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# Asymptotically constant-free, $p$ -robust and guaranteed a posteriori error estimates for the Helmholtz equation

T. Chaumont-Frelet (Inria project-team Nachos, CNRS, UCA LJAD)  
A. Ern (CERMICS, Inria project-team Serena)  
M. Vohralík (Inria project-team Serena, CERMICS)

Numerical methods for wave propagation with applications in electromagnetics and geophysics

Enumath 2019

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# Model problem: the Helmholtz equation

Given  $f$ , we seek  $u$  such that

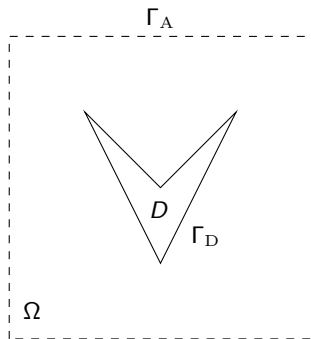
$$\begin{cases} -k^2 u - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} - iku = 0 & \text{on } \Gamma_A, \end{cases}$$

where:

$\Omega \subset \mathbb{R}^d$  is a polytopal domain,

$\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_A}$ ,

$k \geq 0$  is the wavenumber.



After integration by parts, we obtain a weak formulation:

Find  $u \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(u, v) = (f, v) \quad \forall v \in H_{\Gamma_D}^1(\Omega),$$

where

$$b(u, v) := -k^2(u, v) - ik(u, v)_{\Gamma_A} + (\nabla u, \nabla v).$$

The discrete problem consists in finding  $u_h \in V_h$  such that

$$b(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,$$

where  $V_h$  is the usual Lagrange FE discretization subspace of degree  $p \geq 1$

$$V_h := \left\{ v_h \in H_{\Gamma_D}^1(\Omega) \mid v_h|_K \in \mathcal{P}_p(K) \quad \forall K \in \mathcal{T}_h \right\},$$

and where  $\mathcal{T}_h$  is a “shape-regular” mesh of  $\Omega$ .

We would like to estimate the discretization error

$$e_h := u - u_h,$$

in some suitable norm  $\|\cdot\|_{*,\Omega}$ .

We associate with each  $K \in \mathcal{T}_h$  a **computable** quantity

$$\eta_K := \eta_K(u_h, f)$$

that we call a local “estimator”. We gather the contributions

$$\eta := \left( \sum_{K \in \mathcal{T}_h} \eta_K^2 \right)^{1/2},$$

into a “global” estimator.



A “good” a posteriori estimator is **reliable**

$$\|e_h\|_{\star, \Omega} \leq \mathbf{C}\eta,$$

and **locally efficient**

$$\eta_K \leq \mathbf{C}\|e_h\|_{\star, \omega_K},$$

where:

$\mathbf{C}$  is a “nice” constant,

$\mathbf{C}$  only depends on the “shape-regularity” and  $p$ ,

$\omega_K$  is the patch of elements sharing a vertex with  $K$ .

An error estimate is **guaranteed**, if

$$\|e_h\|_{*,\Omega} \leq \mathbf{C}\eta,$$

with a fully computable constant  $\mathbf{C}$ .

An error estimate is **constant-free** if  $\mathbf{C} = 1$ , i.e.

$$\|e_h\|_{*,\Omega} \leq \eta.$$

Finally, we say that an error estimate is  **$p$ -robust**, if

$$\eta_K \leq \mathbf{C}\|e_h\|_{*,\omega_K}$$

with a constant  $\mathbf{C}$  that is independent of  $p$ .

The goal of this talk is to (try to) achieve these properties using an “equilibrated” estimator.

- 1 The low wavenumber regime: the Poisson equation
- 2 The Helmholtz equation in the high-wavenumber regime
- 3 Numerical examples
- 4 Conclusions

- 1 The low wavenumber regime: the Poisson equation
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Let us first set  $k = 0$ . We obtain a “Poisson” problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} = 0 & \text{on } \Gamma_A. \end{cases}$$

Equivalently, we seek  $u \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(u, v) = (f, v) \quad \forall v \in H_{\Gamma_D}^1(\Omega),$$

where now

$$b(u, v) = (\nabla u, \nabla v).$$

The natural norm for the analysis is

$$|v|_{1,\Omega} = \|\nabla v\|_{0,\Omega}.$$

We call “equilibrated flux” an element  $\sigma \in \mathbf{H}(\text{div}, \Omega)$  such that

$$\begin{cases} \nabla \cdot \sigma = f & \text{in } \Omega, \\ \sigma \cdot \mathbf{n} = 0 & \text{on } \Gamma_A. \end{cases}$$

We call “equilibrated flux” an element  $\boldsymbol{\sigma} \in \mathbf{H}(\text{div}, \Omega)$  such that

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = 0 & \text{on } \Gamma_A. \end{cases}$$

Recalling that

$$\begin{cases} \nabla \cdot (-\nabla u) = f & \text{in } \Omega, \\ (-\nabla u) \cdot \mathbf{n} = 0 & \text{on } \Gamma_A, \end{cases}$$

this definition mimics the properties of  $-\nabla u$ .



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this definition mimics the properties of  $-\nabla u$ .

We have

$$\|e_h\|_{1,\Omega} = \|(-\nabla u) + \nabla u_h\|_{0,\Omega} \leq \|\boldsymbol{\sigma} + \nabla u_h\|_{0,\Omega}$$

for any equilibrated flux  $\boldsymbol{\sigma} \in \mathbf{H}(\text{div}, \Omega)$ .

This is known as the Prager-Synge theorem. How do we establish this result?

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We have the following expression for the error:

$$\begin{aligned} |e_h|_{1,\Omega}^2 &= b(e_h, e_h) \\ &= b(u, e_h) - b(u_h, e_h) \\ &= (f, e_h) - (\nabla u_h, \nabla e_h) \end{aligned}$$

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We can rewrite the first term

$$(f, e_h) = (\nabla \cdot \sigma, e_h) = -(\sigma, \nabla e_h).$$

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We can rewrite the first term

$$(f, e_h) = (\nabla \cdot \sigma, e_h) = -(\sigma, \nabla e_h).$$

This leads to

$$|e_h|_{1,\Omega}^2 = -(\nabla u_h + \sigma, \nabla e_h) \leq \|\sigma + \nabla u_h\|_{0,\Omega} |e_h|_{1,\Omega},$$

and the result follows.

Given an “equilibrated flux”  $\boldsymbol{\sigma} \in \mathbf{H}(\text{div}, \Omega)$ , we set

$$\eta_K := \|\boldsymbol{\sigma} + \nabla u_h\|_{0,K} \quad \forall K \in \mathcal{T}_h$$

and

$$\eta^2 := \sum_{K \in \mathcal{T}_h} \eta_K^2 = \|\boldsymbol{\sigma} + \nabla u_h\|_{0,\Omega}^2.$$

Then, we have

$$\|e_h\|_{1,\Omega} \leq \eta = \|\boldsymbol{\sigma} + \nabla u_h\|_{0,\Omega}.$$

Equilibrated fluxes provide constant-free error estimates!

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For the sake of simplicity, we assume that  $f|_K \in \mathcal{P}_{p-1}(K)$  for all  $K \in \mathcal{T}_h$ . All the presented result naturally extends by adding “oscillation terms”

$$\text{osc}_K(f) := \frac{h_K}{\pi} \|f - \pi_K^{p-1} f\|_{0,K}.$$

Let  $\mathbf{W}_h \subset \mathbf{H}(\text{div}, \Omega)$  denote the Raviart-Thomas space of degree  $p$ . There exists  $\boldsymbol{\sigma}_h \in \mathbf{W}_h$  such that

$$\nabla \cdot \boldsymbol{\sigma}_h = f,$$

so that we can construct “discrete” equilibrated fluxes.



We recall that we have the upper bound

$$|e_h|_{1,\Omega} \leq \eta := \|\sigma_h + \nabla u_h\|_{0,\Omega}.$$

Thus the “ideal” discrete flux would be

$$\sigma_h := \underset{\substack{\tau_h \in \mathbf{W}_h \\ \nabla \cdot \tau_h = f \text{ in } \omega_a \\ \tau_h \cdot \mathbf{n} = 0 \text{ on } \partial\omega_a \setminus \Gamma_D}}{\operatorname{argmin}} \|\tau_h + \nabla u_h\|_{0,\Omega}.$$

It is computable, through resolution of a discrete global saddle point problem.

Fortunately, it is possible to localize the computations, vertex by vertex.

# Construction through “patch” problems (hidden magic)

For each vertex  $\mathbf{a} \in \mathcal{V}_h$ , consider the following vertex-patch problem

$$\sigma_h^{\mathbf{a}} := \underset{\substack{\tau_h^{\mathbf{a}} \in \mathbf{W}_h(\omega_{\mathbf{a}}) \\ \nabla \cdot \tau_h^{\mathbf{a}} = f^{\mathbf{a}} \text{ in } \omega_{\mathbf{a}} \\ \tau_h^{\mathbf{a}} \cdot \mathbf{n} = 0 \text{ on } \partial\omega_{\mathbf{a}} \setminus \Gamma_D}}{\operatorname{argmin}} \|\tau_h^{\mathbf{a}} + \psi_{\mathbf{a}} \nabla u_h\|_{0,\Omega},$$

where  $\omega_{\mathbf{a}}$  is the “vertex patch” around  $\mathbf{a}$

$$f^{\mathbf{a}} = \psi_{\mathbf{a}} f - \nabla \psi_{\mathbf{a}} \cdot \nabla u_h,$$

and  $\psi_{\mathbf{a}}$  is the “hat function” associated with  $\mathbf{a}$ .

We obtain an equilibrated flux by setting

$$\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_h^{\mathbf{a}}.$$

This construction only involves patch-wise finite element problems.

These problems involves a small number of dofs and can solved in parallel.

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Our particular construction of  $\sigma_h$  leads to  $p$  robust estimates: we have

$$\eta_K := \|\sigma_h + \nabla u_h\|_{0,K} \leq C |e_h|_{1,\omega_K}$$

where  $C$  is independent of  $p$ .

**D. Braess, V. Pillwein, J. Schöberl**, *CMAME*, 2009:

Equilibrated residual error estimates are  $p$ -robust.

**A. Ern and M. Vohralík**, *Math. Comp.*, 2019:

Stable broken  $H^1$  and  $\mathbf{H}(\text{div})$  polynomial extensions for  $p$ -robust potential and flux reconstruction in 3D.

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We construct an equilibrated flux  $\sigma_h$  based on local finite element problems.

This flux provides a constant-free error estimate

$$|e_h|_{1,\Omega} \leq \|\sigma_h + \nabla u_h\|_{0,\Omega} =: \eta.$$

This estimator is locally efficient and  $p$ -robust since

$$\eta_K := \|\sigma_h + \nabla u_h\|_{0,K} \leq C |e_h|_{1,\omega_K},$$

with a constant  $C$  that is independent of  $p$ .

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Let us now assume that  $k$  is large. We recall that

$$b(u, v) := -k^2(u, v) - ik(u, v)_{\Gamma_A} + (\nabla u, \nabla v) \quad \forall u, v \in H_{\Gamma_D}^1(\Omega).$$

We will consider the “natural” norm

$$\|u\|_{1,k,\Omega}^2 := k^2\|u\|_{0,\Omega}^2 + k\|u\|_{0,\Gamma_A}^2 + \|\nabla u\|_{0,\Omega}^2.$$

In particular, we have

$$|b(u, v)| \leq \|u\|_{1,k,\Omega} \|v\|_{1,k,\Omega},$$

which is interesting for goal-oriented error estimation and adaptivity.

The sesquilinear form is not coercive

$$\operatorname{Re} b(\phi, \phi) = |\phi|_{1,\Omega}^2 - k^2 \|\phi\|_{0,\Omega}^2, \quad \phi \in H_{\Gamma_D}^1(\Omega).$$

Instead, it satisfies a Gårdling-like inequality:

$$\operatorname{Re} b(\phi, \phi) = \|\phi\|_{1,k,\Omega}^2 - \left( 2k^2 \|\phi\|_{0,\Omega}^2 + k \|\phi\|_{0,\Gamma_A}^2 \right), \quad \phi \in H_{\Gamma_D}^1(\Omega).$$

The coercivity loss is more severe for higher wavenumber  $k$ .



For  $\phi \in H_{\Gamma_D}^1(\Omega)$ , there exists a unique  $\mathcal{S}_\phi^* \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(w, \mathcal{S}_\phi^*) = 2k^2(w, \phi) + k(w, \phi)_{\Gamma_A}.$$

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Importantly, we have

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Recalling that

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Recalling that

$$\operatorname{Re} b(\phi, \phi) = \|\phi\|_{1,k,\Omega}^2 - \left( 2k^2\|\phi\|_{0,\Omega}^2 + k\|\phi\|_{0,\Gamma_A}^2 \right),$$

we have

$$\operatorname{Re} b(\phi, \phi + \mathcal{S}_\phi^*) = \|\phi\|_{1,k,\Omega}^2$$

for all  $\phi \in H_{\Gamma_D}^1(\Omega)$ .

We then introduce

$$\gamma_{\text{ba}} := \sup_{\phi \in H_{\Gamma_D}^1(\Omega) \setminus \{0\}} \min_{v_h \in V_h} \frac{\|\mathcal{S}_\phi^* - v_h\|_{1,k,\Omega}}{\|\phi\|_{1,k,\Omega}}.$$

It is the best constant such that

$$\min_{v_h \in V_h} \|\mathcal{S}_\phi^* - v_h\|_{1,k,\Omega} \leq \gamma_{\text{ba}} \|\phi\|_{1,k,\Omega} \quad \forall \phi \in H_{\Gamma_D}^1(\Omega).$$

It measures how well the FE scheme can approximate the “coercivity help”  $\mathcal{S}_\phi^*$ .



In the strong sense,  $\mathcal{S}_\phi^*$  is defined by

$$\begin{cases} -k^2 \mathcal{S}_\phi^* - \Delta \mathcal{S}_\phi^* & = 2k^2 \phi & \text{in } \Omega, \\ \mathcal{S}_\phi^* & = 0 & \text{on } \Gamma_D, \\ \nabla \mathcal{S}_\phi^* \cdot \mathbf{n} + ik \mathcal{S}_\phi^* & = k\phi & \text{on } \Gamma_A. \end{cases}$$

In particular  $\mathcal{S}_\phi^* \in H^{1+\varepsilon}(\Omega)$  for some  $\varepsilon > 0$ , so that

$$\gamma_{\text{ba}} \leq \mathbf{C}(\Omega, \Gamma_D, k) \left(\frac{h}{\rho}\right)^\varepsilon \rightarrow 0.$$

For non-trapping domains, we can show that

$$\gamma_{\text{ba}} \leq \mathbf{C}(\Omega, \Gamma_D) \left( \frac{kh}{\rho} + k \left( \frac{kh}{\rho} \right)^p \right).$$

**J.M. Melenk and S. Sauter**, *SIAM J. Numer. Anal.*, 2011:

Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation.

**T. Chaumont-Frelet and S. Nicaise**, *IMA J. Numer. Anal.*, 2019:

Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problems.

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We observe that  $-\nabla u \in \mathbf{H}(\operatorname{div}, \Omega)$  and

$$\begin{cases} \nabla \cdot (-\nabla u) = f + k^2 u & \text{in } \Omega, \\ (-\nabla u) \cdot \mathbf{n} = -iku & \text{on } \Gamma_A. \end{cases}$$

This motivates the following definition:

we say that  $\boldsymbol{\sigma} \in \mathbf{H}(\operatorname{div}, \Omega)$  is an equilibrated flux if

$$\begin{cases} \nabla \cdot \boldsymbol{\sigma} = f + k^2 u_h & \text{in } \Omega, \\ \boldsymbol{\sigma} \cdot \mathbf{n} = -iku_h & \text{on } \Gamma_A. \end{cases}$$

Can we relate  $\|e_h\|_{1,k,\Omega}$  and  $\|\boldsymbol{\sigma} + \nabla u_h\|_{0,\Omega}$ ?

As a comparison, the starting point was

$$|e_h|_{1,\Omega}^2 = b(e_h, e_h)$$

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Using the definition of  $\mathcal{S}_{e_h}^*$  and Galerkin orthogonality, we have

$$\|e_h\|_{1,k,\Omega}^2 = b(e_h, e_h + \mathcal{S}_{e_h}^*) = b(e_h, e_h + \mathcal{S}_{e_h}^* - v_h) = b(e_h, \psi)$$

where

$$\psi := e_h + \mathcal{S}_{e_h}^* - v_h \quad v_h := \operatorname{argmin}_{v_h \in V_h} \|\mathcal{S}_{e_h}^* - v_h\|_{1,k,\Omega}.$$

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$$\|e_h\|_{1,k,\Omega}^2 = b(e_h, e_h + \mathcal{S}_{e_h}^*) = b(e_h, e_h + \mathcal{S}_{e_h}^* - v_h) = b(e_h, \psi)$$

where

$$\psi := e_h + \mathcal{S}_{e_h}^* - v_h \quad v_h := \operatorname{argmin}_{v_h \in V_h} \|\mathcal{S}_{e_h}^* - v_h\|_{1,k,\Omega}.$$

We can show that

$$\begin{aligned} \|\psi\|_{1,k,\Omega} &\leq \|e_h\|_{1,k,\Omega} + \min_{v_h \in V_h} \|\mathcal{S}_{e_h}^* - v_h\|_{1,k,\Omega} \\ &\leq (1 + \gamma_{\text{ba}}) \|e_h\|_{1,k,\Omega}. \end{aligned}$$

# A type of Prager-Synge theorem

We have

$$\begin{aligned}\|e_h\|_{1,k,\Omega}^2 &= b(e_h, \psi) = b(u, \psi) - b(e_h, \psi) \\ &= (f, \psi) + k^2(u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi) \\ &= (f + k^2 u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi).\end{aligned}$$



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$$\begin{aligned}\|e_h\|_{1,k,\Omega}^2 &= b(e_h, \psi) = b(u, \psi) - b(e_h, \psi) \\ &= (f, \psi) + k^2(u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi) \\ &= (f + k^2 u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi).\end{aligned}$$

For any equilibrated flux  $\sigma \in \mathbf{H}(\operatorname{div}, \Omega)$ , we have

$$(f + k^2 u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} = (\nabla \cdot \sigma, \psi) - (\sigma \cdot \mathbf{n}, \psi)_{\partial\Omega} = -(\sigma, \nabla \psi).$$

We have

$$\begin{aligned}\|e_h\|_{1,k,\Omega}^2 &= b(e_h, \psi) = b(u, \psi) - b(e_h, \psi) \\ &= (f, \psi) + k^2(u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi) \\ &= (f + k^2 u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi).\end{aligned}$$

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It follows that

$$\|e_h\|_{1,k,\Omega}^2 = -(\sigma + \nabla u_h, \nabla \psi) \leq \|\sigma + \nabla u_h\|_{0,\Omega} |\psi|_{1,\Omega},$$

We have

$$\begin{aligned}\|e_h\|_{1,k,\Omega}^2 &= b(e_h, \psi) = b(u, \psi) - b(e_h, \psi) \\ &= (f, \psi) + k^2(u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi) \\ &= (f + k^2 u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} - (\nabla u_h, \nabla \psi).\end{aligned}$$

For any equilibrated flux  $\sigma \in \mathbf{H}(\operatorname{div}, \Omega)$ , we have

$$(f + k^2 u_h, \psi) + ik(u_h, \psi)_{\Gamma_A} = (\nabla \cdot \sigma, \psi) - (\sigma \cdot \mathbf{n}, \psi)_{\partial\Omega} = -(\sigma, \nabla \psi).$$

It follows that

$$\|e_h\|_{1,k,\Omega}^2 = -(\sigma + \nabla u_h, \nabla \psi) \leq \|\sigma + \nabla u_h\|_{0,\Omega} |\psi|_{1,\Omega},$$

and we can conclude since

$$|\psi|_{1,\Omega} \leq \|\psi\|_{1,k,\Omega} \leq (1 + \gamma_{\text{ba}}) \|e_h\|_{1,k,\Omega}.$$

We thus obtain that

$$\|e_h\|_{1,k,\Omega} \leq (1 + \gamma_{ba}) \|\sigma + \nabla u_h\|_{0,\Omega}$$

for any equilibrated flux  $\sigma$ .

This estimate is “asymptotically constant-free”, as  $\gamma_{ba} \rightarrow 0$  as  $(h/\rho) \rightarrow 0$ .

Guaranteed estimates can be obtained if computable upper-bounds

$$\gamma_{ba} \leq \tilde{\gamma}_{ba}$$

are provided, which is possible for non-trapping domains.

**T. Chaumont-Frelet, A. Ern, M. Vohralík**, *submitted*, 2019:

On the derivation of guaranteed and  $p$ -robust a posteriori error estimates for the Helmholtz equation.

- 1 The low wavenumber regime: the Poisson equation
- 2 The Helmholtz equation in the high-wavenumber regime
  - How do we deal with the absence of coercivity?
  - Error upper bound via equilibrated fluxes
  - Practical construction, reliability and  $p$ -robustness
  - Summary
- 3 Numerical examples
- 4 Conclusions

We follow the techniques used for the Poisson problem.

For  $\mathbf{a} \in \mathcal{V}_h$ , we introduce vertex-patch minimization problems

$$\sigma_h^{\mathbf{a}} := \underset{\substack{\tau_h^{\mathbf{a}} \in \mathbf{W}_h(\omega_{\mathbf{a}}) \\ \nabla \cdot \tau_h^{\mathbf{a}} = d^{\mathbf{a}} \text{ in } \omega_{\mathbf{a}} \\ \tau_h^{\mathbf{a}} \cdot \mathbf{n} = b^{\mathbf{a}} \text{ on } \partial\omega_{\mathbf{a}} \setminus \Gamma_D}}{\operatorname{argmin}} \|\tau_h^{\mathbf{a}} - \nabla u_h\|_{0, \omega_{\mathbf{a}}}.$$

where

$$d^{\mathbf{a}} := \psi^{\mathbf{a}}(f + k^2 u_h) - \nabla \psi^{\mathbf{a}} \cdot \nabla u_h \quad b^{\mathbf{a}} := ik \psi^{\mathbf{a}} u_h \mathbf{1}_{\Gamma_A}.$$

We obtain the global equilibrated flux by summation:

$$\sigma_h := \sum_{\mathbf{a} \in \mathcal{V}_h} \sigma_h^{\mathbf{a}}.$$

Thanks to our particular construction, we additionally have

$$\eta_K := \|\boldsymbol{\sigma}_h + \nabla u_h\|_{0,K} \leq \mathbf{C} \left( 1 + \left( \frac{kh}{p} \right)^{1/2} + \frac{kh}{p} \right) \|e_h\|_{1,k,\omega_K},$$

where the constant  $\mathbf{C}$  only depends on the shape-regularity of the mesh.

The estimator is reliable and  $p$ -robust as long as

$$\frac{kh}{p} \leq \mathbf{C}$$

which means that  $N_{\text{dofs}/\lambda} \geq \mathbf{C}$ .

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We can construct an equilibrated flux  $\sigma_h$  using local finite element problems.

The construction is very similar to the coercive case (different rhs).

The error estimates

$$\|e_h\|_{1,k,\Omega} \leq (1 + \gamma_{ba})\eta$$

is asymptotically constant-free since  $\gamma_{ba} \rightarrow 0$  as  $h/p \rightarrow 0$ .

For non-trapping domains,  $\gamma_{ba} \ll 1$  means

$$N_{\text{dofs}/\lambda} \gg C(\Omega, \Gamma_D)(1 + k^{1/p}),$$

i.e. many dofs per wavelength or large  $p$ .

Guaranteed upper bounds can be obtained if computable upper bounds

$$\gamma_{ba} \leq \tilde{\gamma}_{ba},$$

are provided. It is tough, but possible for several configurations of interest!

The  $p$ -robust local lower-bound

$$\eta_K \leq \mathbf{C} \left( 1 + \left( \frac{kh}{p} \right)^{1/2} + \frac{kh}{p} \right) \|e_h\|_{1,k,\omega_K}$$

holds.

Thus, the estimator is efficient and  $p$ -robust when we have sufficiently many dofs per wavelength:

$$\frac{kh}{p} \leq \mathbf{C} \quad \text{or} \quad N_{\text{dofs}/\lambda} \geq \mathbf{C}.$$

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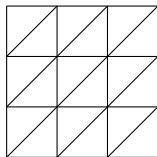
We consider the propagation of a plane wave in  $\Omega = (-1, 1)^2$

$$\begin{cases} -k^2 u - \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} - iku = g & \text{on } \Gamma_A, \end{cases}$$

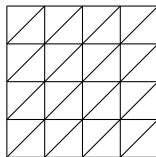
where

$$g := \nabla \xi_\theta \cdot \mathbf{n} - ik\xi_\theta \quad \xi_\theta := e^{ik\mathbf{d} \cdot \mathbf{x}}$$

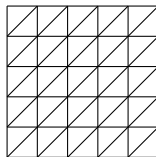
with  $\mathbf{d} := (\cos \theta, \sin \theta)$  and  $\theta = \pi/12$ . The solution is  $u = \xi_\theta$ .



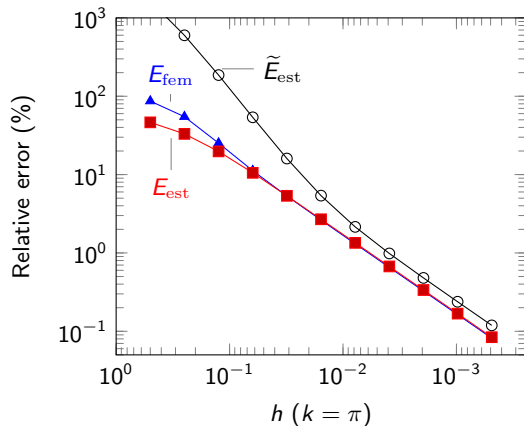
$$h = \sqrt{2} \times 2/3$$



$$h = \sqrt{2} \times 1/2$$



$$h = \sqrt{2} \times 2/5$$

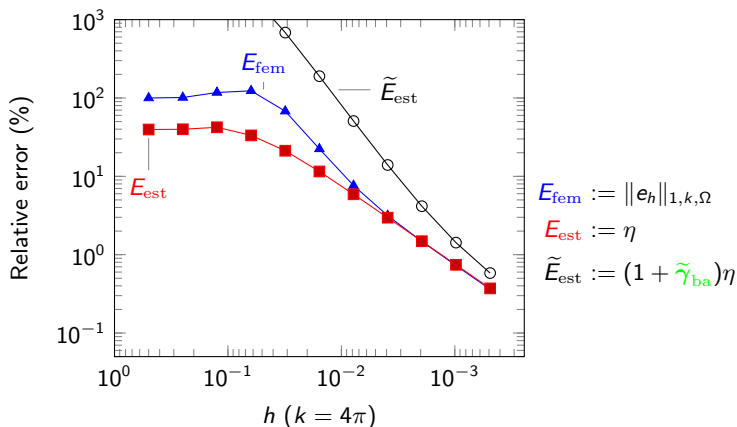


$$E_{fem} := \|e_h\|_{1,k,\Omega}$$

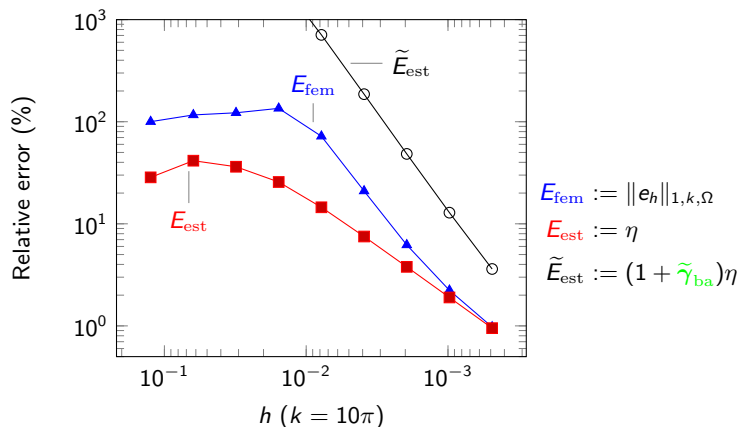
$$E_{est} := \eta$$

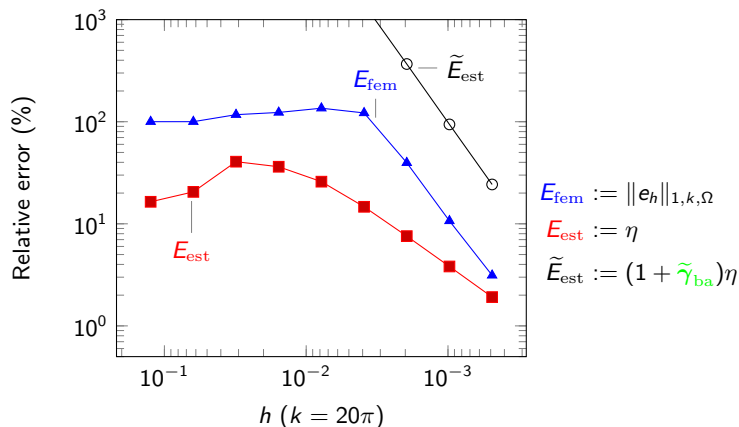
$$\tilde{E}_{est} := (1 + \tilde{\gamma}_{ba})\eta$$

# Plane wave experiment $p = 1$ and $k = 4\pi$

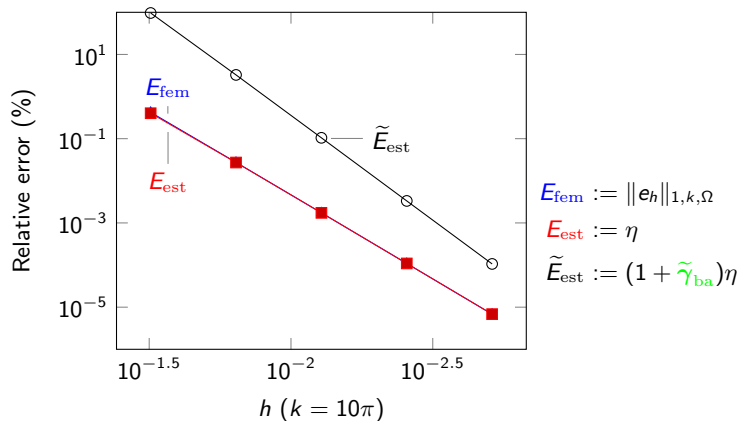


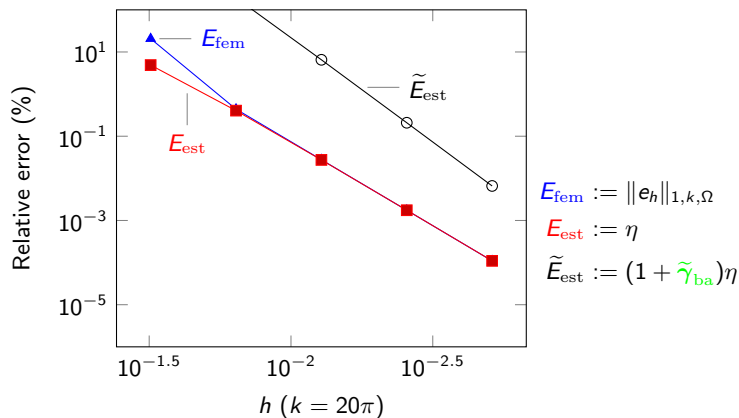


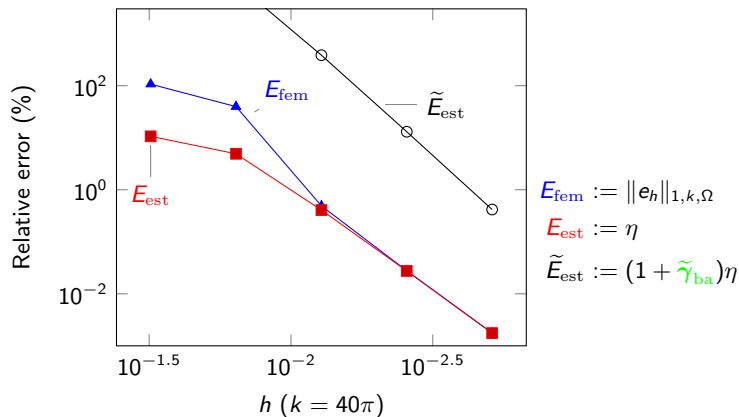


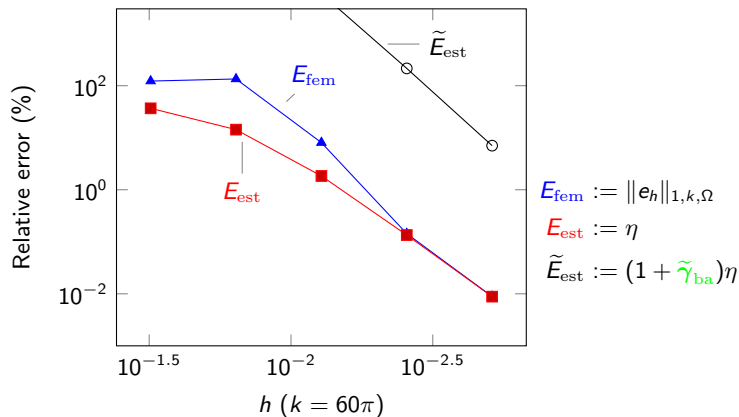


# Plane wave experiment $p = 4$ and $k = 10\pi$







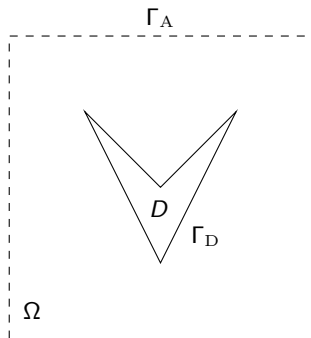


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We now consider a scattering problem

$$\begin{cases} -k^2 u - \Delta u & = 0 & \text{in } \Omega, \\ u & = 0 & \text{on } \Gamma_D, \\ \nabla u \cdot \mathbf{n} - iku & = g & \text{on } \Gamma_A, \end{cases}$$

where again  $g = \nabla \xi_\theta \cdot \mathbf{n} - ik\xi_\theta$ .

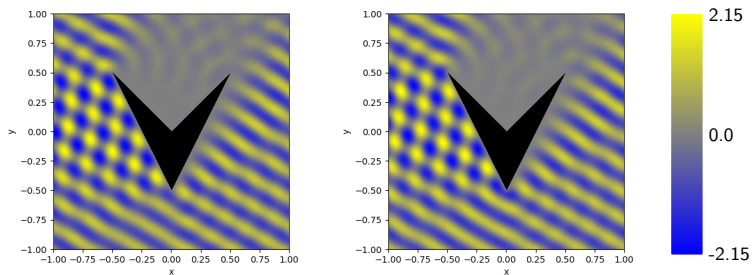


We fix the wavenumber  $k = 10\pi$  and employ  $\mathcal{P}_3$  elements.

We consider a sequence of meshes that are adaptively refined using  $\eta_K$ .

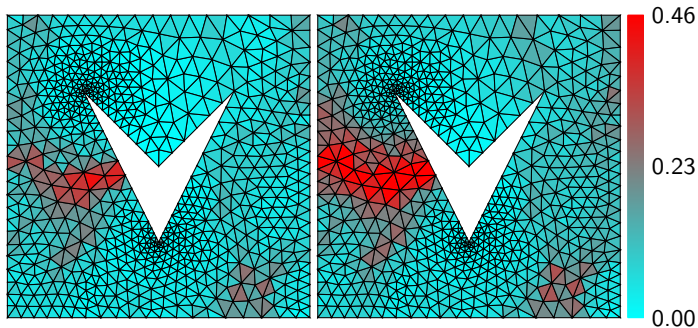


# Solution of the scattering problem

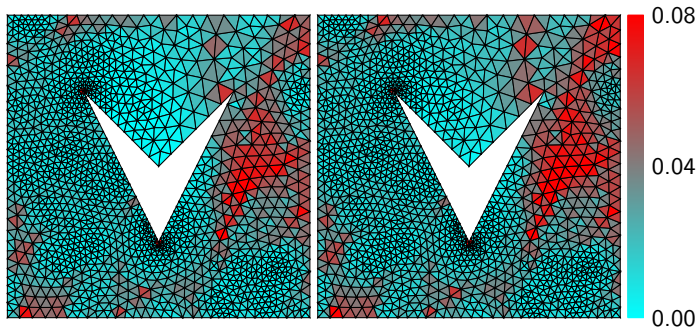


Real (left) and imaginary (right) parts of the solution

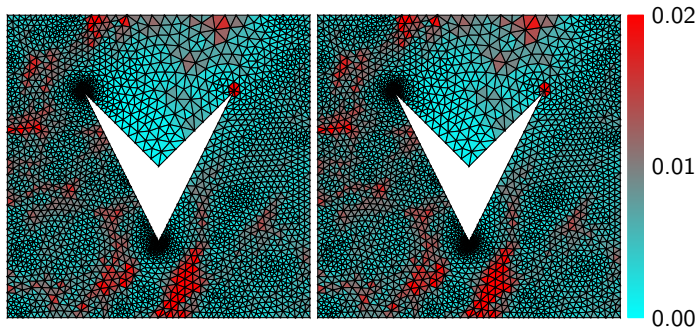
# Estimated error in mesh #1



Estimator  $\eta_K$  (left) and elementwise error  $\|e_h\|_{1,k,K}$  (right)

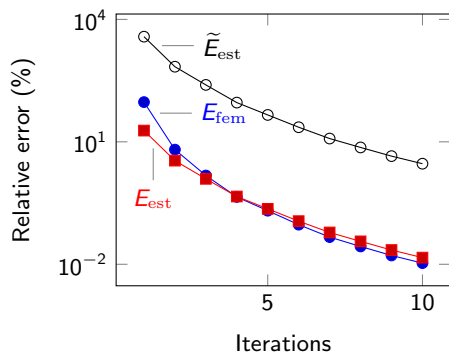


Estimator  $\eta_K$  (left) and elementwise error  $\|e_h\|_{1,k,K}$  (right)



Estimator  $\eta_K$  (left) and elementwise error  $\|e_h\|_{1,k,K}$  (right)

# Behavior of the estimator through the adaptive procedure



$$E_{fem} := \|e_h\|_{1,k,\Omega}$$

$$E_{est} := \eta$$

$$\tilde{E}_{est} := (1 + \tilde{\gamma}_{ba})\eta$$

Behaviors of the estimated and analytical errors in the adaptive procedure

- 1 The low wavenumber regime: the Poisson equation
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The upper bound

$$\|e_h\|_{1,k,\Omega} \leq (1 + \gamma_{\text{ba}})\eta$$

is “asymptotically constant-free” since  $\gamma_{\text{ba}} \rightarrow 0$  as  $h/p \rightarrow 0$ .

We “qualitatively” know the behaviour of  $\gamma_{\text{ba}}$  for non-trapping domains

$$\gamma_{\text{ba}} \leq \mathbf{C}(\Omega, \Gamma_D) \left( \frac{kh}{p} + k \left( \frac{kh}{p} \right)^p \right).$$

It is possible to obtain (coarse) guaranteed upper bounds, by deriving computable estimates  $\gamma_{\text{ba}} \leq \tilde{\gamma}_{\text{ba}}$ .

We obtain lower bounds of the form

$$\eta_{\kappa} \leq \mathbf{C} \left( 1 + \left( \frac{kh}{\rho} \right)^{1/2} + \frac{kh}{\rho} \right) \|e_h\|_{1,k,\Omega},$$

where the constant  $\mathbf{C}$  only depends on the shape-regularity parameter of  $\mathcal{T}_h$ .

The proposed estimator is efficient and  $p$ -robust as long as  $N_{\text{dofs}/\lambda} \geq \mathbf{C}$ .



## Early work in 1D, with first-order FE:

**I. Babuška, F. Ihlenburg, T. Strouboulis and S.K. Gangaraj**, *Int. J. Numer. Meth. Engrg.*, 1997:

A posteriori error estimation for finite element solutions of Helmholtz equation. Part I & II.

## Residual-based estimators in 3D with high-order FE and DG:

**W. Dörfler and S. Sauter**, *Comput. Meth. Appl. Math.*, 2013:

A posteriori error estimation for highly indefinite Helmholtz problems.

**S. Sauter and J. Zech**, *SIAM J. Numer. Anal.*, 2015:

A posteriori error estimation of  $hp - dG$  finite element methods for highly indefinite Helmholtz problems.

## Recent works using equilibrated fluxes:

**S. Congreve, J. Gedicke, I. Perugia**, *SIAM J. Sci. Comp.*, 2019:

Robust adaptive  $hp$ -discontinuous Galerkin finite element methods for the Helmholtz equation.

**T. Chaumont-Frelet, A. Ern, M. Vohralík**, *submitted*, 2019:

On the derivation of guaranteed and  $p$ -robust a posteriori error estimates for the Helmholtz equation.