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DECOMPOSITION OF COMPLETE GRAPHS INTO ISOMORPHIC
SUBGRAPHS WITH FIVE VERTICES

J.-C. Bermond, C. Huang, A. Rosa and D. Sotteau

Abstract

A graph H is said to be G -decomposable if the edges of H can be partitioned into subgraphs isomorphic to G ; this is denoted by $H \rightarrow G$. When H is the complete graph on n vertices, K_n , the decomposition is called a G -design. This paper deals with necessary and sufficient conditions for $K_n \rightarrow G$, where G is a graph on five vertices, none of which is isolated.

I. Introduction.

A graph H is said to be G -decomposable if the edges of H can be partitioned into subgraphs isomorphic to G . This situation is denoted by $H \rightarrow G$; H is also said to admit a G -decomposition. A G -design is a G -decomposition of K_n , the complete graph on n vertices. The existence of a G -design for various graphs G has been studied in literature (for reference see [1]); in particular, the case where G is a graph with at most four vertices has been solved completely [2]. In this paper, we consider the case where G is a graph with five vertices, none of which is isolated.

It is easy to see the following:

PROPOSITION 1.1: Let G be a graph with k vertices and e edges. If $K_n \rightarrow G$ then

- (i) $n \geq k$
- (ii) $n(n-1) \equiv 0 \pmod{2e}$
- (iii) $n-1 \equiv 0 \pmod{f}$

where f is the greatest common divisor of the degrees of the vertices of G .

In [15], it was proved that these necessary conditions are asymptotically sufficient, that is, there exists an integer $n(G)$ such that $K_n \rightarrow G$ for $n \geq n(G)$, and n satisfying the necessary conditions of Proposition 1.1. Our objective is to prove that these necessary conditions are always sufficient except possibly in some cases to be specified below (usually for certain small values of n).

Table 1 exhibits all graphs on five vertices (none of them isolated). Necessary conditions for $K_n \rightarrow G$ are listed as well, and the last column describes the results obtained in each case. In Section II, some general methods for proving $K_n \rightarrow G$ are given while in Section III, each graph G is considered individually.

In the remainder of the paper, n is assumed to be greater than or equal to five.

II. General Construction Methods.

The necessary and sufficient conditions for $K_n \rightarrow G$ have already been obtained for some graphs G with five vertices. The graphs and the respective references are:

- G_3 , a path of length four and G_4 [8],
- G_5 , a star of four edges [12,16],
- G_{10} , a cycle of length five [3,11] and
- G_{23} , the complete graph on five vertices [5].

We will not duplicate the constructions here but we will point out that the methods used in this paper are as a rule simpler.

Generally, two methods are used in construction: the method of differences and the composition method. The first method is well-known (see for example [4] or [1]), hence we will discuss just the second method.

Given a graph G , the necessary conditions for $K_n \rightarrow G$ follow immediately from Proposition 1.1. Notice that the maximum degree of a vertex in a graph with five vertices is four and that there are no 3-regular graphs on five vertices, hence condition (iii) in the Proposition 1.1 applies only to those graphs whose vertex-degrees are multiples of 2, namely G_{10}, G_{15}, G_{17} and G_{23} . We then proceed to investigate whether a necessary condition is also sufficient.

To prove $K_n \rightarrow G$, we will instead show that $K_{n_i} \rightarrow G$ and $K_{n_1, n_2, \dots, n_h} \rightarrow G$ for some values of i , on account of the following two lemmas (for whose proofs see [1]):

LEMMA A. If $K_{n_i} \rightarrow G$ for $1 \leq i \leq h$ and $K_{n_1, n_2, \dots, n_h} \rightarrow G$,

then $K_n \rightarrow G$ where $n = \sum_{i=1}^h n_i$.

LEMMA B. If $K_{n_i+1} \rightarrow G$ for $1 \leq i \leq h$ and $K_{n_1, n_2, \dots, n_h} \rightarrow G$, then

$$K_n \rightarrow G \text{ where } n = \sum_{i=1}^h n_i + 1.$$

The graph K_{n_1, n_2, \dots, n_h} is the complete multipartite graph on vertices $X = \bigcup_{i=1}^h X_i$ where $|X_i| = n_i$ with edges joining each pair of vertices from different sets $X_i, X_j, i \neq j$.

In this paper, the value of h is 2, 3, 4, or 5. Unless otherwise stated, the vertices of K_{n_i} , that is, the elements of X_i will be the elements of a group Γ , which is isomorphic to Z_{n_i} , the group of residues modulo n_i , or $Z_x \times Z_{y_h}$ where $x \cdot y = n_i$; the elements of K_{n_i+1} will be $\Gamma \cup \{\infty\}$. If $n = \sum_{i=1}^h n_i$ (or $n = \sum_{i=1}^h n_i + 1$), the elements of K_n will be $\bigcup_{i=1}^h X_i$ (or $\bigcup_{i=1}^h X_i \cup \{\infty\}$ respectively).

We now have two different types of constructions to consider:

TYPE I: $K_{n_i} \rightarrow G.$

TYPE II: $K_{n_1, n_2, \dots, n_h} \rightarrow G.$

Furthermore, since Lemmas A and B can be applied inductively, only some small values of n_i 's need be considered.

In Type I construction, we use usually the method of differences and we will give only the *base graphs* of the decomposition since the rest of the graphs can be obtained by applying an automorphism of the group Γ on the vertices of the base graph, as illustrated in the following example.

EXAMPLE 1. Let G be the graph $G_1 = \begin{matrix} x & y & z & u & v \\ 0 & \text{---} & 0 & \text{---} & 0 \\ 0 & & & & 0 \end{matrix}$, which is denoted by $[(x,y,z) (u,v)]$. The proof of $K_6 \rightarrow G_1$ is simply:

$V(K_6) = Z_5 \cup \{\infty\}$, $[(0,1,3) (\infty,2)] \pmod 5$. The statement indicates the vertices of K_6 and a base graph of the decomposition. Hence the graphs of the decomposition are the graphs $[(i,1+1,3+i) (\infty,2+i)]$ for $i \in Z_5$.

Similarly for the cases $n = 7, 9$ and 10 :

For $K_7 \rightarrow G_1$, we have $V(K_7) = Z_7$, and the base graph is $[(0,1,4) (3,5)] \pmod 7$, which means that the graphs of the decomposition are $[(i,1+i,4+i) (3+i,5+i)]$ for $i \in Z_7$.

For $K_9 \rightarrow G_1$, $V(K_9) = Z_3 \times Z_3$ and the base graphs are $[((0,0), (0,1), (1,0)) ((1,1), (2,2))] \pmod (3,3)$ and $[((0,0), (1,0), (2,0)) ((0,1), (2,1))] \pmod (-,3)$. Hence the graphs of the decomposition are the graphs $[((i,j), (i,1+j), (1+i,j)) ((1+i,1+j), (2+i,2+j))]$ for all $i, j \in Z_3$ and the graphs $[((0,i), (1,i), (2,i)) ((0,i+1), (2,i+1))]$ for $i \in Z_3$.

Finally, for $K_{10} \rightarrow G_1$, $V(K_{10}) = Z_2 \times Z_5$ and the base graphs are $[((0,1), (0,0), (1,1)) ((1,2), (1,3))] \cup [((1,0), (0,1), (0,3)) ((1,2), (1,4))] \cup [((0,0), (1,0), (0,2)) ((0,1), (1,3))] \pmod (-,5)$. Hence the graphs of the decomposition are the graphs $[((0,1+i), (0,i), (1,1+i)) ((1,2+i), (1,3+i))]$, $[((1,i), (0,1+i), (0,3+i)) ((1,2+i), (1,4+i))]$ and $[((0,i), (1,i), (0,2+i)) ((0,1+i), (1,3+i))]$ for $i \in Z_5$.

Type II construction is similar and as a rule we indicate only the base graphs of a decomposition; only occasionally do we exhibit all the graphs of a decomposition. Take the case of $K_{n_1, n_2, \dots, n_h} \rightarrow G$ with $n_i = r$ for all i ; the construction is again illustrated by an example.

EXAMPLE 2. Let G be $G_{13} = \begin{matrix} u & x \\ 0 & \text{---} & 0 \\ 0 & & 0 \\ z & & y \end{matrix} \text{---} 0v$ and denoted by $[x,y,z,u,v]$.

The proof of $K_{4,4,4} \rightarrow G_{13}$ is simply $[0_1, 2_3, 2_2, 3_3, 0_2] \cup [0_1, 0_3, 3_1, 2_2, 1_2] \pmod 4$. It is understood that $V(K_{4,4,4}) = \bigcup_{i=1}^4 X_i$ where $X_i = \{0_i, 1_i, 2_i, 3_i\}$ and that the graphs of the decomposition are $[i_1, (2+i)_3, (2+i)_2, (3+i)_3, i_2]$ and $[i_1, i_3, (i+3)_1, (2+i)_2, (1+i)_2]$ for $i \in Z_4$.

In the case where $n_i = r$ for $1 \leq i \leq h-1$ and $n_h = r' > r$, the vertices of X_h are g_h for $g \in Z_r$ and $\infty, \infty', \infty'', \dots, \infty^{(r'-r-1)}$.

EXAMPLE 3. The base graphs of $K_{4,4,7} \rightarrow G_{13}$ are $[0_1, 1_2, 0_3, 0_2, \infty] \cup [2_2, 3_3, 3_1, \infty, 0_1] \cup [3_2, 1_3, 3_1, \infty, 0_1] \pmod{4}$. Hence $V(K_{4,4,7}) = \bigcup_{i=1}^3 X_i$

with $X_i = \{0_i, 1_i, 2_i, 3_i\}$ for $i = 1$ or 2 and $X_3 = \{0_3, 1_3, 2_3, 3_3, \infty, \infty, \infty\}$; the graphs of the decomposition are $[i_1, (1+i)_2, i_3, i_2, \infty]$, $[(2+i)_2, (3+i)_3, (3+i)_1, \infty, i_1]$ and $[(3+i)_2, (1+i)_3, (3+i)_1, \infty, i_1]$ for $i \in \mathbb{Z}_4$.

Sometimes a different method is used in proving $K_{r,r,r,\dots,r} \rightarrow G$: we let $X_i = \{i-1, i-1+h, \dots, i-1+(r-1)h\}$ for $i = 1, 2, \dots, h$. Hence $V(K_{r,r,\dots,r}) = \mathbb{Z}_{rh}$ and base graphs of the decomposition will be taken modulo rh , as seen in:

EXAMPLE 4. The proof of $K_{4,4,4,4} \rightarrow G_{13}$ is : $V(K_{4,4,4,4}) = \mathbb{Z}_{16}$ and the base graph is $[0, 10, 9, 14, 7] \pmod{16}$, which means that the graphs of the decomposition are $[i, 10+i, 9+i, 14+i, 7+i]$ for $i \in \mathbb{Z}_{16}$.

To prove that $K_{n_1, n_2, \dots, n_h} \rightarrow G$, Lemmas C, D, E, and F below are also useful.

LEMMA C. If $K_{r_1, r_2} \rightarrow G$ and $K_{r_1, r'_2} \rightarrow G$, then $K_{ar_1, br_2 + cr'_2} \rightarrow G$ for integers $a \geq 1$ and b or $c \geq 1$.

LEMMA D. If $K_{r_1, r_2, r_3} \rightarrow G$ and $K_{r_1, r_2, r'_3} \rightarrow G$, then $K_{ar_1, ar_2, (a-b)r_3 + br'_3} \rightarrow G$ for integers a, b with $0 \leq b \leq a$.

LEMMA E. If $K_{r,r,r} \rightarrow G$ and $K_{r,r,r,r} \rightarrow G$ then $K_{pr, pr, pr, qr} \rightarrow G$ for $p \neq 2, 6$, $0 \leq q \leq p$.

LEMMA F. If $K_{r,r,r,r} \rightarrow G$ and $K_{r,r,r,r,r} \rightarrow G$ then $K_{pr, pr, pr, pr, qr} \rightarrow G$ for $p \neq 2, 3, 6, 10, 14$ and $0 \leq q \leq p$.

Let us remark that in Lemma E, the condition $K_{r,r,r,r} \rightarrow G$ is unnecessary if $q = 0$ and the lemma is true for all $p \geq 1$; the condition $K_{r,r,r} \rightarrow G$ is unnecessary if $q = p$. Similar interpretation can be used in other lemmas, and in the case of Lemma F when $q = 0$, the condition $K_{r,r,r,r,r} \rightarrow G$ is unnecessary and the lemma is true for all $p \neq 2, 6$.

The proofs of Lemmas C and D can be found in [1]. Before we prove Lemmas E and F we need two auxiliary results.

PROPOSITION 2.1. If $p \neq 2, 6$ and $0 \leq q \leq p$ then $K_{p,p,p,q} \rightarrow \{K_4, K_3\}$.

Proof. The existence of a decomposition of $K_{p,p,p,p}$ into K_4 is equivalent to that of a pair of orthogonal Latin squares of order p ; the latter is known to exist for $p \neq 2$ or 6 [14]. If we delete the vertices of $K_{p,p,p,p}$ that are not in $K_{p,p,p,q}$ from the K_4 's which contain them, we get K_3 's. Hence it is possible to partition the edges of $K_{p,p,p,q}$ with $p \neq 2, 6$ into subgraphs isomorphic to K_4 or K_3 .

PROPOSITION 2.2. For $p \neq 2, 3, 6, 10, 14$ and $0 \leq q \leq p$, $K_{p,p,p,p,q} \rightarrow \{K_5, K_4\}$.

The proof is similar to the proof of Proposition 2.1:

$K_{p,p,p,p,q} \rightarrow K_5$ is equivalent to the existence of three pairwise orthogonal Latin squares of order p ; these are known to exist for $p \neq 2, 3, 6, 10, 14$ [13].

Now let G be a graph with vertex set $\{1, 2, \dots, k\}$ and let $G \otimes S_n$ be the lexicographic product of G by an independent set of n elements, that is, a graph with vertex set the disjoint union of k independent sets X_i , where $|X_i| = n$ such that two vertices are joined by an edge if and only if they belong to two different sets X_i, X_j and $\{i, j\}$ is an edge in G .

It is easy to see that $K_{pr,pr,pr,qr} = K_{p,p,p,q} \otimes S_r$, which, by Proposition 2.1, is decomposable into subgraphs isomorphic to $K_4 \otimes S_r$ or $K_3 \otimes S_r$ for $p \neq 2, 6$. Since $K_4 \otimes S_r = K_{r,r,r,r}$ and $K_3 \otimes S_r = K_{r,r,r}$, both of which by the hypothesis are decomposable into subgraphs isomorphic to G , Lemma E is proved.

The proof of Lemma F is quite similar (except that one uses Proposition 2.2).

III. Constructions for particular graphs.

In this section, we prove the results given in Table 1 namely, we find necessary and sufficient conditions for $K_n \rightarrow G$, where G is any graph on five vertices, none of which is isolated. Most of the constructions will be based on the composition method and the lay-out for each graph G will be:

- (a) the graph G , its labels and notation
- (b) necessary conditions for $K_n \rightarrow G$;
- (c) constructions showing $K_{n_i} \rightarrow G$ or $K_{n_1, n_2, \dots, n_h} \rightarrow G$ for some small values of n_i in order to start the composition;
- (d) compositions, with lemmas and parameters used indicated;
- (e) conclusions.

Explanations will be given in the first few cases but after that, only the information is given.

- (1) (a) G_1 : $0 \overset{x}{\text{---}} 0 \overset{y}{\text{---}} 0 \overset{z}{\text{---}} 0 \overset{u}{\text{---}} 0 \overset{v}{\text{---}}$, denoted by $[(x,y,z)(u,v)]$
- (b) Necessary conditions for $K_n \rightarrow G_1$: $n \equiv 0$ or $1 \pmod{3}$
- (c) - $K_{2,3} \rightarrow G_1$: the subgraphs of the decomposition are $[(0_2, 1_1, 2_2)(0_1, 1_2)]$ and $[(0_2, 0_1, 2_2)(1_1, 1_2)]$
- $K_n \rightarrow G_1$ for $n = 6, 7, 9$ or 10 : see Example 1 in Section II.

(d)

| n | $K_n \rightarrow G_1$ (with $h=3$) by Lemma | n_1 | n_2 | $K_{n_1, n_2} \rightarrow G$ by Lemma C with $r_1=2, r_2=3, r'_2=0, a=3$ $c=0$ and b : |
|------------------|---|-------|--------|--|
| $3t, t \geq 4$ | A | 6 | $3t-6$ | $t-2$ |
| $3t+1, t \geq 4$ | B | 6 | $3t-6$ | $t-2$ |

- (e) Theorem: $K_n \rightarrow G_1$ if and only if $n \equiv 0$ or $1 \pmod{3}$, $n \geq 5$.

Explanation: $K_{3t} \rightarrow G_1$ is proved by using Lemma A, which states that if $K_6 \rightarrow G_1$, $K_{3t-6} \rightarrow G_1$ and $K_{6, 3t-6} \rightarrow G_1$, then $K_{3t} \rightarrow G_1$. The case $K_6 \rightarrow G_1$ was proved in part (c); $K_{3t-6} \rightarrow G_1$ is true for $t \geq 4$ by induction, and finally $K_{6, 3t-6} \rightarrow G_1$ is proved by using Lemma C, which in this case states that if $K_{2,3} \rightarrow G$ then $K_{6, 3(t-2)} \rightarrow G_1$ (since $c = 0$). The case $K_{2,3} \rightarrow G$ is proved in (c).

The second line in the array can be interpreted similarly.

- (2) (a) G_2 : $0 \overset{x}{\text{---}} 0 \overset{u}{\text{---}} 0 \overset{v}{\text{---}}$, denoted by $[(x,y,z)(u,v)]$

- (b) Necessary conditions for $K_n \rightarrow G_2$: $n \equiv 0$ or $1 \pmod{8}$

(c) - $K_{2,2,2} \rightarrow G_2$: the base graph is

$$[(0_1, 0_2, 1_3)(1_1, 1_2)] \text{ mod } (-, 3)$$

$$- K_{2,2,2,2} \rightarrow G_2: [(0_1, 0_2, 1_3)(1_1, 0_4)] \cup$$

$$[(0_2, 0_3, 1_4)(0_1, 1_2)] \cup$$

$$[(0_3, 0_4, 0_1)(1_2, 1_4)] \text{ mod } (2, -).$$

$$- K_8 \rightarrow G_2: V(K_8) = Z_7 \cup \{\infty\}, [(0, 1, 3)(\infty, 2)] \text{ mod } 7.$$

$$- K_9 \rightarrow G_2: V(K_9) = Z_9, [(0, 1, 3)(2, 6)] \text{ mod } 9.$$

$$- K_{16} \rightarrow G_2: V(K_{16}) = Z_{15} \cup \{\infty\},$$

$$[(0, 1, 7)(\infty, 2)] \cup [(3, 5, 8)(2, 6)] \text{ mod } 15.$$

$$- K_{17} \rightarrow G_2: V(K_{17}) = Z_{17},$$

$$[(0, 1, 7)(2, 6)] \cup [(3, 5, 8)(2, 10)] \text{ mod } 17.$$

$$- K_{40} \rightarrow G_2: V(K_{40}) = Z_{39} \cup \{\infty\}, [(0, 18, 19)(1, \infty)] \cup$$

$$[(0, 15, 17)(1, 7)] \cup [(0, 13, 16)(1, 9)] \cup$$

$$[(0, 10, 14)(1, 12)] \cup [(0, 7, 12)(1, 10)] \text{ mod } 39.$$

$$- K_{41} \rightarrow G_2: V(K_{41}) = Z_{41}, \text{ the first base graph is}$$

$$[(0, 18, 19)(1, 21)] \text{ the other 4 are just}$$

the last four base graphs for the case

$$K_{40} \rightarrow G_2, \text{ except that the elements are}$$

taken modulo 41.

(d)

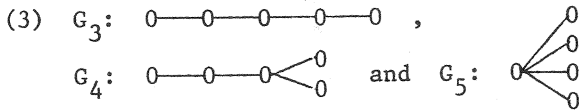
| n | $K_n \rightarrow G_2$ (with $h=4$) by Lemma | $n_1=n_2=n_3$ | n_4 | $K_{n_1, n_2, n_3, n_4} \rightarrow G_2$ by Lemma E with $r=2, p=4t$ and q |
|--------------------|---|---------------|-------|--|
| $24t, t \geq 1$ | A | $8t$ | 0 | 0 |
| $24t+1, t \geq 1$ | B | $8t$ | 0 | 0 |
| $24t+8, t \geq 1$ | A | $8t$ | 8 | 4 |
| $24t+9, t \geq 1$ | B | $8t$ | 8 | 4 |
| $24t+16, t \geq 2$ | A | $8t$ | 16 | 8 |
| $24t+17, t \geq 2$ | B | $8t$ | 16 | 8 |

(e) *Conclusion:* $K_n \rightarrow G_2$ if and only if $n \equiv 0$ or $1 \pmod{8}$.

Explanation: The fourth row of the array means that $K_{24t+9} \rightarrow G_2$ can be proved to be true by Lemma B if $K_{8t+1} \rightarrow G_2$, $K_9 \rightarrow G_2$ and $K_{8t, 8t, 8t, 8} \rightarrow G_2$.

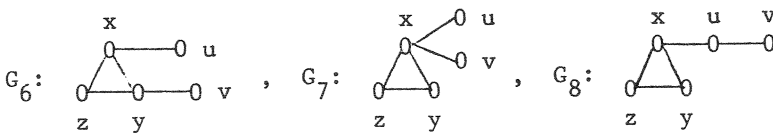
$K_{8t+1} \rightarrow G_2$ can be proved by induction, $K_9 \rightarrow G_2$ is in part (c) and $K_{8t, 8t, 8t, 8} \rightarrow G_2$ is proved by Lemma E, which states that $K_{8t, 8t, 8t, 8} \rightarrow G_2$ if $K_{2, 2, 2, 2} \rightarrow G_2$ and $K_{2, 2, 2, 2} \rightarrow G_2$, which have been proved in part (c).

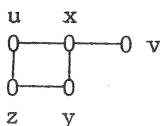
The other rows can be interpreted similarly.



As we mentioned in section II, the necessary and sufficient condition for $K_n \rightarrow G$, $G = G_3, G_4$ or G_5 has already been found to be $n \equiv 0$ or $1 \pmod{8}$.

(4) $G = G_6, G_7, G_8$ or G_9



and G_9 :  and are all denoted by $[x,y,z,u,v]$.

These four cases are considered together because the necessary conditions for $K_n \rightarrow G$ for each G are the same: $n \equiv 0$ or $1 \pmod{5}$; also, they belong to a class of graphs called dragons [11]. It was shown in the same paper [11] that the condition $n \equiv 0$ or $1 \pmod{10}$ is also sufficient for $K_n \rightarrow G$. In a different paper [7] the case $n \equiv 1$ or $5 \pmod{10}$ was considered; it was shown that $K_n \rightarrow G$ exists for $n \geq 5$ when $G = G_6$ and for $n > 5$ when $G = G_7, G_8$ or G_9 . The construction used in [7] is based on the method of differences and an additional condition was also satisfied, namely, that each vertex of K_n occurs in the same number of graphs - such decomposition is usually called a balanced graph design.

We will consider the only unsolved case, $n \equiv 6 \pmod{10}$; we will deal with G_6, G_7 and G_8 jointly, and with G_9 separately. We will use the composition method (which, of course, could be used to deal with the case $n \equiv 0$ or 1 or $5 \pmod{10}$ as well).

(A) $G = G_6, G_7$ or G_8

$$(c) - K_{5,5,5} \rightarrow G: \left\{ \bigcup_{j=1}^3 [0_j, 0_{j+1}, 1_{j+2}, 2_{j+1}, a] \right\} \pmod{5}$$

$$\text{where } a = 3_{j+2}, 3_{j+1} \text{ or } 0_{j+2}$$

$$G = G_6, G_7 \text{ or } G_8 \text{ respectively.}$$

$$- K_{5,5,5,5} \rightarrow G: \left\{ \bigcup_{j=1}^4 [0_j, 1_{j+2}, 0_{j+1}, 2_{j+1}, a_j] \right\} \cup [1_4, 3_2, 0_1, 3_3, b] \cup [3_3, 1_1, 0_2, 2_4, c] \pmod{5}$$

$$\text{where } a_1 = 1_1, 0_3 \text{ or } 2_4$$

$$a_2 = 1_2, 0_4 \text{ or } 2_1$$

$$a_3 = 4_3, 2_1 \text{ or } 0_2$$

$$a_4 = 3_4, 3_2 \text{ or } 0_3$$

$$b = 2_3, 4_1 \text{ or } 4_2$$

$$c = 3_4, 4_2 \text{ or } 0_1$$

$$\text{for } G = G_6, G_7 \text{ or } G_8 \text{ respectively.}$$

- $K_6 \rightarrow G_6$: $[1,5,0,2,4],[4,0,3,1,2],[2,3,5,4,1]$
- $K_6 \rightarrow G_7$: $[1,5,0,2,3],[4,0,3,1,5],[2,3,5,4,0]$
- $K_6 \rightarrow G_8$: $[1,5,0,2,3],[4,1,3,0,2],[5,2,4,3,0]$
- $K_{26} \rightarrow G$: $V(K_{26}) = Z_2 \times Z_{13}$,
 $[(0,0),(0,1),(1,0),(0,4),a] \cup [(0,0),(0,2),(1,3),(0,5),b]$
 $\cup [(0,0),(0,3),(1,5),(1,4),c] \cup [(1,0),(1,1),(0,3),(1,3),d]$
 $\cup [(1,0),(1,2),(0,6),(1,4),e] \pmod{(-,13)}$.

where $a = (0,7), (0,6)$ or $(0,10)$
 $b = (1,8), (1,6)$ or $(1,11)$
 $c = (1,11), (1,8)$ or $(0,9)$
 $d = (1,6), (1,5)$ or $(1,8)$
 $e = (1,8), (1,6)$ or $(1,10)$
for $G = G_6, G_7$ or G_8 respectively.

- $K_{36} \rightarrow G$: $V(K_{36}) = Z_5 \times Z_7 \cup \{\infty\}$,
 $\{ \cup_{j=1}^3 [(0,0),(0,j),(2,q_j),(1,j),a] \} \pmod{(5,7)}$

where $q_1 = 3, q_2 = 6$ and $q_3 = 1$
and $a = (1,3+2j), (1,3+j)$ or $(0,4)$ for
 $G = G_6, G_7$ or G_8 ;

the remaining three base graphs are obtained from $K_6 \rightarrow G$
with ∞ replacing the element 0 and $(i,0)$ replacing i ,
 $i = 1,2,\dots,5$.

For instance, in the case $G = G_6$, the three graphs are
 $[(1,0),(5,0),\infty,(2,0),(4,0)] \cup [(4,0),\infty,(3,0),(1,0),(2,0)]$
 $\cup [(2,0),(3,0),(5,0),(4,0),(1,0)] \pmod{(-,7)}$.

(d)

| n | $K_n \rightarrow G$ by Lemma | $n_1=n_2=n_3$ | n_4 | $K_{n_1,n_2,n_3,n_4} \rightarrow G$ by Lemma E with $r=5$, p and q | |
|------------------------|---------------------------------|---------------|-------|---|---|
| | | | | | |
| $30t+6, t=2, t \geq 4$ | B | $10t$ | 5 | $2t$ | 1 |
| $30t+16, t \geq 0$ | B | $10t+5$ | 0 | $2t+1$ | 0 |
| $30t+26, t \geq 1$ | B | $10t+5$ | 10 | $2t+1$ | 2 |

(B) $G = G_9$

(c) - $K_6 \rightarrow G_9$: Assume that $K_6 \rightarrow G_9$, that means we have 3 graphs in the decomposition, and hence a total of three vertices of degree 3, nine vertices of degree 2 and three vertices of degree 1. However, since a vertex of K_6 has degree 5, three vertices in K_6 must each occur as a vertex of degree 3 in one graph and a vertex of degree 2 in another, while the remaining three must each occur as a vertex of degree 2 in two graphs and vertex of degree 1 in the other. One could, with a bit of effort, show that such a decomposition is impossible.

- $K_{5,5} \rightarrow G_9$: $[0_1, 0_2, 2_1, 1_2, 2_2] \text{ mod } 5$

- $K_{16} \rightarrow G_9$: $V(K_{16}) = Z_5 \times Z_3 \cup \{\infty\}$,

$\cup_{i=0}^2 [(i,0), (i,1), (i+1,1), (i+3,0), \infty] \cup$

$[(0,0), (4,0), (2,0), (4,1), (1,2)] \cup [(0,0), (2,1), (1,0),$

$(2,2), (3,2)] \cup [(4,0), (3,1), (2,0), (3,2), \infty] \cup$

$[(1,0), (0,2), (4,1), (4,2), (3,1)] \text{ mod } (-,3).$

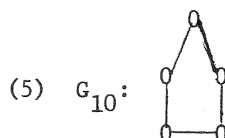
(d)

| | $K_n \rightarrow G_9$ by Lemma | n_1 | n_2 | $K_{n_1, n_2} \rightarrow G_9$ by Lemma C with $r_1=r_2=5, a=2t-1, b=2, c$ |
|-------------------|-----------------------------------|-----------|-------|---|
| $10t+6, t \geq 2$ | B | $(2t-1)5$ | 10 | 0 |

(e) $K_n \rightarrow G_6$ if and only if $n \equiv 0$ or $1 \pmod{5}$

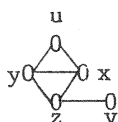
$K_n \rightarrow G_7$ or G_8 if and only if $n \equiv 0$ or $1 \pmod{5}, n \geq 6$

$K_n \rightarrow G_9$ if and only if $n \equiv 0$ or $1 \pmod{5}, n > 6$.



This case was dealt with already in Section II.

Conclusion: $K_n \rightarrow G_{10}$ if and only if $n \equiv 1$ or $5 \pmod{10}$.

(6) (a) G_{11} :  , $[x, y, z, u, v]$.

(b) $n \equiv 0, 1, 4$ or $9 \pmod{12}$.

(c) - $K_{4,4,4} \rightarrow G_{11}$: $[2_2, 0_3, 1_1, 0_1, 0_2]$, $[3_2, 1_3, 0_1, 1_1, 1_2]$,
 $[2_2, 2_3, 2_1, 3_1, 1_2]$, $[3_2, 3_3, 3_1, 2_1, 0_2]$,
 $[2_1, 0_2, 0_3, 1_3, 3_2]$ $[3_1, 1_2, 1_3, 0_3, 2_2]$,
 $[0_1, 0_2, 2_3, 3_3, 3_2]$ and $[1_1, 1_2, 3_3, 2_3, 2_2]$.

- $K_{4,4,4,4} \rightarrow G_{11}$: $V(K_{4,4,4,4}) = Z_{16}$, $[0, 7, 5, 6, 2] \pmod{16}$

- $K_{2,2,2,2} \rightarrow G_{11}$: $[0_1, 1_3, 0_2, 1_4, 0_4]$, $[0_2, 1_4, 1_1, 0_3, 1_3]$,
 $[0_3, 1_1, 1_2, 0_4, 1_4]$ and $[0_4, 1_2, 0_1, 1_3, 0_3]$.

- $K_{4,4,7} \rightarrow G_{11}$: $[i_1, i_3, (1+i)_2, i_2, \infty^*]$,
 $[i_1, (2+i)_2, (3+i)_3, \infty'', (1+i)_1]$ and
 $[i_1, (3+i)_2, \infty^*, (1+i)_3, (2+i)_1]$ for $i \in Z_4$ with
 $\infty^* = \infty$ when $i = 0$ or 1
 ∞'' when $i = 2$ or 3 .

- $K_9 \rightarrow G_{11}$: $V(K_9) = Z_3 \times Z_3$, $[(0,0), (0,1), (1,0),$
 $(2,0), (0,2)] \cup [(1,0), (2,0), (2,1), (1,1), (0,0)] \pmod{(-,3)}$.

- $K_{12} \rightarrow G_{11}$: $V(K_{12}) = Z_{11} \cup \{\infty\}$, $[1, 0, 5, 3, \infty] \pmod{11}$.

- $K_{13} \rightarrow G_{11}$: $V(K_{13}) = Z_{13}$, $[1, 0, 5, 3, 11] \pmod{13}$.

- $K_{16} \rightarrow G_{11}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,
 $[(0,0), (0,1), (1,0), (2,0), (1,1)] \cup$
 $[(0,0), (1,3), (0,2), \infty, (1,4)] \cup$
 $[(1,0), (2,0), (2,1), (1,2), \infty] \cup$
 $[(2,0), (2,2), (0,4), (1,3), (2,1)] \pmod{(-,5)}$.

- $K_{21} \rightarrow G_{11}$: $V(K_{21}) = Z_3 \times Z_7$,
 $[(0,1), (0,0), (1,3), (1,1), (1,6)] \cup$
 $[(1,6), (0,2), (0,0), (1,0), (0,3)] \cup$
 $[(2,2), (2,0), (1,3), (0,2), (2,1)] \cup$
 $[(2,1), (2,0), (0,6), (0,4), (2,5)] \cup$
 $[(1,2), (1,0), (2,3), (2,2), (2,6)] \pmod{(-,7)}$.
- $K_{24} \rightarrow G_{11}$: $V(K_{24}) = Z_{23} \cup \{\infty\}$, $[0,11,4,5,\infty] \cup$
 $[0,10,2,1,5] \pmod{23}$.
- $K_{52} \rightarrow G_{11}$: $V(K_{52}) = Z_3 \times Z_{17} \cup \{\infty\}$,
 $[(0,0), (0,7), (1,7), (1,9), \infty] \cup$
 $[(0,1), (0,0), (1,13), (0,5), (0,10)] \cup$
 $[(0,2), (0,0), (1,6), (0,8), (0,1)] \cup$
 $[(1,1), (0,0), (1,8), (0,3), (2,0)] \cup$
 $[(1,5), (1,0), (2,13), (0,6), (2,3)] \cup$
 $[(1,4), (1,0), (2,6), (0,7), (2,14)] \cup$
 $[(2,0), (1,0), (1,1), (2,5), (1,7)] \cup$
 $[(2,3), (1,0), (1,2), (2,7), (1,10)] \cup$
 $[(1,3), (1,0), (2,15), (2,14), \infty] \cup$
 $[(2,1), (2,0), (0,16), (0,14), (2,4)] \cup$
 $[(2,2), (2,0), (0,10), (0,9), (2,16)] \cup$
 $[(2,0), (2,3), (0,4), (0,5), \infty] \cup$
 $[(2,0), (2,6), (0,6), (1,13), (2,3)] \pmod{(-,17)}$.

(d) We have two arrays:

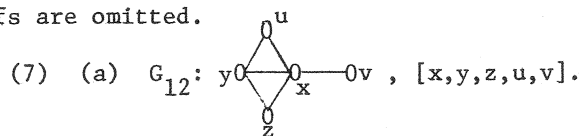
| n | $K_n \rightarrow G_{11}$ by Lemma | $n_1=n_2=n_3$ | n_4 | $K_{n_1, n_2, n_3, n_4} \rightarrow G_{11}$ by Lemma E with r p q |
|---------------|--------------------------------------|---------------|-------|--|
| 33 | B | 8 | 8 | 2 4 4 |
| $36t+13, t=1$ | B | 12 | 12 | 4 3 3 |
| $36t+33, t=1$ | B | 20 | 8 | 4 5 2 |

Note: It is not true that $K_{2,2,2} \rightarrow G_{11}$, however, (see remark after Lemma F), the condition $K_{r,r,r} \rightarrow G$ is not necessary for Lemma E when $p = q$.

| n | $K_n \rightarrow G_{11}$ by Lemma | $n_1=n_2$ | n_3 | $K_{n_1, n_2, n_3} \rightarrow G_{11}$ by Lemma D with $r_1=r_2=r_3=4, r_3'=7,$ | |
|--------------------|--------------------------------------|-----------|----------|--|---|
| | | | | a | b |
| $36t, t \geq 1$ | A | $12t$ | $12t$ | $3t$ | 0 |
| $36t+1, t \geq 1$ | B | $12t$ | $12t$ | $3t$ | 0 |
| $36t+4, t \geq 1$ | B | $12t$ | $12t+3$ | $3t$ | 1 |
| $36t+9, t \geq 1$ | A | $12t$ | $12t+9$ | $3t$ | 3 |
| $36t+12, t \geq 1$ | A | $12t+4$ | $12t+4$ | $3t+1$ | 0 |
| $36t+13, t \geq 2$ | B | $12t$ | $12t+12$ | $3t$ | 4 |
| $36t+16, t \geq 2$ | B | $12t$ | $12t+15$ | $3t$ | 5 |
| $36t+21, t \geq 1$ | A | $12t+4$ | $12t+13$ | $3t+1$ | 3 |
| $36t+24, t \geq 1$ | A | $12t+4$ | $12t+16$ | $3t+1$ | 4 |
| $36t+25, t \geq 0$ | B | $12t+8$ | $12t+8$ | $3t+2$ | 0 |
| $36t+28, t \geq 0$ | B | $12t+8$ | $12t+11$ | $3t+2$ | 1 |
| $36t+33, t \geq 2$ | A | $12t+4$ | $12t+25$ | $3t+1$ | 7 |

(e) $K_n \rightarrow G_{11}$ if and only if $n \equiv 0, 1, 4$ or $9 \pmod{12}$.

The next case G_{12} is similar to the case of G_{11} , hence some proofs are omitted.



(b) $n \equiv 0, 1, 4$ or $9 \pmod{12}$.

(c) - $K_{4,4,4} \rightarrow G_{12}$: If $K_{2,2,2} \rightarrow G_{12}$, then $K_{4,4,4} \rightarrow G_{12}$ by Lemma D with $r_1 = r_2 = r_3 = 2, a = 2$ and $b = 0$;

$K_{2,2,2} \rightarrow G_{12}$ is easy to prove, for example,

$[0_3, 0_1, 0_2, 1_2, 1_1]$ and $[1_3, 1_1, 1_2, 0_2, 0_1]$ are two

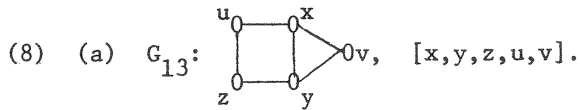
graphs partitioning the edges of $K_{2,2,2}$.

- $K_{4,4,4,4} \rightarrow G_{12}$: $V(K_{4,4,4,4}) = Z_{16}$, $[0,7,5,6,3] \bmod 16$,
- $K_{2,2,2,2} \rightarrow G_{12}$: $[0_1,1_3,0_2,1_4,0_3]$, $[0_2,1_4,0_3,1_1,0_4]$,
 $[1_1,0_3,0_4,1_2,1_3]$ and $[1_2,0_4,0_1,1_3,1_4]$.
- $K_{4,4,7} \rightarrow G_{12}$: $[0_3,0_1,0_2,1_2,2_2] \cup [0_1,2_2,3_3,\infty,1_3] \cup$
 $[0_1,3_2,\infty',\infty'',2_3] \bmod 4$.
- $K_9 \rightarrow G_{12}$: $V(K_9) = Z_3 \times Z_3$,
 $[(0,1), (0,0), (1,0), (2,0), (1,2)] \cup$
 $[(2,0), (1,0), (2,1), (1,1), (0,2)] \bmod (-,3)$.
- $K_{12} \rightarrow G_{12}$: $V(K_{12}) = Z_{11} \cup \{\infty\}$, $[1,0,5,3,\infty] \bmod 11$.
- $K_{13} \rightarrow G_{12}$: $V(K_{13}) = Z_{13}$, $[1,0,5,3,7] \bmod 13$.
- $K_{16} \rightarrow G_{12}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,
 $[(0,1), (0,0), (1,0), (2,0), (1,3)] \cup$
 $[(1,3), (0,0), (0,2), \infty, (1,4)] \cup$
 $[(2,0), (1,0), (2,1), (1,2), \infty] \cup$
 $[(2,0), (2,2), (0,4), (1,3), (0,3)] \bmod (-,5)$.
- $K_{21} \rightarrow G_{12}$: The base graphs are the same as in $K_{21} \rightarrow G_{11}$
with the last elements of each graph replaced by
 $(0,4), (1,2), (2,5), (0,2)$ and $(2,0)$, in the order
given.
- $K_{24} \rightarrow G_{12}$: $V(K_{24}) = Z_{23} \cup \{\infty\}$, $[0,11,4,5,\infty] \cup$
 $[0,10,2,1,3] \bmod 23$.
- $K_{52} \rightarrow G_{12}$: The base graphs are the same as those of
 $K_{52} \rightarrow G_{11}$ with the last elements in each graph replaced
by $\infty, (1,4), (1,7), (2,10), (1,11), (1,12), (2,7), (2,11), \infty,$
 $(0,13), (0,13), \infty$ and $(0,3)$, in the order given.

(d) see the case G_{11} .

- (e) The conclusion is the same as the case G_{11} as well:
 $K_n \rightarrow G_{12}$ if and only if $n \equiv 0, 1, 4$ or $9 \pmod{12}$.

The case $K_n \rightarrow G_{13}$ can be solved similarly to the last two cases, i.e. when $G = G_{11}$ or G_{12} , hence a part of the proof is omitted; however, due to the fact that we do not know whether $K_{24} \rightarrow G_{13}$ some variations have to be made.



- (b) $n \equiv 0, 1, 4$ or $9 \pmod{12}$.

- (c) - $K_{4,4,4} \rightarrow G_{13}$: see Example 2 of Section II.

- $K_{4,4,4,4} \rightarrow G_{13}$: see Example 4 of Section II.

- $K_{2,2,2,2} \rightarrow G_{13}$: $V(K_{2,2,2,2}) = Z_8$, $[0, 2, 5, 3, 1]$,
 $[0, 6, 1, 7, 5]$, $[2, 4, 1, 3, 7]$ and $[6, 4, 5, 7, 3]$.

- $K_{4,4,7} \rightarrow G_{13}$: see Example 3 of Section II.

- $K_9 \rightarrow G_{13}$: $V(K_9) = Z_3 \times Z_3$,
 $[(1, 0), (1, 1), (0, 0), (0, 1), (2, 2)] \cup$
 $[(2, 0), (2, 1), (0, 0), (1, 0), (0, 1)] \pmod{(-, 3)}$.

- $K_{12} \rightarrow G_{13}$: $V(K_{12}) = Z_{12}$, $[3, 7, 8, 2, 10]$,
 $[6, 4, 3, 11, 9]$, $[4, 8, 11, 0, 5]$, $[10, 11, 5, 6, 4]$,
 $[0, 1, 8, 6, 5]$, $[0, 2, 9, 8, 7]$, $[0, 3, 8, 10, 9]$,
 $[1, 3, 5, 7, 6]$, $[1, 9, 7, 11, 10]$, $[2, 4, 7, 6, 1]$
and $[2, 5, 9, 11, 10]$.

- $K_{13} \rightarrow G_{13}$: $V(K_{13}) = Z_{13}$, $[5, 0, 3, 1, 6] \pmod{13}$.

- $K_{16} \rightarrow G_{13}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,
 $[\infty, (1, 2), (0, 3), (2, 0), (0, 0)] \cup$
 $[(0, 1), (1, 2), (1, 0), (0, 0), (0, 4)] \cup$
 $[(2, 2), (2, 0), (1, 0), (1, 1), (0, 1)] \cup$

$$[(2,1), (1,2), (2,4), (0,1), (2,0)] \text{ mod } (-,5).$$

$$- K_{21} \rightarrow G_{13}: V(K_{21}) = Z_3 \times Z_7,$$

$$[(0,0), (0,1), (1,1), (1,4), (0,3)] \cup$$

$$[(0,1), (1,6), (0,0), (1,2), (1,4)] \cup$$

$$[(2,1), (2,4), (1,0), (1,1), (0,4)] \cup$$

$$[(1,1), (2,6), (1,0), (2,2), (2,4)] \cup$$

$$[(0,6), (2,1), (0,2), (2,0), (2,2)] \text{ mod } (-,7).$$

- $K_{24} \rightarrow G_{13}$: we have no proof for this case, but we conjecture that it is true.

$$- K_{52} \rightarrow G_{13}: V(K_{52}) = Z_3 \times Z_{17} \cup \{\infty\},$$

$$[\infty, (1,16), (0,8), (2,8), (0,0)] \cup$$

$$[(0,0), (0,1), (1,4), (1,1), (0,8)] \cup$$

$$[(0,0), (0,2), (1,8), (1,2), (0,6)] \cup$$

$$[(0,1), (1,11), (0,0), (1,5), (0,4)] \cup$$

$$[(0,1), (1,15), (0,0), (1,13), (0,6)] \cup$$

$$[(1,0), (1,1), (2,4), (2,1), (1,5)] \cup$$

$$[(1,0), (1,2), (2,8), (2,2), (1,9)] \cup$$

$$[(1,1), (2,5), (1,0), (2,13), (2,12)] \cup$$

$$[(1,0), (2,8), (1,1), (2,15), (2,16)] \cup$$

$$[(0,15), (2,10), (1,0), (2,9), (2,11)] \cup$$

$$[(2,0), (2,2), (1,2), (0,2), (0,3)] \cup$$

$$[(0,14), (2,1), (0,8), (2,0), (2,5)] \cup$$

$$[(0,16), (2,1), (0,12), (2,0), (2,6)] \text{ mod } (-,17).$$

In addition, we have

$$- K_{12,12,12,12,12,12} \rightarrow G_{13}: V(K_{12,12,12,12,12,12}) = Z_{72},$$

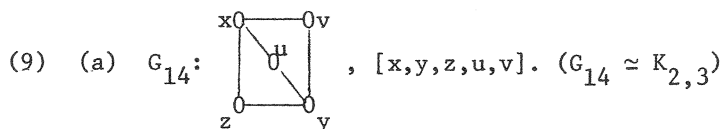
$$[0,3,64,31,1] \cup [0,9,67,35,4] \cup [0,20,1,22,7] \cup$$

$$[0,23,57,10,8] \cup [0,55,71,44,29] \text{ mod } 72.$$

- (d) The two arrays are the same as those in the case of $G = G_{11}$ with the following exception: since we do not know whether $K_{24} \rightarrow G_{13}$ is true, the statements in the longer array for the cases $n = 36t$ and $n = 36t+9$ are true only when $t \geq 1$ and $t \neq 2$. The cases thus not included in the array, namely $n = 36t$ and $n = 36t+9$ for $t = 2$ can be proved as follows:

| n | $K_n \rightarrow G_{13}$ by Lemma | $n_i, i=1,2,\dots,h$ | $K_{n_1, n_2, \dots, n_h} \rightarrow G_{13}$ by Lemma E with $r=4, p=q$ |
|--------------|--------------------------------------|-------------------------|---|
| $36t, t=2$ | A | $n_i=12, i=1,2,\dots,6$ | - (proved in part (c)) |
| $36t+9, t=2$ | B | $n_i=20, i=1,2,3,4$ | 5 |

- (e) $K_n \rightarrow G_{13}$ if and only if $n \equiv 0,1,4$ or $9 \pmod{12}$, except possibly when $n = 24$.



- (b) $n \equiv 0,1,4$ or $9 \pmod{12}$.

- (c) - $K_9 \nrightarrow G_{14}$: The degrees of vertices of G_{14} are 2 and 3, since a vertex of K_9 has degree 8, it must appear either in two subgraphs with degree 3 and in one subgraph with degree 2 (in which case, the vertex is said to be type 1), or in four subgraphs with degree 2 (type 2). Suppose that there are a vertices in K_9 which are type 1 and b vertices which are type 2, then obviously $a + b = 9$. As there are 6 subgraphs isomorphic to G_{14} in the decompositions, by counting in two ways the number of vertices with degree 3 (or 2), we have

$$2a = 12$$

$$a + 4b = 18.$$

Solving the equations, we have $a = 6, b = 3$. Now, in a G_{14} , the vertices of degree 2 are adjacent

only to vertices of degree 3, the three vertices of type 2 can never be joined to each other, a contradiction to $K_9 \rightarrow G_{14}$.

- $K_{12} \nrightarrow G_{14}$: the proof is similar, though longer than the one above, and is hence omitted.

- $K_{13} \rightarrow G_{14}$: $V(K_{13}) = Z_{13}, [0,1,2,4,6] \text{ mod } 13$.

- $K_{16} \rightarrow G_{14}$: $V(K_{16}) = Z_3 \times Z_5 \cup \{\infty\}$,
 $[(1,0), (1,2), (0,0), (2,0), (2,1)] \cup$
 $[\infty, (0,4), (0,0), (1,0), (2,0)] \cup$
 $[(0,0), (2,2), (0,2), (2,3), (2,4)] \cup$
 $[(0,0), (1,0), (1,3), (1,4), (2,2)] \text{ mod } (-,5)$.

- $K_{21} \rightarrow G_{14}$: $V(K_{21}) = Z_3 \times Z_7$,
 $[(0,0), (1,1), (0,1), (0,3), (1,3)] \cup$
 $[(0,0), (2,2), (0,2), (1,2), (1,4)] \cup$
 $[(1,0), (1,2), (0,1), (1,3), (2,6)] \cup$
 $[(1,0), (2,0), (2,1), (2,2), (2,3)] \cup$
 $[(2,0), (2,1), (0,2), (0,4), (0,6)] \text{ mod } (-,7)$.

- $K_{24} \rightarrow G_{14}$: $V(K_{24}) = \{\infty, 0, 1, \dots, 6\} \cup \bigcup_{i=1}^2 X_i$ where

$X_i = \{0_i, 1_i, \dots, 7_i\}$. Since $K_{16} \rightarrow G_{14}$, we can construct 20 subgraphs from the vertices of $X_1 \cup X_2$.

The remaining 26 subgraphs are:

- $[0, 1, 3, 4_1, 4_2], [0_1, 0_2, 1, 2, 3]$
- $[1, 2, 4, 5_1, 5_2], [1_1, 1_2, 0, 2, 3]$
- $[2, 3, 5, 4_1, 4_2], [2_1, 2_2, 0, 1, 3]$
- $[3, 4, 6, 5_1, 5_2], [3_1, 3_2, 0, 1, 2]$
- $[4, 5, 0, 7_1, 7_2], [6_1, 6_2, 0, 1, 2]$
- $[5, 6, 1, 4_1, 4_2], [6_1, 6_2, 3, 4, 5]$
- $[6, 0, 2, 5_1, 5_2], [7_1, 7_2, 0, 1, 2]$
- $[7_1, 7_2, 3, 6, \infty]$

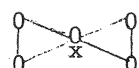
and $[\infty, i, i+1, i_1, i_2]$ for $i = 0, 1, \dots, 6,$

$[j_1, j_2, 4, 5, 6]$ for $j = 0, 1, 2, 3.$

(d)

| n | $K_n \rightarrow G_{14}$ by Lemma | n_1 | n_2 | $K_{n_1, n_2} \rightarrow G_{14}$ by Lemma C with $r_1=2, r_2=3, r_2'=0, c=0,$ | |
|-------------------|--------------------------------------|--------------|-------|---|---|
| | | | | a | b |
| $12t, t \geq 3$ | B | $12t(t-2)+8$ | 15 | $6(t-2)+4$ | 5 |
| $12t+1, t \geq 2$ | B | $12(t-1)$ | 12 | $6(t-1)$ | 4 |
| $12t+4, t \geq 2$ | B | $12(t-1)$ | 15 | $6(t-1)$ | 5 |
| $12t+9, t \geq 2$ | B | $12(t-1)+8$ | 12 | $6(t-1)+4$ | 4 |


(e) $K_n \rightarrow G_{14}$ iff $n \equiv 0, 1, 4, 9 \pmod{12}, n \geq 13.$

(10) (a) G_{15} : 

(b) $n \equiv 1$ or $9 \pmod{12}.$

(c) The case $n \equiv 1 \pmod{12}$ was solved in [7].

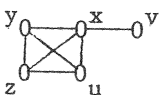
Let $n = 12t+9.$

It is well-known that a resolvable decomposition into triangles exists [12], that is $K_n \rightarrow$  and furthermore, the triangles of the decomposition can be partitioned into $6t + 4$ resolution classes, each containing $4t + 3$ disjoint triangles (and thus covering all the vertices of K_n).

Partition the resolution classes into pairs (there is an even number of them) and consider a particular pair of resolution classes. By the theorem on simultaneous representatives [4, pg.51] applied to the two classes, we get common representatives of the triangles in them. Joining the two triangles which contain the common representative x at x yields a G_{15} . Hence we have

$$K_{12t+9} \rightarrow G_{15}.$$

(d) $K_n \rightarrow G_{15}$ if and only if $n \equiv 1$ or $9 \pmod{12}.$

(11) (a) G_{16} :  , $[x, y, z, u, v]$.

(b) $n \equiv 0$ or $1 \pmod{7}$.

(c) - $K_7 \nrightarrow G_{16}$: use degree argument, similar to the case $K_9 \nrightarrow G_{14}$.

- $K_8 \nrightarrow G_{16}$: use degree argument.

- $K_{7,7,7,7} \rightarrow G_{16}$: $[0_1, 1_2, 1_3, 0_4, 0_2] \cup [0_1, 2_2, 3_3, 6_4, 6_3] \cup [0_1, 3_2, 5_3, 5_4, 1_4] \cup [4_2, 0_1, 0_3, 4_4, 3_3] \cup [5_2, 0_1, 2_3, 3_4, 6_4] \cup [4_3, 0_1, 6_2, 2_4, 6_4] \pmod{7}$.

- $K_{7,7,7,7,7} \rightarrow G_{16}$: $V(K_{7,7,7,7,7}) = Z_{35}$, $[0, 1, 4, 13, 7] \cup [0, 2, 8, 19, 14] \pmod{35}$.

- $K_{14} \rightarrow G_{16}$: $V(K_{14}) = Z_{13} \cup \{\infty\}$, $[0, 1, 4, 6, \infty] \pmod{13}$.

- $K_{15} \rightarrow G_{16}$: $V(K_{15}) = Z_{15}$, $[0, 1, 4, 6, 7] \pmod{15}$.

- $K_{21} \rightarrow G_{16}$: $V(K_{21}) = Z_4 \times Z_5 \cup \{\infty\}$, $[(0, 0), (0, 1), (1, 3), (2, 0), (1, 1)] \cup [(1, 0), (0, 0), (2, 1), (2, 3), (3, 2)] \cup [(0, 0), (0, 2), (3, 2), (3, 3), (2, 2)] \cup [(3, 1), (3, 3), (2, 0), (2, 1), (1, 0)] \cup [(1, 0), (1, 1), (2, 0), (3, 4), (1, 2)] \cup [\infty, (0, 0), (1, 4), (3, 4), (2, 0)] \pmod{(-, 5)}$.

- $K_{22} \rightarrow G_{16}$: $V(K_{22}) = Z_2 \times Z_{11}$, $[(0, 0), (0, 4), (1, 0), (1, 1), (0, 5)] \cup [(1, 5), (0, 0), (0, 2), (0, 3), (1, 1)] \cup [(0, 0), (1, 4), (1, 6), (1, 9), (1, 10)] \pmod{(-, 11)}$.

- $K_{28} \rightarrow G_{16}$: $V(K_{28}) = Z_{27} \cup \{\infty\}$, $[0, 3, 4, 14, \infty] \cup [0, 6, 8, 15, 5] \pmod{27}$.

- $K_{29} \rightarrow G_{16}$: $V(K_{29}) = Z_{29}$, $[0,1,3,11,6] \cup [0,4,9,16,14] \pmod{29}$.
- $K_{35} \rightarrow G_{16}$: $V(K_{35}) = Z_2 \times Z_{17} \cup \{\infty\}$,
 $[(0,0), (0,6), (0,7), (1,8), (0,4)] \cup [(0,0), (0,3), (0,5), (1,9), (1,13)] \cup [(0,0), (1,5), (1,11), (1,14), \infty] \cup [(1,0), (0,0), (1,10), (1,15), (1,1)] \cup [(1,0), (1,4), (0,5), (0,14), \infty] \pmod{(-,17)}$.
- $K_{36} \rightarrow G_{16}$: $V(K_{36}) = Z_4 \times Z_9$,
 $[(0,0), (0,2), (1,8), (2,4), (0,1)] \cup [(0,0), (0,4), (1,7), (2,0), (0,3)] \cup [(1,0), (1,1), (2,0), (3,6), (1,2)] \cup [(1,0), (1,3), (2,4), (3,3), (1,4)] \cup [(0,3), (2,0), (2,4), (3,5), (1,4)] \cup [(0,5), (1,5), (3,0), (3,4), (1,7)] \cup [(2,0), (2,1), (2,3), (3,3), (0,1)] \cup [(0,0), (3,0), (3,1), (3,3), (3,6)] \cup [(1,5), (0,0), (2,3), (3,7), (2,2)] \cup [(1,4), (0,0), (2,7), (3,5), (3,2)] \pmod{(-,9)}$.
- $K_{42} \rightarrow G_{16}$: $V(K_{42}) = Z_{41} \cup \{\infty\}$, $[0,3,4,20,15] \cup [0,8,10,19,14] \cup [0,5,12,18, \infty] \pmod{41}$.
- $K_{43} \rightarrow G_{16}$: $V(K_{43}) = Z_{43}$, $[0,8,9,19,6] \cup [0,5,7,20,12] \cup [0,3,17,21,16] \pmod{43}$.
- $K_{49} \rightarrow G_{16}$: $V(K_{49}) = Z_7 \times Z_7$,
 $[(i,0), (i,1), (i+1,5), (i+3,3), (i+2,j)]$
with $j = \begin{cases} 4 & \text{for } i = 0,1,2,3,4. \\ 0 & \text{for } i = 5,6. \end{cases}$
 $\cup [(i,0), (i+1,1), (i+1,3), (i+3,4), (i+3,j)]$

with $j = \begin{cases} 6 & \text{for } i = 0,1,2,3,4,6. \\ 0 & \text{for } i = 5. \end{cases}$

$\cup [(i,0), (i+1,6), (i+3,1), (i+3,5), (k,j)]$

with $k = 6, j = 0$ for $i = 0$

$k = i+1, j = 2$ for $i = 1,2,3,4$

$k = 2, j = 0$ for $i = 5,6$

$\cup [(1,0), (2,0), (3,0), (0,0), (4,0)]$

$\cup [(4,0), (5,0), (6,0), (3,0), (2,0)]$

$\cup [(0,4), (1,6), (5,0), (6,2), (4,4)] \text{ mod } (-,7).$

- $K_{50} \rightarrow G_{16}: V(K_{50}) = Z_2 \times Z_{25},$

$[(0,0), (0,1), (1,1), (1,3), (0,8)] \cup$

$[(0,0), (0,2), (0,5), (1,10), (0,12)] \cup$

$[(0,0), (0,4), (0,11), (1,15), (1,9)] \cup$

$[(0,0), (0,6), (0,15), (1,22), (1,12)] \cup$

$[(0,0), (1,19), (1,20), (1,24), (1,18)] \cup$

$[(1,0), (1,3), (1,9), (0,11), (1,11)] \cup$

$[(1,0), (1,7), (1,15), (0,19), (1,12)] \text{ mod } (-,25).$

- $K_{56} \rightarrow G_{16}: V(K_{56}) = Z_5 \cup \{\infty\}, [0,5,6,26,3] \cup$

$[0,11,13,25,17] \cup [0,10,28,32,19] \cup$

$[0,8,15,24,\infty] \text{ mod } 55.$

- $K_{57} \rightarrow G_{16}: V(K_{57}) = Z_{57}, [0,9,10,26,5] \cup$

$[0,11,13,25,8] \cup [0,3,23,27,18] \cup$

$[0,6,21,28,19] \text{ mod } 57.$

- $K_{63} \rightarrow G_{16}: V(K_{63}) = Z_2 \times Z_{31} \cup \{\infty\},$

$[(0,0), (0,1), (0,13), (1,0), (1,11)] \cup$

$[(0,0), (0,2), (0,6), (1,8), (1,12)] \cup$

$[(0,0), (0,3), (0,10), (1,13), (1,16)] \cup$

$[(0,0), (0,5), (0,16), (1,20), (1,19)] \cup$

$[(0,0), (0,8), (0,17), (1,22), \infty] \cup$

$$\begin{aligned}
& [(1,0), (1,1), (1,4), (1,10), (0,8)] \cup \\
& [(1,0), (1,2), (1,13), (0,4), (0,6)] \cup \\
& [(1,0), (1,7), (1,15), (0,14), (0,3)] \cup \\
& [(1,0), (1,5), (1,17), (1,10), \infty] \text{ mod } (-,31).
\end{aligned}$$

$$\begin{aligned}
- K_{64} \rightarrow G_{16}: \quad V(K_{64}) &= Z_7 \times Z_9 \cup \{\infty\}, \\
& [(i,0), (i,1), (i+1,1), (i+3,1), (i+2,6)] \quad i = 0,1,\dots,6 \\
& \cup [(i,0), (i,4), (i+1,7), (i+3,2), (i+2,j)] \\
& \text{with } \begin{cases} j = 1 & \text{for } i = 0,1,2,3,4,5 \\ j = 3 & \text{for } i = 6 \end{cases} \\
& \cup [(i,0), (i+1,6), (i+1,8), (i+3,4), (k,j)] \\
& \text{with } \begin{aligned} k &= 2, j = 3, \quad \text{for } i = 0 \\ k &= i+1, j = 5 \quad \text{for } i = 1,2 \\ k &= i+1, j = 2 \quad \text{for } i = 3,4,5,6 \end{aligned} \\
& \cup [(i,0), (i+1,4), (i+3,3), (i+3,6), (k,j)] \\
& \text{with } \begin{aligned} k &= 3, j = 8 \quad \text{for } i = 0 \\ k &= 5, j = 4 \quad \text{for } i = 1 \\ k &= 6, j = 4 \quad \text{for } i = 2 \\ k &= 6, j = 5 \quad \text{for } i = 3 \\ k &= 0, j = 5 \quad \text{for } i = 4 \\ k &= 0, j = 3 \quad \text{for } i = 5 \\ k &= 2, j = 8 \quad \text{for } i = 6 \end{aligned} \\
& \cup [(4,1), (0,0), (1,2), (3,5), \infty] \\
& \cup [(4,5), (1,0), (2,2), (5,1), (6,8)] \\
& \cup [(5,5), (2,0), (3,2), \infty, (6,1)] \\
& \cup [(6,4), (0,0), (1,5), \infty, (3,5)] \text{ mod } (-,9).
\end{aligned}$$

$$\begin{aligned}
- K_{70} \rightarrow G_{16}: \quad V(K_{70}) &= Z_{69} \cup \{\infty\}, [0,11,13,25,1] \\
& \cup [0,3,29,34,20] \cup [0,10,28,32,30] \\
& \cup [0,17,23,50,21] \cup [0,8,15,24,\infty] \text{ mod } 69.
\end{aligned}$$

$$\begin{aligned}
- K_{71} \rightarrow G_{16}: \quad V(K_{71}) &= Z_{71}, [0,11,13,25,1] \\
& \cup [0,3,29,34,20] \cup [0,10,28,32,30] \\
& \cup [0,17,23,50,19] \cup [0,8,15,24,35] \text{ mod } 71.
\end{aligned}$$

- $K_{77} \rightarrow G_{16}: V(K_{77}) = Z_4 \times Z_{19} \cup \{\infty\}$,
 $[(0,0), (1,0), (2,0), (3,0), \infty]$
 $\cup [(0,\alpha), (1,2\alpha), (2,3\alpha), (3,4\alpha), (0,2\alpha)]$
with $\alpha = 1, 2, 3, 4, 5, 6, 7, 8, 9$
 $\cup [(1,2\alpha), (0,\alpha), (2,3\alpha), (3,4\alpha), (1,3\alpha-3)]$
with $\alpha = 10, 11, 12$
 $\cup [(2,3\alpha), (0,\alpha), (1,2\alpha), (3,4\alpha), (2,4\alpha-6)]$
with $\alpha = 13, 14, 15$
 $\cup [(3,4\alpha), (0,\alpha), (1,2\alpha), (2,3\alpha), (3,5\alpha-9)]$
with $\alpha = 16, 17, 18$
 $\cup [(i,0), (i,1), (i,4), (i,6), \infty]$
for $i = 1, 2, 3 \pmod{(-, 19)}$.
- $K_{78} \rightarrow G_{16}: V(K_{78}) = Z_2 \times Z_{39}$,
 $[(0,0), (0,16), (0,18), (0,19), (1,4)]$
 $\cup [(0,0), (0,8), (0,15), (1,35), (1,6)]$
 $\cup [(0,0), (0,4), (0,13), (1,26), (1,7)]$
 $\cup [(0,0), (0,6), (0,11), (1,3), (1,11)]$
 $\cup [(0,0), (0,10), (0,22), (1,1), (1,14)]$
 $\cup [(1,0), (1,16), (1,18), (1,19), (0,24)]$
 $\cup [(1,0), (1,8), (1,15), (0,37), (0,23)]$
 $\cup [(1,0), (1,4), (1,14), (0,34), (0,15)]$
 $\cup [(1,0), (1,6), (1,11), (0,16), (0,11)]$
 $\cup [(1,0), (1,9), (1,22), (0,1), (0,7)]$
 $\cup [(0,0), (0,14), (1,0), (1,12), (1,33)] \pmod{(-, 39)}$.
- $K_{91} \rightarrow G_{16}: V(K_{91}) = Z_2 \times Z_{45} \cup \{\infty\}$,
 $[(0,0), (0,1), (0,4), (0,20), \infty]$
 $\cup [(0,0), (0,2), (0,9), (1,10), (1,6)]$
 $\cup [(0,0), (0,5), (0,15), (1,40), (1,11)]$

$\cup [(0,0), (0,8), (0,22), (1,44), (1,17)]$
 $\cup [(0,0), (0,11), (0,24), (1,27), (1,19)]$
 $\cup [(1,0), (1,1), (1,4), (1,20), \infty]$
 $\cup [(1,0), (1,2), (1,8), (0,4), (0,21)]$
 $\cup [(1,0), (1,5), (1,15), (0,27), (0,15)]$
 $\cup [(0,0), (0,6), (0,18), (1,20), (1,13)]$
 $\cup [(0,0), (0,17), (1,0), (1,9), (1,42)]$
 $\cup [(1,0), (1,7), (1,24), (0,40), (0,14)]$
 $\cup [(1,0), (1,11), (1,23), (0,30), (0,13)]$
 $\cup [(1,0), (1,13), (1,27), (0,6), (0,11)] \pmod{(-,45)}.$

$- K_{92} \rightarrow G_{16}: V(K_{92}) = Z_4 \times Z_{23},$
 $[(0,\alpha), (1,2\alpha), (2,3\alpha), (3,4\alpha), (0,2\alpha+1)]$
 with $\alpha = 0,1,\dots,10$
 $\cup [(1,2\alpha), (0,\alpha), (2,3\alpha), (3,4\alpha), (1,3\alpha-4)]$
 with $\alpha = 11,12,13,14$
 $\cup [(2,3\alpha), (0,\alpha), (1,2\alpha), (3,4\alpha), (2,4\alpha-8)]$
 with $\alpha = 15,16,17,18$
 $\cup [(3,4\alpha), (0,\alpha), (1,2\alpha), (2,3\alpha), (3,5\alpha-12)]$
 with $\alpha = 19,20,21,22$
 $\cup [(i,0), (i,1), (i,4), (i,6), (i,11)]$
 for $i = 1,2,3 \pmod{(-,23)}.$

$- K_{148} \rightarrow G_{16}: V(K_{148}) = Z_4 \times Z_{37},$
 $[(0,\alpha), (1,2\alpha), (2,3\alpha), (3,4\alpha), (0,2\alpha+1)]$
 with $\alpha = 0,1,2,\dots,17$
 $\cup [(1,2\alpha), (0,\alpha), (2,3\alpha), (3,4\alpha), (1,3\alpha-10)]$
 with $\alpha = 18,19,\dots,28$
 $\cup [(2,3\alpha), (0,\alpha), (1,2\alpha), (3,4\alpha), (2,4\alpha-14)]$
 with $\alpha = 29,30,31,32$

$$\cup [(3, 4\alpha), (0, \alpha), (1, 2\alpha), (2, 3\alpha), (3, 5\alpha-18)]$$

$$\text{with } \alpha = 33, 34, 35, 36$$

$$\cup [(1, 0), (1, 1), (1, 4), (1, 6), (1, 7)]$$

$$\cup [(i, 0), (i, 1), (i, 7), (i, 11), (i, 8)]$$

$$\cup [(i, 0), (i, 2), (i, 5), (i, 14), (i, 13)]$$

$$i = 2, 3 \pmod{(-, 37)}.$$

(d)

| n | $K_n \rightarrow G_{16}$ by Lemma | $n_1 = n_2 = n_3 = n_4$ | n_5 | $K_{n_1, n_2, n_3, n_4, n_5} \rightarrow G_{16}$ by Lemma F with $r = 7$ | |
|---|--------------------------------------|-------------------------|-------|--|----|
| | | | | p= | q= |
| $28t, t \geq 3,$ $t \neq 6, 17, 21, 29$ | A | $7t$ | 0 | t | 0 |
| $28t+1, t \geq 3,$ $t \neq 6, 17, 29$ | B | $7t$ | 0 | t | 0 |
| $28t+7, t \geq 6$ $t \neq 7, 11, 15, 18, 22, 30$ | A | $7t-7$ | 35 | $t-1$ | 5 |
| $28t+8, t \geq 6,$ $t \neq 7, 11, 15, 18, 30$ | B | $7t-7$ | 35 | $t-1$ | 5 |
| $28t+14, t \geq 4,$ $t \neq 6, 10, 14, 17, 21, 29$ | A | $7t$ | 14 | t | 2 |
| $28t+15, t \geq 4,$ $t \neq 6, 10, 14, 17, 29$ | B | $7t$ | 14 | t | 2 |
| $28t+21, t \geq 4,$ $t \neq 6, 10, 14, 17, 21, 29$ | A | $7t$ | 21 | t | 3 |
| $28t+22, t \geq 4,$ $t \neq 6, 10, 14, 17, 29$ | B | $7t$ | 21 | t | 3 |
| $28t,$ $t = 6, 17, 21, 29$ | A | $7t-7$ | 28 | $t-1$ | 4 |
| $28t+1,$ $t = 6, 17, 29$ | B | $7t-7$ | 28 | $t-1$ | 4 |
| $28t+7,$ $t = 11, 15, 18, 22, 30$ | A | $7t-14$ | 63 | $t-2$ | 9 |

| | | | | | |
|-----------------------------|---|-------|----|-----|---|
| 28t+8, t=11,15,18,30 | B | 7t-14 | 63 | t-2 | 9 |
| 28t+14, t=10,14,17,21,29 | A | 7t-7 | 42 | t-1 | 6 |
| 28t+15, t=10,14,17,29 | B | 7t-7 | 42 | t-1 | 6 |
| 28t+21, t=10,14,17,21,29 | A | 7t-7 | 49 | t-1 | 7 |
| 28t+22, t=10,14,17,29 | B | 7t-7 | 49 | t-1 | 7 |

The array is not complete and the missing cases are considered below.

(i) Whether $K_n \rightarrow G_{16}$ for $n = 28t+7$ and $t = 4, 5$, or 7 , and $n = 28t+8$ for $t = 4$ or 7 is still undecided.

(ii) $n = 28t+14$ or $28t+15$ for $t = 3$: if we have $K_{2,2,2,2,2,2,2,2} \rightarrow K_4$ then it can easily be seen that $K_{14,14,14,14,14,14,14,14} \rightarrow K_{7,7,7,7} \rightarrow G_{16}$, and, by Lemma A and B, respectively, we get $K_{98} \rightarrow G_{16}$ and $K_{99} \rightarrow G_{16}$. The decomposition of $K_{2,2,2,2,2,2,2,2} \rightarrow G_{16}$ is

$$\left\{ \bigcup_{j=1}^7 [0_j, 0_{j+1}, 0_{j+3}, 1_{j+6}] \right\} \pmod{2}.$$

(iii) $n = 28t+14$ or $28t+15$ for $t = 6$: similarly, if $K_{2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2} \rightarrow K_4$ then trivially $K_{14,14,14,14,14,14,14,14,14,14,14,14,14,14,14,14} \rightarrow K_{7,7,7,7} \rightarrow G_{16}$ and thus $K_{182} \rightarrow G_{16}$ and $K_{183} \rightarrow G_{16}$. $K_{2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2} \rightarrow K_4$ is given by

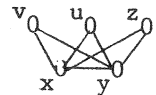
$$\left\{ \bigcup_{j=1}^{13} [0_{j+2}, 0_{j+5}, 0_{j+6}, 1_j] \cup [0_j, 0_{j+5}, 0_{j+7}, 1_{j+4}] \right\} \pmod{2}.$$

(iv) $n = 28t+21$ or $28t+22$ for $t = 3$: if we have $K_{3,3,3,3,3} \rightarrow K_4$, then $K_{21,21,21,21,21} \rightarrow K_{7,7,7,7} \rightarrow G_{16}$ and by Lemma A and B, respectively, we have $K_{105} \rightarrow G_{16}$ and $K_{106} \rightarrow G_{16}$. The case $K_{3,3,3,3,3} \rightarrow K_4$ is proved by letting $V(K_{3,3,3,3,3}) = Z_{15}$ and the base graph $[0,1,3,7]$ mod 15.

(v) $n = 28t+21$ or $28t+22$ for $t = 6$: if we have

$K_{3,3,3,3,3,3,3,3} \rightarrow K_4$, then $K_{21,21,21,21,21,21,21,21} \rightarrow K_{7,7,7,7} \rightarrow G_{16}$ and we have both $K_{189} \rightarrow G_{16}$ and $K_{190} \rightarrow G_{16}$. $K_{3,3,3,\dots,3} \rightarrow K_4$ is proved by letting $V(K_{3,3,\dots,3}) = Z_3 \times Z_3 \times Z_3$ and the base graphs $[(0,2,1), (0,0,1), (1,1,2), (1,1,0)] \cup [(0,2,0), (1,2,2), (0,0,2), (1,0,0)] \pmod{(3,3,3)}$.

(e) $K_n \rightarrow G_{16}$ if and only if $n \equiv 0$ or $1 \pmod{7}$, $n \neq 7, 8$ except possibly when $n = 119, 120, 147, 203, 204$.

(12) (a) G_{17} :  , $[x, y, z, u, v] (G_{17} \cong K_{1,1,3})$.

(b) $n \equiv 1$ or $7 \pmod{14}$.

(c) - $K_{7,7,7} \rightarrow G_{17}$: $[0_2, 0_3, 1_1, 2_1, 3_1] \cup [0_3, 0_1, 1_2, 2_2, 3_2] \cup [0_1, 0_2, 1_3, 2_3, 3_3] \pmod{7}$.

- $K_{7,7,21} \rightarrow G_{17}$: since $G_{17} = K_{1,1,3}$, the decomposition can be obtained by applying Lemma D with $r_1 = r_2 = 1$, $r_3 = 3$, $a = 7$ and $b = 0$.

- $K_7 \rightarrow G_{17}$: $V(K_7) = Z_2 \times Z_3 \cup \{\infty\}$, $[(0,0), (1,0), (0,1), (1,1), \infty] \pmod{(-,3)}$.

- $K_{15} \rightarrow G_{17}$: $V(K_{15}) = Z_{15}$, $[0, 1, 3, 5, 7] \pmod{15}$.

- $K_{29} \rightarrow G_{17}$: $V(K_{29}) = Z_{29}$, $[0, 1, 4, 6, 8] \cup [0, 2, 11, 12, 15] \pmod{29}$.

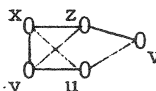
- $K_{49} \rightarrow G_{17}$: since $K_7 \rightarrow K_3$ (equivalent to a Steiner triple system of order 7) then $K_{7,7,7,7,7,7,7} \rightarrow K_{7,7,7} \rightarrow G_{17}$ and, since it has been proved that $K_7 \rightarrow G_{17}$, we have $K_{49} \rightarrow G_{17}$ by Lemma A.

(d)

| n | $K_n \rightarrow G_{17}$ by Lemma | $n_1=n_2$ | n_3 | $K_{n_1, n_2, n_3} \rightarrow G_{17}$ by Lemma D with $r_1=r_2=r_3=7, r'_3=21,$ | |
|--------------------|--------------------------------------|-----------|----------|---|---|
| | | | | a | b |
| $42t+1, t \geq 1$ | B | $14t$ | $14t$ | $2t$ | 0 |
| $42t+7, t \geq 2$ | A | $14t-7$ | $14t+21$ | $2t-1$ | 2 |
| $42t+15, t \geq 1$ | B | $14t$ | $14t+14$ | $2t$ | 1 |
| $42t+21, t \geq 0$ | A | $14t+7$ | $14t+7$ | $2t+1$ | 0 |
| $42t+29, t \geq 1$ | B | $14t$ | $14t+28$ | $2t$ | 2 |
| $42t+35, t \geq 0$ | A | $14t+7$ | $14t+21$ | $2t+1$ | 1 |

(e) $K_n \rightarrow G_{17}$ if and only if $n \equiv 1$ or $7 \pmod{14}$.

Remark: The case $K_n \rightarrow G_{17}$ for $n \equiv 1 \pmod{14}$ has already been solved, using a different method, in [6], furthermore, the decomposition satisfies an extra condition and is called a balanced G-design for $G = G_{17}$.

(13) (a) G_{18} :  , $[x, y, z, u, v]$.

(b) $n \equiv 0$ or $1 \pmod{7}$.

(c) - $K_8 \nrightarrow G_{18}$: We use degree argument again. The degree of a vertex in K_8 is 7 and since the degrees of vertices in G_{18} are 2 or 3, any vertex of K_8 must appear in one graph of the decomposition with degree 3 and in two others with degree 2. However, this contradicts the fact that in the four graphs of the decomposition, there are altogether 4×4 vertices of degree 3 and 4×1 vertices of degree 2.

- $K_{14} \nrightarrow G_{18}$: use similar argument as above.

- $K_{7,7,7} \rightarrow G_{18}$: $[0_1, 0_2, 0_3, 1_3, 2_2]$
 $\cup [0_1, 4_3, 1_2, 2_2, 5_1] \cup [5_2, 2_3, 0_1, 6_1, 5_3] \pmod{7}$.

- $K_{7,7,14} \rightarrow G_{18}$: A slightly different notation is used here, namely, though the elements of the first two sets of elements X_1, X_2 are still $g_i, g \in Z_7, i = 1, 2$ the third set is now assumed to be a union of two sets X_3 and X'_3 with elements g_3 and g'_3 respectively, $g \in Z_7$. The base graphs are

$$\begin{aligned} & [0_1, 0_2, 0_3, 1_3, 2_2] \cup [0_1, 2_2, 4_3, 0'_3, 1_2] \\ & \cup [0_1, 1_2, 1'_3, 4'_3, 0_2] \cup [5_2, 2_3, 0_1, 6_1, 5_3] \\ & \cup [3_2, 5'_3, 0_1, 6_1, 2'_3] \pmod{7}. \end{aligned}$$

- $K_{7,7,7,7} \rightarrow G_{18}$: $[0_1, 0_3, 0_2, 1_2, 1_4]$
 $\cup [0_1, 2_2, 3_3, 4_3, 4_4] \cup [1_3, 3_4, 0_1, 6_1, 5_2]$
 $\cup [0_3, 2_1, 3_4, 4_4, 5_2] \cup [3_2, 6_4, 0_1, 1_3, 4_2]$
 $\cup [3_4, 4_3, 5_1, 1_2, 5_4] \pmod{7}.$

- $K_7 \rightarrow G_{18}$: $V(K_7) = Z_2 \times Z_3 \cup \{\infty\}$:
 $[(0,0), (1,0), (0,1), (1,1), \infty] \pmod{(-,3)}.$

- $K_{15} \rightarrow G_{18}$: $V(K_{15}) = Z_{15}, [0,1,3,6,10] \pmod{15}.$

- $K_{22} \rightarrow G_{18}$: $V(K_{22}) = Z_2 \times Z_{11}$,
 $[(0,0), (1,6), (1,3), (1,10), (1,5)]$
 $\cup [(0,0), (0,1), (0,3), (1,1), (1,0)]$
 $\cup [(1,2), (0,4), (0,0), (0,9), (1,5)] \pmod{(-,11)}.$

- $K_{29} \rightarrow G_{18}$: $V(K_{29}) = Z_{29}, [0,1,10,13,24]$
 $\cup [0,2,8,5,12] \pmod{29}.$

- $K_{35} \rightarrow G_{18}$: $V(K_{35}) = Z_2 \times Z_{17} \cup \{\infty\}$,
 $[(0,9), (1,9), (0,0), (1,2), \infty]$
 $\cup [(1,2), (1,8), (0,0), (0,1), (0,7)]$
 $\cup [(1,4), (1,12), (0,0), (0,1), (1,14)]$
 $\cup [(0,2), (0,5), (0,1), (0,0), (1,16)]$
 $\cup [(1,5), (1,2), (1,1), (1,0), (0,12)] \pmod{(-,17)}.$

(d)

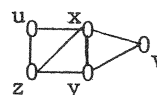
| n | $K_n \rightarrow G_{18}$ by Lemma | $n_1=n_2$ | n_3 | $K_{n_1, n_2, n_3} \rightarrow G_{18}$ by Lemma D with $r_1=r_2=r_3=7$; $r'_3=14$, a b | |
|--|--------------------------------------|-----------|----------|--|---|
| | | | | a | b |
| (PART A) | | | | | |
| $42t, t \geq 2,$ $t \neq 3, 4, 7$ | A | $14t$ | $14t$ | $2t$ | 0 |
| $42t+1, t \geq 1$ | B | $14t$ | $14t$ | $2t$ | 0 |
| $42t+7, t \geq 2,$ $t \neq 3, 4, 7$ | A | $14t$ | $14t+7$ | $2t$ | 1 |
| $42t+8, t \geq 1,$ $t \neq 2, 6, 8$ | B | $14t$ | $14t+7$ | $2t$ | 1 |
| $42t+14, t \geq 5,$ $t \neq 6, 7$ | A | $14t$ | $14t+14$ | $2t$ | 2 |
| $42t+15, t \geq 1$ | B | $14t$ | $14t+14$ | $2t$ | 2 |
| $42t+21, t \geq 0$ | A | $14t+7$ | $14t+7$ | $2t+1$ | 0 |
| $42t+22, t \geq 1,$ $t \neq 2, 6, 8$ | B | $14t+7$ | $14t+7$ | $2t+1$ | 0 |
| $42t+28, t \geq 1,$ $t \neq 2, 3, 6$ | A | $14t+7$ | $14t+14$ | $2t+1$ | 1 |
| $42t+29, t \geq 1,$ $t \neq 2, 6, 8$ | B | $14t+7$ | $14t+14$ | $2t+1$ | 1 |
| $42t+35, t \geq 1$ | A | $14t+7$ | $14t+21$ | $2t+1$ | 2 |
| $42t+36, t \geq 3,$ $t \neq 5, 6, 7, 8$ | B | $14t+7$ | $14t+21$ | $2t+1$ | 2 |
| (PART B) | | | | | |
| $42t,$ $t=4, 7$ | A | $14t-7$ | $14t+14$ | $2t-1$ | 3 |
| $42t+7,$ $t=3, 4, 7$ | A | $14t-7$ | $14t+21$ | $2t-1$ | 4 |
| $42t+8,$ $t=6, 8$ | B | $14t-7$ | $14t+21$ | $2t-1$ | 4 |
| $42t+14,$ $t=3, 4, 6, 7$ | A | $14t-7$ | $14t+28$ | $2t-1$ | 5 |
| $42t+22,$ $t=2, 6, 8$ | B | $14t$ | $14t+21$ | $2t$ | 3 |

| | | | | | |
|------------------------|---|---------|----------|--------|---|
| $42t+28,$ $t=6$ | A | $14t$ | $14t+28$ | $2t$ | 4 |
| $42t+29,$ $t=2,6,8$ | B | $14t$ | $14t+28$ | $2t$ | 4 |
| $42t+36,$ $t=5,7,8$ | B | $14t$ | $14t+35$ | $2t$ | 5 |
| $42t+36,$ $t=6$ | B | $14t-7$ | $14t+49$ | $2t-1$ | 8 |
| (PART C) | | | | | |
| $42 \times 3 = 126$ | A | 35 | 21 | 5 | 3 |
| 28 | A | 7 | 7 | 1 | 1 |
| $2+42+28=112$ | A | 35 | 7 | 5 | 1 |
| $3 \times 42+28=154$ | A | 49 | 7 | 7 | 1 |
| $42+36=78$ | B | 21 | 14 | 3 | 2 |

Finally, for $n = 49$, since $K_{49} \rightarrow \{K_7, K_{7,7,7,7,7,7,7}\}$,
 $K_7 \rightarrow G_{18}$, $K_{7,7,7} \rightarrow G_{18}$ and $K_{7,7,7,7,7,7,7} \rightarrow K_{7,7,7}$ (due
to $K_7 \rightarrow K_3$) we have $K_{49} \rightarrow G_{18}$.

(e) $K_n \rightarrow G_{18}$ if and only if $n \equiv 0$ or $1 \pmod{7}$, $n \geq 7$,
 $n \neq 8, 14$ except possibly when $n = 36, 42, 56, 92, 98, 120$.

Remark: if we had been able to show that $K_n \rightarrow G_{18}$ for
 $n = 36, 42$ and 56 , then the existence of $K_n \rightarrow G_{18}$ for
 $n = 92, 98$ or 120 would be implied and Parts (B) and (C)
of the array in (d) could have been incorporated into Part (A).

(14) (a) G_{19} :  , $[x, y, z, u, v]$.

(b) $n \equiv 0$ or $1 \pmod{7}$.

(c) - $K_8 \nrightarrow G_{19}$: again by degree argument.

- $K_{7,7,7} \rightarrow G_{19}$: $V(K_{7,7,7}) = Z_{21}, [5, 7, 0, 1, 15] \pmod{21}$.

- $K_{7,7,7,7} \rightarrow G_{19}$: $[5_2, 0_1, 1_3, 2_1, 0_3]$
 $\cup [2_3, 4_1, 0_4, 0_1, 3_4] \cup [1_2, 1_4, 6_1, 3_4, 0_1]$
 $\cup [4_2, 0_1, 3_3, 3_4, 5_4] \cup [0_2, 1_3, 3_4, 0_3, 5_4]$
 $\cup [0_3, 2_2, 3_1, 3_2, 6_4] \pmod{7}$.
- $K_{14,14,14,14} \rightarrow G_{19}$: $V(K_{14,14,14,14}) = Z_{56}$,
 $[0, 13, 27, 10, 18] \cup [0, 15, 26, 19, 9]$
 $\cup [0, 22, 25, 23, 21] \pmod{56}$.
- $K_7 \rightarrow G_{19}$: $V(K_7) = Z_2 \times Z_3 \cup \{\infty\}$,
 $[(0, 0), (1, 0), (1, 2), (0, 1), \infty] \pmod{(-, 3)}$.
- $K_{14} \rightarrow G_{19}$: $V(K_{14}) = Z_{14}$, $[8, 9, 0, 7, 10]$,
 $[0, 3, 2, 1, 12]$, $[0, 6, 5, 4, 13]$, $[11, 2, 10, 0, 4]$,
 $[8, 4, 3, 1, 6]$, $[5, 13, 8, 2, 11]$, $[11, 1, 12, 8, 6]$,
 $[5, 9, 12, 7, 1]$, $[3, 7, 10, 5, 13]$, $[9, 3, 11, 7, 6]$,
 $[1, 4, 10, 13, 7]$, $[6, 12, 2, 7, 10]$ and $[13, 4, 9, 2, 12]$.
- $K_{15} \rightarrow G_{19}$: $V(K_{15}) = Z_{15}$, $[0, 5, 7, 4, 6] \pmod{15}$.
- $K_{22} \rightarrow G_{19}$: $V(K_{22}) = Z_2 \times Z_{11}$,
 $[(0, 0), (0, 1), (1, 10), (1, 0), (0, 3)]$
 $\cup [(1, 7), (0, 4), (0, 0), (1, 4), (1, 9)]$
 $\cup [(0, 0), (1, 2), (1, 6), (0, 5), (1, 8)] \pmod{(-, 11)}$.
- $K_{29} \rightarrow G_{19}$: $V(K_{29}) = Z_{29}$, $[1, 0, 5, 8, 10]$
 $\cup [2, 0, 8, 19, 15] \pmod{29}$.
- $K_{36} \rightarrow G_{19}$: $V(K_{36}) = Z_4 \times Z_9$,
 $[(0, 0), (0, 2), (1, 5), (1, 8), (1, 0)]$
 $\cup [(1, 2), (0, 0), (0, 1), (2, 8), (3, 5)] \pmod{(4, 9)}$ and
 $[(0, 0), (0, 4), (2, 5), (2, 0), (2, 8)]$
 $\cup [(1, 0), (1, 4), (3, 5), (3, 0), (3, 8)] \pmod{(-, 9)}$.

- $K_{50} \rightarrow G_{19}$: $V(K_{50}) = Z_2 \times Z_{25}$
 $[(0,0), (1,13), (1,12), (0,12), (1,2)]$
 $\cup [(1,16), (1,14), (0,0), (0,11), (1,24)]$
 $\cup [(1,17), (1,20), (0,0), (0,16), (1,8)]$
 $\cup [(0,0), (1,22), (1,18), (0,8), (0,15)]$
 $\cup [(0,0), (0,1), (1,24), (1,19), (0,5)]$
 $\cup [(0,0), (1,21), (1,15), (0,6), (0,18)]$
 $\cup [(0,0), (1,4), (1,11), (0,3), (0,23)] \pmod{(-,25)}$.

- $K_{71} \rightarrow G_{19}$: $V(K_{71}) = Z_{71}$: $[0,17,35,10,2]$
 $\cup [3,7,0,29,30] \cup [0,14,30,31,38]$
 $\cup [21,9,0,34,41] \cup [11,5,0,19,33] \pmod{71}$.

- $K_{92} \rightarrow G_{19}$: $V(K_{92}) = \bigcup_{i=1}^{13} V_i \cup \{\infty\}$ with $|V_i| = 7$

for $i = 1, 2, \dots, 13$.

K_{92} is the edge-disjoint union of

(i) K_{29} on the elements of $\bigcup_{i=1}^4 V_i \cup \{\infty\}$

(ii) three K_{22} 's on the elements of $V_{5+i} \cup V_{6+i} \cup V_{7+i} \cup \{\infty\}$ for $i = 0, 3$, or 6 .

(iii) the graph $K_{3,3,3,4} \otimes S_7$.

If we can prove that $K_{3,3,4} \rightarrow \{K_3, K_4\}$ then
 $K_{3,3,3,4} \otimes S_7 \rightarrow \{K_{7,7,7}, K_{7,7,7,7}\}$ and since $K_{7,7,7} \rightarrow G_{19}$,
 $K_{7,7,7,7} \rightarrow G_{19}$, we have $K_{92} \rightarrow G_{19}$. Now $K_{3,3,3,4} \rightarrow \{K_3, K_4\}$
with base graphs $[0_4, 2_1, 1_2] \cup [0_4, 2_2, 1_3] \cup [0_4, 2_3, 1_1] \cup [0_4, 0_1, 0_2, 0_3] \cup [\infty, 0_1, 1_2, 2_3] \pmod{3}$.

(d)

| n | $K_n \rightarrow G_{19}$ by Lemma | $n_1=n_2=n_3$ | n_4 | $K_{n_1, n_2, n_3, n_4} \rightarrow G_{19}$ by Lemma E, with | | |
|--------------------------------------|--------------------------------------|---------------|---------------|---|-----|---|
| | | | | r | p | q |
| $21t, t \geq 1$ | A | $7t$ | 0 | 7 | t | 0 |
| $21t+1, t \geq 2$ | B | $7t$ | 0 | 7 | t | 0 |
| $21t+7, t \geq 1,$ $t \neq 2, 6$ | A | $7t$ | 7 | 7 | t | 1 |
| $21t+8, t \geq 5,$ $t \neq 7$ | B | $7(t-1)$ | 28 | 7 | t-1 | 4 |
| $21t+14, t \geq 2,$ $t \neq 2, 6$ | A | $7t$ | 14 | 7 | t | 2 |
| $21t+15, t \geq 2,$ $t \neq 2, 6$ | B | $7t$ | 14 | 7 | t | 2 |
| $21t+8,$ $t=7$ | B | 14×3 | 14×2 | 14 | 3 | 2 |
| $21t+14,$ $t=2$ | A | 14 | 14 | 14 | 1 | 1 |
| $21t+14,$ $t=6$ | A | 14×3 | 14 | 14 | 3 | 1 |
| $21t+15,$ $t=2$ | B | 14 | 14 | 14 | 1 | 1 |
| $21t+15,$ $t=6$ | B | 14×3 | 14 | 14 | 3 | 1 |

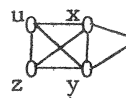
Some cases are missing from the array and are considered here.

$K_{21t+7} \rightarrow G_{19}$ for $t = 2$ and 6 . The proofs of both cases are similar, hence only one is mentioned here: as $K_7 \rightarrow K_3$, we have $K_{7,7} \rightarrow \{K_7, K_{7,7,7}\}$, both of which are decomposable into G_{19} , hence $K_{49} \rightarrow G_{19}$; for $K_{133} \rightarrow G_{19}$, the only difference in the proof is the fact that $K_{19} \rightarrow K_3$.

$K_{14,14,14} \rightarrow K_{7,7,7} \rightarrow G_{19}$, hence $K_{14,14,14} \rightarrow G_{19}$.

$K_{35} \rightarrow G_{19}$ if $K_{7,7,7,7,7} \rightarrow G_{19}$, but the latter can be proved: let $V(K_{7,7,7,7,7}) = Z_{35}$ then $[0,1,4,13,8] \cup [2,0,19,8,14] \pmod{35}$ are the base graphs.

(e) $K_n \rightarrow G_{19}$ if and only if $n \equiv 0$ or $1 \pmod{7}$, $n \neq 8$.

(15) (a) G_{20} :  $[x, y, z, u, v]$.

(b) $n \equiv 0$ or $1 \pmod{16}$.

(c) - $K_{4,4,4,4,4} \rightarrow G_{20}$: $V(K_{4,4,4,4,4}) = Z_{20}$, $[0, 1, 7, 3, 9] \pmod{20}$.

- $K_{16,16,16,16} \rightarrow G_{20}$: $V(K_{16,16,16,16}) = Z_{64}$,

$[0, 1, 15, 6, 34] \cup [13, 2, 23, 0, 40]$

$\cup [3, 45, 28, 10, 0] \pmod{64}$.

- $K_{17} \rightarrow G_{20}$: $V(K_{17}) = Z_{17}$, $[0, 5, 7, 1, 8] \pmod{17}$.

- $K_{33} \rightarrow 20$: $V(K_{33}) = Z_{33}$, $[0, 16, 14, 1, 22]$

$\cup [0, 3, 8, 12, 10] \pmod{33}$.

- $K_{49} \rightarrow G_{20}$: $V(K_{49}) = Z_{49}$, $[0, 1, 20, 24, 12]$

$\cup [0, 8, 17, 2, 22] \cup [0, 3, 16, 21, 10] \pmod{49}$.

- $K_{97} \rightarrow G_{20}$: $V(K_{97}) = Z_{97}$, $[1, 21, 36, 0, 39]$

$\cup [0, 2, 42, 72, 33] \cup [0, 4, 47, 84, 28]$

$\cup [71, 94, 0, 8, 80] \cup [16, 45, 91, 0, 26]$

$\cup [0, 90, 85, 32, 49] \pmod{97}$.

- $K_{113} \rightarrow G_{20}$: $V(K_{113}) = Z_{113}$, $[0, 1, 36, 43, 56]$

$\cup [0, 3, 16, 108, 14] \cup [0, 9, 48, 98, 49]$

$\cup [27, 31, 68, 0, 65] \cup [81, 91, 93, 0, 109]$

$\cup [47, 53, 0, 30, 99] \cup [29, 54, 0, 80, 73] \pmod{113}$.

- $K_{177} \rightarrow G_{20}$: $V(K_{177}) = Z_{177}$, $[0, 5, 3, 26, 55]$

$\cup [0, 6, 10, 52, 67] \cup [12, 20, 104, 0, 63]$

$\cup [0, 31, 24, 40, 58] \cup [48, 62, 80, 0, 87]$

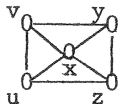
$\cup [0, 124, 160, 96, 102] \cup [0, 15, 71, 143, 69]$

$\cup [109, 142, 0, 30, 9] \cup [41, 60, 107, 0, 119]$

$$\cup [0, 120, 82, 37, 29] \cup [74, 164, 0, 63, 75] \pmod{177}.$$

(e) $K_n \rightarrow G_{20}$ for $n = 17, 33, 49, 97, 113$ and 177 .

Remark: We can prove $K_n \rightarrow G_{20}$ for $n = 16t$ or $16t+1$ if $K_n \rightarrow G_{20}$ for $t = 1, 2, 3, 6, 7, 8, 9, 11, 26$ or 27 . We have been able to solve only six of the twenty cases.

(16) (a) G_{21} :  , $[x, y, z, u, v]$. ($G_{21} \simeq K_{1,2,2}$)

(b) $n \equiv 0$ or $1 \pmod{16}$.

(c) $- K_{16} \nrightarrow G_{21}$: by degree argument.

$- K_{8,8,8} \rightarrow G_{21}$: $V(K_{8,8,8}) = Z_{24}$, $[0, 1, 5, 22, 14] \pmod{24}$.

$- K_{16,16,16} \rightarrow G_{21}$: $K_{16,16,16} \rightarrow K_{8,8,8} \rightarrow G_{21}$.

$- K_{17} \rightarrow G_{21}$: $V(K_{17}) = Z_{17}$, $[0, 1, 4, 9, 7] \pmod{17}$.

$- K_{33} \rightarrow G_{21}$: $V(K_{33}) = Z_{33}$, $[0, 2, 19, 32, 24]$

$\cup [0, 5, 23, 29, 26] \pmod{33}$.

$- K_{65} \rightarrow G_{21}$: $V(K_{65}) = Z_{65}$, $[0, 8, 35, 64, 52]$

$\cup [0, 7, 24, 63, 16] \cup [0, 6, 40, 62, 51]$

$\cup [0, 5, 28, 61, 15] \pmod{65}$.

(d) Proof of $K_{16+1} \rightarrow G_{21}$, $t \geq 1$.

| n | $K_n \rightarrow G_{21}$ by Lemma | $n_1 = n_2$ | n_3 | $K_{n_1, n_2, n_3} \rightarrow G_{21}$ by Lemma D with $r_1=r_2=r_3=16, r'_3=8$ | |
|--------------------|--------------------------------------|-------------|----------|--|---|
| | | | | a | b |
| $48t+1, t \geq 1$ | B | $16t$ | $16t$ | t | 0 |
| $48t+17, t \geq 3$ | B | $16t+16$ | $16t-16$ | t+1 | 4 |
| $48t+33, t \geq 1$ | B | $16t+16$ | $16t$ | t+1 | 2 |

We need the following decompositions for the induction used in the array.

$$- K_{16,16,8} \rightarrow G_{21}: \text{ by Lemma D with } r_1 = r_2 = 2, \\ r_3 = r'_3 = 1, a = 8 \text{ and } b = 0,$$

$$K_{113} \rightarrow G_{21}: V(K_{113}) = \bigcup_{i=1}^7 Z_{16}^i \cup \{\infty\}$$

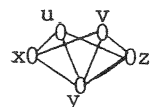
- $K_{113} \rightarrow \{K_{17}, K_{16,16,16,16,16,16,16}\}$. It has been shown that $K_{17} \rightarrow G_{21}$ and also $K_{16,16,16} \rightarrow G_{21}$, the last decomposition with $K_7 \rightarrow K_3$ implies that $K_{16,16,16,16,16,16,16} \rightarrow G_{21}$.

As for the other infinite family, $n \equiv 0 \pmod{16}$, if we can prove that $K_n \rightarrow G_{21}$ for $n = 16t$, $t = 2, 3, 4, 5$ or 7 , then we would have $K_n \rightarrow G_{21}$ for the whole family by applying Lemma A and using the same method as displayed in the array. However, of the five values, a decomposition has been obtained only for one, namely when $t = 4$.

$$- K_{64} \rightarrow G_{21}: V(K_{64}) = \bigcup_{i=1}^3 V_i \cup \{\infty\},$$

$$V_i = \{0_i, 1_i, \dots, 20_i\}, [10_1, 0_1, 2_1, 7_1, 15_2] \\ \cup [20_2, 1_1, 0_1, 4_1, 19_3] \cup [9_1, 0_1, 9_2, 2_1, 6_2] \\ \cup [19_3, 0_1, 6_1, 8_2, 14_2] \cup [1_2, \infty, 0_1, 11_2, 14_3] \\ \cup [8_2, 0_2, 1_2, 3_2, 10_3] \cup [0_2, 3_2, 4_3, 9_2, 18_3] \\ \cup [13_2, 0_1, 0_3, 13_3, 17_2] \cup [12_3, 0_1, 6_3, 7_3, 16_3] \\ \cup [7_3, 0_1, 5_3, 3_1, 14_3] \cup [12_2, 0_1, 3_3, 2_1, 10_3] \\ \cup [0_1, 3_2, 9_3, 20_3, 17_3] \pmod{21}.$$

(e) $K_n \rightarrow G_{21}$ if $n \equiv 1 \pmod{16}$ and $n = 64$, $K_{16} \nrightarrow G_{21}$.

(17) (a) G_{22} :  $[x, y, z, u, v]$. ($G_{22} \simeq K_{1,1,1,2}$).


(b) $n \equiv 0$ or $1 \pmod{9}$.

(c) - $K_n \nrightarrow G_{22}$ for $n = 9, 10, 18$: by degree argument.

$$- K_{19} \rightarrow G_{22}: V(K_{19}) = Z_{19}, [0, 1, 3, 9, 15] \pmod{19}.$$

- $K_{6,6,6,6} \rightarrow G_{22}$: $V(K_{6,6,6,6}) = Z_{24}$, $[0,3,18,1,13] \pmod{24}$.

(e) $K_n \rightarrow G_{22}$ for $n = 9,10,18$; however, $K_{19} \rightarrow G_{22}$.

(18) (a) G_{23} : 

G_{23} is the complete graph with five vertices and, as mentioned in Section II, $K_n \rightarrow G_{23}$ has already been shown to hold [5]; that is,

$K_n \rightarrow G_{23}$ if and only if $n \equiv 1$ or $5 \pmod{20}$.

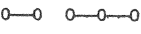
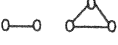
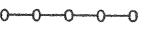
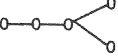

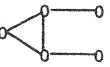

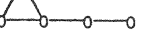

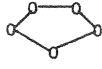
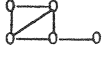
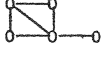


(Equivalently, a balanced incomplete block design, BIBD $(n,5,1)$ exists if and only if $n \equiv 1$ or $5 \pmod{20}$.)


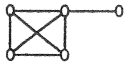






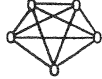
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Table 1: on $K_n + G_i$, where G_i is a graph with five vertices, none of which are isolated and $n \geq 5$.

| Graph | Necessary Condition | Result on Sufficient Condition |
|--|--|--|
| G_1 :  | $n \equiv 0 \text{ or } 1 \pmod{3}$ | same* |
| G_2 :  | $n \equiv 0 \text{ or } 1 \pmod{8}$ | same |
| G_3 :  | | same |
| G_4 :  | | same |
| G_5 :  | | same |
| G_6 :  | | $n \equiv 0 \text{ or } 1 \pmod{5}$ |
| G_7 :  | $n \equiv 0 \text{ or } 1 \pmod{5}$ | $n \equiv 0 \text{ or } 1 \pmod{5}, n \geq 6^+$ |
| G_8 :  | | $n \equiv 0 \text{ or } 1 \pmod{5}, n \geq 6^+$ |
| G_9 :  | | $n \equiv 0 \text{ or } 1 \pmod{5}, n > 6^+$ |
| G_{10} :  | $n \equiv 1 \text{ or } 5 \pmod{10}$ | same |
| G_{11} :  | $n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}$ | same |
| G_{12} :  | | same |
| G_{13} :  | | $n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}$, except possibly when $n = 24^x$ |
| G_{14} :  | | $n \equiv 0, 1, 4 \text{ or } 9 \pmod{12}, n \geq 13^+$ |

| Graph | Necessary Condition | Result on Sufficient Condition |
|--|--------------------------------------|--|
| G_{15} :  | $n \equiv 1 \text{ or } 9 \pmod{12}$ | same |
| G_{16} :  | $n \equiv 0 \text{ or } 1 \pmod{7}$ | $n \equiv 0 \text{ or } 1 \pmod{7}$, $n \neq 7, 8$ except possibly when $n = 119, 120, 147, 203, 204^x$ |
| G_{17} :  | $n \equiv 1 \text{ or } 7 \pmod{14}$ | same |
| G_{18} :  | $n \equiv 0 \text{ or } 1 \pmod{7}$ | $n \equiv 0 \text{ or } 1 \pmod{7}$, $n \neq 8, 14$ except possibly when $n = 36, 92, 56, 92, 98, 120^x$ |
| G_{19} :  | | |
| G_{20} :  | $n \equiv 0 \text{ or } 1 \pmod{16}$ | $n = 17, 33, 49, 97, 113 \text{ and } 117^x$ |
| G_{21} :  | | |
| G_{22} :  | $n \equiv 0 \text{ or } 1 \pmod{9}$ | $n \neq 9, 10, 18, n = 19^x$ |
| G_{23} :  | $n \equiv 1 \text{ or } 5 \pmod{20}$ | same |

*same: The necessary conditions are also sufficient.

+ : The necessary conditions are not always sufficient.

x : Complete solution has not been obtained.