Finite element approximation of Maxwell's equations with unfitted meshes for borehole simulations

Théophile Chaumont-Frelet, Serge Nicaise, David Pardo

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ICIAM - July 17, 2019

Novel computational methods for electromagnetic problems
in complex nonlinear materials

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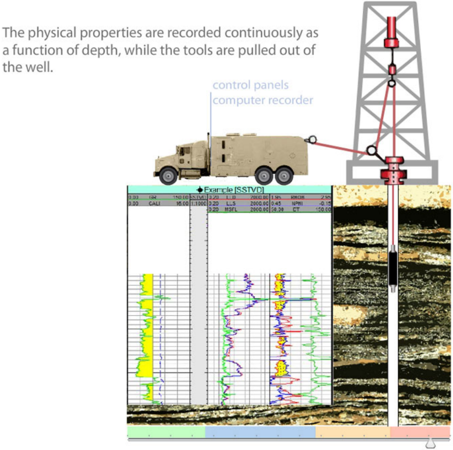
~~**Novel**~~ computational methods for electromagnetic problems
in ~~**complex**~~ ~~**non**~~linear materials

Geosteering

Guide the drilling trajectory using “real-time” magnetic measurements

Geosteering



Reproduce the measurements using finite element discretizations

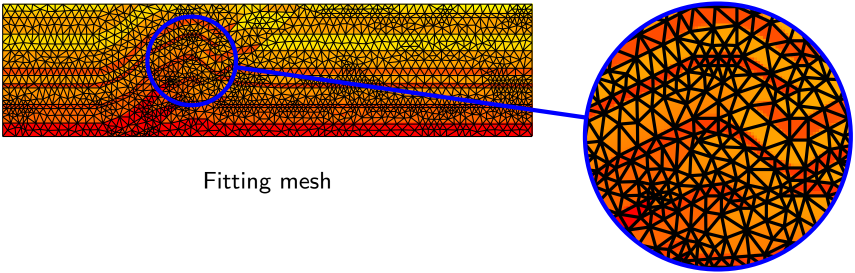
Non-fitting meshes (aka unfitted meshes)

Finite element discretizations are build on a mesh of the domain.

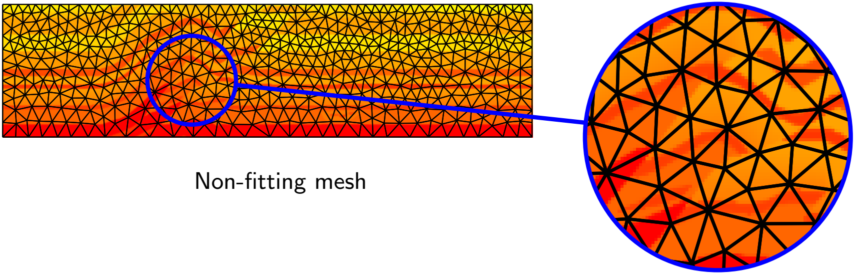
A mesh is fitting if the physical interfaces are aligned with faces.
The physical coefficients are constant (or smooth) inside each element.

A non-fitting mesh is generated independently of the physical coefficients.

Fitting mesh vs non-fitting mesh

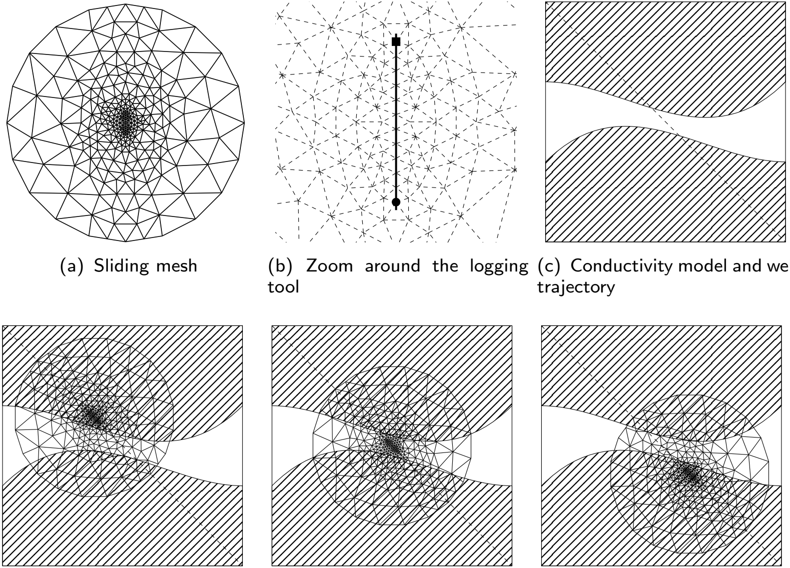


Fitting mesh



Non-fitting mesh

Non-fitting mesh for geosteering



(a) Sliding mesh

(b) Zoom around the logging tool

(c) Conductivity model and well trajectory

(d) Position 1

(e) Position 2

(f) Position 3

Pros and cons of non-fitting meshes

Main advantages of non-fitting meshes:

- simplified mesh generation (Cartesian grids)
- minimize remeshing
- larger elements

Possible drawbacks of non-fitting meshes:

- quadrature schemes to integrate the linear system coefficients
- possible accuracy loss

Pros and cons of non-fitting meshes

Main advantages of non-fitting meshes:

- simplified mesh generation (Cartesian grids)
- minimize remeshing
- larger elements

Possible drawbacks of non-fitting meshes:

- quadrature schemes to integrate the linear system coefficients
- **possible accuracy loss**

Outline

Find $\boldsymbol{E}, \boldsymbol{H} : \Omega \to \mathbb{C}^3$ such that

$$\left\{ \begin{array}{rcrcl} \nabla \times \boldsymbol{E} & - & i\omega\mu_0 \boldsymbol{H} & = & \boldsymbol{M}, \\ \nabla \times \boldsymbol{H} & + & \sigma \boldsymbol{E} & = & \boldsymbol{J}, \end{array} \right.$$

where

- $\boldsymbol{M}, \boldsymbol{J}$ are given load terms (electric and/or magnetic dipoles),
- ω is the operating frequency,
- $\mu_0 = 4\pi \times 10^{-7}$ NA^{-2} is the vacuum permeability,
- $\sigma : \Omega \to \mathbb{R}$ is the conductivity model.

The medium is **strongly dissipative** ($i\epsilon\omega + \sigma \simeq \sigma$).
The conductivity σ is piecewise constant.

The permeability μ_0 is **constant**.
We are mostly interested in **magnetic measurements**.

In the following, we focus on the case where:

- Ω is convex,
- the medium is layered ($\sigma \simeq \sigma(z)$).

Magnetic field regularity

$$\boldsymbol{H} \in \left(H^1(\Omega)\right)^3$$

Electric field regularity

$$\boldsymbol{E}|_{\Omega_l} \in \left(H^1(\Omega_l)\right)^3 \qquad \boldsymbol{E} \in \left(H^{1/2-\varepsilon}(\Omega)\right)^3$$

Approximation by polynomials (quick analysis)

Approximation of $\boldsymbol{H}$

$$\|\boldsymbol{H} - \boldsymbol{\pi}_h \boldsymbol{H}\|_{0,\Omega} \simeq h$$

Approximation of $\boldsymbol{E}$ with fitting meshes

$$\|\boldsymbol{E} - \boldsymbol{\pi}_h \boldsymbol{E}\|_{0,\Omega} \simeq h$$

Approximation of $\boldsymbol{E}$ with non-fitting meshes

$$\|\boldsymbol{E} - \boldsymbol{\pi}_h \boldsymbol{E}\|_{0,\Omega} \simeq h^{1/2-\varepsilon}$$

The key question

Standard approximation theory (Céa's Lemma) yields:

Fitting meshes

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{0,\Omega}+\|\boldsymbol{H}-\boldsymbol{H}_h\|_{0,\Omega}\simeq h$$

Non-fitting meshes

$$\|\boldsymbol{E}-\boldsymbol{E}_h\|_{0,\Omega}+\|\boldsymbol{H}-\boldsymbol{H}_h\|_{0,\Omega}\simeq h^{1/2}$$

Given that $\boldsymbol{H}$ is **globally regular**, is the convergence rate of $\boldsymbol{H}_h$ **sharp**?

Remark on the general case

In general, the regularity of $\boldsymbol{E}$ depends on σ

Regularity of $\boldsymbol{E}$

$$\boldsymbol{E} \in \left(H^{\tau(\sigma)}(\Omega)\right)^3$$

where $0 < \tau(\sigma) < 1/2$.

Interpolation errors

$$\|\boldsymbol{H} - \boldsymbol{\pi}_h \boldsymbol{H}\|_{0,\Omega} \simeq h \qquad \|\boldsymbol{E} - \boldsymbol{\pi}_h \boldsymbol{E}\|_{0,\Omega} \simeq h^{\tau(\sigma)}$$

Interpolation errors

$$\|\boldsymbol{E} - \boldsymbol{\pi}_h \boldsymbol{E}\|_{0,\Omega} + \|\boldsymbol{H} - \boldsymbol{\pi}_h \boldsymbol{H}\|_{0,\Omega} \simeq h^{\tau(\sigma)}$$

Outline

The magnetic formulation

We recast Maxwell's equation into a second-order system:

$$\left\{ \begin{array}{rcll} -i\omega\mu_0 \boldsymbol{H} + \boldsymbol{\nabla} \times \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H}\right) & = & \boldsymbol{M} & \text{in } \Omega, \\ \left(\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H}\right) \times \boldsymbol{n} & = & \boldsymbol{0} & \text{on } \partial\Omega. \end{array} \right.$$

We obtain $\boldsymbol{E} \simeq \boldsymbol{\nabla} \times \boldsymbol{H}$ by post-processing.

Variational formulation

Find $\boldsymbol{H} \in H(\mathbf{curl}, \Omega)$ such that

$$b(\boldsymbol{H}, \boldsymbol{v}) = (\boldsymbol{M}, \boldsymbol{v})$$

for all $\boldsymbol{v} \in H(\mathbf{curl}, \Omega)$.

$$b(\boldsymbol{H}, \boldsymbol{v}) = -i\omega\mu_0(\boldsymbol{H}, \boldsymbol{v}) + (\sigma^{-1} \boldsymbol{\nabla} \times \boldsymbol{H}, \boldsymbol{\nabla} \times \boldsymbol{v})$$

We consider first-order Nédélec elements on a mesh $\mathcal{T}_h = \{K\}$:

$$\boldsymbol{V}_h = \left\{ \boldsymbol{v}_h \in H(\mathbf{curl}, \Omega) \mid \boldsymbol{v}_h|_K = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b},\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3 \right\}.$$

Discrete variational formulation

Find $\boldsymbol{H}_h \in \boldsymbol{V}_h$ such that

$$b(\boldsymbol{H}_h, \boldsymbol{v}_h) = (\boldsymbol{M}, \boldsymbol{v}_h)$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

Error estimates

Assuming that $\boldsymbol{M} \in H(\mathbf{div}, \Omega)$:

Regularity

$$\boldsymbol{H} \in \left(H^1(\Omega)\right)^3 \qquad \boldsymbol{\nabla} \times \boldsymbol{H} \in \left(H^{1/2-\varepsilon}(\Omega)\right)^3$$

Interpolation error

$$\|\boldsymbol{H} - \boldsymbol{\pi}_h \boldsymbol{H}\|_{0,\Omega} \lesssim h \|\boldsymbol{M}\|_{\mathbf{div},\Omega} \quad \|\boldsymbol{\nabla} \times (\boldsymbol{H} - \boldsymbol{\pi}_h \boldsymbol{H})\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \|\boldsymbol{M}\|_{0,\Omega}$$

Error estimate (Céa's Lemma)

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{0,\Omega} + \|\boldsymbol{\nabla} \times (\boldsymbol{H} - \boldsymbol{H}_h)\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \|\boldsymbol{M}\|_{\mathbf{div},\Omega}$$

Error estimates

We have

Interpolation error

$$\|\boldsymbol{H} - \boldsymbol{\pi}_h \boldsymbol{H}\|_{0,\Omega} \lesssim h \|\boldsymbol{M}\|_{\mathbf{div},\Omega}$$

but only

Error estimate

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \|\boldsymbol{M}\|_{\mathbf{div},\Omega}$$

Can we improve this error estimate? **Yes!**

The Poisson equation

Consider the simpler problem

$$\begin{cases} u - \Delta u &= f & \text{in } \Omega, \\ \boldsymbol{\nabla} u \cdot \boldsymbol{n} &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$.

Using Lagrange elements, we are in the similar situation where:

Interpolation error

$$\|u - \pi_h u\|_{0,\Omega} \lesssim h^2 \|f\|_{0,\Omega}$$

Error estimate (Céa's lemma)

$$\|u - u_h\|_{0,\Omega} \lesssim h \|f\|_{0,\Omega}$$

Duality technique: the "Aubin-Nitsche trick"

Introduce ξ as the unique solution to

$$a(w, \xi) = (w, u - u_h), \quad \forall w \in H^1(\Omega).$$

Then, selecting $w = u - u_h$

$$\|u - u_h\|_{0,\Omega}^2 = a(u - u_h, \xi) = a(u - u_h, \xi - \pi_h \xi) \lesssim \|u - u_h\|_{1,\Omega} \|\xi - \pi_h \xi\|_{1,\Omega}.$$

We observe that ξ is solution to

$$\begin{cases} \xi - \Delta \xi &= u - u_h & \text{in } \Omega, \\ \nabla \xi \cdot \boldsymbol{n} &= 0 & \text{on } \partial\Omega, \end{cases}$$

and $\xi \in H^2(\Omega)$ with

$$|\xi|_{2,\Omega} \lesssim \|u - u_h\|_{0,\Omega}$$

since $u - u_h \in L^2(\Omega)$.

Duality technique: the "Aubin-Nitsche trick"

Hence,

$$\begin{aligned}\|u - u_h\|_{0,\Omega}^2 &\lesssim \|u - u_h\|_{1,\Omega}\|\xi - \pi_h\xi\|_{1,\Omega} \\ &\lesssim h\|u - u_h\|_{1,\Omega}|\xi|_{2,\Omega} \\ &\lesssim h\|u - u_h\|_{1,\Omega}\|u - u_h\|_{0,\Omega}.\end{aligned}$$

Error estimate

$$\|u - u_h\|_{0,\Omega} \lesssim h\|u - u_h\|_{1,\Omega} \lesssim h^2\|f\|_{0,\Omega}.$$

Direct application fails. Indeed, if we let $\boldsymbol{\xi}$ solve

$$b(\boldsymbol{w}, \boldsymbol{\xi}) = (\boldsymbol{w}, \boldsymbol{H} - \boldsymbol{H}_h), \quad \forall \boldsymbol{w} \in H(\mathbf{curl}, \Omega),$$

then $\boldsymbol{\xi} \notin \left(H^1(\Omega)\right)^3$ because $\boldsymbol{H} - \boldsymbol{H}_h \notin H(\mathbf{div}, \Omega)$.

The key idea is to introduce the Helmhotlz-Hodge decomposition

$$\boldsymbol{H} - \boldsymbol{H}_h = \boldsymbol{\phi} + \boldsymbol{\nabla} p.$$

with $\boldsymbol{\nabla} \cdot \boldsymbol{\phi} = 0$.

We employ the "Aubin-Nitsche trick" to estimate $\|\boldsymbol{\phi}\|_{0,\Omega}$.

Additional "orthogonality" arguments are used to estimate $\|\boldsymbol{\nabla} p\|_{0,\Omega}$.

Details of the proof

T. Chaumont-Frelet, S. Nicaise and D. Pardo
SIAM J. Numerical Analysis 2018

For regular solution $\tau(\sigma) > 1/2$:

L. Zhong, S. Shu, G. Wittum and J. Xu
J. Computational Mathematics 2009

General case (done in the same time):

A. Ern and J.L. Guermond
Computer & Mathematics with Applications 2018

Error estimate

Error estimate

$$\|\boldsymbol{H} - \boldsymbol{H}_h\|_{0,\Omega} \lesssim h^{1-2\varepsilon} \|\boldsymbol{M}\|_{\mathbf{div},\Omega}$$

We can remove the "-2ε" if we employ "non-Sobolev" interpolation spaces.

The convergence rate is linear, as in the case of fitting meshes.

Outline

Another option is to consider the second-order system:

$$\begin{cases} -i\omega\mu_0\sigma \boldsymbol{E} + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} &= \boldsymbol{J} \quad \text{in } \Omega, \\ \boldsymbol{E} \times \boldsymbol{n} &= \boldsymbol{0} \quad \text{on } \partial\Omega, \end{cases}$$

and post-process $\boldsymbol{H} \simeq \boldsymbol{\nabla} \times \boldsymbol{E}$.

Variational formulation

Find $\boldsymbol{E} \in H_0(\mathbf{curl}, \Omega)$ such that

$$b(\boldsymbol{E}, \boldsymbol{v}) = (\boldsymbol{J}, \boldsymbol{v})$$

for all $\boldsymbol{v} \in H_0(\mathbf{curl}, \Omega)$.

$$b(\boldsymbol{E}, \boldsymbol{v}) = -i\omega\mu_0(\sigma \boldsymbol{E}, \boldsymbol{v}) + (\boldsymbol{\nabla} \times \boldsymbol{E}, \boldsymbol{\nabla} \times \boldsymbol{v})$$

Finite element discretization

We consider first-order Nédélec elements on a mesh $\mathcal{T}_h = \{K\}$:

$$\boldsymbol{V}_h = \left\{ \boldsymbol{v}_h \in H_0(\mathbf{curl}, \Omega) \mid \boldsymbol{v}_h|_K = \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{b},\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3 \right\}.$$

Discrete variational formulation

Find $\boldsymbol{E}_h \in \boldsymbol{V}_h$ such that

$$b(\boldsymbol{E}_h, \boldsymbol{v}_h) = (\boldsymbol{J}, \boldsymbol{v}_h)$$

for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$.

Error estimate

Assuming that $\boldsymbol{J} \in H(\mathbf{div}, \Omega)$

Regularity

$$\boldsymbol{E} \in \left(H^{1/2-\varepsilon}(\Omega)\right)^3 \qquad \boldsymbol{\nabla} \times \boldsymbol{E} \in \left(H^1(\Omega)\right)^3$$

Interpolation error

$$\|\boldsymbol{E} - \boldsymbol{\pi}_h \boldsymbol{E}\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \|\boldsymbol{J}\|_{\mathbf{div},\Omega} \quad \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{\pi}_h \boldsymbol{E})\|_{0,\Omega} \lesssim h \|\boldsymbol{J}\|_{0,\Omega}$$

Error estimate (Céa's Lemma)

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{0,\Omega} + \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \|\boldsymbol{J}\|_{\mathbf{div},\Omega}$$

Error estimate

We have

Interpolation error

$$\|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{\pi}_h \boldsymbol{E})\|_{0,\Omega} \lesssim h \|\boldsymbol{J}\|_{\mathbf{div},\Omega}$$

but only

Error estimate

$$\|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \|\boldsymbol{J}\|_{\mathbf{div},\Omega}$$

Can we improve this error estimate? **Yes!**

A modified “Aubin-Nitsche trick” (Step 1)

Let $\phi \in H_0(\mathbf{curl}, \Omega)$ solve

$$b(\boldsymbol{w}, \boldsymbol{\phi}) = (\nabla \times \boldsymbol{w}, \nabla \times (\boldsymbol{E} - \boldsymbol{E}_h)), \quad \forall \boldsymbol{w} \in H_0(\mathbf{curl}, \Omega),$$

so that

$$\|\nabla \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega}^2 = b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\phi}) = b(\boldsymbol{E} - \boldsymbol{E}_h, \boldsymbol{\phi} - \boldsymbol{\pi}_h \boldsymbol{\phi}).$$

Intermediate error estimate

$$\|\nabla \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega}^2 \lesssim h^{1/2-\varepsilon} \left(\|\boldsymbol{\phi} - \boldsymbol{\pi}_h \boldsymbol{\phi}\|_{0,\Omega} + \|\nabla \times (\boldsymbol{\phi}_h - \boldsymbol{\pi}_h \boldsymbol{\phi})\|_{0,\Omega} \right) \|\boldsymbol{J}\|_{\mathbf{div},\Omega}.$$

It remains to estimate the interpolation errors in the right-hand side.

A modified "Aubin-Nitsche trick" (Step 2)

Since $\boldsymbol{\phi} \in H_0(\mathbf{curl}, \Omega)$ is solution to

$$b(\boldsymbol{w}, \boldsymbol{\phi}) = (\boldsymbol{\nabla} \times \boldsymbol{w}, \boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)), \quad \forall \boldsymbol{w} \in H_0(\mathbf{curl}, \Omega),$$

we can show that $\boldsymbol{\phi} \in \left(H^{1/2-\varepsilon}(\Omega)\right)^3$ with

$$|\boldsymbol{\phi}|_{1/2-\varepsilon,\Omega} \lesssim \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega}.$$

First interpolation error estimate

$$\|\boldsymbol{\phi} - \boldsymbol{\pi}_h \boldsymbol{\phi}\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega}$$

A modified “Aubin-Nitsche trick” (Step 3)

We have

$$\|\boldsymbol{\nabla} \times \boldsymbol{\phi}\|_{0,\Omega} \lesssim \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega},$$

but we cannot extend this result to

$$|\boldsymbol{\nabla} \times \boldsymbol{\phi}|_{s,\Omega}$$

for $s > 0$, because $\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h) \notin H(\mathbf{curl}, \Omega)$.

We cannot estimate

$$\|\boldsymbol{\nabla} \times (\boldsymbol{\phi} - \boldsymbol{\pi}_h \boldsymbol{\phi})\|_{0,\Omega}$$

directly.

A modified "Aubin-Nitsche trick" (Step 3)

We introduce $\boldsymbol{\psi} = \boldsymbol{\phi} - (\boldsymbol{E} - \boldsymbol{E}_h)$. We see that

$$\begin{aligned} b(\boldsymbol{v}, \boldsymbol{\psi}) &= b(\boldsymbol{v}, \boldsymbol{\phi}) - b(\boldsymbol{v}, \boldsymbol{E} - \boldsymbol{E}_h) \\ &= (\boldsymbol{\nabla} \times \boldsymbol{v}, \boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)) - b(\boldsymbol{v}, \boldsymbol{E} - \boldsymbol{E}_h) \\ &= -i\omega\mu_0(\boldsymbol{v}, \sigma(\boldsymbol{E} - \boldsymbol{E}_h)). \end{aligned}$$

We can show that $\boldsymbol{\nabla} \times \boldsymbol{\psi} \in (H^1(\Omega))^3$ with

$$|\boldsymbol{\nabla} \times \boldsymbol{\psi}|_{1,\Omega} \lesssim \|\boldsymbol{E} - \boldsymbol{E}_h\|_{0,\Omega}.$$

Intermediate interpolation result

$$\|\boldsymbol{\nabla} \times (\boldsymbol{\psi} - \boldsymbol{\pi}_h \boldsymbol{\psi})\|_{0,\Omega} \lesssim h \|\boldsymbol{E} - \boldsymbol{E}_h\|_{0,\Omega} \lesssim h^{3/2-\varepsilon} \|\boldsymbol{J}\|_{\mathbf{div},\Omega}.$$

A modified "Aubin-Nitsche trick" (Step 3)

We have $\boldsymbol{\phi} = \boldsymbol{E} - \boldsymbol{E}_h + \boldsymbol{\psi}$, thus

$$\boldsymbol{\phi} - \boldsymbol{\pi}_h\boldsymbol{\phi} = \boldsymbol{E} - \boldsymbol{\pi}_h\boldsymbol{E} - \underbrace{(\boldsymbol{E}_h - \boldsymbol{\pi}_h\boldsymbol{E}_h)}_{=0} + \boldsymbol{\psi} - \boldsymbol{\pi}_h\boldsymbol{\psi},$$

and

$$\|\boldsymbol{\nabla} \times (\boldsymbol{\phi} - \boldsymbol{\pi}_h\boldsymbol{\phi})\|_{0,\Omega} \lesssim \|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{\pi}_h\boldsymbol{E})\|_{0,\Omega} + \|\boldsymbol{\nabla} \times (\boldsymbol{\psi} - \boldsymbol{\pi}_h\boldsymbol{\psi})\|_{0,\Omega}.$$

Second interpolation error estimate

$$\|\boldsymbol{\nabla} \times (\boldsymbol{\phi} - \boldsymbol{\pi}_h\boldsymbol{\phi})\|_{0,\Omega} \lesssim \left(h + h^{3/2-\varepsilon}\right) \|\boldsymbol{J}\|_{\mathbf{div},\Omega}.$$

A modified "Aubin-Nitsche trick" (Step 4)

We regroup the intermediate results:

$$\|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega} \lesssim h^{1/2-\varepsilon} \left(h^{1/2-\varepsilon} + h + h^{3/2-\varepsilon} \right) \|\boldsymbol{J}\|_{\mathbf{div},\Omega}.$$

Error estimate

$$\|\boldsymbol{\nabla} \times (\boldsymbol{E} - \boldsymbol{E}_h)\|_{0,\Omega} \lesssim h^{1-2\varepsilon} \|\boldsymbol{J}\|_{\mathbf{div},\Omega}$$

We can remove the "-2ε" if we employ "non-Sobolev" interpolation spaces.

The convergence rate is linear, as in the case of fitting meshes.

Surprisingly $\boldsymbol{\nabla} \times \boldsymbol{E}_h$ converges faster than $\boldsymbol{E}_h$ in $L^2(\Omega)$-norm!

Outline

2.5D Maxwell's equations

We assume that $\sigma(x, y, z) = \sigma(x, z)$.

$$\hat{H}_a^{\xi}(x, z) = \int_{y=-\infty}^{+\infty} \boldsymbol{H}_a(x, y, z) e^{-i\xi y} dy, \quad \hat{\boldsymbol{H}}^{\xi} = (\hat{H}_x^{\xi}, \hat{H}_y^{\xi})$$

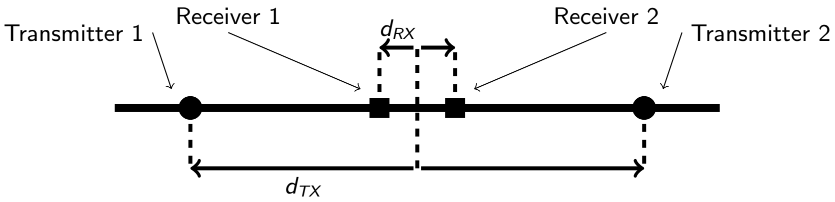
2.5D formulation

Find $(\hat{H}_y^{\xi}, \hat{\boldsymbol{H}}^{\xi}) \in H^1(\hat{\Omega}) \times H(\mathrm{curl}, \hat{\Omega})$

$$\left\{ \begin{array}{rcl} (i\omega\mu_0 + \xi^2\sigma^{-1})\hat{\boldsymbol{H}}^{\xi} + \mathbf{curl}\left(\sigma^{-1}\,\mathrm{curl}\,\hat{\boldsymbol{H}}^{\xi}\right) - i\xi\sigma^{-1}\boldsymbol{\nabla} H_y^{\xi} & = & \boldsymbol{M}^{\xi}, \\ i\omega\mu_0 H_y^{\xi} - \boldsymbol{\nabla}\cdot(\sigma^{-1}\boldsymbol{\nabla} H_y^{\xi}) + i\xi\boldsymbol{\nabla}\cdot(\sigma^{-1}\boldsymbol{H}^{\xi}) & = & M_y^{\xi}. \end{array} \right.$$

$$\boldsymbol{H}_a(x, y, z) \simeq \sum_{j=1}^{N} w_j \hat{\boldsymbol{H}}_a(x, z) e^{i\xi_j y}$$

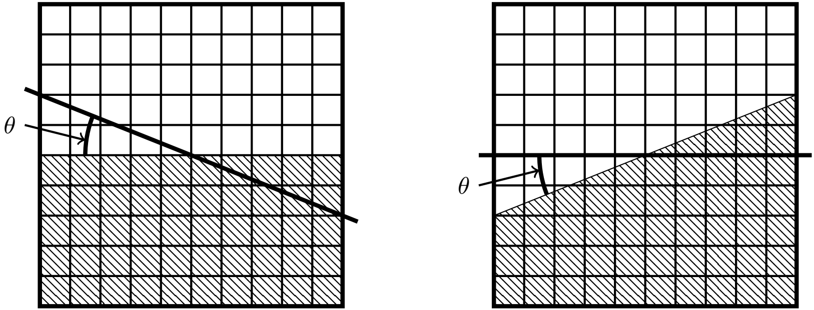
Logging tool



Logging tool
($d_{TX} = 0.6$ m, $d_{RX} = 0.1$ m)

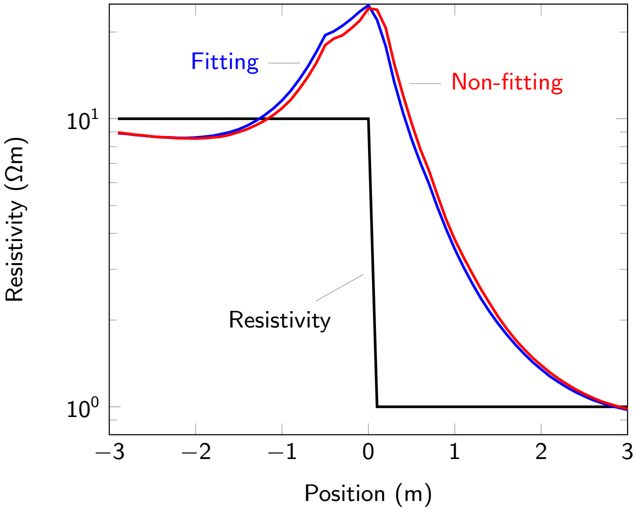
$$P_{log} = \frac{1}{2}\,\mathrm{Im}\left(\log\frac{\boldsymbol{H}_{zz}^{11}}{\boldsymbol{H}_{zz}^{12}} + \log\frac{\boldsymbol{H}_{zz}^{22}}{\boldsymbol{H}_{zz}^{21}}\right) \quad A_{log} = \frac{1}{2}\,\mathrm{Re}\left(\log\frac{\boldsymbol{H}_{zz}^{11}}{\boldsymbol{H}_{zz}^{12}} + \log\frac{\boldsymbol{H}_{zz}^{22}}{\boldsymbol{H}_{zz}^{21}}\right)$$

Benchmark

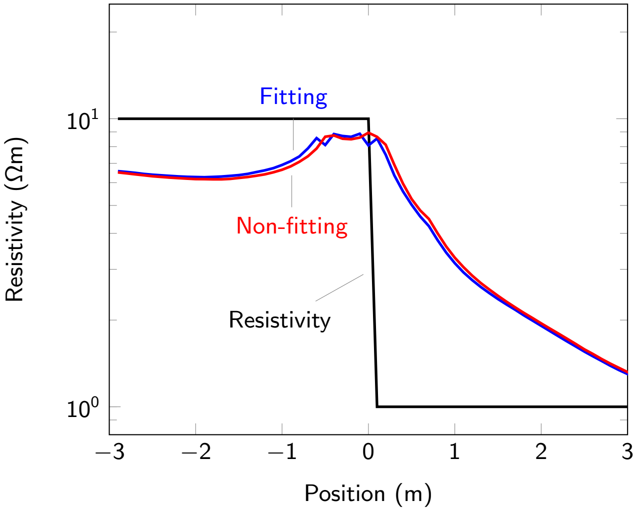


Fitting (left) and non-fitting (right) meshes

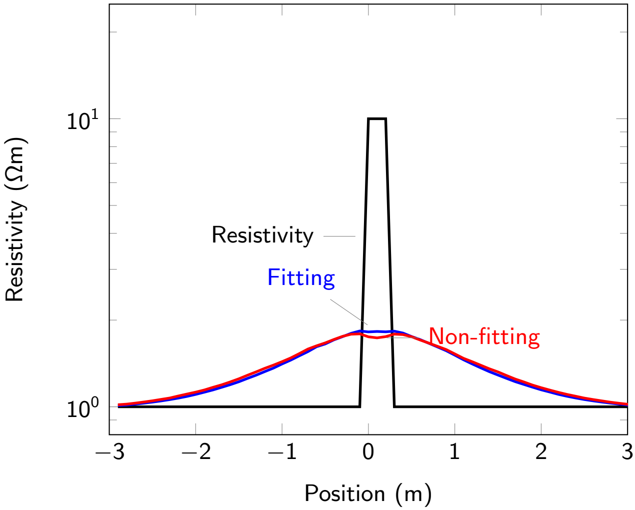
Two layers: Apparent resistivity (P-log)



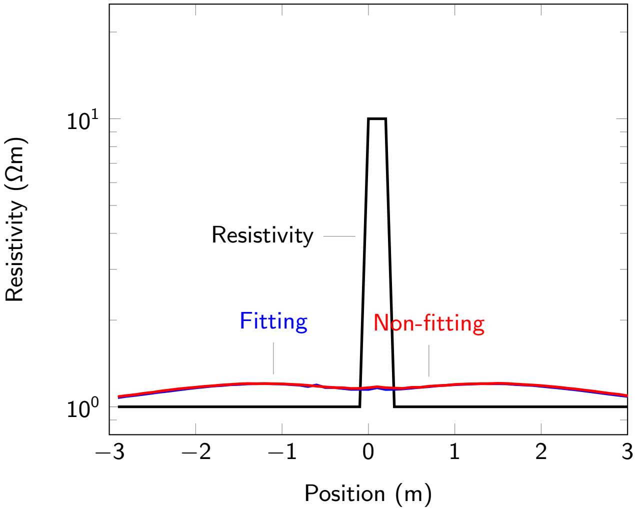
Two layers: Apparent resistivity (A-log)



Thin layer: Apparent resistivity (P-log)



Thin layer: Apparent resistivity (A-log)



Outline

Main abstract results

Assume that $\mu = \mu_0$ is constant.
Assume that σ represents a layered media.
Assume that $\boldsymbol{J}$ and $\boldsymbol{M}$ are sufficiently smooth.
Assume that the linear system coefficients are exactly integrated.

Error estimates for fitting meshes

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{L^2(\Omega)} = \mathcal{O}(h) \qquad \|\boldsymbol{H} - \boldsymbol{H}_h\|_{L^2(\Omega)} = \mathcal{O}(h)$$

Error estimates for non-fitting meshes

$$\|\boldsymbol{E} - \boldsymbol{E}_h\|_{L^2(\Omega)} = \mathcal{O}(h^{1/2}) \qquad \|\boldsymbol{H} - \boldsymbol{H}_h\|_{L^2(\Omega)} = \mathcal{O}(h)$$

Main abstract results

The use of non-fitting meshes deteriorates the convergence rate of $\boldsymbol{E}_h$.

The convergence rate of $\boldsymbol{H}_h$ is preserved.

T. Chaumont-Frelet, S. Nicaise and D. Pardo
SIAM J. Numerical Analysis 2018

Non-fitting meshes can simulate tools based on magnetic field measurements.

Main numerical results

We have tested different quadrature schemes.

Non-fitting meshes provide accurate results with an appropriate scheme.

T. Chaumont-Frelet, D. Pardo and A. Rodrìguez-Rozàs
Computational Geosciences 2018