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Chiralities in topological vector spaces

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Abstract

Differential Linear Logic extends Linear Logic by allowing the differentiation of proofs. Trying to interpret this proof-theoretical notion of differentiation by traditional analysis, one faces the fact that analysis badly accommodates with the very basic layers of Linear Logic. Indeed, tensor products are seldom associative and spaces stable by double duality enjoy very poor stability properties. In this work, we unveil the polarized settings lying beyond several models of Differential Linear Logic. By doing so, we identify chiralities - a categorical axiomatic developed from game semantics - as an adequate setting for expressing several results from the theory of topological vector spaces. In particular, complete spaces provide an interpretation for negative connectives, while barrelled or bornological spaces provide an interpretation for positive connectives.

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1 Introduction

Linear Logic (LL) is the result of a decomposition of Intuitionistic Logic via an involutive *linear* negation. This linear negation takes its root in semantics: the linear negation of a formula is interpreted as the *dual*¹ of the vector space interpreting the formula. While LL's primary intuitions lie in algebra, the study of vectorial models [9, 10] of it led to the introduction of Differential Linear Logic [14] (DiLL). This new proof system introduces the possibility to differentiate proofs and led to advances in the semantics of probabilistic and differentiable programming [11, 6].

Infinite dimensional vector spaces are necessary to interpret all proofs of DiLL. However these spaces are seldom isomorphic to their double dual. The class of all *reflexive* topological vector spaces, that is of spaces invariant via double-dual, moreover enjoys poor stability properties. More crucially, duality in topological vector spaces does not define a closure operator: simply considering E'' does not produce a reflexive space. Thus historical models of Linear Logic traditionally interpret formulas via very specific vector spaces: vector spaces of sequences [9], vector spaces over discrete field [10]. How close is the differentiation at stakes in DiLL from the one of real analysis? Denotational models of DiLL in real-analysis either don't interpret the involutivity of linear negation [4] or imply a certain discretisation for the interpretation of non-linear proofs [19, 8].

Polarization is a syntactical refinement of Linear logic arising for matters of proof-search [1, 16]. By making vary the topology on the dual, this paper unveils polarized models behind preexisting models of DiLL and construct new ones. Meanwhile, it attaches topological notions to the concept of polarity in proof theory. We first revisit the poor stability properties of reflexive spaces by decomposing it in a polarized version model of MLL. We also revisit the notion of *bornological* spaces persistent in DiLL's denotational semantics [32, 4] as

¹ The dual of a (topological) \mathbb{K} -vector space is the space of all linear continuous linear forms on it: $E' := \mathcal{L}(E, \mathbb{K})$



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43 an interpretation for positives. In a nutshell, we show that while (different variants of)
 44 *complete* spaces interpret *negative* connectives, barrelled spaces (as introduced by Bourbaki)
 45 or bornological spaces are the good interpretation for *positive*. While we acknowledge the
 46 poor computational value of these models - as only the multiplicative part of Linear Logic
 47 is properly interpreted here- we believe that our setting will extend to exponentials and
 48 non-linearity, as indicated in the perspective. Indeed, this paper unifies the duality at stakes
 49 in Linear Logic with the central notion of duality in functional analysis.

50 Smooth and polarized differential linear logic

51 Before diving into more details, we give a few intuitions to the categorical semantics of
 52 (Differential) Linear Logic. We refer to the literature for a detailed introduction [27, 13].
 53 Linear Logic is constructed on a fundamental duality between linear and non-linear proofs.
 54 It features two conjunctions \otimes and \times , two disjunctions \wp and \oplus , as well as exponential
 55 connectives $?$ and $!$ on which structural rules are defined. The exponential $!$ encodes non-
 56 linearity: in the call-by-name translation of Intuitionistic Logic to Linear Logic, traditional
 57 non-linear implications are translated as linear implications from the exponential: $A \Rightarrow B =$
 58 $!A \multimap B$. An involutive linear negation $(-)^{\perp}$ is defined inductively on formulas.

59 As such, a categorical model of Linear Logic is constituted of a linear-non-linear adjunction [27]
 60 between two categories. A monoidal closed category $(\mathcal{L}, \otimes, 1)$ interprets linear proofs and
 61 the multiplicative connectives, while a cartesian closed category $(\mathcal{C}, \times, 0)$ interprets non-
 62 linear proofs. To interpret the involutive linear negation of LL, the category \mathcal{L} must be
 63 $*$ -autonomous. The exponential $!$ is a co-monad on \mathcal{L} , coming from a strong monoidal
 64 adjunction: $! := \mathcal{E}' \circ U$ and $\mathcal{E}' : \mathcal{C} \rightarrow \mathcal{L} \vdash U : \mathcal{L} \rightarrow \mathcal{C}$. On top of that, interpreting DiLL
 65 necessitates an additive categorical structure on \mathcal{L} and a natural transformation $\bar{d} : ! \rightarrow Id$
 66 enabling the linearization of proof (hence their differentiation).

67 Topological vector spaces, to be defined precisely afterwards, are a generalization of normed
 68 or metric spaces necessary to higher-order functions. Smooth functions between topological
 69 vector spaces are those functions which can be infinitely or everywhere differentiated. To
 70 handle composition or differentiation of smooth functions, the topology of their codomain
 71 must verify some completeness property². However, this requirement for completeness mixes
 72 badly with reflexive spaces (those interpreting an involutive linear negation). Hence the
 73 difficulty to construct smooth models of DiLL.

74 Beyond the distinction between linear and non-linear proofs, *polarization* in LL [25]
 75 distinguishes between *positive* and *negative* formulas.

76 Negative Formulas: $N, M := a \mid ?P \mid \uparrow P \mid N \wp M \mid \perp \mid N \& M \mid \top$.
 77 Positive Formulas: $P, Q := a^{\perp} \mid !N \mid \downarrow N \mid P \otimes Q \mid 0 \mid P \oplus Q \mid 1$.

78 Semantically, polarization splits \mathcal{L} in two categories \mathcal{P} and \mathcal{N} ³. The developments of this
 79 paper all take place in the categorical setting developed by Mellies: chiralities [27]⁴ are a
 80 decomposition of $*$ -autonomous categories in two adjunctions. A strong monoidal adjunction

² As an example, Mackey-Completeness is a minimal completeness condition used by Frölicher, Kriegel and Michor [15, 24] to develop a theory of higher-order smooth functions

³ Beware that the name polarity is employed with its proof theory meaning: polarity describes a proof-theoretic behaviour of a formulas and their interpretation. In the theory of topological vector spaces, the polar of a set denotes the set of all linear forms which are bounded by 1 on this subsets. The two meaning of polarity are not unrelated in the light of Proposition 26, as in barrelled spaces the polar to a neighbourhood is bounded.

⁴ We warn the reader that chiralities have no obvious link with the orientation-related chiralities in physics

81 $(-)^{\perp_L} : \mathcal{P} \longrightarrow \mathcal{N}^{op} \vdash (-)^{\perp_N} : \mathcal{N}^{op} \longrightarrow \mathcal{P}$ interprets negations, accompanied with an
 82 adjunction interpreting shifts $\uparrow : \mathcal{P} \longrightarrow \mathcal{N} \vdash \downarrow : \mathcal{N} \longrightarrow \mathcal{P}$. This semantics enables an
 83 internal interpretation of polarized connectives - thus refining traditional interpretation in
 84 terms of dual pairs.

85 Organisation

86 We begin this paper by an introduction to topological vector spaces in section 2, leading to
 87 the introduction of two basic *-autonomous categories of vector spaces factorising through
 88 dual pairs. In section 3, we introduce chiralities as a categorical model of polarized MLL.
 89 In section 4 we decompose the notion of reflexivity in a chirality of barrelled or weakly
 90 quasi-complete topological vector spaces - thus showing that chiralities are a relevant setting
 91 to the intricate theory of topological vector spaces. The last section 5 we give two chiralities
 92 based on bornological spaces, refining existing models of DiLL. The first one in section
 93 5.2 refines the model based on convenient spaces [22, 4], while the second in section 5.3
 94 refines the models based on Schwartz ε tensor product. Most proofs are quite direct for the
 95 reader familiar with the theory of topological vector spaces. Those for which we didn't find
 96 a reference in the literature are given in appendix.

97 2 *-autonomous categories of topological vector spaces

98 This preliminary section presents a rapid introduction to the various topologies on vector
 99 spaces and spaces of linear maps between them. We introduce in particular the weak and
 100 Mackey topology which both leads to *-autonomous DiLL, resulting respectively into a
 101 negative and positive interpretation of DiLL.

102 ▶ **Definition 1.** *A Hausdorff and locally convex topological vector space is a vec-*
 103 *tor space endowed with a Hausdorff topology making scalar multiplication and addition*
 104 *continuous, and such that every point has a basis of convex 0-neighbourhoods.*

105 We abbreviate by lcs the term locally convex and Hausdorff topological vector space.
 106 We denote by TOPVEC the category of lcs and linear continuous maps between them. The
 107 topology of a topological vector space E is thus described by the set $\mathcal{V}_E(0)$ of all its 0-
 108 neighbourhoods. From now on we work with locally convex Hausdorff topological vector
 109 spaces on \mathbb{R} and denote them by lcs. Working with these object, we will be confronted to
 110 two definitions of equality:

- 111 ▶ **Definition 2.** 1. *Two lcs E and F might have the same algebraic structure. The*
 112 *existence of a linear isomorphism between E and F will be denoted: $E \sim F$.*
 113 2. *Two linearly isomorphic lcs E and F might have the same topological structure. The*
 114 *existence of a linear homeomorphism between E and F is stronger than algebraic equality*
 115 *will be denoted: $E \simeq F$.*

116 Functional analysis is basically the study of spaces of (linear) functions as objects of the
 117 same class as their codomains. To construct a topology on a space of linear function, one
 118 must decide of a bornology, that is of the class of sets on which convergence must be uniform.

119 ▶ **Definition 3.** *The space of all linear continuous functions between lcs E and F is denoted*
 120 *$\mathcal{L}(E, F)$. The dual of a lcs E is denoted $E' := \mathcal{L}(E, \mathbb{R})$.*

121 ▶ **Definition 4.** *Several bornologies (that is, total collections of sets closed by finite union,*
 122 *arbitrary intersection and inclusion) can be defined on a lcs E . The following ones will be*
 123 *used in this article:*

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- 124 1. $\sigma(E)$, the bornology of all finite subsets of E .
 125 2. $\beta(E)$, the bornology of all \mathcal{T}_E sets absorbed by any 0-neighbourhood of E .
 126 3. $\mu(E)$, the bornology of all absolutely convex compact sets in E_σ , that is of all the weakly
 127 compact absolutely convex sets.

Any bornology α on E defines a topology on $\mathcal{L}(E, F)$, referred to as the topology of uniform convergence on α . It is generated by following sub-basis of 0-neighbourhoods:

$$\mathcal{W}_{B,U} = \{\ell | \ell(B) \subset U\}$$

128 for $B \in \alpha$ and $U \in \mathcal{V}_E(0)$. We will denote by $\mathcal{L}_\alpha(E, F)$ the vector space $\mathcal{L}(E, F)$ endowed
 129 with this topology. All of the bornologies σ, μ, β make $\mathcal{L}_\alpha(E, F)$ and thus E'_α a lcs.

130 Thus any bornology α defines in particular a topological dual $(-)'_\alpha$. The duals $E'_\sigma, E'_\mu,$
 131 E'_β are called respectively the *weak*, *Mackey* and *strong dual*.

132 ► **Definition 5.** Any lcs E can be seen as a space of linear forms, through the following
 133 continuous linear injection:

$$134 \quad ev_E : \begin{cases} E & \longrightarrow & (E'_\alpha)' \\ x & \mapsto & \delta_x : (f \longrightarrow f(x)) \end{cases}$$

135 Thus the topologies constructed above on spaces of linear forms can be defined on any lcs E ,
 136 for which the dual E' has been computed first hand. In particular, a lcs will be said:

- 137 1. **Mackey** when it is endowed with the topology induced by $(E'_{\mu(E)})'_{\mu(E')}$,
 138 2. **weak** when it is endowed with the topology induced by $(E'_{\sigma(E)})'_{\sigma(E')}$,
 139 3. **barrelled** when it is endowed with the topology induced by $(E'_{\beta(E)})'_{\beta(E')}$. Barrelled spaces
 140 are in particular Mackey [18, 11.1].

141 We will denote respectively by E_μ and E_σ the lcs E endowed with the Mackey and the weak
 142 topology described above.

143 ► **Remark 6.** The weak topology is a very particular topology with a discrete flavour. On the
 144 contrary, examples of Mackey spaces are easy to find: as soon as a space is metrisable, it is
 145 Mackey. The basic example of metrisable spaces are the finite dimensional vector spaces or
 146 the normed spaces. For an example of a spaces with is metrisable and not normed, consider
 147 the space of smooth functions $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$ endowed with the topology on uniform convergence
 148 of the iterated derivative on compact subsets of \mathbb{R}^n . Examples of barrelled lcs then include
 149 complete metrisable spaces, and as such Banach spaces. Example of non-metrisable spaces
 150 include spaces of distributions as $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})'_\beta$

151 ► **Definition 7.** E is said to be **semi-reflexive** when $E \sim (E'_\beta)'$, and **reflexive** when
 152 $E \simeq (E'_\beta)'_\beta$. As a corollary, a lcs E is reflexive if and only if it is barrelled and semi-reflexive.

153 Reflexive spaces are stable by product or direct sums. Thus using the strong dual as
 154 interpretation for the negation of linear logic gives us very little chance to construct a model
 155 of DiLL without strongly restricting the kind of vector spaces one handles. On the contrary,
 156 any space is invariant under double weak or Mackey dual.

157 When a monoidal category resists *-autonomy, the traditional solution is to consider
 158 pairs of objects of this category, and interpret negation as the switching of position inside a
 159 pair. This way, one can enforce the dual of constructions which do not preserve reflexivity
 160 - typically tensor products. Chu categories of vector spaces as defined by Barr [2] are a
 161 categorical axiomatization of the notion of dual pairs [26].

► **Definition 8** (Chu categories of vector spaces). *Object of CHU are pairs of vector spaces (E_1, E_2) equipped with a symmetric non-degenerate linear form $\langle \cdot, \cdot \rangle : E \times F \rightarrow \mathbb{R}$. Morphisms of CHU are pairs of linear maps:*

$$(f_1, f_2) : (E_1, E_2) \longrightarrow (F_1, F_2)$$

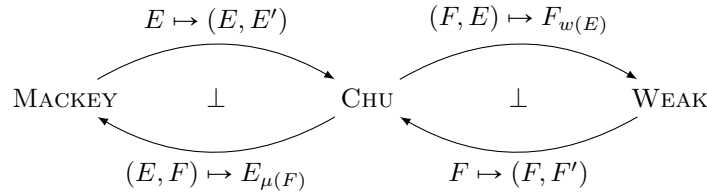
162 with $f_1 : E_1 \rightarrow F_1$ and $f_2 : F_2 \rightarrow E_2$ such that for every $x \in E_1, y \in F_2$ one has
 163 $\langle f_1(x)|y \rangle = \langle x|f_2(y) \rangle$. CHU is a $*$ -autonomous category when endowed with the following
 164 constructions:

- 165 ■ $(E_1, E_2)^\perp = (E_2, E_1)$
- 166 ■ $(E_1, E_2) \otimes (F_1, F_2) = (E_1 \otimes F_1, \mathcal{L}(E_2, F_2))$
- 167 ■ $(E_1, E_2) \multimap (F_1, F_2) = (L(E_1, F_1), E_1 \otimes F_2)$

► **Theorem 9** (The Mackey-Arens theorem). *The weak topology on E is the coarsest locally convex topology on E which preserves the dual, while the Mackey topology is the finest. In particular:*

$$(E_{\sigma(E')})' \sim E' \sim (E_{\mu(E')})'$$

168 Work by Barr [2] reinterprets this theorem in terms of dual pairs: the Mackey Topology
 169 induces a right adjoint to the functor $\mathcal{D} : E \mapsto (E, E')$ from the category TOPVEC of lcs and
 170 continuous linear maps to the category of dual pairs, while the weak topology induces the left
 171 adjoint to this functor.



172

173 These adjunctions naturally result in $*$ -autonomous categories over WEAK and MACKEY.
 174 However these constructions are *saturated*: topologies on tensor products or hom-sets are
 175 defined from the dual and are in no way internal. We showed in previous work that WEAK
 176 spaces provide a *negative interpretation* of DiLL [19], in the sense that negative connective
 177 preserve weak topologies (see proposition 17).

178 Likewise, as hinted by the above diagram, we will show that MACKEY spaces provide a
 179 positive interpretation of MLL - which could be extended to LL also by formal power series.
 180 To show this, we need to dive into topological tensor products. Here again, topologies on
 181 vector spaces introduce a variety of distinct notions of continuity.

182 ► **Definition 10.** *Consider E, F and G three lcs. We denote:*

- 183 1. $\mathcal{B}(E \times F, G)$ the vector space of all **continuous** bilinear functions from $E \times F$ (endowed
 184 with the product topology) to G .
- 185 2. $\mathcal{HB}_\alpha(E \times F, G)$ the vector space of all α -**hypocontinuous** bilinear functions from $E \times F$
 186 to G , where $\alpha \in \{\sigma, \mu, \beta\}$. These are the bilinear maps h such that for any $B_E \in \alpha(E)$
 187 and $B_F \in \alpha(F)$, the families of linear functions $\{y \in F \mapsto H(x, y) | x \in B_E\}$ and $\{x \in E \mapsto$
 188 $h(x, y) | y \in B_F\}$ are equicontinuous.
- 189 3. $\mathcal{B}(E \times F, G)$ the vector space of all **separately continuous** bilinear functions from
 190 $E \times F$ to G .

191 Continuity implies α -hypocontinuity, which in turns implies separate continuity. While
 192 separate continuity is too weak to be compatible with a fine topology on vector spaces,

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193 continuity is in general too strong to ensure the monoidal closedness of our models. *Hypo-*
 194 *continuity turns out to be the good notion to work with*, as in historical models of DiLL [12].
 195 For concision, we respectively denote as $\mathcal{B}(E \times F)$, $\mathcal{HB}_\alpha(E \times F)$ and $\mathcal{B}(E \times F)$ the spaces
 196 of scalar bilinear forms $\mathcal{B}(E \times F, \mathbb{R})$, $\mathcal{HB}_\alpha(E \times F, \mathbb{R})$ and $\mathcal{B}(E \times F, \mathbb{R})$.

197 ▶ **Definition 11.** *The projective tensor product $E \otimes_\pi F$ is the finest topology on $E \otimes F$ making*
 198 *the canonical bilinear map $E \times F \rightarrow E \otimes F$ continuous. The α -tensor product $E \otimes_\alpha F$*
 199 *is the finest topology on $E \otimes F$ making the canonical bilinear map $h : E \times F \rightarrow E \otimes F$*
 200 *α -hypocontinuous.*

201 The projective tensor product is commutative and associative [18, 15] on lcs and preserves
 202 this class of topological vector spaces. So does the weak tensor product $E \otimes_\sigma F$ [19, 2.12].
 203 For wider bornologies, commutativity is immediate but *associativity becomes more specific*, as
 204 its asks to have a good knowledge of the bornology α on $E \otimes_\alpha F$. This question is sometimes
 205 called as "Grothendieck' problème des topologies".

206 ▶ **Proposition 12.** *For any lcs E, F and G we have a linear isomorphism $\mathcal{L}(E \otimes_\alpha F, G) \sim$*
 207 *$\mathcal{HB}_\alpha(E \times F, G)$. In particular, $(E \otimes_\alpha F)' \sim \mathcal{HB}_\alpha(E \times F)$.*

3 Chiralities as polarized models of MLL_{pol}

209 We now detail what we believe to be the relevant setting to express the internal stability of
 210 polarized models of DiLL. Chiralities were introduced by Mellies [29] after an investigation in
 211 game semantics. In this section we recall the definitions of dialogue chiralities and introduce
 212 several versions, according to the involutivity of negations functors.

213 ▶ **Definition 13** ([28]). *A mixed chirality consists in two symmetric monoidal categories*
 214 *$(\mathcal{P}, \otimes, 1)$ and $(\mathcal{N}, \wp, \perp)$, between which there are two adjunctions, one of which being strong*
 215 *monoidal:*

$$\begin{array}{ccc}
 & \xrightarrow{(-)^{\perp P}} & \\
 (\mathcal{P}, \otimes, 1) & \perp & (\mathcal{N}^{op}, \wp, \perp) \\
 & \xleftarrow{(-)^{\perp N}} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \uparrow & \\
 \mathcal{P} & \perp & \mathcal{N} \\
 & \downarrow &
 \end{array}
 \quad (1)$$

217 with a family of natural bijections:

$$\chi_{p,n,m} : \mathcal{N}(\uparrow p, n \wp m) \sim \mathcal{N}(\uparrow(p \otimes n^{\perp N}), m) \quad (\text{curryfication}) \quad (2)$$

219 The natural bijections χ account for the lost monoidal closedness. They must respect the
 220 various associativity morphisms by making the following diagrams commute:

$$\begin{array}{ccc}
 \mathcal{N}(\uparrow((m \wp n) \wp a), b) & \xrightarrow{\quad \chi \quad} & \mathcal{N}(\uparrow a, (m \wp n)^* \otimes b) \\
 \downarrow \text{assoc} & & \uparrow \text{assoc., monoidality of negation} \\
 \mathcal{N}(\uparrow(m \wp (n \wp a)), b) & \xrightarrow{\quad \chi \quad} & \mathcal{N}(\uparrow(n \wp a), m^* \otimes b) \xrightarrow{\quad \chi \quad} \mathcal{N}(\uparrow a, n^* \otimes (m^* \otimes b))
 \end{array} \quad (3)$$

222 ▶ **Definition 14.** *A **dialogue chirality** is a mixed chirality in which the monoidal adjunction*
 223 *is an equivalence. A **negative chirality** is a mixed chirality in which the monoidal adjunction*
 224 *is reflective. A **positive chirality** is a mixed chirality in which the monoidal adjunction is*
 225 *co-reflective.*

226 Multiplicative Linear Logic is the subpart of Linear Logic constructed from the \otimes and
 227 \wp connectives, which is traditionnaly interpreted in a $*$ -autonomous category. Interpreting
 228 polarized Multiplicative Linear Logic in Chiralities requires an additional family of morphisms,
 229 which basically says that there is only one closure operation between the interpretation of
 230 negative and positive formulas. Thus one asks for a family of natural isomorphisms in \mathcal{P} :

$$231 \quad \text{clos}_p : \downarrow(p^{\perp P}) \simeq (\uparrow p)^{\perp N}. \quad \text{closure} \quad (4)$$

232 The categorical semantics of Linear Logic interprets formulas as objects of a certain
 233 category, and proofs as morphisms. Positive formulas of Multiplicative Linear Logic are
 234 interpreted in \mathcal{P} , negative formulas are interpreted in \mathcal{N} . In a negative or dialogue chirality,
 235 a proof of $\vdash n_1, \dots, n_n, p$ is interpreted as an arrow in $\mathcal{N}(p^{\perp P}, n_1 \wp \dots n_n)$, and a proof of
 236 $\vdash n_1, \dots, n_n$ as an arrow in $\mathcal{N}(\uparrow 1, n_1 \wp \dots n_n)$. Symmetrically, in a positive chirality proofs
 237 should be interpreted as arrows in $\mathcal{P}((n_1 \wp \dots \wp n_n)^{\perp N}, p)$ or $\mathcal{P}((n_1 \wp \dots \wp n_n)^{\perp N}, \downarrow(\perp))$.
 238 We refer to [3] for details on the invariance by cut-elimination of this procedure.

239 **► Theorem 15.** *Dialogue, negative and positive chiralities provide a categorical semantics*
 240 *for polarized MLL.*

241 In TOPVEC there may not be a shift from positive to negative describing exactly what a
 242 double negation would do to an object of \mathcal{P}^5 . We thus introduce the following generalisation
 243 for chiralities:

244 **► Definition 16.** *A topological chirality takes place between two adjoint categories \mathcal{T} and*
 245 *\mathcal{C} . It adds to this first adjunction a strong monoidal contravariant adjunction between a full*
 246 *subcategory of \mathcal{T} and a full subcategory of \mathcal{C} ,*

$$247 \quad \begin{array}{ccc} & \xrightarrow{(-)^{\perp P}} & \\ (\mathcal{P}, \otimes, 1) & \perp & (\mathcal{N}^{op}, \wp, \perp) \\ & \xleftarrow{(-)^{\perp N}} & \end{array} \quad \begin{array}{ccc} & \uparrow & \\ \mathcal{T} & \perp & \mathcal{C} \\ & \downarrow & \end{array} \quad (5)$$

248 *and such that equations 2, 3 and 4 are still validated in \mathcal{T} and \mathcal{C} respectively.*

249 In topological chiralities, proofs of MLL_{pol} are interpreted exactly following the pattern
 250 described previously. The category \mathcal{T} might be the category of lcs and \mathcal{C} the reflective
 251 subcategory of complete lcs (see proposition 18 or theorem 26). However, we also present
 252 chiralities in which we have a non-transparent interpretation for \downarrow , or in which \mathcal{C} is not even
 253 a subcategory of \mathcal{T} .

254 As an example, we briefly revisit existing models of MLL, inherited from models of DiLL,
 255 in terms of chiralities. In earlier work [19], the author built a model of DiLL in which
 256 formulas were interpreted by weak spaces. We argued that the fact that spaces of linear maps
 257 endowed with the pointwise convergence topology preserve weak spaces gave this model a
 258 polarized flavour. The space $E \wp_\sigma F := \mathcal{L}_\sigma(E'_w, F)$ is always endowed with its weak topology
 259 ([18, 15.4.7]) and the MLL model described in [19] easily refines in a chirality:

260 **► Proposition 17.** *The following adjunctions define a negative chirality:*

⁵ For example when negatives are interpreted by metrisable spaces (Proposition 18), there is no operation on TOPVEC making a space metrisable.

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$$\begin{array}{ccc}
 & \xrightarrow{(-)'_{\sigma}} & \\
 261 & (\text{TOPVEC}, \otimes_w, \mathbb{R}) \perp (\text{WEAK}^{op}, \mathfrak{Y}_{\sigma}, \mathbb{R}) & \text{TOPVEC} \perp \text{WEAK} \\
 & \xleftarrow{(-)'_{\sigma}} & \xleftarrow{\iota}
 \end{array}$$

262 *in which ι denotes the inclusion functor.*

263 More recently, in order to find the good setting in which to interpret non-linear proofs as
 264 the usual smooth function of real analysis, we constructed a polarized model of DiLL [20]
 265 [21] in which positive formulas are interpreted as complete nuclear DF spaces and negative
 266 formulas are constructed as nuclear Fréchet spaces⁶.

267 **► Proposition 18.** *For its multiplicative part, the distribution model of DiLL organises into*
 268 *the following negative topological chirality:*

$$\begin{array}{ccc}
 & \xrightarrow{(-)'_{\beta}} & \\
 269 & (\text{NDF}, \tilde{\otimes}_{\pi}, \mathbb{R}) \perp (\text{NF}^{op}, \hat{\otimes}, \mathbb{R}) & \text{TOPVEC} \perp \text{COMPL} \\
 & \xleftarrow{(-)'_{\beta}} & \xleftarrow{\iota}
 \end{array}$$

270 *in which $\tilde{}$ denotes the completion of a lcs, $\tilde{\otimes}_{\pi}$ denotes the completion of the projective*
 271 *tensor product, NDF the category of nuclear DF spaces, NF the category of Nuclear Fréchet*
 272 *spaces, COMPL the category of complete lcs and \tilde{E} the completion of the lcs E .*

273 Curryfication (Equation 2) is indeed verified, due to the fact that on nuclear complete
 274 DF spaces (that is duals of nuclear fréchet spaces), separate continuity implies continuity
 275 [23, 40.2.11]. Closure (Equation 4) is exactly the fact that completion preserve the dual.
 276 Compared to the model previously exposed[20], the interpretation of the shift to negatives
 277 \uparrow as a completion procedure allows to relax the condition on complete Nuclear DF spaces.
 278 Positives formulas are interpreted as Nuclear DF spaces and need not to be completed.

279 Guided by intuitions of theorem 9, we show that Mackey spaces leave stable the positive
 280 constructions, and in particular a certain topological tensor product. The proof is provided
 281 in the appendix.

282 **► Proposition 19.** *Consider $E, F \in \text{MACKEY}$. Then $E \otimes_{\mu} F$ is Mackey.*

283 It is however not enough to construct a positive chirality. Consider the following adjunc-
 284 tions, in which ι denotes the inclusion functor and $N \mathfrak{Y}_{\mu} M := (N'_{\mu} \otimes_{\mu} M'_{\mu})'_{\mu}$:

$$\begin{array}{ccc}
 & \xrightarrow{(-)'_{\mu}} & \\
 285 & (\text{MACKEY}, \otimes_{\mu}, \mathbb{R}) \perp (\text{TOPVEC}, \mathfrak{Y}_{\mu}, \mathbb{R}) & \text{MACKEY} \perp \text{TOPVEC} \\
 & \xleftarrow{(-)'_{\mu}} & \xleftarrow{(-)_{\mu}}
 \end{array}$$

⁶ Fréchet spaces are metrisable complete lcs, while DF spaces describe their strong duals. Nuclear spaces are the lcs on which several different topological tensor product correspond. Precise definitions can be found in the litterature [18, 12.4, 21.1]

286 They would define a positive chirality if we had a good characterization of weakly compact
 287 sets on $E \otimes_\mu F$, allowing us to prove the associativity of \mathfrak{A}_μ . As of today, it is however not
 288 the case.

289 Thus we investigate the interpretation of positive formulas MLL in Mackey spaces. This
 290 leads to three models: a first one based on barrelled spaces (section 4) and two others refining
 291 it with the notion of *bornological spaces* (section 5). The goal now is to handle as negatives
 292 spaces with *some completeness*, in order to work with smooth functions and differentiability.

293 4 Decomposing reflexivity through polarization

294 In this section we show that reflexive spaces decompose in a dialogue chirality. Remember that
 295 a lcs E is reflexive if and only if is barrelled and semi-reflexive (definition 7). Semi-reflexivity
 296 can in fact be characterized in terms of completeness:

297 ▶ **Proposition 20.** [18, 11.4.1] *E is semi-reflexive iff it is weakly quasi-complete: any*
 298 *bounded Cauchy filter in E converges in E_σ . Thus E is reflexive iff it is barrelled and weakly*
 299 *quasi-complete.*

300 These requirements enjoy *antagonist stability properties*: barrelled spaces are stable
 301 under inductive limits while weak completeness is preserved by projective limits. In fact,
 302 barrelledness and weak quasi-completeness are in duality:

303 ▶ **Proposition 21.** [18, 11.1.4] *A Mackey space is barrelled if and only if its weak dual is*
 304 *quasi-complete.*

305 This proposition is the backbone for the construction of a new chirality between BARR, the
 306 full subcategory of barrelled lcs and WQCOMPL, the full subcategory of weak quasi-complete
 307 lcs. We first retrieve a fundamental proposition allowing to prove curryfication (equation 2)
 308 and then state the necessary stability, associativity and monoidality lemmas. The proofs are
 309 given in appendix.

310 ▶ **Proposition 22.** [5, III.5.3.6] *When E and F are barrelled, every separately continuous*
 311 *bilinear map on $E \times F$ is β -hypocontinuous.*

312 ▶ **Proposition 23.** *The bounded tensor product \otimes_β preserves barrelled spaces and is associative*
 313 *on BARR.*

314 ▶ **Proposition 24.** *Consider E and F two barrelled lcs. Then $(E \otimes_\beta F)' \sim \mathcal{L}(E, F'_\beta)$.*

315 We denote by $(\mathcal{L}(E, F))_\mu$ the space $\mathcal{L}(E, F)$ endowed with the Mackey topology induced
 316 by its predual $(E \otimes_\beta F')$.

317 ▶ **Proposition 25.** 1. *The Mackey dual of a weak quasi-complete space is barrelled and the*
 318 *weak dual of a barrelled space is quasi-complete.*

319 2. *Consider F a weak and quasi-complete space, and E a barrelled space. Then for any lcs*
 320 *E , $\mathcal{L}_\sigma(E, F)$ is quasi-complete and endowed with its weak topology.*

321 3. *Consider E, F barrelled spaces. Then $(E \otimes_\beta F)'_w \simeq \mathcal{L}_\sigma(E, F'_w)$.*

322 4. *Consider E, F two weak and quasi complete lcs. Then $(\mathcal{L}_\sigma(E'_\mu, F))_\mu \simeq E'_\mu \otimes_\beta F'_\beta$.*

323 5. *The binary operation $\mathfrak{A}_w : (E, F) \mapsto \mathcal{L}_\sigma(E'_\mu, F)$ is associative and commutative on*
 324 *WQCOMPL.*

325 6. *Consider $F \in \text{WEAK}$ and $E \in \text{MACKKEY}$. Then $\mathcal{L}(E_w, F) \sim \mathcal{L}(E, F_\mu)$ and $\mathcal{L}(E'_\sigma, F) \sim$
 326 $\mathcal{L}(F'_\mu, E)$*

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7. Consider $E \in \text{BARR}$ and $F \in \text{WQCOMPL}$ and $G \in \text{Weak}$. Then:

$$\mathcal{L}(E, \mathcal{L}_\sigma(F'_\mu, G)) \sim \mathcal{L}(E \otimes_\beta F'_\mu, G)$$

8. For any $E \in \text{MACKEY}$ and thus any $E \in \text{BARR}$, $(E_w)'_\mu \simeq (E'_w)_\mu$.

As the others, the proof of the preceding proposition is detailed in the appendix. Let us however insist on the fact that the remarkable stability properties are quite inherent to barrelledness : for example, the second point is proven thanks to Banach-Steinhaus theorem, which precisely holds for function with barrelled spaces as codomains.

► **Theorem 26.** *Barrelled spaces and weak quasi-complete spaces organise in the following topological dialogue chirality:*

$$\begin{array}{ccc} & \xrightarrow{(-)'_\sigma} & \\ (\text{BARR}, \otimes_\beta, \mathbb{R}) & \perp & (\text{WQCOMPL}^{op}, \mathfrak{Y}_w, \mathbb{R}) \\ & \xleftarrow{(-)'_\mu} & \end{array} \quad \begin{array}{ccc} & \xrightarrow{(-)_\sigma} & \\ \text{MACKEY} & \perp & \text{WEAK} \\ & \xleftarrow{(-)_\mu} & \end{array}$$

in which curryfication (Equation 2) is given by proposition 25.7 and closure (Equation 4) by proposition 25.8.

► **Remark 27.** As indicated to the author by Y. Dabrowski, there is in fact of a closure operation making lcs barrelled [30, 4.4.10], which would give a dialogue chirality and not a topological dialogue one. It is however not needed here to interpret proofs of polarized MLL.

5 Duality with bornological spaces

Bornological spaces were at the heart of the duality in vectorial models of LL [32, III.5], and in the first smooth intuitionistic model of DiLL [4]. However, it was shown that in the context of intuitionistic smooth models, bornological topologies were unnecessary, and the first model made of bornological and Mackey-complete spaces was refined into a model made only of Mackey-complete space [22]. We show that *bornologicality is in fact the key to make smooth models classical*, through polarization.

In this section, we describe two topological chiralities based on bornological spaces. Section 5.2 offers a polarized extension to Intuitionistic Models of DiLL [4], while in section 5.3 we describe a chirality refining [8] which could lead to a more satisfactory interpretation of differentiation.

5.1 Bornologies and bounded linear maps

In this section, we recall preliminary material on the more specific subject of vector spaces endowed with bornologies, as exposed in the litterature [17, 15]. So far, we worked with topological vector spaces, on which the canonical bounded subsets are the one of $\beta(E)$. One can also work with bornologies as the primary structure, and from that construct 0-neighbourhoods as those which absorb any element of the bornology.

► **Definition 28.** *Consider E a vector space. A bornology on a vector space E is a **vector bornology** if it is stable under addition and scalar multiplication. It is **convex** if it is stable under convex closure, and **Hausdorff** if the only bounded sub-vector space in \mathcal{B} is $\{0\}$.*

360 ▶ **Definition 29.** A bounded map is a map for which the image of a bounded set is bounded.
 361 We denote by $\mathbf{L}(E, F)$ the vector space of all bounded linear maps between $E, F \in \text{Bornvec}$.
 362 Is is endowed with the bornology of all equibounded sets of functions, that is sets of functions
 363 sending uniformly a bounded set in E to a bounded set in F .

364 ▶ **Definition 30.** We consider the category BORNVEC of vector spaces endowed with a convex
 365 Hausdorff vector bornology, with linear bounded maps as arrows.

366 While the converse is not true, a linear continuous map is always bounded. Thus we
 367 have a functor $\text{Born} : \text{TOPVEC} \rightarrow \text{BORNVEC}$ mapping any lcs E to the same vector space
 368 endowed with its bornology $\beta(E)$, and a linear continuous function to itself.

369 ▶ **Definition 31.** Consider $E \in \text{BORNVEC}$ with bornology \mathcal{B}_E . Then a subset $U \subset E$ is said
 370 to be bornivorous if for every $B \in \mathcal{B}_E$ there is a scalar $\lambda \in \mathbb{K}$ such that $B \subset \lambda U$.

371 We consider the functor $\text{Top} : \text{BORNVEC} \rightarrow \text{TOPVEC}$ which maps E to the lcs E with
 372 the topology of generated by bornivorous subsets, and which is the identity on linear bounded
 373 functions.

374 ▶ **Proposition 32.** A linear bounded map between two vector spaces E and F endowed with
 375 respective bornologies \mathcal{B}_E and \mathcal{B}_F defines a linear continuous maps between E endowed with
 376 $\text{Top}(\mathcal{B}_E)$ and F endowed with $\text{Top}(\mathcal{B}_F)$.

377 The interaction between bornologies and topologies is best described through the following
 378 adjunction [15, 2.1.10]:



380 In the light of section 3, as the domain of a left-adjoint functor, spaces with bornologies
 381 should interpret positive connectives while lcs are better suited to interpret negatives. We
 382 will refine this intuition through the category of bornological lcs, which is the co-reflective
 383 category arising through the previous adjunction.

384 ▶ **Proposition 33.** [18, 13.1.1] A lcs E is said to be **bornological** if one of these following
 385 equivalent propositions is true:

- 386 1. For any other lcs F , any bounded linear map $f : E \rightarrow F$ is continuous, that is $\mathcal{L}(E, F) =$
 387 $\mathbf{L}(E, F)$,
 388 2. E is endowed with the topology $\text{Top} \circ \text{Born}(E)$,
 389 3. E is Mackey, and any bounded linear form $f : E \rightarrow \mathbb{K}$ is continuous.

390 We denote by BTOPVEC the category of bornological lcs and continuous (equivalently
 391 bounded) linear maps between them⁷. Equivalently to BTOPVEC , one can consider topological
 392 spaces in BORNVEC , that is spaces in BORNVEC which are invariant under $\text{Born} \circ \text{Top}$. This
 393 are the vector spaces with a convex vector bornology which consists exactly of all the sets
 394 absorbed by all the bornivorous subsets.

⁷ Beware of the difference between spaces of BORNVEC which are not endowed with a canonical bornology, and bornological lcs of BTOPVEC .

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395 ▶ **Proposition 34.** [15] BTOPVEC is a co-reflective category in Top and TBORNVEC is
 396 reflective in BORNVEC .

$$\begin{array}{ccc}
 & U & \\
 \text{BTOPVEC} & \xrightarrow{\quad} & \text{TOPVEC} \\
 & \text{Top} \circ \text{Born} & \\
 \end{array}
 \quad
 \begin{array}{ccc}
 & \text{Born} \circ \text{Top} & \\
 \text{BORNVEC} & \xrightarrow{\quad} & \text{TBORNVEC} \\
 & \iota & \\
 \end{array}$$

398 in which U and ι denotes forgetful functors, leaving objects and maps unchanged.

399 ▶ **Proposition 35.** [30, 11.3] Consider E and F two bornological lcs. Then $E \otimes_{\beta} F$ is
 400 bornological.

401 \otimes_{β} is associative and commutative on TBORNVEC (see [15, 3.8.1] or [4, 3.1]), $(\text{BTOPVEC}, \otimes_{\beta}, \mathbb{R})$
 402 is a monoidal category.

403 ▶ **Proposition 36.** As bornological lcs are in particular Mackey, we have a contravariant
 404 adjunction and a coreflection:

$$\begin{array}{ccc}
 & (-)'_{\mu} & \\
 (\text{BTOPVEC}, \otimes_{\beta}, \mathbb{R}) & \xrightarrow{\quad} & (\text{MACKEY}^{op}) \\
 & \text{Top} \circ \text{Born}((-)'_{\mu}) & \\
 \end{array}
 \quad
 \begin{array}{ccc}
 & \iota & \\
 \text{BTOPVEC} & \xrightarrow{\quad} & \text{MACKEY} \\
 & \text{Top} \circ \text{Born} & \\
 \end{array}$$

406 This however is not enough to have a chirality: we do not have a suitable interpretation for
 407 the dual of \otimes_{β} which would be associative on all Mackey spaces, and not just on duals of
 408 bornological spaces. More generally, bornological spaces do not verify a duality theorem
 409 with some kind of complete spaces, or at least not some kind involving duals which preserves
 410 reflexivity [18, 13.2.4]. One solution detailed in section 5.2, is to add a suitable
 411 notion of completeness. The other solution, in section 5.3 is to refine our setting, and consider
 412 ultrabornological spaces.

413 5.2 Convenient vector spaces classically

414 To the notion of bornology corresponds a good notion of completeness, enforcing the conver-
 415 gence of Cauchy sequences with respect the norms generated by bounded subsets.

▶ **Definition 37.** [15] Consider V an absolutely convex and bounded subspace of a $E \in$
 BORNVEC . We denote by E_V the vector space generated by V . It is a normed vector space
 when endowed with the gauge:

$$p_V : x \in E_V \mapsto \sup\{\lambda > 0 \mid \lambda x \in V\}.$$

416 An absolutely convex and bounded subset V is said to be a **Banach disk** when E_V is complete
 417 for its norm. E is said to be **Mackey-Complete** when every absolutely convex and bounded
 418 subset is a banach disk.

419 Equivalently, the definition of Mackey-Completeness extends to TOPVEC when one
 420 considers the bounded sets of $\beta(E)$. We choose the notation MCO to denote the full
 421 subcategory of TOPVEC made of Mackey-complete lcs. Mackey-complete spaces are the heart
 422 of several smooth models of DiLL [4, 8, 22].

423 In particular, in work by Blute Ehrhard and Tasson [4] formulas were interpreted by
 424 **convenient spaces**, that is bornological lcs which are also Mackey-complete⁸. We denote
 425 by CONV the full subcategory of bornological and Mackey-complete lcs, endowed with linear
 426 bounded (equivalently continuous) maps.

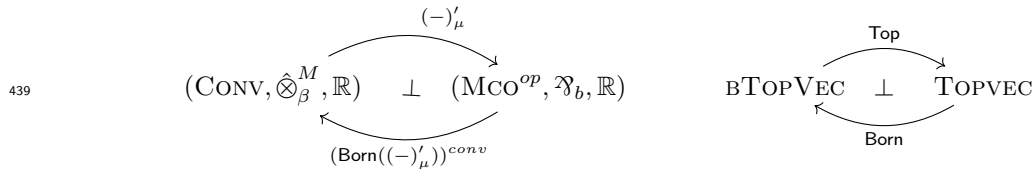
427 ▶ **Proposition 38.** [15, 2.6.5] *The full subcategory CONV ⊂ BTOPVEC of Mackey-complete*
 428 *bornological lcs is a reflective subcategory with the Mackey completion $\widehat{_}^M$ as left adjoint to*
 429 *inclusion.*

430 The Mackey-completed B tensor product $\widehat{\otimes}_B^M$ is easily proved to be commutative and
 431 associative on CONV [15, 3.8]. For $F \in \text{TOPVEC}$, let us denote $(F)^{conv} := \text{Top} \circ \widehat{\text{Born}}(F)^M$
 432 the completion of the bornologification of F .

433 ▶ **Definition 39.** *For $E, F \in \text{MCO}$, $E \mathfrak{Y}_b F := ((\text{Born}(E'_\mu) \widehat{\otimes}_B^M \text{Born}(F'_\mu))^{conv})'_\mu$.*

434 This operation preserve Mackey-completeness (see the proof of theorem 40), and is com-
 435 mutative by [18, 8.6.5]. We acknowledge that this definition lacks simplicity, and ideally
 436 polarization should allow for less completions on the \mathfrak{Y} .

437 ▶ **Theorem 40.** *Convenient spaces and Mackey-Complete spaces organise in the following*
 438 *topological positive chirality:*



440 5.3 Ultrabornological and Schwartz spaces

441 In this section, we refine the previous chirality into a finer one, to get closer to objects
 442 used in the first classical non-polarized smooth models of DiLL [8]. Convenient spaces are a
 443 particular case of ultrabornological spaces, that is spaces which are bornological with respect
 444 to a stricter class of bounded subsets.

445 ▶ **Definition 41.** [18, 11.1] *A lcs E is said to be **ultrabornological** when its 0-neighbourhoods*
 446 *are exactly the one absorbing all Banach disks.*

447 Let us denote by UBTOPVEC the full subcategory of ultrabornological spaces. If we
 448 denote by $\text{uBorn} : \text{TOPVEC} \rightarrow \text{BORNVEC}$ the functor mapping a lcs E to the same vector
 449 space endowed with the bornology of its Banach disks, we have a coreflective subcategory:



451 Ultrabornological spaces are in particular barrelled [18, 13.1.3], and offer a fine duality theory
 452 related to Schwartz spaces. For a lcs E , let us denote by B_0 the bornology consisting of the
 453 absolutely convex and weakly closed closure of the set of maps converging 0 in some E_B , for
 454 B an absolutely convex and weakly closed bounded subset of B .

⁸ Mackey-completeness in fact is what makes bornological lcs ultrabornological, and in particular barrelled. This work can be seen as an adaptation of convenient spaces to the chirality of barrelled spaces

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455 ▶ **Definition 42.** *Schwartz spaces* are those lcs which are endowed with the topology of
 456 uniform convergence on the sequences in their dual which converges to 0 in some E'_V , where
 457 V stands for an equicontinuous subset of E^9 (proposition [18, 10.4.1]). We denote by SCHW
 458 the full subcategory of Schwartz lcs.

459 We denote by $\mathcal{S} : \text{TOPVEC} \rightarrow \text{SCHW}$ the functor mapping a lcs to the same lcs endowed
 460 with the topology of uniform convergence on the sequences in E' which converge equicontinu-
 461 ously to 0, and by SCOMPL subcategory of Schwartz and complete lcs. Although there is
 462 not a unique Schwartz topology on a space preserving the dual, $\mathcal{S}(E)$ is the finest Schwartz
 463 topology which is coarsest than the original topology of E .

464 ▶ **Proposition 43.** [18, 13.2.6] *A lcs E is ultrabornological if and only if the schwartzification*
 465 $\mathcal{S}(E'_\mu)$ *of its Mackey-dual is complete.*

466 Through this dual characterization, we can offer a refinement of the smooth unpolarized
 467 model of DiLL [8], in which formulas are interpreted by so-called ρ -reflexive spaces¹⁰. These
 468 are a **reflexive version of Schwartz Mackey-Complete spaces**. Indeed, Schwartz
 469 Mackey Complete spaces were introduced as a refinement of quasi-complete spaces, on which
 470 a good interpretation for the \mathfrak{A} would still be associative, and which could offer some hope
 471 for reflexivity. We recall the following characterization of ρ -reflexive spaces:

472 ▶ **Proposition 44.** [8, Thm 5.9] *A Hausdorff locally convex space is ρ -reflexive, if and only*
 473 *if it is Mackey complete, has its Schwartz topology associated to the Mackey topology of its*
 474 *dual $\mu_{(s)}(E, E')$ and its dual is also Mackey complete with its Mackey topology.*

475 Thus this model really is a negative interpretation of DiLL, and we will emphasize this
 476 point of view by refining it into a negative chirality. Negative formulas are interpreted in the
 477 category **Compl μ Sch** of Complete spaces which are endowed with the finest Schwartz topo-
 478 logy preserving the dual¹¹: $E \simeq \hat{E}^M$ and $E \simeq \mathcal{S}(E'_\mu)$. The following chirality corresponds
 479 to the decomposition of η -reflexivity as described by Jarchow [18, 13.4.6].

480 ▶ **Theorem 45.** *Ultrabornological spaces and complete spaces which have the Schwartz topology*
 481 *associated to their Mackey topology organise in the following topological dialogue chirality:*

$$\begin{array}{ccc}
 & \mathcal{S}((-)'_\mu) & \\
 \curvearrowright & & \curvearrowleft \\
 \text{(UBTOPVEC, } \hat{\otimes}_\beta^M, \mathbb{R}) & \perp & (\text{Compl}\mu\text{Sch}^{op}, \epsilon, \mathbb{R}) \\
 \curvearrowleft & & \curvearrowright \\
 & (-)'_\mu & \\
 & & \text{TOPVEC } \perp \text{ Sch} \\
 & & \mathcal{S}(-) \\
 & & \downarrow \\
 & & \iota
 \end{array}$$

483 in which ϵ refers to Schwartz' ϵ product [31].

484 The cartesian closed category of smooth maps, interpreting the non-linear proofs of
 485 Linear Logic in [8], was based on the same pattern than the smooth maps in the bornological
 486 setting [24]. In that context, differentiation leads to a bounded linear function, and not

⁹ Note that equicontinuity only depends on the topology of E and not on the choice of a bornology on E

¹⁰ For a Mackey space, being ultrabornological is also equivalent for the strong nuclearification of its Mackey dual to be complete. Thus everything done here in terms of Schwartz spaces could be done in term of nuclear spaces, as for unpolarized smooth models of DiLL

¹¹ A lcs can be endowed with several Schwartz topologies which preserve the dual, while bornologification for example depends only of the dual pair (E, E') as bounded and weakly bounded set correspond.

487 necessarily a continuous one. The previous adjunction should lead to a polarized model of
488 DiLL with Complete Schwartz spaces, in which smooth maps are defined on ultrabornological
489 spaces, and their differential is thus immediately continuous. Indeed, lifting this model -to
490 higher-order will lead to functions having an ultrabornological space as codomain, and thus
491 to bounded functions to be continuous.

492 **6 Conclusion**

493 This work presented several chiralities of topological vector spaces, and refines four preexisting
494 smooth models of Differential Linear Logic. We show that chiralities are a good setting for
495 mostly preexisting yet intricate results in the theory of topological spaces, and that this
496 mathematical theory sheds light on previously unseen computational behaviours. Indeed,
497 the following features are observed here:

- 498 ■ Two distinct negations (in Theorem 26, Theorem 40 and Theorem 45).
- 499 ■ A non-transparent interpretation of the positive shift \downarrow in Theorem 26 and Theorem 40.
- 500 ■ Chiralities which are not dialogue chiralities but which feature a negation involutive only
501 on the negatives (Proposition 18) or on the positive (Theorem 40).

502 The chirality of barrelled spaces and weakly quasi-complete lcs is the most elegant one,
503 as any topological operation corresponds to a logical operation. The situation is less clear
504 in the case of bornological spaces, which are a good interpretation for the positives but on
505 which the interpretation of the negatives undergoes closure operations. In particular, the
506 role of Mackey-completion is not clear. While it allows to interpret positive formulas by
507 ultrabornological (thus barrelled) spaces, it results in a completeness condition on positive
508 formulas, while completeness is usually understood as the characteristic of negative formulas.

509 The results exposed here could lead to developments in the theory of programming
510 languages involving linear negations [7] [6]. But most of all, we believe that chiralities are a
511 good setting for the non-linear part of Differential Linear, and the models presented here
512 would serve as a basis for models of higher-order differential computations. Indeed, there is no
513 categorical semantics of DiLL reflecting the symmetry of its exponential laws. We conjecture
514 that chiralities should model the interaction between positives and negatives, as well as the
515 interaction between linear proofs and non-linear proofs in DiLL. In the linear-non-linear
516 chirality, the exponential would model the strong monoidal adjunction, while dereliction and
517 codereliction would be modelled as shifts.

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580 A Proofs omitted in the paper

581 **Proof of Proposition 12.** Let us denote $h_\alpha : E \times F \rightarrow E \otimes_\alpha F$ the canonical α -hypocontinuous
582 bilinear mapping. By precomposition with h_α , any continuous linear map in $\mathcal{L}(E \otimes_\alpha F)$
583 results in a α -hypocontinuous map. Consider $h \in \mathcal{HB}_\alpha(E \times F, G)$ and suppose that its
584 algebraic factorisation to a linear map \tilde{h} on $E \otimes F$ is not continuous on the α -tensor product:
585 then there is a 0-neighbourhood V in G such that $\tilde{h}^{-1}(V)$ is not a 0-neighbourhood in $E \otimes_\alpha F$.
586 The topology on $E \otimes F$ generated by the α topology still makes h_α α -hypocontinuous as
587 $h_\alpha^{-1}(\tilde{h}^{-1}(V)) = h^{-1}(V)$, and we obtain a contradiction. ◀

588 **Proof of Proposition 19.** For the purpose of this proof, we will denote by $(E \otimes F)_\mu$ the tensor
589 product of two lcs, endowed with the Mackey topology induced by its dual $\mathcal{HB}_\mu(E \times F)$. By
590 proposition 12, we have that $(E \otimes_\mu F)' = \mathcal{HB}_\mu(E \times F)$, thus the topology of $E \otimes_\mu F$ is coarser
591 than the one of $(E \otimes F)_\mu$. Let us show that the canonical bilinear map $h : E \times F \rightarrow (E \otimes F)_\mu$
592 is μ -hypocontinuous. Then as $(E \otimes_\mu F)$ is defined as the finest topology on the tensor product
593 making h μ -hypocontinuous, we will have that $E \otimes_\mu F$ is finer than the one of $(E \otimes F)_\mu$
594 and our proof will follow.

595 Consider K a weakly compact absolutely convex subset of E . Let us show that the family
596 of functions $h(K, -)$ is equicontinuous. As F is endowed with its Mackey topology, continuity
597 from F to $(E \otimes F)_\mu$ is equivalent to weak continuity from F to $E \otimes F$ endowed with the
598 weak topology induced by $\mathcal{HB}_\mu(E \times F)$. Consider $\ell \in (E \otimes F)_\mu'$. By proposition 9 we have
599 that $\ell \circ h \in \mathcal{HB}_\mu(E \times F)$, and thus the family $\ell \circ h(K, _)$ is equicontinuous. Equicontinuity
600 of f over weakly compact and absolutely convex sets in F is treated symmetrically. ◀

601 **Proof of Proposition 23.** Consider E and F two barrelled spaces. Let us show that $E \otimes_\beta F$
602 is barrelled. As on barrelled spaces the bornologies μ and β correspond, and as the Mackey
603 tensor product preserves Mackey spaces (proposition 19), we just need to show that the weak
604 dual $(\mathcal{HB}_\beta(E \times F))_\sigma$ of $E \otimes_\beta F$ is quasi-complete. However, we also know that on barrelled
605 spaces, the β -hypocontinuous bilinear maps are exactly the separately continuous ones. Thus
606 we just need to show that any simply bounded Cauchy-Filter $(f_\gamma)_{\gamma \in \Gamma}$ simply converges to a
607 separately continuous bilinear maps. This follows from the quasi-completeness of E'_w and
608 F'_w .

609 Let us show associativity Consider E, F and G three barrelled spaces. As on barrelled
610 spaces the μ -tensor product corresponds with the β tensor product, and as the first one
611 preserve the Mackey topology, we just need to show that $((E \otimes_\beta F) \otimes_\beta G)$ and $(E \otimes_\beta (F \otimes_\beta G))$
612 have the same dual. These dual are respectively $\mathcal{HB}_\beta((E \otimes_\beta F) \times G)$ and $\mathcal{HB}_\beta(E \otimes_\beta (F \times G))$.
613 By the fact that on barrelled spaces, β -hypocontinuity and separate continuity correspond,
614 we have that these two space are linearly isomorphic. ◀

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615 **Proof of Proposition 24.** By proposition 12 we have $(E \otimes_{\beta} F)' \sim \mathcal{HB}_{\beta}(E \times F)$. Let us show
 616 that $\mathcal{HB}_{\beta}(E \times F) \sim \mathcal{L}(E, F'_{\beta})$. Consider $h \in \mathcal{HB}_{\beta}(E \times F)$. For any $x \in E$, we have by the
 617 fact that hypocontinuity implies separate continuity that $h(x, _) \in F'$, and by hypocontinuity
 618 that $x \mapsto h(x, _)$ is continuous from E to F'_{β} . Conversely, any $\ell \in \mathcal{L}(E, F'_{\beta})$ is hypocontinuous
 619 by proposition 22. \blacktriangleleft

620 **Proof of Proposition 25. 1.** This follows from proposition 21 and from the fact that weak
 621 duals are weak spaces, and Mackey duals are Mackey spaces by proposition 9.

622 2. That $\mathcal{L}_{\sigma}(E, F'_W)$ is endowed with its weak topology follows from [18, 15.4.7]. Quasi-
 623 completeness follows from the fact that bounded sets of $\mathcal{L}_{\sigma}(E, F'_W)$ are the simply
 624 bounded ones, that bounded Cauchy filters converge pointwise thanks to the quasi-
 625 completeness of F , and that this limit function defined pointwise is continuous thanks to
 626 the Banach-Steinhaus theorem [18, 11.1.3] applied to E .

627 3. By proposition 12 we have that $(E \otimes_{\beta} F)' \sim \mathcal{HB}_{\beta}(E \times)$. As on barrelled spaces β -
 628 hypocontinuous functions and separately continuous functions correspond, we have in
 629 turn that $(E \otimes_{\beta} F)' \sim \mathcal{HB}_{\beta}(E \times) \sim \mathcal{B}(E \times F) \sim \mathcal{L}(E, F'_w)$. The linear homeomorphism
 630 $(E \otimes_{\beta} F)'_w \simeq \mathcal{L}_{\sigma}(E, F'_w)$ follows from the fact that the latter space is endowed with its
 631 weak topology [18, 15.4.7].

632 4. As $\mathcal{L}_{\sigma}(E'_{\mu}, F)$ is induced by the weak topology induced by $E' \otimes F'$, we have that
 633 $(\mathcal{L}_{\sigma}(E'_{\mu}, F))_{\mu} \simeq (E' \otimes F')_{\mu(\mathcal{L}(E'_{\mu}, F))}$. As E'_{μ} and F'_{μ} are both barrelled spaces, it fol-
 634 lows from propositions 19 and 23 that $E'_{\mu} \otimes_{\beta} F'_{\beta}$ is Mackey and linearly homeomorphic
 635 to $(E' \otimes F')_{\mu(\mathcal{L}(E'_{\mu}, F))}$.

636 5. Associativity and commutativity follow from the fact that $\mathcal{L}_{\sigma}(E', F)$ is endowed with the
 637 weak topology induced by $E' \otimes F'$.

638 6. The first point follows from proposition 9 (and is in fact part of the proof to the adjunction
 639 9). The second point follows from [18, 8.6.1, 8.6.5].

640 7. By proposition 12, we have that $\mathcal{L}(E \otimes_{\beta} F'_{\mu}, G) \sim \mathcal{HB}_{\beta}(E \times F'_{\mu}, G)$. As F'_{μ} is barrelled
 641 we have by proposition 22 that $\mathcal{HB}_{\beta}(E \times F'_{\mu}, G)$ is isomorphic to $\mathcal{B}(E \times F'_{\mu}, G)$, and our
 642 result follow easily.

643 8. Both $(E_w)'_{\mu}$ and $(E'_w)_{\mu}$ correspond algebraically to the vector space E' . The former is
 644 endowed with the topology of uniform convergence of $\sigma(E_w)$ compact subsets of E . The
 645 second is endowed with the topology of uniform convergence on $\sigma((E'_w)')$ compact subsets
 646 of E . As $E \sim (E'_w)$, both topologies correspond. \blacktriangleleft

648 **Proof of Theorem 40. ■** The Mackey dual $(\text{Top}(E))'_{\mu}$ of a convenient lcs is always Mackey-
 649 complete. Indeed, bounded sets of $(\text{Top}(E))'_{\mu}$ are the scalarly bounded ones ([18, 8.3.4]),
 650 thus these are the simply bounded ones, sending a point in E to a bounded set in
 651 \mathbb{R} . However, as bornological Mackey-Complete lcs are barrelled, we have by Banach-
 652 Steinhaus theorem that simply bounded sets of $\text{Top}(E)'$ are equicontinuous, and thus
 653 equibounded as E is bornological. Equibounded sets of $\text{Top}(E)'$ are easily shown to be
 654 Banach disks.

655 ■ As bornological lcs are in particular Mackey, we have that $(\text{Born}(\text{Top}(-)'_{\mu})'_{\mu})^{\text{conv}}$ is the
 656 identity on CONV .

657 ■ For $E \in \text{Conv}$ and $F \in \text{MCO}$, one has by the diverse adjunctions at stakes: $\mathcal{L}(\text{Top}(E)'_{\mu}, F) \simeq$
 658 $\mathcal{L}(F'_{\mu}, (\text{Top}(E)'_{\mu})'_{\mu}) \simeq \mathcal{L}(F'_{\mu}, \text{Top}(E)) \simeq \mathbf{L}(\text{Born}(F'_{\mu}), E) \simeq \mathbf{L}(\widehat{\text{Born}(F'_{\mu})}^M, E)$. Thus the
 659 contravariant adjunction is proved.

660 ■ One has easily that $(E \hat{\otimes}_B^M F)'_B \simeq \mathbf{L}_{\beta}(\text{Top}(E), F'_B)$, and thus $(-)'_B$ is indeed a strong
 661 monoidal functor.

662 ■ Let us prove equation 2, in the case this time of a positive chirality. Thus we need
 663 to give natural bijections for $E, F \in \mathbf{Conv}$ and $G \in \mathbf{COMPL}$: $\mathbf{L}(E \hat{\otimes}_B^M F, (G)^{conv}) \simeq$
 664 $\mathbf{L}(E, \mathbf{L}(F, (G)^{conv}))$ This follows from the reflection of Proposition 38, and the monoidality
 665 of $\hat{\otimes}_B$ in bornological vector spaces.

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667 **Proof of Theorem 40.** As ultrabornological spaces are in particular barrelled, the associ-
 668 ativity and commutativity of $\hat{\otimes}_B^{ub}$ holds. The functors $\mathcal{S}((-)'_\mu)$ and $(-)'_\mu$ well defined by
 669 proposition 43. Consider in particular $F \in \mathbf{Mc}\mu\mathbf{Sch}$. Then F'_μ is Mackey, and by definition
 670 $\mathcal{S}(\widehat{(F'_\mu)'_\mu}) \simeq F$ is complete. As ultrabornological spaces are Mackey, the duality functors
 671 define an equivalence of categories. The adjunction follows from proposition 9 and from
 672 the that the schwartification of a lcs preserves its dual [18, 10.4.4], and thus its Mackey
 673 topology. As ultrabornological spaces are barrelled, curryfication (equation 2) is inherited
 674 from theorem 26.

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