]$ and $\gamma(n, r) = [0, 0]$, for
 393 all node $n \neq n_0$ and all $r \in R_n$. Then, $\text{mintime}(\mathcal{N}_\phi) \leq 2$ iff ϕ has a satisfiable
 394 assignment: if $\text{mintime}(\mathcal{N}_\phi) \leq 2$, there is a run with decision r taken at n_0
 395 which is final. But existence of any such final run implies satisfiability of ϕ . For
 396 reverse implication, if ϕ is satisfiable, then the corresponding run for satisfying
 397 assignment takes 2 units time, which means that $\text{mintime}(\mathcal{N}_\phi) \leq 2$.

Similarly, we can prove that the MaxTime problem is co-NP complete by
 changing $\gamma(n_0, r') = [1, 1]$ and asking if $\text{maxtime}(\mathcal{N}_\phi) > 1$ for the new \mathcal{N}_ϕ . The
 answer will be yes iff ϕ is satisfiable. \square

398 We now consider the related problem of checking if $\text{mintime}(\mathcal{N}) = T$ (or if
 399 $\text{maxtime}(\mathcal{N}) = T$). These problems are harder than their threshold variant un-
 400 der usual complexity assumptions: they are DP-complete (Difference Polynomial

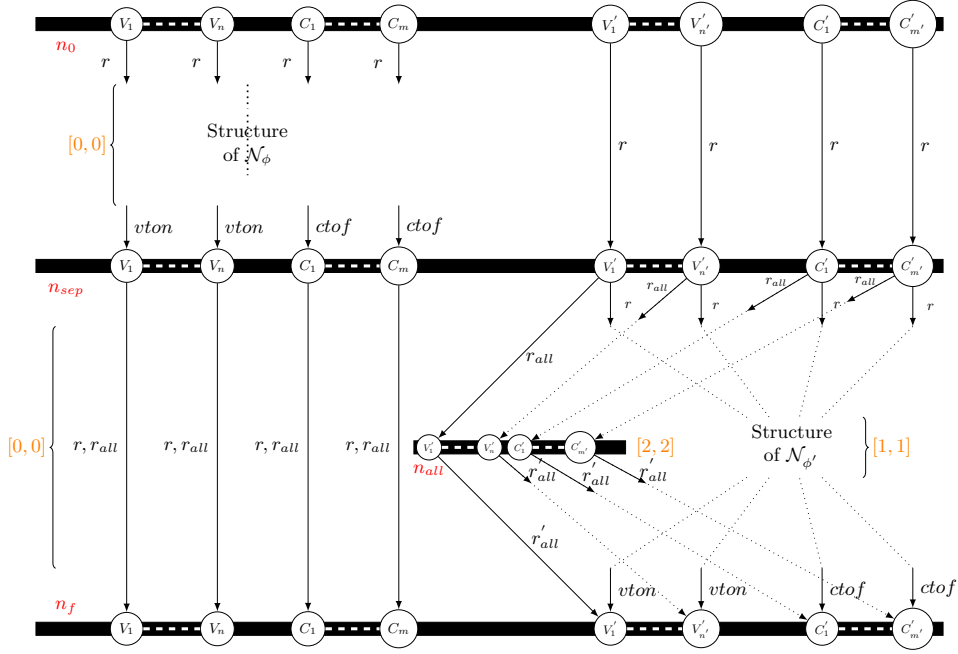


Fig. 4. Structure of $\mathcal{N}_{\phi, \phi'}$

401 time class, i.e., second level of the Boolean Hierarchy, defined as intersection of
 402 a problem in NP and one in co-NP [16]).

403 **Proposition 2.** *The $\text{mintime}(\mathcal{N}) = T$ and $\text{maxtime}(\mathcal{N}) = T$ decision prob-*
 404 *lems are DP-complete for acyclic deterministic negotiations.*

405 *Proof.* We only give the proof for mintime (the proof for maxtime is given in
 406 Appendix A). Indeed, it is easy to see that this problem is in DP, as it can be
 407 written as $\text{mintime}(\mathcal{N}) \leq T$ which is in NP and $\neg(\text{mintime}(\mathcal{N}) \leq T - 1)$,
 408 which is in co-NP. To show hardness, we use the negotiation constructed in the
 409 above proof as a gadget, and show a reduction from the SAT-UNSAT problem
 410 (a standard DP-complete problem).

411 The SAT-UNSAT Problem asks given two Boolean expressions ϕ and ϕ' , both
 412 in CNF forms with three literals per clause, is it true that ϕ is satisfiable and ϕ'
 413 is unsatisfiable? SAT-UNSAT is known to be DP-complete [16]. We reduce this
 414 problem to $\text{mintime}(\mathcal{N}) = T$.

415 Given ϕ, ϕ' , we first make the corresponding negotiations \mathcal{N}_ϕ and $\mathcal{N}_{\phi'}$ as in
 416 the previous proof. Let n_0 and n_f be the initial and final nodes of \mathcal{N}_ϕ and n'_0
 417 and n'_f be the initial and final nodes of $\mathcal{N}_{\phi'}$. (Similarly, for other nodes we write
 418 ' above the nodes to signify they belong to $\mathcal{N}_{\phi'}$).

In the negotiation $\mathcal{N}_{\phi, \phi'}$, we introduce a new node n_{all} , in which all the pro-
 cesses participate (see Figure 4). The node n_{all} has a single outcome r'_{all} which

sends all the processes to n_f . Also, for node n'_0 , apart from the outcome r which sends all processes to different nodes, there is another outcome r_{all} which sends all the processes to n_{all} . Now we merge the nodes n_f and n'_0 and call the merged node n_{sep} . Also nodes n_0 and n'_f now have all the processes of \mathcal{N}_ϕ and $\mathcal{N}_{\phi'}$ participating in them. This merged process gives us a new negotiation $\mathcal{N}_{\phi, \phi'}$ in which the structure above n_{sep} is same as \mathcal{N}_ϕ while below it is same as $\mathcal{N}_{\phi'}$. Node n_{sep} now has all the processes of \mathcal{N}_ϕ and $\mathcal{N}_{\phi'}$ participating in it. The outcomes of n_{sep} will be same as that of n'_0 (r_{all}, r). For both the outcomes of n_{sep} the processes corresponding to \mathcal{N}_ϕ directly go to n_f of the $\mathcal{N}_{\phi, \phi'}$. Similarly n_0 of $\mathcal{N}_{\phi, \phi'}$ which is same n_0 of \mathcal{N}_ϕ , sends processes corresponding to $\mathcal{N}_{\phi'}$ directly to n_{sep} for all its outcomes. We now define timing function γ for $\mathcal{N}_{\phi, \phi'}$ which is as follows: $\gamma(Lone'_{v_i}, r) = [1, 1]$ for all $v_i \in \phi'$ and $r \in \{\mathbf{true}, \mathbf{false}\}$, $\gamma(n_{all}, r'_{all}) = [2, 2]$ and $\gamma(n, r) = [0, 0]$ for all other outcomes of nodes. With this construction, one can conclude that $mintime(\mathcal{N}_{\phi, \phi'}) = 2$ iff ϕ is satisfiable and ϕ' is unsatisfiable (see Appendix for details). This completes the reduction and hence proves DP-hardness. \square

419 Finally, we consider a related problem of computing the min and max time.
 420 To consider the decision variant, we rephrase this problem as checking whether
 421 an arbitrary bit of the minimum execution time is 1. Perhaps surprisingly, we
 422 obtain that this problem goes even beyond DP, the second level of the Boolean
 423 Hierarchy and is in fact hard for Δ_2^P (second level of the *polynomial* hierarchy),
 424 which contains the entire Boolean Hierarchy. Formally,

425 **Theorem 4.** *Given an acyclic deterministic timed negotiation and a positive*
 426 *integer k , computing the k^{th} bit of the maximum/minimum execution time is*
 427 *Δ_2^P -complete.*

428 Finally, we remark that if we were interested in the optimization variant and
 429 not the decision variant of the problem, the above proof can be adapted to show
 430 that these variants are OptP-complete (as defined in [14]). But as optimization
 431 is not the focus of this paper, we avoid formal details of this proof.

432 6 Sound Negotiations

433 Sound negotiations are negotiations in which every run can be extended to
 434 a final run, as in Fig. 1. In this section, we show that $maxtime(\mathcal{N})$ can be
 435 computed in PTIME for sound negotiations, hence giving PTIME complexi-
 436 ties for the $maxtime(\mathcal{N}) \leq T?$ and $maxtime(\mathcal{N}) = T?$ questions. However, we
 437 show that $mintime(\mathcal{N}) \leq T$ is NP-complete for sound negotiations, and that
 438 $mintime(\mathcal{N}) = T$ is DP-complete, even if T is given in unary.

439 Consider the graph $G_{\mathcal{N}}$ of a negotiation \mathcal{N} . Let $\pi = (n_0, (p_0, r_0), n_1) \cdots$
 440 $(n_k, (p_k, r_k), n_{k+1})$ be a path of $G_{\mathcal{N}}$. We define the *maximal execution time* of
 441 a path π as the value $\delta^+(\pi) = \sum_{i \in 0..k} \gamma^+(n_i, r_i)$. We say that a path $\pi =$
 442 $(n_0, (p_0, r_0), n_1) \cdots (n_\ell, (p_\ell, r_\ell), n_{\ell+1})$ is a path of some run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r'_1)}$
 443 $\cdots (M_k, \mu_k)$ if r_0, \dots, r_ℓ is a subword of r'_1, \dots, r'_k .

444 **Lemma 1.** *Let \mathcal{N} be an acyclic and sound timed negotiation. Then $\text{maxtime}(\mathcal{N})$*
 445 *$= \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.*

446 *Proof.* Let us first prove that $\text{maxtime}(\mathcal{N}) \geq \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.
 447 Consider any path π of $G_{\mathcal{N}}$, ending in some node n . First, as \mathcal{N} is sound, we can
 448 compute a run ρ_π such that π is a path of ρ_π , and ρ_π ends in a configuration
 449 in which n is enabled. We associate with ρ_π the timed run ρ_π^+ which asso-
 450 ciates to every node the latest possible execution date. We have easily $\delta(\rho_\pi^+) \geq$
 451 $\delta(\pi)$, and then we obtain $\max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta(\rho_\pi^+) \geq \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta(\pi)$. As
 452 $\text{maxtime}(\mathcal{N})$ is the maximal duration over all runs, it is hence necessarily greater
 453 than $\max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta(\rho_\pi^+) + \gamma^+(n_f, r_f)$.

454 We now prove that $\text{maxtime}(\mathcal{N}) \leq \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. Take
 455 any timed run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r_1)} \dots (M_k, \mu_k)$ of \mathcal{N} with a unique maximal node
 456 n_k . We show that there exists a path π of ρ such that $\delta(\rho) \leq \delta^+(\pi)$ by induction
 457 on the length k of ρ . The initialization is trivial for $k = 1$. Let $k \in \mathbb{N}$. Because n_k
 458 is the unique maximal node of ρ , we have $\delta(\rho) = \max_{p \in P_{n_k}} \mu_{k-1}(p) + \gamma^+(n_k, r_k)$.
 459 We choose one p_{k-1} maximizing $\mu_{k-1}(p)$. Let $\ell < k$ be the maximal index of a
 460 decision involving process p_{k-1} (i.e. $p_{k-1} \in P_{n_\ell}$). Now, consider the timed run
 461 ρ' subword of ρ , but with n_ℓ as unique maximal node (that is, it is ρ where
 462 nodes $n_i, i > \ell$ has been removed, but also where some nodes $n_i, i < \ell$ have been
 463 removed if they are not causally before n_ℓ (in particular, $P_{n_i} \cap P_{n_\ell} = \emptyset$).

464 By definition, we have that $\delta(\rho) = \delta(\rho') + \gamma^+(n_\ell, r_\ell) + \gamma^+(n_k, r_k)$. We ap-
 465 ply the induction hypothesis on ρ' , and obtain a path π' of ρ' ending in n_ℓ
 466 such that $\delta(\rho') + \gamma^+(n_\ell, r_\ell) \leq \delta^+(\pi')$. It suffices to consider the path $\pi =$
 467 $\pi'.(n_\ell, (p_{k-1}, r_\ell), n_k)$ to prove the inductive step $\delta(\rho) \leq \delta^+(\pi) + \gamma^+(n_k, r_k)$.

Thus $\text{maxtime}(\mathcal{N}) = \max \delta(\rho) \leq \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. \square

468 Lemma 1 gives a way to evaluate the maximal execution time. This amounts
 469 to finding a path of maximal weight in an acyclic graph, which is a standard
 470 PTIME problem that can be solved using standard max-cost calculation.

471 **Proposition 3.** *Computing the maximal execution time for an acyclic sound*
 472 *negotiation $\mathcal{N} = (N, n_0, n_f, \mathcal{X})$ can be done in time $O(|N| + |\mathcal{X}|)$.*

473 A direct consequence is that $\text{maxtime}(\mathcal{N}) \leq T$ and $\text{maxtime}(\mathcal{N}) = T$ prob-
 474 lems can be solved in polynomial time when \mathcal{N} is sound. Notice that if \mathcal{N} is
 475 deterministic but not sound, then Lemma 1 does not hold: we only have an
 476 inequality.

477 We now turn to $\text{mintime}(\mathcal{N})$. We show that it is strictly harder to compute
 478 for sound negotiations than $\text{maxtime}(\mathcal{N})$.

479 **Theorem 5.** *$\text{mintime}(\mathcal{N}) \leq T$ is NP-complete in the strong sense for sound*
 480 *acyclic negotiations, even if \mathcal{N} is very weakly non-deterministic.*

481 *Proof (sketch).* First, we can decide $\text{mintime}(\mathcal{N}) \leq T$ in NP. Indeed, one can
 482 guess a final (untimed) run ρ of size $\leq |N|$, consider ρ^- the timed run corre-
 483 sponding to ρ where all outcomes are taken at the earliest possible dates, and
 484 compute in linear time $\delta(\rho^-)$, and check that $\delta(\rho^-) \leq T$.

The hardness part is obtained by reduction from the **Bin Packing** problem. The reduction is similar to Knapsack, that we will present in Thm. 7. The difference is that we use ℓ bins in parallel, rather than 2 process, one for the weight and one for the value. The hardness is thus strong, but the negotiation is not k -layered for a bounded k (It is $2\ell + 1$ bounded, with ℓ depending on the input). A detailed proof is given in Appendix B. \square

485 We show that $\text{mintime}(\mathcal{N}) = T$ is harder to decide than $\text{mintime}(\mathcal{N}) \leq T$,
486 with a proof similar to Prop. 2.

487 **Proposition 4.** *The $\text{mintime}(\mathcal{N}) = T$? decision problem is DP-complete for*
488 *sound acyclic negotiations, even if it is very weakly non-deterministic.*

489 An open question is whether the minimal execution time can be computed in
490 PTIME if the negotiation is both sound and deterministic. The reduction from
491 Bin Packing does not work with deterministic (and sound) negotiations.

492 7 k -Layered Negotiations

493 In the previous sections, we have considered sound negotiations, and determinis-
494 tic negotiations. For both classes, computing the minimal execution time cannot
495 be done in PTIME (unless NP=PTIME), even if constants are given in unary.
496 In this section, we consider k -layeredness (see Section 2), a syntactic property
497 that can be efficiently verified (it suffices to compute the depth of each node,
498 which can be done in polynomial time).

499 7.1 Algorithmic properties

500 Let k be a fixed integer. We first show that the maximum execution time can
501 be computed in PTIME for k -layered negotiations. Let N_i be the set of nodes
502 at layer i . We define for every layer i the set S_i of subsets of nodes $X \subseteq N_i$
503 which can be jointly enabled and such that for every process p , there is exactly
504 one node $n(X, p)$ in X with $p \in n(X, p)$. Formally, we define S_i inductively. We
505 start with $S_0 = \{n_0\}$. We then define S_{i+1} from the contents of layer S_i : we have
506 $Y \in S_{i+1}$ iff $\bigcup_{n \in Y} P_n = P$ and there exist $X \in S_i$ and an outcome $r_m \in R_m$ for
507 every $m \in X$, such that $n \in \mathcal{X}(n(X, p), p, r_m)$ for each $n \in Y$ and $p \in P_n$.

508 **Theorem 6.** *Let $k \in \mathbb{N}^+$. Computing the maximum execution time for a k -*
509 *layered acyclic negotiation \mathcal{N} can be done in PTIME. More precisely, the worst-*
510 *case time complexity is $O(|P| \cdot |\mathcal{N}|^{k+1})$.*

511 *Proof (Sketch).* The first step is to compute S_i layer by layer, by following its
512 inductive definition. The set S_i is of size at most 2^k , as $|N_i| < k$ by definition
513 of k -layeredness. Knowing S_i , it is easy to build S_{i+1} by induction. This takes
514 time in $O(|P||\mathcal{N}|^{k+1})$: We need to consider all k -tuple of outcomes for each layer.
515 There can be $|\mathcal{N}|^k$ such tuples. We need to do that for all processes ($|P|$), and
516 for all layers (at most $|\mathcal{N}|$).

We then keep for each subset $X \in S_i$ and each node $n \in X$, the maximal time $f_i(n, X) \in \mathbb{N}$ associated with n and X . From S_{i+1} and f_i , we inductively compute f_{i+1} in the following way: for all $X \in S_i$ with successor $Y \in S_{i+1}$ for outcomes $(r_p)_{p \in P}$, we denote $f_{i+1}(Y, n, X) = \max_{p \in P(n)} f_i(X, n(X, p)) + \gamma^+(n(X, p), r_p)$. If there are several choices of $(r_p)_{p \in P}$ leading to the same Y , we take r_p with the maximal $f_i(X, n(X, p)) + \gamma^+(n(X, p), r_p)$. We then define $f_{i+1}(Y, n) = \max_{X \in S_i} f_{i+1}(Y, n, X)$. Again, the initialization is trivial, with $f_0(\{n_0\}, n_0) = 0$. The maximal execution time of \mathcal{N} is $f(\{n_f\}, n_f)$. \square

517 We can bound the complexity precisely by $O(d(\mathcal{N}) \cdot C(\mathcal{N}) \cdot \|R\|^{k^*})$, with:
518 – $d(\mathcal{N}) \leq |\mathcal{N}|$ the depth of n_f , that is the number of layers of \mathcal{N} , and $\|R\|$ is
519 the maximum number of outcomes of a node,
520 – $C(\mathcal{N}) = \max_i |S_i| \leq 2^k$, which we will call the *number of contexts of \mathcal{N}* , and
521 which is often much smaller than 2^k .
522 – $k^* = \max_{X \in \cup_i S_i} |X| \leq k$. We say that \mathcal{N} is *k^* -thread bounded*, meaning
523 that there cannot be more that k^* nodes in the same context X of any layer.
524 Usually, k^* is strictly smaller than $k = \max_i |N_i|$, as $N_i = \bigcup_{X \in S_i} X$.

525 Consider again the Brexit example Figure 1. We have $(k + 1) = 7$, while
526 we have the depth $d(\mathcal{N}) = 6$, the negotiation is $k^* = 3$ -thread bounded (k^* is
527 bounded by the number of processes), $\|R\| = 2$, and the number of contexts is
528 at most $C(\mathcal{N}) = 4$ (EU chooses to enforce backstop or not, and Pa chooses to
529 go to court or not).

530 7.2 Minimal Execution Time

531 As with sound negotiations, computing minimal time is much harder than com-
532 puting the maximal time for k -layered negotiations:

533 **Theorem 7.** *Let $k \geq 6$. The $\text{Min} \leq T$ problem is NP-Complete for k -layered*
534 *acyclic negotiations, even if the negotiation is sound and very weakly non-deterministic.*

535 *Proof.* One can guess in polynomial time a final run of size $\leq |\mathcal{N}|$. If the exe-
536 cution time of this final run is smaller than T then we have found a final run
537 witnessing $\text{mintime}(\mathcal{N}) \leq T$. Hence the problem is in NP.

538 Let us now show that the problem is NP-hard. We proceed by reduction from
539 the **Knapsack** decision problem. Let us consider a set of items $U = \{u_1, \dots, u_n\}$
540 of respective values v_1, \dots, v_n and weight w_1, \dots, w_n and a knapsack of maximal
541 capacity W . The knapsack problem asks, given a value V whether there exists a
542 subset of items $U' \subseteq U$ such that $\sum_{u_i \in U'} v_i \geq V$ and such that $\sum_{u_i \in U'} w_i \leq W$.

543 We build a negotiation with $2n$ processes $P = \{p_1, \dots, p_{2n}\}$, as shown in
544 Fig. 5. Intuitively, $p_i, i \leq n$ will serve to encode the value of selected items as
545 timing, while $p_i, i > n$ will serve to encode the weight of selected items as timing.

546 Concerning timing constraints for outcomes we do the following: Outcomes
547 0, yes and no are associated with $[0, 0]$. Outcome c_i is associated with $[w_i, w_i]$,
548 the weight of u_i . Last, outcome b_i is associated with a more complex function,

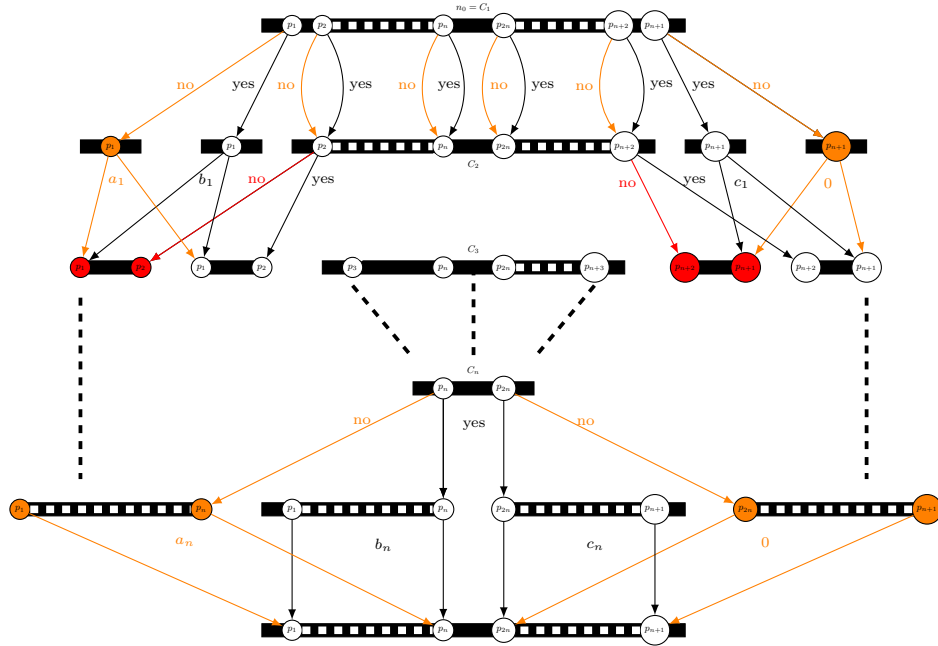


Fig. 5. The negotiation encoding Knapsack

549 such that $\sum_i b_i \leq W$ iff $\sum_i v_i \geq V$. For that, we set $[\frac{(v_{max}-v_i)W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for
 550 outcome b_i , where v_{max} is the largest value of an item, and V is the total value
 551 we want to reach at least. Also, we set $[\frac{(v_{max})W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for outcome a_i . We
 552 set $T = W$, the maximal weight of the knapsack.

553 Now, consider a final run ρ in \mathcal{N} . The only choices in ρ are outcomes *yes* or
 554 *no* from C_1, \dots, C_n . Let I be the set of indices such that *yes* is the outcome from
 555 all C_i in this path. We obtain $\delta(\rho) = \max(\sum_{i \notin I} a_i + \sum_{i \in I} b_i, \sum_{i \in I} c_i)$. We have
 556 $\delta(\rho) \leq T = W$ iff $\sum_{i \in I} w_i \leq W$, that is the sum of the weights is lower than
 557 W , and $\sum_{i \notin I} \frac{(v_{max})W}{n \cdot v_{max}-V} + \sum_{i \in I} \frac{(v_{max}-v_i)W}{n \cdot v_{max}-V} \leq W$. That is, $n \cdot v_{max} - \sum_{i \in I} v_i \leq$
 558 $n \cdot v_{max} - V$, i.e. $\sum_{i \in I} v_i \geq V$. Hence, there exists a path ρ with $\delta(\rho) \leq T = W$
 559 iff there exists a set of items of weight less than W and of value more than V . \square

560 It is well known that Knapsack is weakly NP-hard, that is, it is NP-hard only
 561 when weights/values are given in binary. This means that Thm. 7 shows that
 562 minimum execution time $\leq T$ is NP-hard only when T is given in binary. We
 563 can actually show that for k -layered negotiations, the $\text{mintime}(\mathcal{N}) \leq T$ problem
 564 can be decided in PTIME if T is given in unary (i.e. if T is not too large):

565 **Theorem 8.** *Let $k \in \mathbb{N}$. Given a k -layered negotiation \mathcal{N} and T written in*
 566 *unary, one can decide in PTIME whether the minimum execution time of \mathcal{N} is*
 567 *$\leq T$. The worst-case time complexity is $O(|\mathcal{N}| \cdot |P| \cdot (T \cdot |\mathcal{N}|)^k$).*

568 *Proof.* We will remember for each layer i a set \mathcal{T}_i of functions τ from nodes N_i
569 of layer i to a value in $\{1, \dots, T, \perp\}$. Basically, we have $\tau \in \mathcal{T}_i$ if there exists a
570 path ρ reaching $X = \{n \in N_i \mid f(n) \neq \perp\}$, and this path reaches node $n \in X$
571 after $\tau(n)$ time units. As for S_i , for all p , we should have a unique node $n(\tau, p)$
572 such that $p \in n(f, p)$ and $\tau(n(\tau, p)) \neq \perp$. Again, it is easy to initialize $\mathcal{T}_0 = \{\tau_0\}$,
573 with $\tau_0(n_0) = 0$, and $\tau_0(n) = \perp$ for all $n \neq n_0$.

574 Inductively, we build \mathcal{T}_{i+1} in the following way: $\tau_{i+1} \in \mathcal{T}_{i+1}$ iff there exists a
575 $\tau_i \in \mathcal{T}_i$ and $r_p \in R_{n(\tau_i, p)}$ for all $p \in P$ such that for all n with $\tau_{i+1}(n) \neq \perp$, we
576 have $\tau_{i+1}(n) = \max_p \tau_i(n(\tau_i, p)) + \gamma(n(\tau_i, p), r_p)$.

We have that the minimum execution time for \mathcal{N} is $\min_{\tau \in \mathcal{T}_n} \tau(n_\tau)$, for n the
depth of n_f . There are at most T^k functions τ in any \mathcal{T}_i , and there are at most
 $|\mathcal{N}|$ layers to consider, giving the complexity. \square

577 As with Thm. 6, we can more accurately state the complexity as $O(d(\mathcal{N}) \cdot$
578 $C(\mathcal{N}) \cdot \|R\|^{k^*} \cdot T^{k^* - 1})$. The $k^* - 1$ is because we only need to remember minimal
579 functions $\tau \in \mathcal{T}_i$: if $\tau'(n) \geq \tau(n)$ for all n , then we do not need to keep τ' in \mathcal{T}_i .
580 In particular, for the knapsack encoding in the proof of Thm. 7, we have $k^* = 3$,
581 $\|R\| = 2$ and $C(\mathcal{N}) = 4$.

582 Notice that if k is part of the input, then the problem is strongly NP-hard,
583 even if T is given in unary, as e.g. encoding bin packing with ℓ bins result to a
584 $2\ell + 1$ -layered negotiations.

585 8 Conclusion

586 In this paper, we considered timed negotiations. We believe that time is of the
587 essence in negotiations, as exemplified by the Brexit negotiation. It is thus im-
588 portant to be able to compute in a tractable way the minimal and maximal
589 execution time of negotiations.

590 We showed that we can compute in PTIME the maximal execution time for
591 acyclic negotiations that are either sound or k -layered, for k fixed. We showed
592 that we cannot compute in PTIME the maximal execution time for negotiations
593 that are not sound nor k -layered, even if they are deterministic and acyclic
594 (unless NP=PTIME). We also showed that surprisingly, computing the minimal
595 execution time is much harder, with strong NP-hardness results in most of the
596 classes of negotiations, contradicting a claim in [10]. We came up with a new
597 reasonable class of negotiations, namely k -layered negotiations, which enjoys
598 a pseudo PTIME algorithm to compute the minimal execution time. That is,
599 the algorithm is PTIME when the timing constants are given in unary. We
600 showed that this restriction is necessary, as the problem becomes NP-hard for
601 constants given in binary, even when the negotiation is sound and very weakly
602 non-deterministic. The problem to know whether the minimal execution time
603 can be computed in PTIME for deterministic and sound negotiation remains
604 open.

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658 Appendix A: Deterministic Negotiations

659 We start by considering the class of deterministic acyclic negotiations. We show
660 that both maximal and minimal execution time cannot be computed in PTIME
661 (unless NP=PTIME), as the threshold problems are (co-)NP-complete.

662 **Theorem 3.** *The $\text{mintime}(\mathcal{N}) \leq T$ decision problem is NP complete, and the*
663 *$\text{maxtime}(\mathcal{N}) \leq T$ decision problem is co-NP complete for acyclic deterministic*
664 *timed negotiations.*

665 *Proof.* For $\text{mintime}(\mathcal{N}) \leq T$, containment in NP is easy: we just need to guess
666 a run ρ (of polynomial size as \mathcal{N} is acyclic), consider the associate timed run ρ^-
667 where all decisions are taken at their earliest possible dates, and check whether
668 $\delta(\rho^-) \leq T$, which can be done in time $O(|\mathcal{N}| + \log T)$.

669 For the hardness, we give the proof in two steps. First, we start with a proof
670 of Proposition 1 that reachability problem is NP-hard using reduction of 3-CNF
671 SAT, i.e., given a formula ϕ , we build a deterministic negotiation \mathcal{N}_ϕ s.t. ϕ is
672 satisfiable iff \mathcal{N}_ϕ has a final run. In a second step, we introduce timings on this
673 negotiation and show that $\text{mintime}(\mathcal{N}_\phi) \leq T$ iff ϕ is satisfiable.

674 Step 1: Reducing 3-CNF-SAT to Reachability problem.

675 Given a boolean formula ϕ with variables v_i , $1 \leq i \leq n$ and clauses c_j , $1 \leq j \leq$
676 m , for each variable v_i we define the sets of clauses $S_{i,\text{t}} = \{c_j | v_i \text{ is present in } c_j\}$
677 and $S_{i,\text{f}} = \{c_j | \neg v_i \text{ is present in } c_j\}$. Clauses in $S_{i,\text{t}}$ and $S_{i,\text{f}}$ are naturally ordered:
678 $c_i < c_j$ iff $i < j$. We denote these elements $S_{i,\text{t}}(1) < S_{i,\text{t}}(2) < \dots$
679 Similarly for set $S_{i,\text{f}}$.

680 Now, we construct a negotiation \mathcal{N}_ϕ with a process V_i for each variable v_i
681 and a process C_j for each clause c_j :

- 682 – Initial node n_0 has a single outcome r taking each process C_j to node Lone_{c_j} ,
683 and each process V_i to node Lone_{v_i} .
- 684 – Lone_{c_j} has three outcomes: if literal $v_i \in c_j$, then t_i is an outcome, taking
685 V_i to Pair_{c_j, v_i} , and if literal $\neg v_i \in c_j$, then f_i is an outcome, taking V_i to
686 $\text{Pair}_{c_j, \neg v_i}$.
- 687 – The outcomes of Lone_{v_i} are **true** and **false**. Outcome **true** brings v_i to
688 node $\text{Tlone}_{v_i, 1}$ and outcome **false** brings v_i to node $\text{Flone}_{v_i, 1}$.
- 689 – We have a node $\text{Tlone}_{v_i, j}$ for each $j \leq |S_{i,\text{t}}|$ and $\text{Flone}_{v_i, j}$ for each $j \leq |S_{i,\text{f}}|$,
690 with V_i as only process. Let $c_r = S_{i,\text{t}}(j)$. Node $\text{Tlone}_{v_i, j}$ has two outcomes
691 vton bringing V_i to $\text{Tlone}_{v_i, j+1}$ (or n_f if $j = |S_{i,\text{t}}|$), and $\text{vtoc}_{i,r}$ bringing V_i
692 to Pair_{c_r, v_i} . The two outcomes from $\text{Flone}_{v_i, j}$ are similar.
- 693 – Node Pair_{c_r, v_i} has V_i and C_r as its processes and one outcome ctof which
694 takes process C_j to final node n_f and process V_i to $\text{Tlone}_{v_i, j+1}$ (with $c_r =$
695 $S_{i,\text{t}}(j)$), or to n_f if $j = |S_{i,\text{t}}|$. Node $\text{Pair}_{c_r, \neg v_i}$ is defined in the same way
696 from $\text{Flone}_{v_i, j}$.

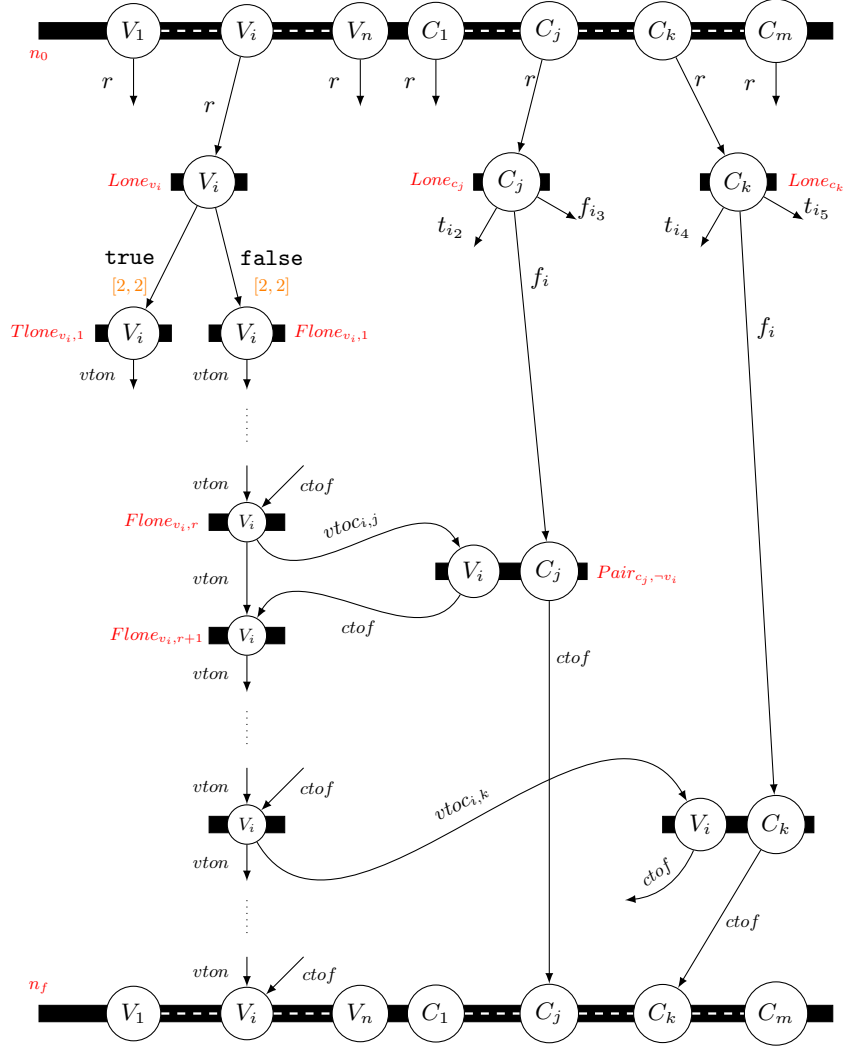


Fig. 3. A part of \mathcal{N}_ϕ where clause c_j is $(i_2 \vee \neg i \vee \neg i_3)$ and clause c_k is $(i_4 \vee \neg i \vee i_5)$. Timing is $[0, 0]$ wherever not mentioned

697 *Claim.* \mathcal{N}_ϕ has a final run iff ϕ is satisfiable.

698 *Proof.* First we show that if there is a run ρ from n_0 to n_f then ϕ is satisfiable.
 699 In ρ , all processes reached n_f . So each process V_i takes either outcome **true** or
 700 **false** in ρ . Let val the valuation associated each variable v_i with the choice
 701 **true** or **false** by V_i . We now show that all clause c_r have at least one literal
 702 true in val . In ρ , process C_r reaches the final node n_f : it must have gone via
 703 one node either $Pair_{c_r, v_i}$ or $Pair_{c_r, \neg v_i}$, for some i . Wlog, let us assume that C_r
 704 went to $Pair_{c_r, \neg v_i}$. The only way it is possible is for process V_i to have been in
 705 $Flone_{v_i, j}$, with $c_r = S_{i, f}(j)$. This is possible only if V_i decided outcome **false** at
 706 $Lone_{v_i}$. So this implies that literal $\neg v_i$ of c_j is true in val . Hence ϕ is satisfiable.

707 Conversely, we show that if ϕ is satisfiable then \mathcal{N}_ϕ has final run. Let val a
 708 satisfiable assignment $val : V \rightarrow \{\mathbf{true}, \mathbf{false}\}$ for ϕ . We build a run ρ which is
 709 final. After reaching $Lone_{v_i}$, V_i will decide the outcome according to the value
 710 of $val(v_i)$ and reach $Flone_{v_i, 1}$ or $Tlone_{v_i, 1}$ accordingly. Let $G_i(val)$ be the set
 711 of clause c_j such that i is the minimal literal of c_j true under val . When there is
 712 a choice between two outcomes $vtoc$ and $vtoc_{i, k}$ for process V_i , the run chooses
 713 $vtoc_{i, k}$ iff $k \in G_i(val)$. Concerning C_j , it appears in exactly one $G_i(val)$, because
 714 val satisfies ϕ . If $val(v_i) = \mathbf{true}$, run ρ chooses outcome t_i for V_i in node $Lone_{c_j}$,
 715 and outcome f_i if $val(v_i) = \mathbf{false}$. Observe that the same variable v_i can be
 716 associated with several clauses c_j , but then all these clauses go to the same type
 717 of nodes i.e. $Pair_{c_j, v_i}$ if $val(v_i) = \mathbf{true}$ and $Pair_{c_j, \neg v_i}$ if $val(v_i) = \mathbf{false}$.

This run ρ is final: Every process C_j reaches n_f after participating in exactly
 one node $Pair_{c_j, v_i}$ or $Pair_{c_j, \neg v_i}$. Every process V_i reaches n_f after participating
 in zero or more node $Pair_{c_j, v_i}$ or $Pair_{c_j, \neg v_i}$ (it participates in exactly $|G_i|$ such
 nodes). \square

718 With this we claim that \mathcal{N}_ϕ has a final run iff ϕ is satisfiable which com-
 719 pletes the first step of the proof. Observe that the negotiation \mathcal{N}_ϕ constructed
 720 is deterministic and acyclic (but it is not sound).

721 **Step 2:** Before we introduce timing on \mathcal{N}_ϕ , we introduce a new outcome r'
 722 at n_0 which takes all processes to n_f . Now, the timing function γ associated
 723 with the \mathcal{N}_ϕ is: $\gamma(n_0, r) = [2, 2]$ and $\gamma(n_0, r') = [3, 3]$ and $\gamma(n, r) = [0, 0]$, for
 724 all node $n \neq n_0$ and all $r \in R_n$. Then, $\text{mintime}(\mathcal{N}_\phi) \leq 2$ iff ϕ has a satisfiable
 725 assignment: if $\text{mintime}(\mathcal{N}_\phi) \leq 2$, there is a run with decision r taken at n_0
 726 which is final. But existence of any such final run implies satisfiability of ϕ . For
 727 reverse implication, if ϕ is satisfiable, then the corresponding run for satisfying
 728 assignment takes 2 units time, which means that $\text{mintime}(\mathcal{N}_\phi) \leq 2$.

729 Similarly, we can prove that the MaxTime problem is Co-NP complete by
 730 changing $\gamma(n_0, r') = [1, 1]$ and asking if $\text{maxtime}(\mathcal{N}_\phi) > 1$ for the new \mathcal{N}_ϕ . The
 731 answer will be yes iff ϕ is satisfiable.

As a side note, we observe that the NP-hardness for mintime could also have
 been proved without introducing the new result r' but then it would have been
 possible that \mathcal{N}_ϕ had no final run making $\text{mintime}(\mathcal{N}_\phi) \leq 2$ vacuous. \square

732 We now consider the related problem of checking if $\text{mintime}(\mathcal{N}) = T$ (or if
 733 $\text{maxtime}(\mathcal{N}) = T$). These problems are harder than their threshold variant un-

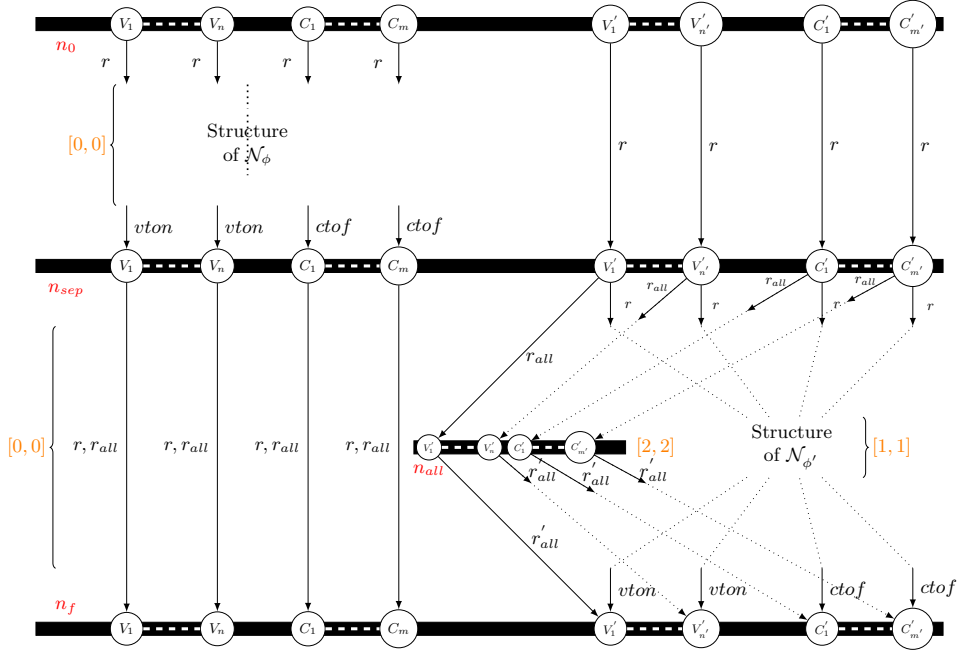


Fig. 4. Structure of $\mathcal{N}_{\phi, \phi'}$

734 der usual complexity assumptions: they are DP-complete (Difference Polynomial
735 time class, i.e., second level of the Boolean Hierarchy, defined as intersection of
736 a problem in NP and co-NP [16]).

737 **Proposition 2.** *The $\text{mintime}(\mathcal{N}) = T$ and $\text{maxtime}(\mathcal{N}) = T$ decision prob-*
738 *lems are DP-complete for acyclic deterministic negotiations.*

739 *Proof.* Indeed, it is easy to see that this problem is in DP, as it can be written
740 as $\text{mintime}(\mathcal{N}) \leq T$ which is in NP and $\neg(\text{mintime}(\mathcal{N}) \leq T - 1)$, which is in
741 co-NP. To show hardness, we use the negotiation constructed in the above proof
742 as a gadget, and show a reduction from the SAT-UNSAT problem (a standard
743 DP-complete problem).

744 SAT-UNSAT Problem : Given two Boolean expressions ϕ and ϕ' , both in
745 CNF forms with three literals per clause, is it true that ϕ is satisfiable and ϕ'
746 is unsatisfiable? SAT-UNSAT is known to be DP-Complete [16]. We reduce this
747 problem to $\text{mintime}(\mathcal{N}) = T$.

748 Given ϕ, ϕ' , we first make the corresponding negotiations \mathcal{N}_ϕ and $\mathcal{N}_{\phi'}$ as in
749 the previous proof. Let n_0 and n_f be the initial and final nodes of \mathcal{N}_ϕ and n'_0 and
750 n'_f be the initial and final nodes of $\mathcal{N}_{\phi'}$. (Similarly, for other nodes we write ' $'$
751 above the nodes to signify they belong to $\mathcal{N}_{\phi'}$). In the negotiation $\mathcal{N}_{\phi, \phi'}$, we introduce
752 a new node n_{all} (see Figure 4), in which all the processes participate. The node
753 n_{all} has a single outcome r'_{all} which sends all the processes to n_f . Also, for node

754 n'_0 , apart from the outcome r which sends all processes to different nodes, there
 755 is another outcome r_{all} which sends all the processes to n_{all} .

756 Now we merge the nodes n_f and n'_0 and call the merged node n_{sep} . Also nodes
 757 n_0 and n'_f now have all the processes of \mathcal{N}_ϕ and $\mathcal{N}_{\phi'}$ participating in them.

758 This merged process gives us a new negotiation $\mathcal{N}_{\phi,\phi'}$ in which the structure
 759 above n_{sep} is same as \mathcal{N}_ϕ while below it is same as $\mathcal{N}_{\phi'}$. Node n_{sep} now has all the
 760 processes of \mathcal{N}_ϕ and $\mathcal{N}_{\phi'}$ participating in it. The outcomes of n_{sep} will be same
 761 as that of n'_0 (r_{all}, r). For both the outcomes of n_{sep} the processes corresponding
 762 to \mathcal{N}_ϕ directly go to n_f of the $\mathcal{N}_{\phi,\phi'}$. Similarly n_0 of $\mathcal{N}_{\phi,\phi'}$ which is same n_0 of
 763 \mathcal{N}_ϕ , sends processes corresponding to $\mathcal{N}_{\phi'}$ directly to n_{sep} for all its outcomes.
 764 We now define timing function γ for $\mathcal{N}_{\phi,\phi'}$ which is as follows:

- 765 - $\gamma(Lone'_{v_i,r}) = [1, 1]$ for all $v_i \in \phi'$ and $r \in \{\mathbf{true}, \mathbf{false}\}$,
- 766 - $\gamma(n_{all}, r'_{all}) = [2, 2]$ and
- 767 - $\gamma(n, r) = [0, 0]$ for all other outcomes of nodes.

768 The claim is that

769 *Claim.* $mintime(\mathcal{N}_{\phi,\phi'}) = 2$ iff ϕ is satisfiable and ϕ' is unsatisfiable.

770 *Proof.* If $mintime(\mathcal{N}_{\phi,\phi'}) = 2$, this implies that ϕ is satisfiable, for if it was not
 771 satisfiable then for no run, all the processes corresponding to ϕ could reach n_{sep}
 772 and therefore the negotiation could not complete and hence MinTime would be
 773 infinite. Also ϕ' is unsatisfiable because if it would have been satisfiable then
 774 there would have been a final run in which the processes after reaching n_{sep}
 775 choose the outcome r from n_{sep} and complete the negotiation. The time for that
 776 run would be 1 unit and therefore the $mintime(\mathcal{N}_{\phi,\phi'}) \neq 2$.

For the other side of the implication, we can argue similarly that if ϕ is
 satisfiable then the processes of \mathcal{N}_ϕ would complete the structure above n_{sep}
 and reach n_{sep} in 0 units of time. From there the processes would have to choose
 the outcome r_{all} to reach n_f because otherwise, the run would not be final. The
 time taken for the path would be 2 units. So total time associated with this run
 will be 2 units which will also be the $mintime(\mathcal{N}_{\phi,\phi'})$. \square

For equality decision problem of MaxTime, the proof is similar; only the
 $\gamma(Lone'_{v_i,r}) = [2, 2]$ for all $v_i \in \phi'$, $\gamma(n_{all}, r'_{all}) = [1, 1]$ and $\gamma(n, r) = [0, 0]$ for
 all other nodes. The question asked is $maxtime(\mathcal{N}_{\phi,\phi'}) = 2$ which is true if only
 if ϕ is satisfiable and ϕ' is unsatisfiable. \square

777 Finally, we consider a related problem of deciding if a bit of $mintime(\mathcal{N})$ is
 778 1 (or similarly with $maxtime(\mathcal{N})$). Perhaps surprisingly, we obtain that these
 779 problems goes even beyond DP (the second level of the Boolean Hierarchy) and
 780 is in fact hard for Δ_2^P , which contains the whole Boolean Hierarchy:

781 **Theorem 4.** *Given an acyclic deterministic timed negotiation and a positive*
 782 *integer k , computing the k^{th} bit of the maximum/minimum execution time is*
 783 *Δ_2^P complete.*

784 *Proof.* Containment is again relatively easy. Given an acyclic deterministic timed
785 negotiation, we can compute the largest possible time attainable as a function
786 of the number of nodes and maximal constant in each node. Now guess the
787 min/max time (in binary) and then check it using NP-oracle or equivalently
788 Co-NP oracle calls.

789 The hardness is not so simple to obtain. We first notice that it suffices to
790 show the problem of whether $\text{maxtime}/\text{mintime}(\mathcal{N}) = \text{odd} ?$ is Δ_2^P hard. This is
791 because odd or even is the same as the last bit. We first show that $\text{maxtime}(\mathcal{N})$
792 $= \text{odd}$ is Δ_2^P complete.

793 Consider the following problem: Given a Boolean formula $\phi(x_1, x_2, \dots, x_n)$, is
794 $x_n = 1$ in the lexicographically largest satisfying assignment of ϕ ?

795 The above problem is known to be Δ_2^P complete [14] and we reduce it to the
796 decision problem of $\text{maxtime}(\mathcal{N}) = \text{odd} ?$ First, we convert ϕ to 3-CNF form us-
797 ing Tseitin transformation. Let the new variables introduced be called t_1, t_2, \dots, t_k .
798 So $\phi(x_1, x_2, \dots, x_n)$ is equisatisfiable to 3-CNF $\phi'(v_1, v_2, \dots, v_n, v_{n+1}, \dots, v_{n+k})$ where
799 $v_i = x_i$ for $i \leq n$ and $v_i = t_i$ for $i > n$. We convert ϕ' to a negotiation $\mathcal{N}_{\phi'}$. $\mathcal{N}_{\phi'}$
800 has the same structure as that of \mathcal{N}_{ϕ} which was constructed in Theorem 3 apart
801 from some change in arcs and participation of processes in nodes.

802 Participation changes are the following : The node Lone_{v_i} associated with each
803 variable v_i of ϕ' now involve two processes namely V_i and V_{i-1} . (Lone_{v_1} has only
804 V_1 as process). Both of the outcomes, **true** and **false** associated with Lone_{v_i}
805 take V_{i-1} to n_f while **true** takes V_i to $\text{TLone}_{v_i,1}$ and **false** takes V_i to $\text{Flone}_{v_i,1}$.
806 Change in arcs is the following: The outcome vton of $\text{FLone}_{v_i,r}$ where $r = |S_{i,t}|$
807 and $\text{TLone}_{v_i,r'}$ where $r' = |S_{i,t}|$ takes V_i to $\text{Lone}_{v_{i+1}}$ (Except for $i = n + k$ for
808 which there is no change). We now define timing function γ as follows:

- 809 - $\gamma(\text{Lone}_{v_i}, \text{true}) = [2^{n-i}, 2^{n-i}]$ for all $i \leq n$ and
- 810 - $\gamma(n, r) = 0$ for all other combination of nodes and outcomes.

811 The claim is that $\text{maxtime}(\mathcal{N}_{\phi'}) = \text{odd}$ iff $x_n = 1$ in the lexicographically
812 largest satisfying assignment of ϕ .

813

814 To prove the claim, we prove a stronger outcome that there is a run which is
815 final and takes time t iff there is a satisfying assignment to ϕ whose lexicographic
816 value is same as t in binary.

817

818 To prove the forward implication, consider any run σ which is final. Now,
819 just like the proof in 3, the process V_i must have chosen either **true** or **false** at
820 node Lone_{v_i} . The assignment f , corresponding to this outcome chosen by each
821 V_i is essentially the one whose lexicographic value is same as t . The fact that
822 this assignment is satisfiable follows from the proof of theorem 3. To show that
823 that lexicographic value is same, first of all the observe that time taken t can be
824 written as $2^{n-i_1} + 2^{n-i_2} + \dots + 2^{n-i_k}$ where V_{i_j} are those processes which chose
825 **true** at Lone_{v_i} . Moreover $i_j \leq n$, which implies that all these variables are also
826 present in ϕ . Also the contribution of a variable x_{i_j} (which is same as v_{i_j})
827 in lexicographic value will be 2^{n-i_j} which is same as its contribution in t . Hence
828 the forward implication.

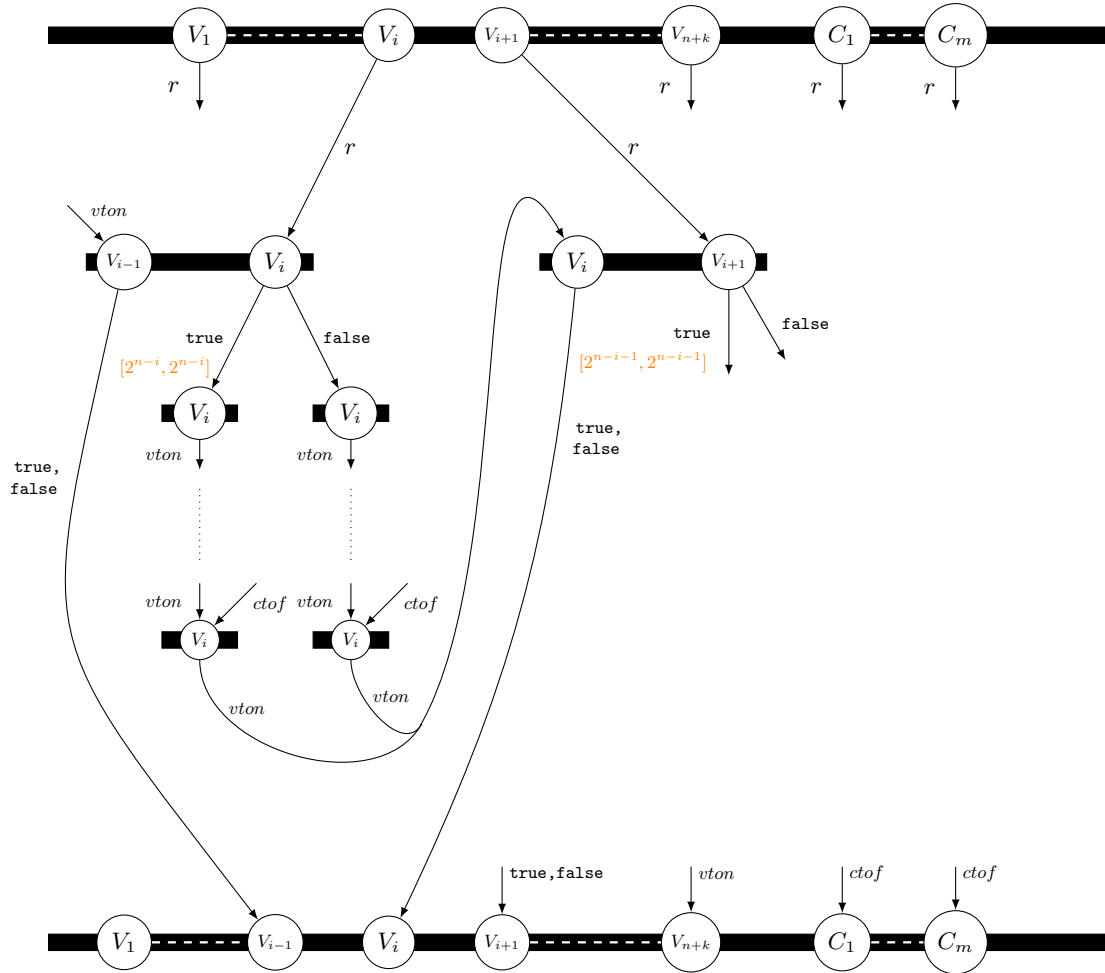


Fig. 5. A part of \mathcal{N}_ϕ . Here if $i > n$, then timing with arcs **true** and **false** will be $[0, 0]$.

829 For backward implication, consider any satisfiable assignment f of ϕ . Since
 830 ϕ and ϕ' are equisatisfiable hence there will exist a satisfiable assignment f' to
 831 ϕ' , such that $f'(x_i) = f(v_i)$ for $i \leq n$. Now following the proof of

832 Thm. 3, it is easy to see that the run σ corresponding to the assignment f' will
 833 be final. Moreover the time taken for the path will be $2^{n-i_1} + 2^{n-i_2} + \dots + 2^{n-i_k}$
 834 where $f'(v_{i_k}) = \mathbf{true}$. Since all these $i_j \leq n$, these variables will also be present
 835 in ϕ and their contribution in lexicographic value of f would also be 2^{n-i_j} . And
 836 hence the backward implication.

This proves the claim and shows that $\text{maxtime}(\mathcal{N}_{\phi'}) = \text{odd}$ iff $x_n = 1$ in the lexicographically largest satisfying assignment of ϕ . \square

837 Finally, we note that if we were interested in the optimization and not the
 838 decision variant of the problem, the above proof can be adapted to show that
 839 the optimization variants are **OptP-Complete** (as defined in [14]).

840 Appendix B: Sound Negotiations

841 Sound negotiations are negotiations in which every run can be extended to
 842 a final run, as in Fig. 1. In this section, we show that $\text{maxtime}(\mathcal{N})$ can be
 843 computed in PTIME for sound negotiations, hence giving PTIME complexities
 844 for the $\text{maxtime}(\mathcal{N}) \leq T?$ and $\text{maxtime}(\mathcal{N}) = T?$ questions. However, we
 845 show that $\text{mintime}(\mathcal{N}) \leq T$ is NP-complete for sound negotiations, and that
 846 $\text{mintime}(\mathcal{N}) = T$ is DP-complete, even if T is given in unary.

847 Consider the graph $G_{\mathcal{N}}$ of a negotiation \mathcal{N} . Let $\pi = (n_0, (p_0, r_0), n_1) \dots$
 848 $(n_k, (p_k, r_k), n_{k+1})$ be a path of $G_{\mathcal{N}}$. We define the *maximal execution time* of
 849 a path π as the value $\delta^+(\pi) = \sum_{i \in 0..k} \gamma^+(n_i, r_i)$. We say that a path $\pi =$
 850 $(n_0, (p_0, r_0), n_1) \dots (n_\ell, (p_\ell, r_\ell), n_{\ell+1})$ is a path of some run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r'_1)}$
 851 $\dots (M_k, \mu_k)$ if r_0, \dots, r_ℓ is a subword of r'_1, \dots, r'_k .

852 **Lemma 1.** *Let \mathcal{N} be an acyclic and sound timed negotiation. Then $\text{maxtime}(\mathcal{N})$*
 853 $= \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.

854 *Proof.* Let us first prove that $\text{maxtime}(\mathcal{N}) \geq \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.
 855 Consider any path π of $G_{\mathcal{N}}$, ending in some node n . First, as \mathcal{N} is sound, we can
 856 compute a run ρ_π such that π is a path of ρ_π , and ρ_π ends in a configuration
 857 in which n is enabled. We associate with ρ_π the timed run ρ_π^+ which associ-
 858 ates to every node the latest possible execution date. We have easily $\delta(\rho_\pi^+) \geq$
 859 $\delta(\pi)$, and then we obtain $\max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta(\rho_\pi^+) \geq \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta(\pi)$. As
 860 $\text{maxtime}(\mathcal{N})$ is the maximal duration over all runs, it is hence necessarily greater
 861 than $\max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta(\rho_\pi^+) + \gamma^+(n_f, r_f)$

862 We now prove that $\text{maxtime}(\mathcal{N}) \leq \max_{\pi \in \text{Paths}(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$.

863 Take any timed run $\rho = (M_1, \mu_1) \xrightarrow{(n_1, r'_1)} \dots (M_k, \mu_k)$ of \mathcal{N} with a unique
 864 maximal node n_k . We show that there exists a path π of ρ such that $\delta(\rho) \leq \delta^+(\pi)$
 865 by induction on the length k of ρ . The initialization is trivial for $k = 1$. Let $k \in \mathbb{N}$.
 866 Because n_k is the unique maximal node of ρ , we have $\delta(\rho) = \max_{p \in P_{n_k}} \mu_{k-1}(p) +$

867 $\gamma^+(n_k, r_k)$. We choose one p_{k-1} maximizing $\mu_{k-1}(p)$. Let $\ell < k$ be maximal index
868 of a decision involving process p_{k-1} (i.e. $p_{k-1} \in P_{n_\ell}$). Now, consider the timed
869 run ρ' subword of ρ , but with n_ℓ as unique maximal node (that is, it is ρ where
870 nodes $n_i, i > \ell$ has been removed, but also where some nodes $n_i, i < \ell$ have been
871 removed if they are not causally before n_ℓ (in particular, $P_{n_i} \cap P_{n_\ell} = \emptyset$).

872 By definition, we have that $\delta(\rho) = \delta(\rho') + \gamma^+(n_\ell, r_\ell) + \gamma^+(n_k, r_k)$. We ap-
873 ply the induction hypothesis on ρ' , and obtain a path π' of ρ' ending in n_ℓ
874 such that $\delta(\rho') + \gamma^+(n_\ell, r_\ell) \leq \delta^+(\pi')$. It suffices to consider the path $\pi =$
875 $\pi'(n_\ell, (p_{k-1}, r_\ell), n_k)$ to prove the inductive step $\delta(\rho) \leq \delta^+(\pi) + \gamma^+(n_k, r_k)$.

Thus $\maxtime(\mathcal{N}) = \max \delta(\rho) \leq \max_{\pi \in Paths(G_{\mathcal{N}})} \delta^+(\pi) + \gamma^+(n_f, r_f)$. \square

876 Lemma 1 gives a way to evaluate the maximal execution time. This amounts
877 to finding a path of maximal weight, which is a standard PTIME graph problem
878 that can be solved using standard max-cost calculation.

879 **Proposition 3.** *Computing the maximal execution time for an acyclic sound*
880 *negotiation $\mathcal{N} = (N, n_0, n_f, \mathcal{X})$ can be done in time $O(|N| + |\mathcal{X}|)$.*

881 *Proof.* First of all, we compute a topological order $<$ on nodes of the graph $G_{\mathcal{N}}$,
882 that is for all $n' \in \mathcal{X}(n, r)$, we have $n < n'$. This can be done in $O(|N| + |\mathcal{X}|)$ [3].
883 Then, we follow the total order $<$ on nodes of $G_{\mathcal{N}}$ and attach to each node n a
884 maximal time $\delta^+(n)$ for runs ending at node n in the following way: $\delta^+(n_0) = 0$
885 and for each node n , we let $\delta^+(n) = \max_{n' | (n', (p, r), n) \in G_{\mathcal{N}}} (\gamma^+(n', r) + \delta^+(n'))$. It
886 is easy to see that $\delta^+(n)$ is the maximal $\delta(\pi)$ over all paths π from n_0 to n . As
887 every transition of $G_{\mathcal{N}}$ is considered only once, the computation of δ^+ can be
888 done in $O(|N| + |\mathcal{X}|)$. It then suffices to return $\delta^+(n_f) + \gamma^+(n_f, r_f)$. \square

889 A direct consequence is that $\maxtime(\mathcal{N}) \leq T$ and $\maxtime(\mathcal{N}) = T$ prob-
890 lems can be solved in polynomial time when \mathcal{N} is. Notice that if \mathcal{N} is determi-
891 nistic but not sound, then Lemma 1 does not hold: we only have an inequality.

892 We now turn to $\mintime(\mathcal{N})$. We show that it is strictly harder to compute
893 for sound negotiations than $\maxtime(\mathcal{N})$.

894 **Theorem 5.** *$\mintime(\mathcal{N}) \leq T$ is NP-complete in the strong sense for sound*
895 *acyclic negotiations, even if \mathcal{N} is very weakly non-deterministic.*

896 *Proof.* First, we can decide $\mintime(\mathcal{N}) \leq T$ in NP. Indeed, one can guess a
897 final (untimed) run ρ of size $\leq |N|$, consider ρ^- the timed run corresponding to
898 ρ where all outcomes are taken at the earliest possible dates, and compute in
899 linear time $\delta(\rho^-)$, and check that $\delta(\rho^-) \leq T$.

900 The hardness part is obtained by reduction from the **Bin Packing** problem.
901 We give a set U of items, a size $s(u) \in \mathbb{N}$ for each $u \in U$, a positive integer
902 B defining a bin capacity. The bin packing problem asks whether there exists
903 a partition of U into k disjoint subsets $U_1, U_2 \dots U_k$ such that the sum of sizes
904 of items in each U_i is smaller or equal to B . Bin Packing is known to be NP-
905 Complete [11] in the strong sense, that is even if the constants are given in
906 unary. Let us now show that every instance of Bin Packing can be reduced to a
907 min-time problem for very-weakly non-deterministic sound negotiations.

908 Given a set U of items, a bin capacity B and number k of bins, we build a
 909 timed negotiation $\mathcal{N}_{U,k}$ with k processes $u_{i,1}, u_{i,2}, \dots, u_{i,k}$ for each item $u_i \in U$,
 910 and k additional processes v_1, v_2, \dots, v_k . The timing of a process v_i will encode the
 911 total size of items put in the bin i . We then show that Bin Packing with items
 912 U , k bins, and a bound B has a solution iff $\text{mintime}(\mathcal{N}_{U,k}) \leq B$.

913 We describe the negotiation $\mathcal{N}_{U,k}$ layer by layer. In total we will have $|U| + 1$
 914 layers: intuitively, we will consider one item in each layer, and make one global
 915 decision to decide in which bin this item goes. The first layer has only the initial
 916 node n_0 . The set of processes involved in n_0 is the set of all processes. The
 917 outcomes from the initial node are $r_{1,1}, \dots, r_{1,k}$, which tell in which bin $1, \dots, k$
 918 the first item is placed. Outcome $r_{1,i}$ leads process $u_{i,1}$ and v_i to node YES_i^1 .
 919 It leads processes $u_{j,1}$ and v_j to NO_j^1 for every $j \neq i$. Last, it leads all other
 920 processes in $\{u_{j,m} \mid j > 1, 1 \leq m \leq k\}$ to node n_1 . Intuitively, moving to node
 921 YES_i^1 means that item u_1 is placed in bin i . The second layer has $2k + 1$ nodes:
 922 $\text{YES}_1^2 \dots \text{YES}_k^2, \text{NO}_1^2 \dots \text{NO}_k^2$ and n_1 . The timing of outcome $r_{1,i}$ from node n_0
 923 is $\gamma(n_0, r_{1,i}) = [0, 0]$.

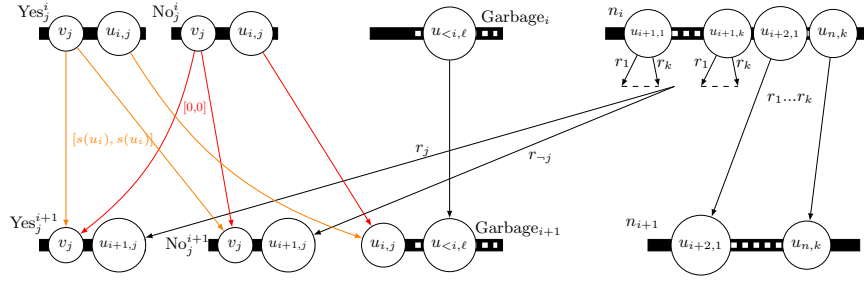


Fig. 6. Layer i of the very weakly non-deterministic $\mathcal{N}(U, k)$

924 Inductively, layer i is defined as in Fig 6. Node n_i contains processes $u_{j,\ell}$ for
 925 all $j > i$ and all ℓ . It is similar to n_0 , with outcome $r_{i+1,1}, \dots, r_{i+1,k}$. Outcome
 926 $r_{i+1,\ell}$ leads process $u_{i+1,\ell}$ to node Yes_ℓ^{i+1} , and process $u_{i+1,j}$ to No_j^{i+1} for all
 927 $j \neq \ell$. Other processes $u_{i',j}$ with $i' > i + 1$ are sent to n_{i+1} . The associated
 928 timings are $[0, 0]$.

929 Node Garbage_i collects all nodes $u_{\ell,j}$ with $\ell < i$. There is a unique outcome,
 930 with associated timing $[0, 0]$, leading all processed to Garbage_{i+1} .

931 Node YES_j^i has a unique outcome r , with timing $\gamma(\text{YES}_j^i, r) = [s(u_i), s(u_i)]$,
 932 and with $\mathcal{X}(\text{YES}_j^i, r) = \{\text{YES}_j^{i+1}, \text{NO}_j^{i+1}\}$. That is, node YES_j^i is non-deterministic,
 933 and it awaits the decision from $u_{i+1,j}$ to know whether it will go to YES_j^{i+1}
 934 or to NO_j^{i+1} . Last, $u_{i,j}$ is sent to node Garbage_{i+1} . This allows each node to
 935 have at least one deterministic process, as v_i only are non-deterministic.

936 In the same way, NO_j^i has a unique outcome r , timed with $\gamma(\text{NO}_j^i, r) = [0, 0]$,
 937 and with $\mathcal{X}(\text{NO}_j^i, r) = \{\text{YES}_j^{i+1}, \text{NO}_j^{i+1}\}$. It sends process $u_{j,i}$ to Garbage_{i+1} .

938 The last layer has only node n_f . Nodes Yes_i^k and No_i^k both have a single
 939 outcome which take all their processes to n_f .

940 The timing function γ is defined as follows: $\gamma(Yes_j^i, r_i) = [s(u_i), s(u_i)]$ and
 941 $\gamma(n, r) = [0, 0]$ for all other node and outcome r .

942 We now prove that $MinTime(\mathcal{N}_{U,k}) \leq B$ iff the answer to Bin Packing is
 943 positive. The maximal execution time over runs ρ of $\mathcal{N}_{U,k}$ is the maximal value
 944 of all valuations $\mu(v_j)$ and $\mu(u_{i,j})$, with $i \in 1..|U|, j \in 1..k$. Take the valuation
 945 μ at the last step before (n_f, r_f) . Consider $t = \max_j \mu(v_j)$. We have easily that
 946 $\mu(u_{i,j}) \leq t$ for all i, j by construction, because each $u_{i,j}$ had the same timing
 947 as v_j before reaching a garbage node. Now, we have $\mu(v_j) = \sum_{(Yes_j^i, r_i) \in \rho} s(u_i)$.
 948 Hence, $\delta(\rho) = \max_{j \in 1..k} \mu(v_j)$. That is, $mintime(\mathcal{N}(U, B, k)) \leq B$ iff there is
 949 a path ρ such that $\mu(v_j) = \sum_{(Yes_j^i, r_i) \in \rho} s(u_i) \leq B$ for all j , ie there exists a
 950 valuation such that each item is in one bin, and no bin exceeds its bound B .

951 Last, we now show that $\mathcal{N}_{U,k}$ is a very weakly non-deterministic, sound and
 952 layered negotiations. First, the only processes that have non-deterministic transi-
 953 tions are processes v_1, \dots, v_k , from Yes_j^i and No_j^i nodes. However, both nodes
 954 also have the same deterministic process u_j^i . Thus $\mathcal{N}_{U,k}$ is very weakly non-
 955 deterministic. Let us now prove soundness. The only choices are made from
 956 node n_i , the rest just follow in a unique way. From any configuration M , let i
 957 such that $M(u_{i+1}, j) = \{n_i\}$ for some j . By construction, i is unique. We can
 958 then do steps $r_{i+1,1} \dots r_{n,1}$, that is choosing to place items $i+1, \dots, n$ to the first
 959 bin. The steps from other processes are uniquely derived, and all processes reach
 960 n_f . The layeredness comes from the definition. Actually, $\mathcal{N}_{U,k}$ is $2k+2$ -layered,
 961 for k the number of bins. However, as k is part of the input, it does not fall in
 962 our k -layered restriction. \square

963 We show that $mintime(\mathcal{N}) = T$ is harder to decide than $mintime(\mathcal{N}) \leq T$:

964 **Proposition 4.** *The $mintime(\mathcal{N}) = T$? decision problem is DP-complete for*
 965 *sound acyclic negotiations, even if it is very weakly non-deterministic.*

966 *Proof.* The reduction is very similar to proof of Proposition 2. First, we define
 967 the complement of Bin-Packing Problem, **Non-Bin-Packing Problem**:

968 Given a set U of items, a size $s(u) \in \mathbb{N}$ for each $u \in U$, a positive integer bin
 969 capacity B , does for any partition U into k disjoint subsets $U_1, U_2 \dots U_k$ there
 970 exist a subset U_i such that the sum of sizes of the items in U_i is more than B ?
 971 Since the Bin-Packing Problem is NP-Complete, so the Non-Bin-Problem is co-
 972 NP Complete. Now consider the following **Bin-Non-Bin Problem** :

973 Given two instances of Bin-Packing parameters, $P_1 = (U_1, s_1, B_1, k_1)$ and
 974 $P_2 = (U_2, s_2, B_2, k_2)$, does P_1 satisfy Bin-Packing Problem and P_2 satisfy Non-
 975 Bin-Packing Problem?

976 Bin-Non-Bin Problem is DP-Complete, so we reduce it to our equality deci-
 977 sion problem of min time. First, we construct the negotiations $\mathcal{N}_{U_1, B_1, k_1}'$ and
 978 $\mathcal{N}_{U_2, B_2, k_2}'$ like in proof of Theorem 5, but only after tripling each $s(u)$ in U_1 and
 979 doubling each $s(u)$ in U_2 . Likewise we triple B_1 and double B_2 , so that new

980 $B'_1 = 3 * B_1$ and $B'_2 = 2 * B_2$.
981 In $\mathcal{N}_{U'_1, B'_1, k_1}$, we add a new node n_0 with a single outcome r which now acts
982 as the first node. The older n_0 is now called n'_0 . We also add a new process a_1 ,
983 which goes to another new node n_{a_1} (has only a_1 as process) from n_0 for its single
984 outcome r . Outcome r sends all other processes from n_0 to n'_0 . Node n_{a_1} has a
985 single outcome r_1 which takes a_1 to n_f . Also, $\gamma(n_{a_1}, r_1) = [3 * B_1 + 1, 3 * B_1 + 1]$
986 while $\gamma(n_0, r) = [0, 0]$.
987 Similarly in $\mathcal{N}_{U'_2, B'_2, k_2}$, we add a new node n_0 with two outcomes r and r_{new}
988 which now acts as the first node. The older n_0 is now called n'_0 . We also add a
989 new process a_2 , which goes to another new node n_{a_2} (has only a_2 as process) from
990 n_0 for its outcome r . Outcome r sends all other processes from n_0 to n'_0 . Node
991 n_{a_2} has a single outcome r_2 which takes a_2 to n_f . Also, $\gamma(n_{a_2}, r_2) = [2 * B_1, 2 * B_1]$
992 while $\gamma(n_0, r) = [0, 0]$. For outcome r_{new} of n_0 , all processes (including a_2) di-
993 rectly go to n_f . Also, $\gamma(n_0, r_{new}) = [2 * B_2 + 1, 2 * B_2 + 1]$.
994 Now we merge the two negotiations $\mathcal{N}_{U'_1, B'_1, k_1}$ and $\mathcal{N}_{U'_2, B'_2, k_2}$ in the same way
995 as we merged in Corollary 2, merging the n_f of $\mathcal{N}_{U'_1, B'_1, k_1}$ with n_0 of $\mathcal{N}_{U'_2, B'_2, k_2}$
996 and making other similar changes we did in Corollary 2. We call this new ne-
997 gotiation $\mathcal{N}_{P'_1, P'_2}$. Note the negotiation $\mathcal{N}_{P'_1, P'_2}$ is sound as well as very weakly
998 non-deterministic.

999 The claim is that $\text{mintime}(\mathcal{N}_{P'_1, P'_2}) = 3 * B_1 + 2 * B_2 + 2$ iff (P_1, P_2) satisfy
1000 Bin-Non-Bin Problem.

1001 We first show the reverse implication i.e if (P_1, P_2) satisfy Bin-Non-Bin Problem,
1002 then $\text{mintime}(\mathcal{N}_{P'_1, P'_2}) = 3 * B_1 + 2 * B_2 + 2$. Since P_1 is satisfiable, so the mintime
1003 to complete the structure above n_{sep} of $\mathcal{N}_{P'_1, P'_2}$ is $3 * B_1 + 1$. This is because all the
1004 processes corresponding to $\mathcal{N}_{U'_1, B'_1, k_1}$ take $(\leq 3 * B)$ time to reach n_{sep} (because
1005 P_1 is satisfies Bin-Packing) while a_1 takes $3 * B_1 + 1$ units of time. After reaching
1006 n_{sep} , processes can now take either outcome r_2 or r_{new} . If processes choose
1007 outcome r_2 , then the timetaken by any final run will be $(\geq 2 * (B_2 + 1))$ because
1008 P_2 satisfies Non-Bin-Packing. On the other hand, if processes choose r_{new} to
1009 reach n_f , then the time taken will be $2 * B_2 + 1$. So it is clear mintime for part
1010 below n_{sep} is $2 * B_2 + 1$. So, overall the $\text{mintime}(\mathcal{N}_{P'_1, P'_2}) = 3 * B_1 + 2 * B_2 + 2$.

1011 For forward implication, we consider all four scenerios of (P_1, P_2) and argue that
1012 P_1 satisfies Bin-Packing and P_2 satisfies Non-Bin-Packing is the only possibility.
1013 First let's assume that P_1 does not satisfy Bin-Packing. Then the mintime to
1014 complete the structure above n_{sep} is $(\geq 3 * (B_1 + 1))$. This is beacuse processes
1015 corresponding to $\mathcal{N}_{U'_1, B'_1, k_1}$ take at least $3 * (B_1 + 1)$ time to reach n_{sep} while
1016 a_1 take $3 * B_1 + 1$. Now since the mintime which can be taken to reach n_f from
1017 n_{sep} in either case whether P_2 satisfies Non-Bin-Packing or not is $(\geq 2 * B_2)$ so
1018 the min time to complete $(\mathcal{N}_{P'_1, P'_2}) \geq 3 * B_1 + 2 * B_2 + 3$. Hence this shows that
1019 P_1 satisfies Bin-Packing. This also shows the final run corresponding to mintime
1020 of $\mathcal{N}_{P'_1, P'_2}$ takes exactly $3 * B_1 + 1$ units of time to reach n_{sep} from n_0 (i.e. all
1021 processes have reached n_{sep}) if $\text{mintime}(\mathcal{N}_{P'_1, P'_2}) = 3 * B_1 + 2 * B_2 + 2$.

1022 Now if we assume the P_2 does not satisfy Non-Bin-Packing, then the mintime to
1023 reach n_f from n_{sep} is $2 * B_2$. And we already know that mintime to reach n_{sep}

1024 from n_0 is $3 * B_1 + 1$. So $\text{mintime}(\mathcal{N}_{P'_1, P'_2}) = 2 * B_2 + 3 * B_1 + 1$. Hence this leaves
 1025 us with the only case when P_1 satisfies Bin-Packing and P_2 satisfies Non-Bin-
 1026 Packing for which we already know that the min time taken is $3 * B_1 + 2 * B_2 + 2$
 1027 from the reverse implication. \square

1028 An open question is whether the minimal execution time can be computed
 1029 in PTIME if the negotiation is both sound and deterministic. The reduction to
 1030 bin packing does not work with deterministic (and sound) negotiations.

1031 Appendix C: k -Layered Negotiations

1032 In the previous sections, we have considered sound negotiations, and determinis-
 1033 tic negotiations. For both classes, computing the minimal execution time cannot
 1034 be done in PTIME (unless NP=PTIME), even if constants are given in unary.
 1035 In this section, we consider k -layeredness (see Section 2), a syntactic property
 1036 that can be efficiently verified (it suffices to compute the depth of each node,
 1037 which can be done in polynomial time).

1038 8.1 Algorithmic properties

1039 Let k be a fixed integer. We first show that Reachability, Soundness and max-
 1040 imum execution time can be checked in PTIME for k -layered negotiations (the
 1041 two first results were stated in Section 2). Let N_i be the set of nodes at layer
 1042 i . We define for every layer i the set S_i of subsets of nodes $X \subseteq N_i$ which can
 1043 be jointly enabled and such that for every process p , there is exactly one node
 1044 $n(X, p)$ in X with $p \in n(X, p)$. Formally, we define S_i inductively. We start with
 1045 $S_0 = \{n_0\}$. We then define S_{i+1} from the contents of layer S_i : we have $Y \in S_{i+1}$
 1046 iff $\bigcup_{n \in Y} P_n = P$ and there exist $X \in S_i$ and an outcome $r_m \in R_m$ for every
 1047 $m \in X$, such that $n \in \mathcal{X}(n(X, p), p, r_m)$ for each $n \in Y$ and $p \in P_n$.

1048 **Theorem 6.** *Let $k \in \mathbb{N}^+$. Checking reachability, soundness and computing the*
 1049 *maximum execution time for a k -layered acyclic negotiation \mathcal{N} can be done in*
 1050 *PTIME. More precisely, the worst-case time complexity is $O(|P| \cdot |\mathcal{N}|^{k+1})$.*

1051 *Proof (Sketch of Proof).* The algorithm has the same form for all problems. The
 1052 basis is to compute S_i layer by layer, by following its inductive definition. The
 1053 set S_i is of size at most 2^k , as $|N_i| < k$ by definition of k -layeredness. Knowing S_i ,
 1054 it is easy to build S_{i+1} by induction. This takes time at most $O(|P| |\mathcal{N}|^{k+1})$: We
 1055 need to consider all k -uple of outcomes for each layer. There can be $|\mathcal{N}|^k$ such
 1056 tuples. We need to do that for all processes ($|P|$), and for all layers (at most
 1057 $|\mathcal{N}|$).

1058 For reachability, we just need to check whether layer $S_d = \{n_f\}$, where d is
 1059 the depth of n_f .

1060 For soundness, let us denote by $\text{Next}(X, (r_n)_{n \in X})$ the set of nodes that
 1061 are successors of nodes in X after outcomes $(r_n)_{n \in X}$. We need to check that
 1062 for all layer i , for all set $X \in S_i$ and all tuple of outcomes $(r_n)_{n \in X}$, there

1063 is a $Y \subseteq \text{Next}(X, (r_n)_{n \in X})$ such that every process p is in exactly one node
 1064 $n(Y, p)$ of Y . All nodes of $\text{Next}(X, (r_n)_{n \in X})$ are at depth $i + 1$, and thus there
 1065 are at most k nodes in $\text{Next}(X, (r_n)_{n \in X})$. There are thus at most 2^k subset
 1066 $Y \subseteq \text{Next}(X, (r_n)_{n \in X})$ and we can check them one by one.

For maximal execution time, we keep for each subset $X \in S_i$ and each
 node $n \in X$, the maximal time $f_i(n, X) \in \mathbb{N}$ associated with n and X . From
 S_{i+1} and f_i , we inductively compute f_{i+1} in the following way: for all $X \in S_i$
 with successor $Y \in S_{i+1}$ for outcomes $(r_p)_{p \in P}$, we denote $f_{i+1}(Y, n, X) =$
 $\max_{p \in P(n)} f_i(X, n(X, p)) + \gamma^+(n(X, p), r_p)$. If there are several choices of $(r_p)_{p \in P}$
 leading to the same Y , we take r_p with the maximal $f_i(X, n(X, p)) + \gamma^+(n(X, p), r_p)$.
 We then define $f_{i+1}(Y, n) = \max_{X \in S_i} f_{i+1}(Y, n, X)$. Again, the initialization is
 trivial, with $f_0(\{n_0\}, n_0) = 0$. The maximal execution time of \mathcal{N} is $f(\{n_f\}, n_f)$.
 That is, for all nodes (at most $|\mathcal{N}|$), we have to consider every k -uple of out-
 comes, and there are at most $|\mathcal{N}|^k$ of them, and every process to compute the
 max, and the complexity is still in $O(|P| \cdot |\mathcal{N}|^{k+1})$. \square

1067 We can bound the complexity precisely by $O(d(\mathcal{N}) \cdot C(\mathcal{N}) \cdot \|R\|^{k^*})$, with:
 1068 – $d(\mathcal{N}) \leq |\mathcal{N}|$ the depth of n_f , that is the number of layers of \mathcal{N} , and $\|R\|$ is
 1069 the maximum number of outcomes of a node,
 1070 – $C(\mathcal{N}) = \max_i |S_i| \leq 2^k$, which we will call the *number of contexts of \mathcal{N}* , and
 1071 which is often much smaller than 2^k .
 1072 – $k^* = \max_{X \in \cup_i S_i} |X| \leq k$. We say that \mathcal{N} is *k^* -thread bounded*, meaning
 1073 that there cannot be more than k^* nodes in the same context X of any layer.
 1074 Usually, k^* is strictly smaller than $k = \max_i |N_i|$, as $N_i = \bigcup_{X \in S_i} X$.

1075 Consider again the Brexit example Figure 1. We have $(k + 1) = 7$, while
 1076 we have the depth $d(\mathcal{N}) = 6$, the negotiation is $k^* = 3$ -thread bounded (k^* is
 1077 bounded by the number of processes), and the number of contexts is at most
 1078 $C(\mathcal{N}) = 4$ (EU chooses to enforce backstop or not, and Pa chooses to go to court
 1079 or not).

1080 8.2 Minimal Execution Time

1081 As with sound negotiations, computing minimal time is much harder than com-
 1082 puting the maximal time for k -layered negotiations:

1083 **Theorem 7.** *Let $k \geq 6$. The $\text{Min} \leq T$ problem is NP-Complete for k -layered*
 1084 *acyclic negotiations, even if the negotiation is sound and very weakly non-deterministic.*

1085 *Proof.* One can guess in polynomial time a final run of size $\leq |\mathcal{N}|$. If the exe-
 1086 cution time of this final run is smaller than T then we have found a final run
 1087 witnessing $\text{Min}(\mathcal{N}) \leq T$. Hence the problem is in NP.

1088 Let us now show that the problem is NP-hard. We proceed by reduction from
 1089 the knapsack decision problem. Let us consider a set of items $U = \{u_1, \dots, u_n\}$
 1090 of respective values v_1, \dots, v_n and weight w_1, \dots, w_n and a knapsack of maximal
 1091 capacity W . The knapsack problem asks, given a value V whether there exists a
 1092 subset of items $U' \subseteq U$ such that $\sum_{u_i \in U'} v_i \geq V$ and such that $\sum_{u_i \in U'} w_i \leq W$.

1093 We build a negotiation with $2n$ processes $P = \{p_1, \dots, p_{2n}\}$. Intuitively, $p_i, i \leq$
1094 n will serve to encode the value as timing, while $p_i, i > n$ will serve to encode
1095 the weight as timing. We set the set of nodes $N = \{n_0, n_f\} \cup \{C_i \mid i \in 1..n\} \cup$
1096 $\{n_{L,0,i}, n_{L,1,i}, n_{R,0,i}, n_{R,1,i} \mid i \in 1..n\}$. Intuitively, node $n_{L,1,i}$ (resp $n_{R,1,i}$) will
1097 be used to remember that item i is placed in the knapsack and that its value
1098 (resp. weight) needs to be added. For all i , node $n_{L,1,i}$ (resp. $n_{R,1,i}$) has a unique
1099 possible outcome, b_i (resp. c_i). Nodes of the form $n_{L,0,i}$ remember that item i
1100 has not been placed in the knapsack, and they have outcome a_i . Nodes of the
1101 form $n_{R,0,i}$ remember that item i has not been placed in the knapsack, and they
1102 all have outcome 0. This outcome does not change the execution time, matching
1103 the fact that the current weight and value of the knapsack is not increased.

1104 Last, nodes of the form C_i will just remember the items that have already
1105 been considered. These nodes have two outputs, yes and no, telling whether the
1106 item i should be placed in the knapsack or not, consistently for weight and value
1107 processes.

1108 We set $P_{n_0} = P_{n_f} = P$, and for other nodes $n_{L,0,i}$, $P_{n_{L,0,i}} = P_{n_{L,1,i}} =$
1109 $\{p_1 \dots p_i\}$ and $P_{n_{R,0,i}} = P_{n_{R,1,i}} = \{p_{n+1} \dots p_{n+i}\}$. Last $P_{C_i} = \{p_{i+1} \dots p_n \cdot p_{n+i} \dots p_{2n}\}$.

1110 We define the transition relation as follows: $\mathcal{X}(n_0, \text{yes}, p_1) = \{n_{L,1,i}\}$, and
1111 $\mathcal{X}(n_0, \text{no}, p_1) = \{n_{L,0,1}\}$, such that process p_1 remembers that the item is picked/notpicked.
1112 In the same way, $\mathcal{X}(n_0, \text{no}, p_{n+1}) = \{n_{R,0,1}\}$ and $\mathcal{X}(n_0, \text{yes}, p_{n+1}) = \{n_{R,1,1}\}$ for
1113 process p_{i+1} . Hence both process p_1, p_{n+1} will have the same information about
1114 whether the first item is picked or not. Finally, for every $k \in 2..n$, we define
1115 $\mathcal{X}(n_0, \text{no}, p_k) = \mathcal{X}(n_0, \text{no}, p_{k+n}) = \mathcal{X}(n_0, \text{yes}, p_k) = \mathcal{X}(n_0, \text{no}, p_{k+n}) = \{C_1\}$.

1116 Other layers are similar: for $i \in 1..n$, we have $\mathcal{X}(C_i, \text{no}, p_i) = \{n_{L,0,i+1}\}$
1117 $\mathcal{X}(C_i, \text{yes}, p_i) = \{n_{L,1,i+1}\}$, Similarly, for every $i \in 1..n$, $\mathcal{X}(C_i, \text{no}, p_{i+n}) =$
1118 $\{n_{R,0,i+1}\}$, and $\mathcal{X}(C_i, \text{yes}, p_{i+n}) = \{n_{R,1,i+1}\}$. We set $\mathcal{X}(C_i, \text{no}, p_j) = \mathcal{X}(C_i, \text{yes}, p_j) =$
1119 $\{C_{i+1}\}$ for every $j \in [i+1, n-1] \cup [n+i+1, 2n]$.

1120 The most interesting set of transitions are to interface $n_{L,0,i}, n_{L,1,i}, n_{R,0,i}, n_{R,1,i}$
1121 with the next layer, in a non deterministic way because they dont know whether
1122 the next item will be picked or not: $\mathcal{X}(n_{L,0,i}, a_i, p_j) = \mathcal{X}(n_{L,1,i}, b_i, p_j) = \{n_{L,0,i+1}, n_{L,1,i+1}\}$
1123 for $j \in 1..i$ and, $\mathcal{X}(n_{R,0,i}, 0, p_j) = \mathcal{X}(n_{R,1,i}, c_i, p_j) = \{n_{R,0,i+1}, n_{R,1,i+1}\}$ for
1124 $j \in n+1..n+i$.

1125 Last, all processes synchronize on n_f by setting $\mathcal{X}(n_{L,0,n}, 0, p_j) = \mathcal{X}(n_{L,1,n}, b_n, p_j) =$
1126 $\mathcal{X}(n_{R,0,n}, 0, p_j) = \mathcal{X}(n_{R,1,n}, c_n, p_j) = \{n_f\}$

1127 We now have to set timing constraints for outcomes. Outcomes 0, yes and
1128 no are associated with $[0, 0]$. Outcome c_i is associated with $[w_i, w_i]$, the weight
1129 of u_i . Last, outcome b_i is associated with a more complex function, such that
1130 $\sum_i b_i \leq W$ iff $\sum_i v_i \geq V$. For that, we set $[\frac{(v_{max}-v_i)W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for outcome
1131 b_i , where v_{max} is the largest value of an item, and V is the total value we want to
1132 reach at least. Also, we set $[\frac{(v_{max})W}{n \cdot v_{max}-V}, \frac{v_{max}W}{n \cdot v_{max}-v_i}]$ for outcome a_i . We set $T = W$,
1133 the maximal weight of the knapsack.

1134 Now, consider a final run ρ in \mathcal{N} . The only choice is about *yes, no* from
1135 C_i . Let I be the set of indices such that *yes* is the outcome from all C_i in
1136 this path. We obtain $\delta(\rho) = \max(\sum_{i \notin I} a_i + \sum_{i \in I} b_i, \sum_{i \in I} c_i)$. We have $\delta(\rho) \leq$
1137 $T = W$ iff $\sum_{i \in I} w_i \leq W$, that is the sum of the weight are lower than W , and

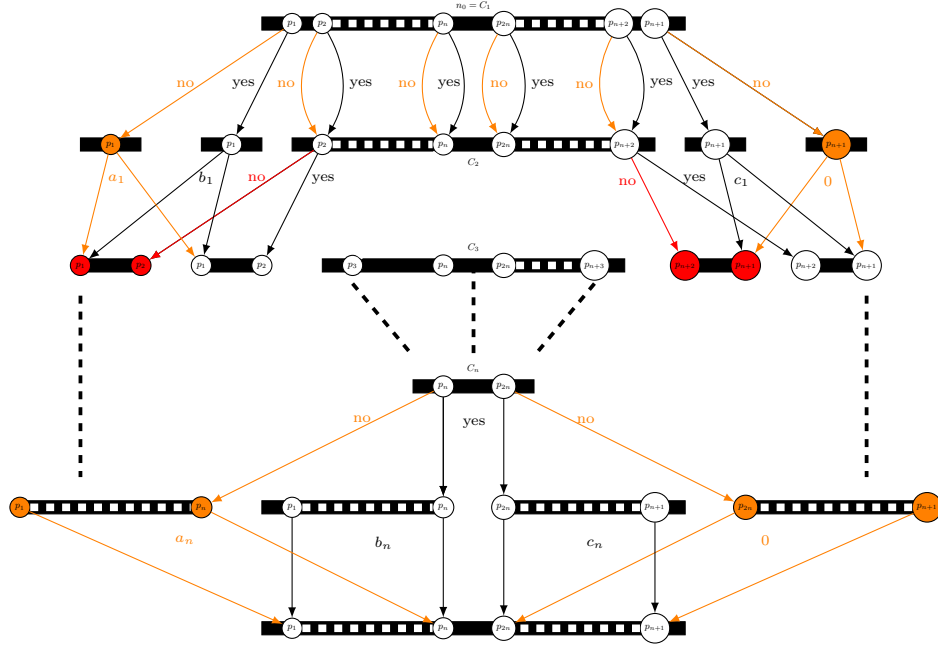


Fig. 5. The negotiation encoding Knapsack

1138 $\sum_{i \notin I} \frac{(v_{max})W}{n \cdot v_{max} - V} + \sum_{i \in I} \frac{(v_{max} - v_i)W}{n \cdot v_{max} - V} \leq W$. That is, $n \cdot v_{max} - \sum_{i \in I} v_i \leq n \cdot v_{max} - V$,
 1139 i.e. $\sum_{i \in I} v_i \geq V$. Hence, there exists a path ρ with $\delta(\rho) \leq T = W$ iff there exists
 1140 a set of items of weight less than W and of value more than V .

1141 So, given a knapsack of size n , a value V and a weight limit W one can build
 1142 a negotiation $\mathcal{N}_V^{Knapsack}$ with $O(3n + 2)$ nodes. We can encode all weights with
 1143 $\log(v_{max} \cdot W) + \log(n \cdot v_{max})$ bits. One can notice that $\mathcal{N}_V^{Knapsack}$ is 5-layered and
 1144 sound.

1145 However, it is not (weakly) non-deterministic because of nodes $n_{L,0,i}, n_{L,1,i}, n_{R,0,i}, n_{R,1,i}$.
 1146 It is easy to add two processes V (resp. W), present in all nodes $n_{L,0,i}, n_{L,1,i}$
 1147 (resp. $n_{R,0,i}, n_{R,1,i}$), and make process P_i (resp. P_{n+i}) leave these nodes, deter-
 1148 ministically leading to a new node $garbage_{i+1}$ at layer $i + 1$. Then the negotiation
 1149 is very weakly deterministic, and 6-layered. \square

1150 Following the same lines as for the proofs of Propositions 2 and 4, a conse-
 1151 quence of Theorem 7 is that the $Min = T$ problem is in DP for k -layered acyclic
 1152 negotiations.

1153 It is well known that Knapsack is weakly NP-hard, that is it NP-hard only
 1154 when weights/values are given in binary. This means that Thm. 7 shows that
 1155 minimum execution time $\leq T$ is NP-hard only when T is given in binary. We
 1156 can actually show that for k -layered negotiations, the $mintime(\mathcal{N}) \leq T$ problem
 1157 can be decided in PTIME if T is given in unary (i.e. if T is not too large):

1158 **Theorem 8.** *Let $k \in \mathbb{N}$. Given a k -layered negotiation \mathcal{N} and T written in*
 1159 *unary, one can decide in PTIME whether the minimum execution time of \mathcal{N} is*
 1160 *$\leq T$. The worst-case time complexity is $O(|\mathcal{N}| \cdot |P| \cdot (T \cdot |\mathcal{N}|)^k$).*

1161 *Proof.* We will remember for each layer i a set \mathcal{T}_i of functions τ from nodes N_i
 1162 of layer i to a value in $\{1, \dots, T, \perp\}$. Basically, we have $\tau \in \mathcal{T}_i$ if there exists a
 1163 path ρ reaching $X = \{n \in N_i \mid f(n) \neq \perp\}$, and this path reaches node $n \in X$
 1164 after $\tau(n)$ time units. As for S_i , for all p , we should have a unique node $n(\tau, p)$
 1165 such that $p \in n(f, p)$ and $\tau(n(\tau, p)) \neq \perp$. Again, it is easy to initialize $\mathcal{T}_0 = \{\tau_0\}$,
 1166 with $\tau_0(n_0) = 0$, and $\tau_0(n) = \perp$ for all $n \neq n_0$.

1167 Inductively, we build \mathcal{T}_{i+1} in the following way: $\tau_{i+1} \in \mathcal{T}_{i+1}$ iff there exists a
 1168 $\tau_i \in \mathcal{T}_i$ and $r_p \in R_{n(\tau_i, p)}$ for all $p \in P$ such that for all n with $\tau_{i+1}(n) \neq \perp$, we
 1169 have $\tau_{i+1}(n) = \max_p \tau_i(n(\tau_i, p)) + \gamma(n(\tau_i, p), r_p)$.

We have that the minimum execution time for \mathcal{N} is $\min_{\tau \in \mathcal{T}_n} \tau(n_\tau)$, for n the
 depth of n_f . There are at most T^k functions τ in any \mathcal{T}_i , and there are at most
 $|\mathcal{N}|$ layers to consider, giving the complexity. \square

1170 As with Thm. 6, we can more accurately state the complexity as $O(d(\mathcal{N}) \cdot$
 1171 $C(\mathcal{N}) \cdot \|R\|^{k^*} \cdot T^{k^* - 1})$. The $k^* - 1$ is because we only need to remember minimal
 1172 functions $\tau \in \mathcal{T}_i$: if $\tau'(n) \geq \tau(n)$ for all n , then we do not need to keep τ' in \mathcal{T}_i .
 1173 In particular, for the knapsack encoding in the proof of Thm. 7, we have $k^* = 3$,
 1174 $\|R\| = 2$ and $C(\mathcal{N}) = 4$.

1175 Notice that if k is part of the input, then the problem is strongly NP-hard,
 1176 even if T is given in unary, as e.g. encoding bin packing with k bins result to a
 1177 $k + 1$ -layered negotiations.