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# Sharp stability analysis for high-order finite element discretizations of general wave propagation problems

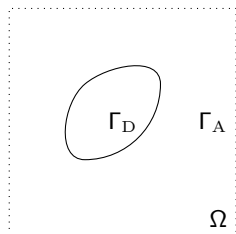
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S. Nicaise (Univ. Valenciennes, LAMAV)

ICIAM - July 17, 2019

Recent advances on computational wave propagation

Given  $f$ , we seek  $u$  such that

$$\left\{ \begin{array}{l} -\frac{\omega^2}{\kappa} u - \nabla \cdot \left( \frac{1}{\rho} \nabla u \right) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u \cdot \mathbf{n} - \frac{i\omega}{\sqrt{\kappa\rho}} u = 0 \quad \text{on } \Gamma_A, \end{array} \right.$$



where:

$\Omega$  is a (sufficiently) smooth domain,  $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_A}$ ,

$\kappa$  and  $\rho$  are (sufficiently) smooth strictly positive functions,

$\omega \geq 0$  is the angular frequency.

Classically, assuming  $f \in L^2(\Omega)$ , we recast the original problem into:

### Variational formulation

Find  $u \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(u, v) = (f, v),$$

where

$$b(u, v) = -\omega^2 \left( \frac{1}{\kappa} u, v \right) - i\omega \left\langle \frac{1}{\sqrt{\kappa\rho}} u, v \right\rangle_{\Gamma_A} + \left( \frac{1}{\rho} \nabla u, \nabla v \right).$$

We equip the space  $H_{\Gamma_D}^1(\Omega)$  with the following “energy” norm

$$\|v\|_{1,\omega,\Omega}^2 = \omega^2 \|v\|_{0,\Omega}^2 + |v|_{1,\Omega}^2,$$

that is motivated by the coefficients in  $b(\cdot, \cdot)$ .

We introduce a conforming finite element subspace

$$V_h = \left\{ v_h \in H_{\Gamma_D}^1(\Omega) \mid v_h|_K \in \mathcal{P}_p(K) \forall K \in \mathcal{T}_h \right\} \subset H_{\Gamma_D}^1(\Omega).$$

that is build on polynomials of degree  $p$ .

## Discrete solution

Find  $u_h \in V_h$  such that

$$b(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

We are interested in stability properties of FEM in the high-frequency regime.

Under which condition is the finite element method stable?

What is the influence of the polynomial degree  $p$ ?

## These questions are linked to the “approximation factor”

For  $\phi \in L^2(\Omega)$ , define  $u_\phi^*$  as the unique element of  $H_{\Gamma_D}^1(\Omega)$  such that

$$b(w, u_\phi^*) = (w, \phi) \quad \forall w \in H_{\Gamma_D}^1(\Omega).$$

### Approximation factor

$$\eta = \sup_{\phi \in L^2(\Omega) \setminus \{0\}} \frac{\|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1, \omega, \Omega}}{\|\phi\|_{0, \Omega}}$$

It is the best constant such that

$$\|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1, \omega, \Omega} \leq \eta \|\phi\|_{0, \Omega} \quad \forall \phi \in L^2(\Omega).$$

$\eta$  illustrate how well  $V_h$  can represent continuous solutions.  
Most of this talk is devoted to a careful analysis of this quantity.

- 1 Relationship between approximation factor and stability of FEM
  - Warm up exercise: the zero frequency case
  - The high-frequency case
- 2 Upper bounds for the approximation factor and stability conditions
  - Settings
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  - A regularity splitting for general media
- 3 Numerical experiments and sharpness of the main results
  - A methodology to illustrate the main results
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We set  $\omega = 0$ . Our problem becomes:

## Variational formulation

Find  $u \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(u, v) = (f, v) \quad \forall v \in H_{\Gamma_D}^1(\Omega),$$

with

$$b(u, v) = \left( \frac{1}{\rho} \nabla u, \nabla v \right).$$

This is a text-book problem, often called the Poisson problem.

The sesquilinear form is continuous and coercive

$$|b(u, v)| \lesssim \|u\|_{1, \omega, \Omega} \|v\|_{1, \omega, \Omega} \quad \operatorname{Re} b(v, v) \gtrsim \|v\|_{1, \omega, \Omega}^2.$$

We immediatly obtain quasi-optimality (Céa's lemma):

$$\begin{aligned}\|u - u_h\|_{1,\omega,\Omega}^2 &\lesssim b(u - u_h, u - u_h) && \text{(Coercivity)} \\ &= b(u - u_h, u - \mathcal{I}_h u) && \text{(Galerkin's orthogonality)} \\ &\lesssim \|u - u_h\|_{1,\omega,\Omega} \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}. && \text{(Continuity)}\end{aligned}$$

## Quasi-optimality

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

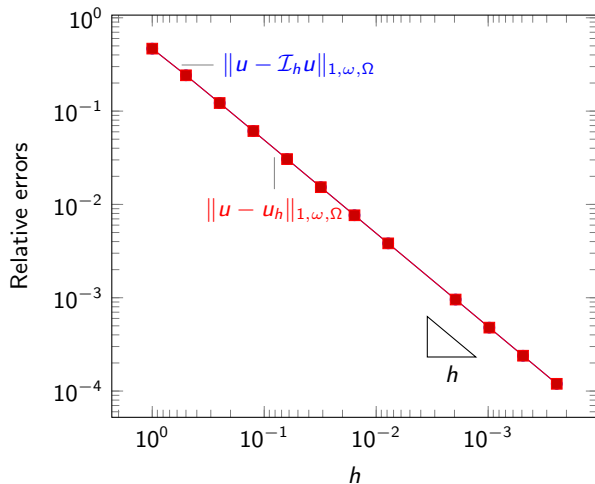
We consider the following 1D toy problem

$$\begin{cases} -u''(x) &= (\pi/2)^2 \sin(\pi x/2) \\ u(0) &= 0, \\ u'(1) &= 0, \end{cases}$$

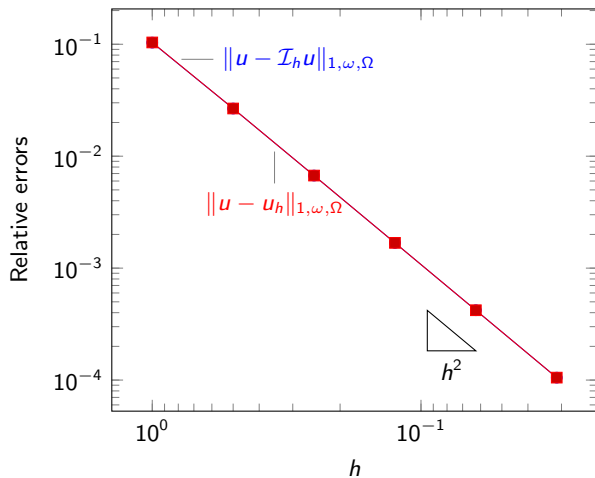
whose solution is given by

$$u(x) = \sin\left(\frac{\pi x}{2}\right).$$

# Convergence of the $\mathcal{P}_1$ FEM



# Convergence of the $\mathcal{P}_2$ FEM



The discrete solution is quasi-optimal for arbitrary meshes.

The FEM is stable on arbitrary meshes:  
up to constant, we get the best possible discrete representation.

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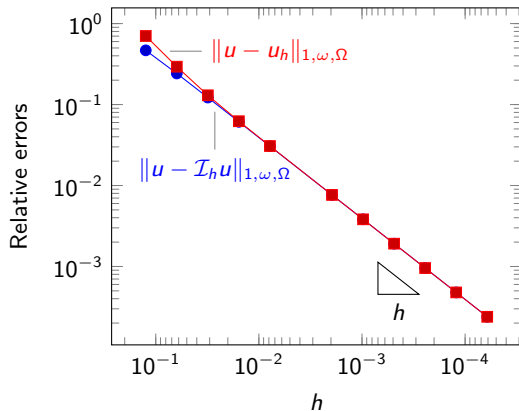
To do so, let us consider the following 1D toy problem

$$\begin{cases} -\omega^2 u(x) - u''(x) & = 1, & x \in (0, 1), \\ u(0) & = 0, \\ u'(1) - iku(1) & = 0, \end{cases}$$

whose solution is given by

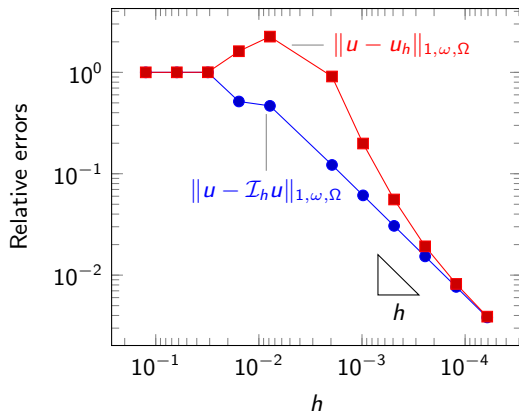
$$u(x) = \frac{1}{\omega^2} (1 - \cos(\omega x)).$$

A low frequency case with  $\mathcal{P}_1$  elements:  $\omega = 4\pi$



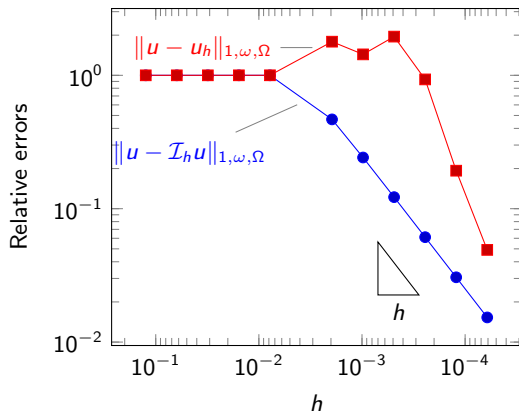
The behaviour is comparable to the coercive case.

A high frequency case with  $\mathcal{P}_1$  elements:  $\omega = 64\pi$



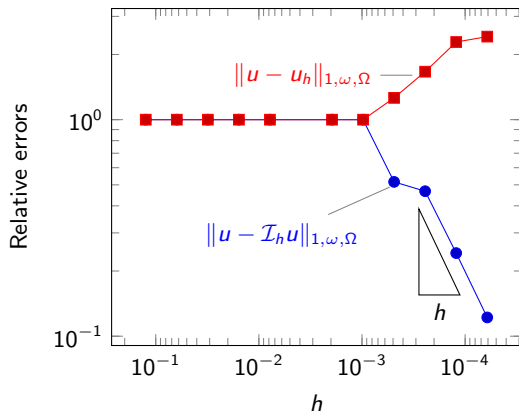
We observe a “gap” between the FEM and the interpolation error.

A high frequency case with  $\mathcal{P}_1$  elements:  $\omega = 256\pi$



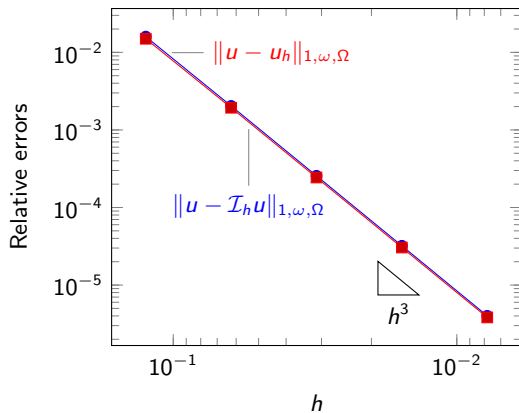
The “gap” is more important for this higher frequency.

A high frequency case with  $\mathcal{P}_1$  elements:  $\omega = 2048\pi$



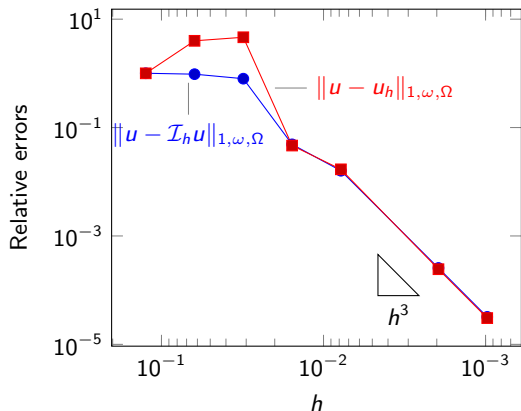
It looks like the FEM is not even converging!

A low frequency case with  $\mathcal{P}_3$  elements:  $\omega = 4\pi$



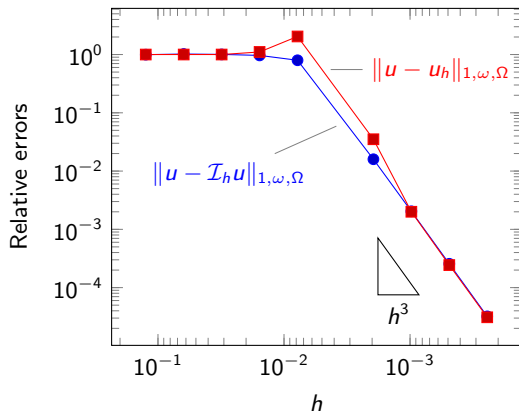
The behaviour is comparable to the coercive case.

A high frequency case with  $\mathcal{P}_3$  elements:  $\omega = 64\pi$



The FEM look stable.

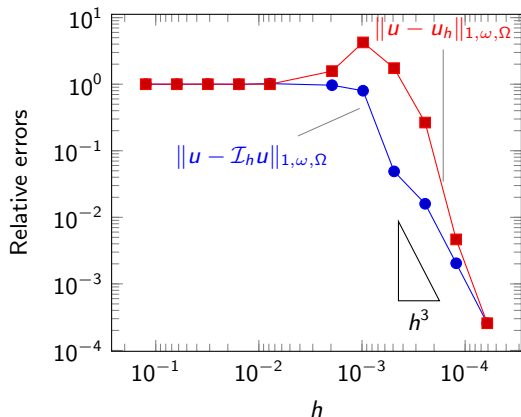
A high frequency case with  $\mathcal{P}_3$  elements:  $\omega = 256\pi$



The FEM still look stable.



A high frequency case with  $\mathcal{P}_3$  elements:  $\omega = 2048\pi$



We finally see the gap.

The FEM is not stable for all meshes:  
there is a “gap” between the interpolation and finite element errors.

This gap increases with the frequency,  
but seems less important for  $p = 3$  than  $p = 1$ .

It is often called “pollution effect” in the literature.  
The remaining of this talk is devoted to a precise analysis of this phenomenon.

The sesquilinear form reads

$$b(u, v) = -\omega^2 \left( \frac{1}{\kappa} u, v \right) - i\omega \left\langle \frac{1}{\sqrt{\kappa\rho}} u, v \right\rangle_{\Gamma_A} + \left( \frac{1}{\rho} \nabla u, \nabla v \right),$$

where  $\omega$  is large.

Hence, it is continuous

$$|b(u, v)| \lesssim \|u\|_{1, \omega, \Omega} \|v\|_{1, \omega, \Omega},$$

but not coercive

$$\operatorname{Re} b(v, v) \gtrsim \|v\|_{1, \omega, \Omega}^2 - \omega^2 \|v\|_{0, \Omega}^2.$$

We would like to employ Céa's Lemma, but we only have a Gårdling inequality:

$$|b(u - u_h, u - u_h)| \gtrsim \|u - u_h\|_{1,\omega,\Omega}^2 - \omega^2 \|u - u_h\|_{0,\Omega}^2.$$

We cannot employ Céa's Lemma directly, as we are missing coercivity.

We can tackle this problem using the "Schatz argument".

**A.H. Schatz, 1974**

*An observation concerning Ritz-Galerkin methods with indefinite bilinear forms.*

**J. Douglas, J.E. Santos, D. Sheen, L.S. Bennethum, 1993,**

*Frequency domain treatment of one-dimensional scalar waves.*

**F. Ihlenburg, I. Babuška, 1995,**

*Finite element solution of the Helmholtz equation with high wave number. Part I: The h-version of the FEM.*

Set  $\phi = u - u_h \in L^2(\Omega)$ , and define  $u_\phi^* \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(w, u_\phi^*) = (w, \phi) = (w, u - u_h) \quad \forall w \in H_{\Gamma_D}^1(\Omega),$$

Then, picking the test function  $w = u - u_h$ , we have

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 &= b(u - u_h, u_\phi^*) = b(u - u_h, u_\phi^* - \mathcal{I}_h u_\phi^*) \\ &\lesssim \|u - u_h\|_{1,\omega,\Omega} \|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1,\omega,\Omega}. \end{aligned}$$

On the other hand by definition of  $\eta$ :

$$\|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1,\omega,\Omega} \leq \eta \|\phi\|_{0,\Omega} = \eta \|u - u_h\|_{0,\Omega}.$$

## Aubin-Nitsche trick

$$\|u - u_h\|_{0,\Omega} \lesssim \eta \|u - u_h\|_{1,\omega,\Omega}.$$

We have established that  $\|u - u_h\|_{0,\Omega} \leq C\eta\|u - u_h\|_{1,\omega,\Omega}$ . Thus:

$$\begin{aligned} b(u - u_h, u - u_h) &\gtrsim \|u - u_h\|_{1,\omega,\Omega}^2 - \omega^2 \|u - u_h\|_{0,\Omega}^2 \\ &\gtrsim \left(1 - C\omega^2\eta^2\right) \|u - u_h\|_{1,\omega,\Omega}^2. \end{aligned}$$

Assuming that  $\omega\eta$  is “sufficiently small”, we have

$$b(u - u_h, u - u_h) \gtrsim \|u - u_h\|_{1,\omega,\Omega}^2,$$

and we can employ Céa's Lemma as before!

## Asymptotic quasi-optimality

Under the assumption that  $\omega\eta$  is sufficiently small, we have

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

This result is not really satisfactory, as we don't know what " $\omega\eta$  is sufficiently small" means!

It motivates careful analysis to develop sharp upper bounds on  $\eta$ . Specifically, explicit estimates in terms of  $\omega$ ,  $h$  and  $p$  are required.

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For the sake of simplicity, we consider a non-trapping problem.

## Non-trapping problem

For all  $\phi \in L^2(\Omega)$ , for all  $\omega \geq 0$  there exists a unique  $u_\phi^* \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(w, u_\phi^*) = (w, \phi) \quad \forall w \in H_{\Gamma_D}^1(\Omega).$$

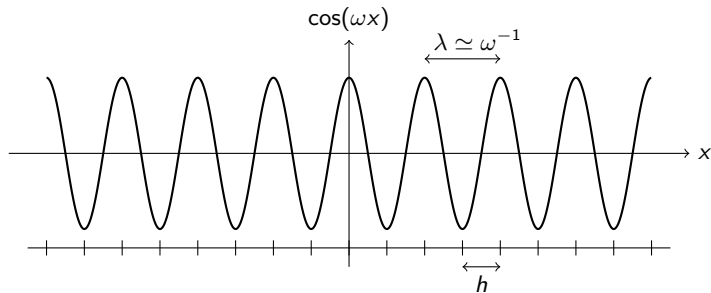
In addition, it holds that

$$\|u_\phi^*\|_{1,\omega,\Omega} \lesssim \|\phi\|_{0,\Omega},$$

uniformly in  $\omega$ .

## Some vocabulary: the number of dofs per wavelength

$N_\lambda = \lambda/h \simeq (\omega h)^{-1}$  is a measure of the number of dofs per wavelength.



In the above picture,  $\lambda = 2h$ : there are two dofs per wavelength.

In the following, we assume that  $\omega \gtrsim 1$  and  $N_\lambda \gtrsim 1$  for the sake of simplicity.

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Recall that

$$\eta = \sup_{\phi \in L^2(\Omega) \setminus \{0\}} \frac{\|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1, \omega, \Omega}}{\|\phi\|_{0, \Omega}}.$$

Consider an arbitrary  $\phi \in L^2(\Omega)$ . There exists a unique  $u_\phi^* \in H_{\Gamma_D}^1(\Omega)$  such that

$$b(w, u_\phi^*) = (w, \phi) \quad \forall w \in H_{\Gamma_D}^1(\Omega).$$

We want to estimate the interpolation error  $u_\phi^* - \mathcal{I}_h u_\phi^*$ .

To do so, we are going to show that  $u_\phi^* \in H^2(\Omega)$ .

In strong form, we have

$$\left\{ \begin{array}{l} -\frac{\omega^2}{\kappa} u_\phi^* - \nabla \cdot \left( \frac{1}{\rho} \nabla u_\phi^* \right) = \phi \quad \text{in } \Omega, \\ u_\phi^* = 0 \quad \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_\phi^* \cdot \mathbf{n} + \frac{i\omega}{\sqrt{\kappa\rho}} u_\phi^* = 0 \quad \text{on } \Gamma_A, \end{array} \right.$$

and we can write

$$\left\{ \begin{array}{l} -\nabla \cdot \left( \frac{1}{\rho} \nabla u_\phi^* \right) := F = \phi + \frac{\omega^2}{\kappa} u_\phi^* \quad \text{in } \Omega, \\ u_\phi^* = 0 \quad \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_\phi^* \cdot \mathbf{n} := G = -\frac{i\omega}{\sqrt{\kappa\rho}} u_\phi^* \quad \text{on } \Gamma_A. \end{array} \right.$$

Then, standard elliptic regularity results show that

$$\|u_\phi^*\|_{2,\Omega} \lesssim \|F\|_{0,\Omega} + \|G\|_{1/2,\Gamma_A}.$$

## A naïve approach: $H^2(\Omega)$ -norm estimate

Since the problem is non-trapping  $\|u_\phi\|_{1,\omega,\Omega} \lesssim \|\phi\|_{0,\Omega}$ , and we have

$$\|F\|_{0,\Omega} = \left\| \phi + \frac{\omega^2}{\kappa} u_\phi^* \right\|_{0,\Omega} \lesssim \|\phi\|_{0,\Omega} + \omega \|u_\phi\|_{1,\omega,\Omega} \lesssim \omega \|\phi\|_{0,\Omega},$$

and

$$\|G\|_{0,\Omega} = \left\| \frac{-i\omega}{\sqrt{\kappa\rho}} u_\phi^* \right\|_{1/2,\Gamma_A} \lesssim \omega \|u_\phi^*\|_{1/2,\Gamma_A} \lesssim \omega \|u_\phi^*\|_{1,\omega,\Omega} \lesssim \omega \|\phi\|_{0,\Omega}.$$

### $H^2(\Omega)$ -norm estimate

$$\|u_\phi^*\|_{2,\Omega} \lesssim \omega \|\phi\|_{0,\Omega}.$$

At that point, standard interpolation theory shows that

$$\|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1,\omega,\Omega} \lesssim h |u_\phi^*|_{2,\Omega} \lesssim \omega h \|\phi\|_{0,\Omega}$$

Recalling that

$$\eta = \sup_{\phi \in L^2(\Omega) \setminus \{0\}} \frac{\|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1,\omega,\Omega}}{\|\phi\|_{0,\Omega}},$$

we obtain:

Upper bound for the approximation factor

$$\eta \lesssim \omega h.$$



We have  $\eta \lesssim \omega h$ . Then, the condition “ $\omega \eta$  small”, means  $\omega^2 h \lesssim 1$ .

As  $\omega^2 h = (N_\lambda)^{-1} \omega$ , the condition

A stability condition

$$N_\lambda \gtrsim \omega$$

guaranties stability of the FEM for all frequencies.

We derived a stability condition of the form  $N_\lambda \gtrsim \omega$ .

This stability condition is valid for any polynomial degree  $p$ . However, we do not see improvements when  $p$  increases, which is not in agreement with numerical experiments.

Actually, this stability condition is optimal if  $p = 1$ , but not when  $p > 1$ .

It seems difficult to obtain an improvement with  $p$ , since we only have  $u_\phi^* \in H^2(\Omega)$  as  $\phi \in L^2(\Omega)$  only.

We cannot simply assume more regularity on  $\phi$ , since we are employing a duality argument.

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We cannot expect more than

$$\eta \simeq \mathcal{O}(h),$$

asymptotically, as the right-hand sides are in  $L^2(\Omega)$ .

The breakthrough idea is to introduce a clever splitting of the solution as

$$u_\phi^* = u_0 + \tilde{u},$$

where  $u_0 \in H^2(\Omega)$  “behaves well” at high-frequency and  $\tilde{u}$  is more regular.

So far, such splitting have only been obtained in homogeneous media.

**J.M. Melenk and S.A. Sauter, 2010**

*Convergence analysis for finite element discretizations of the Helmholtz equation with Dirichlet-to-Neumann boundary conditions.*

**J.M. Melenk and S.A. Sauter, 2011**

*Wavenumber explicit convergence analysis for Galerkin discretizations of the Helmholtz equation.*

The key idea (with a lot of hand-waving!) is the formal expansion

$$u_\phi^\star = \sum_{j \geq 0} \omega^j u_j,$$

where we hope that the iterates are independent of  $\omega$  with increasing regularity.

This expansion is purely formal.

However, plugging it in the PDE problem solved by  $u_\phi^\star$ , we obtain an actual definition for the  $u_j$ .

T. Chaumont-Frelet, S. Nicaise, 2019

*Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problems.*

The key idea (with a lot of **hand-waving!**) is the **formal expansion**

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T. Chaumont-Frelet, S. Nicaise, 2019

*Wavenumber explicit convergence analysis for finite element discretizations of general wave propagation problems.*

Identifying powers of  $\omega$ , we introduce the following definition

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho} \nabla u_0 \right) = \phi & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_0 \cdot \mathbf{n} = 0 & \text{on } \Gamma_A, \end{cases}$$

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho} \nabla u_1 \right) = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_1 \cdot \mathbf{n} = \frac{1}{\sqrt{\kappa\rho}} u_0 & \text{on } \Gamma_A, \end{cases}$$

and

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho} \nabla u_j \right) = \frac{1}{\kappa} u_{j-2} & \text{in } \Omega, \\ u_j = 0 & \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_j \cdot \mathbf{n} = \frac{1}{\sqrt{\kappa\rho}} u_{j-1} & \text{on } \Gamma_A. \end{cases}$$

We have

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho} \nabla u_0 \right) = \phi & \text{in } \Omega, \\ u_0 = 0 & \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_0 \cdot \mathbf{n} = 0 & \text{on } \Gamma_A, \end{cases}$$

so that  $u_0 \in H^2(\Omega)$  with

$$\|u_0\|_{2,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$

Similarly, since

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho} \nabla u_1 \right) = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_1 \cdot \mathbf{n} = \frac{1}{\sqrt{\kappa\rho}} u_0 & \text{on } \Gamma_A. \end{cases}$$

we have

$$\|u_1\|_{3,\Omega} \lesssim \|u_0\|_{3/2,\Gamma_A} \lesssim \|u_0\|_{2,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$



Finally, by induction, we have

$$\left\{ \begin{array}{ll} -\nabla \cdot \left( \frac{1}{\rho} \nabla u_j \right) = \frac{1}{\kappa} u_{j-2} & \text{in } \Omega, \\ u_j = 0 & \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla u_j \cdot \mathbf{n} = \frac{1}{\sqrt{\kappa \rho}} u_{j-1} & \text{on } \Gamma_A, \end{array} \right.$$

so that  $u_j \in H^{j+2}(\Omega)$  with

$$\|u_j\|_{j+2,\Omega} \lesssim \|u_j\|_{j-2,\Omega} + \|u_{j-1}\|_{j-1,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$

We have introduced a sequence  $u_j$  so that  $u_j \in H^{j+2}(\Omega)$  and

$$\|u_j\|_{j+2,\Omega} \lesssim \|\phi\|_{0,\Omega}.$$

The definition of the sequence was motivated by the formal expansion

$$u_\phi^\star = \sum_{j \geq 0} \omega^j u_j.$$

This expansion is purely formal, and actually does not converge.  
We need to thoroughly examine the residuals.

We thus introduce, for  $p \geq 1$  the residuals

$$r_p = u_\phi^* - \sum_{j=0}^{p-2} \omega^j u_j,$$

so that

$$u_\phi^* = \left( \sum_{j=0}^{p-2} \omega^j u_j \right) + r_p.$$

We need to study the regularity and frequency behaviour of  $r_p$ .

We have  $r_1 = u_\phi^*$ . Thus,  $r_1 \in H^2(\Omega)$  with

$$|r_1|_{2,\Omega} \lesssim |u_\phi^*|_{2,\Omega} \lesssim \omega \|\phi\|_{0,\Omega}.$$

Then, we see that  $r_2$  satisfies

$$\begin{cases} -\nabla \cdot \left( \frac{1}{\rho} \nabla r_2 \right) = \frac{\omega^2}{\kappa} u_\phi^* & \text{in } \Omega, \\ r_2 = 0 & \text{on } \Gamma_D, \\ \frac{1}{\rho} \nabla r_2 \cdot \mathbf{n} = -\frac{i\omega}{\sqrt{\kappa\rho}} r_1. \end{cases}$$

Hence,  $r_2 \in H^3(\Omega)$  with

$$|r_2|_{3,\Omega} \lesssim \omega^2 \|u_\phi^*\|_{1,\Omega} + \omega \|r_1\|_{2,\Omega} \lesssim \omega^2 \|\phi\|_{0,\Omega}.$$

More generally, we have

$$\left\{ \begin{array}{ll} -\nabla \cdot \left( \frac{1}{\rho} \nabla r_p \right) = \frac{\omega^2}{\kappa} r_{p-2} & \text{in } \Omega, \\ r_p = 0 & \text{on } \Gamma_A, \\ \frac{1}{\rho} \nabla r_p = -\frac{i\omega}{\sqrt{\kappa\rho}} r_{p-1} & \text{on } \Gamma_D. \end{array} \right.$$

By induction, we show that  $r_p \in H^{p+1}(\Omega)$  with

$$\|r_p\|_{p+1,\Omega} \lesssim \omega^2 \|r_{p-2}\|_{p-1,\Omega} + \omega \|r_{p-1}\|_{p,\Omega} \lesssim \omega^p \|\phi\|_{0,\Omega}.$$

We have shown that for all  $p \geq 1$ , we have

$$u_\phi^\star = \left( \sum_{j=0}^{p-1} \omega^j u_j \right) + r_p,$$

with  $u_j \in H^{j+2}(\Omega)$ ,  $r_p \in H^{p+1}(\Omega)$

$$\|u_j\|_{j+2,\Omega} \lesssim \|\phi\|_{0,\Omega} \quad \|r_p\|_{p+1,\Omega} \lesssim \omega^p \|\phi\|_{0,\Omega}.$$

The residual  $r_p$  behaves as a solution with smooth right-hand side. It is similar to the “regular part” of the Melnik-Sauter splitting.

The  $u_j$  have increasing regularity, and behave “nicely” at high frequencies.

## Upper bound for the approximation factor

We have

$$\begin{aligned}\|u_\phi^* - \mathcal{I}_h u_\phi^*\|_{1,\omega,\Omega} &\lesssim \sum_{j=0}^{p-1} \omega^j \|u_j - \mathcal{I}_h u_j\|_{1,\omega,\Omega} + \|r_p - \mathcal{I}_h r_p\|_{1,\omega,\Omega} \\ &\lesssim \sum_{j=0}^{p-1} \omega^j h^{j+1} |u_j|_{j+2,\omega,\Omega} + h^p |r_p|_{p+1,\omega,\Omega} \\ &\lesssim h \sum_{j=0}^{p-1} (\omega h)^j \|\phi\|_{0,\Omega} + \omega^p h^p \|\phi\|_{0,\Omega} \\ &\lesssim (h + \omega^p h^p) \|\phi\|_{0,\Omega}. \quad ((\omega h)^j = N_\lambda^{-j} \lesssim 1)\end{aligned}$$

Upper bound for the approximation factor

$$\eta \lesssim h + \omega^p h^p.$$

We have shown that  $\omega\eta \lesssim \omega h + \omega^{p+1}h^p \simeq (N_\lambda)^{-1} + \omega(N_\lambda)^{-1/p}$ .

It follows that for any fixed  $p$ , the FEM is stable if

### Stability condition

$$N_\lambda \gtrsim \omega^{1/p}$$

For any fixed  $p$ ,  $N_\lambda$  must be increased to preserve stability, but the increase rate is lower for larger  $p$ .

High order methods require less dofs per wavelength to achieve stability.



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- 2 Upper bounds for the approximation factor and stability conditions
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Our key results state that if  $N_\lambda \gtrsim \omega^{1/p}$ , then

$$\|u - u_h\|_{1,\omega,\Omega} \lesssim \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

To illustrate this, we would like to compute, for a fixed  $p$ , what is the minimal value  $N_\lambda^*(\omega)$  such that the FEM is stable when  $N_\lambda \geq N_\lambda^*(\omega)$ .

We consider a fixed domain and a fixed right-hand side, and solve the Helmholtz problem for several frequencies  $\omega$ .

For each frequency, we approximate the problem for different mesh sizes  $h$ , and record the convergence history of

$$\|u - u_h\|_{1,\omega,\Omega} \quad \text{and} \quad \|u - \mathcal{I}_h u\|_{1,\omega,\Omega}.$$

For each frequency, we denote by  $N_\lambda^*(\omega)$  the smallest value such that

$$\|u - u_h\|_{1,\omega,\Omega} \leq 2\|u - \mathcal{I}_h u\|_{1,\omega,\Omega} \quad \forall N_\lambda \geq N_\lambda^*(\omega),$$

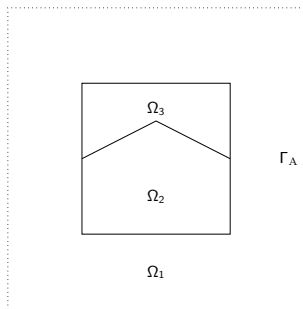
where the constant 2 is chosen arbitrarily.

This  $N_\lambda^*(\omega)$  then defines a sufficient number of dofs per wavelength to ensure stability.

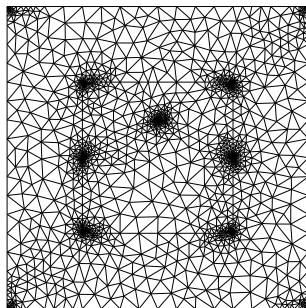
According to our main result, we shall observe that

$$N_\lambda^*(\omega) \lesssim \omega^{1/p}.$$

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Heterogeneous domain



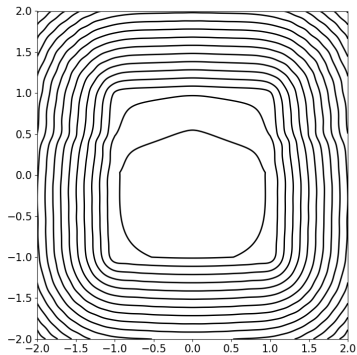
Locally refined mesh

Piecewise constant coefficients

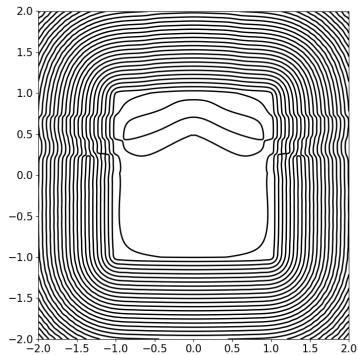
$$\begin{aligned} \kappa_1 &= 1, & \kappa_2 &= 10, & \kappa_3 &= 1000, \\ \rho_1 &= 1, & \rho_2 &= 0.5, & \rho_3 &= 0.1. \end{aligned}$$

The right-hand side is Gaussian load term centered at the origin.

# Zero-levelset curves of the real parts of solutions

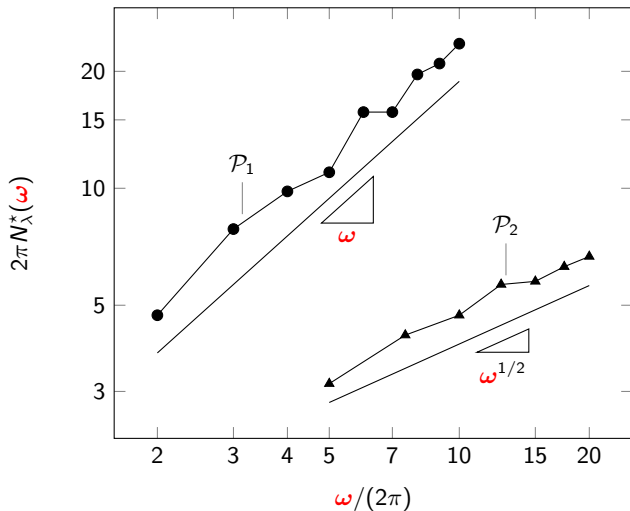


$$\omega = 10\pi$$



$$\omega = 20\pi$$

# Required dofs per wavelength $N_\lambda^*(\omega)$



We have  $N_\lambda^*(\omega) \simeq \omega^{1/p}$ , which indicates that our stability condition is sharp.



We derived a novel frequency-explicit stability condition for heterogeneous domains with smooth coefficients.

With slight modifications, we can actually take into account piecewise smooth coefficients, so that our analysis applies to a wide range of problems.

The derived stability condition is valid for any fixed polynomial degree  $p$ , and numerical experiments indicate that it is sharp.

For non-trapping domains, this stability condition is:  $N_\lambda \gtrsim \omega^{1/p}$ .

This analysis strongly encourages the use of high order FEM, as they exhibit an improved stability.