



HAL
open science

Dynamic equivalence and flatness of control systems: some results and open questions

Jean-Baptiste Pomet

► **To cite this version:**

Jean-Baptiste Pomet. Dynamic equivalence and flatness of control systems: some results and open questions. Dynamics, Control, and Geometry, Sep 2018, Warsaw, Poland. hal-02314768

HAL Id: hal-02314768

<https://inria.hal.science/hal-02314768>

Submitted on 13 Oct 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Dynamic equivalence and flatness of control systems: some results and open questions

Jean-Baptiste Pomet

INRIA Sophia Antipolis, France
Université Côte d'Azur

September 15, 2018

Slides presented at :

Dynamics, Control, and Geometry
conference in honor of Bronisław Jakubczyk
Banach Center, Warsaw

Dynamics, Control, and Geometry – Warsaw – September 12 to 15, 2018

J.-B. Pomet

0/18

Everything is C^ω .

Control systems

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, 0 < m \leq n, \quad (*)$$

studied locally around some $(\bar{x}, \bar{u}) \in \mathbb{R}^{n+m}$,
or around some germ/jet of solution $t \mapsto (\bar{x}(t), \bar{u}(t))$.

“**Geometry**”: f fiber preserving map $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{T}\mathbb{R}^n$.

$f(\mathbb{T}\mathbb{R}^n)$ is a

sub-bundle of $\mathbb{T}\mathbb{R}^n$ if $\text{Rank } \frac{\partial f}{\partial u}$ constant,

affine sub-bundle if $f(x, u) = X_0(x) + \sum_1^m u_k X_k(0)$,

distribution if $X_0 = 0$...

Underdetermined system of ODEs: the general solution
 $t \mapsto (x(t), u(t))$ depends on m arbitrary functions of time $u(\cdot)$ and
 n arbitrary constants $x(0)$. Important object:

“Set of solutions” for under-determined ODEs

\mathcal{B} = set of all (germs of) $t \mapsto (x(t), u(t))$ solution of (*).

(behavior?) (tribute to J. Willems)

“Static” equivalence

$$\begin{aligned}(\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\(\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, v \in \mathbb{R}^{m'}\end{aligned}$$

Definition (local static equivalence)

(Σ) and (Σ') are locally static equivalent at $(\bar{x}, \bar{u})/(\bar{z}, \bar{v})$ iff there is a diffeo $(\varphi, \psi) : \mathcal{O}_{(\bar{x}, \bar{u})} \rightarrow \mathcal{O}_{(\bar{z}, \bar{v})}$, of the form $(x, u) \mapsto (\varphi(x), \psi(x, u))$, (bundle isomorphism) that conjugates f to g .

If control affine & constant control rank, then

- local in state only,
- $\varphi : \mathcal{O}_{\bar{x}} \rightarrow \mathcal{O}_{\bar{z}}$ such that φ_* conjugates affine sub-bundles.

Alternative definition:

$(\varphi, \psi) : \mathcal{O}_{(\bar{x}, \bar{u})} \rightarrow \mathcal{O}_{(\bar{z}, \bar{v})}$ maps (by composition), germs of solutions of Σ to germs of solutions of Σ' , and vice versa.

Deciding static equivalence

It is feasible. Given Σ and Σ' , there is in principle a finite algorithm to write PDEs for φ, ψ and decide (differential elimination) whether there is an obstruction of system formally integrable, Cauchy-Kovalevska, analytic solution.

Geometric study. Invariants of distributions, affine sub-bundles. In non-affine case, affine geometry for submanifolds of $\mathbb{T}\mathbb{R}^n$.

A lot of litterature.

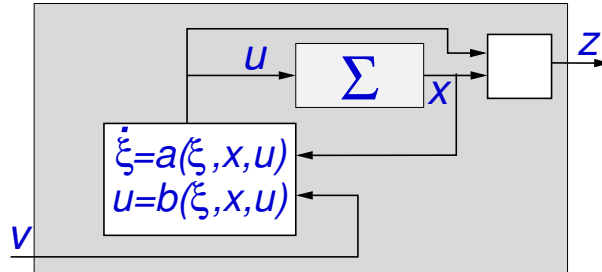
Functionnal moduli, hence equivalence is a “rare” property, or classes are very thin.

Exact linearization: characterization of static equivalence to a linear controllable system [Jakubczyk-Respondek, 80].

Dynamic feedback linearization

Since Static linearization is very restrictive, try dynamic ! (1980's)

Pre-compensator such that $v \rightarrow z$ linear controllable:



[Isidori-Moog-de Luca 86]: performing dynamic decoupling, full linearization may occur.

Decide pre-compensator, check for static feedback linearizability

[Charlet-Lévine-Marino, 91]

► Sufficient conditions.

Linear controllable systems, linearization

Linear controllable system (Σ) : $\dot{x} = Ax + Bu$.

$$\text{Rank}\{B, AB, \dots\} = n.$$

Transformation $z = Px$, $v = Kx + Qu$

with $z = (z_{1,1}, \dots, z_{1,r_1}, \dots, z_{m,1}, \dots, z_{m,r_m})$ $\sum r_k = n$

yields $z_{k,1}^{(r_k)} = v_k$, $1 \leq k \leq m$. Brunovsky canonical form.

The general solution is uniquely defined by m arbitrary functions of time $z_{1,1}, z_{2,1}, \dots, z_{m,1}$ (and no initial conditions).

[Fliess-Lévine-Martin-Rouchon 91]: system $\dot{x} = f(x, u)$ is “**flat**”

(at $\bar{x}, \bar{u}, \dots, \bar{u}^{(j)}$) iff there is a formula giving the general solution a function of arbitrary y_1, \dots, y_m and time-derivatives **and** y_1, \dots, y_m may also be recovered from x, u and derivatives.

(y_1, \dots, y_m) is called a **flat output**.

This yields the dynamic precompensator.

Equivalence

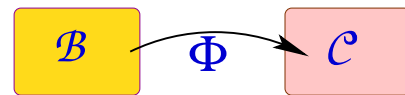
[É. Cartan, “sur l'équivalence absolue...”, 1914]:

La première idée qui vient à l'esprit, et qu'il s'agira de préciser, est la suivante : *deux systèmes seront dits « absolument équivalents » lorsqu'on pourra établir une correspondance univoque (au moins dans un champ fonctionnel suffisamment petit) entre les solutions de ces deux systèmes.*

$$\begin{aligned}
 (\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad \mathcal{B} = \{\text{solutions}\} \\
 (\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, \quad v \in \mathbb{R}^{m'}, \quad \mathcal{C} = \{\text{solutions}\}
 \end{aligned}$$

Definition

Two systems are “**equivalent**” iff their (germs of) solutions are in **one-to-one correspondence**.



The nature of Φ matters a lot !!

Dynamic Feedback Transformations

$$\begin{array}{ccc}
 \Sigma & & \Sigma' \\
 \dot{x} = f(x, u) & \begin{array}{c} \left(\begin{array}{c} x \\ u \\ \dot{u} \\ \ddot{u} \\ \vdots \end{array} \right) & \begin{array}{c} (z, v) = \phi(x, u, \dots, u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x, u) = \psi(z, v, \dots, v^{(K')}) \end{array} & \left(\begin{array}{c} z \\ v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{array} \right) \\
 & & & & \dot{z} = g(z, v)
 \end{array}$$

(Local) dynamic equivalence: $\exists \varphi : \mathcal{O}_{(\bar{x}, \bar{u}, \bar{\dot{u}}, \dots, \bar{u}^{(j)})} \rightarrow \mathbb{R}^{n'} \times \mathbb{R}^{m'}$,
 $\psi : \mathcal{O}_{(\bar{z}, \bar{v}, \bar{\dot{v}}, \dots, \bar{v}^{(j')})} \rightarrow \mathbb{R}^n \times \mathbb{R}^m$ such that the above induces a univocal transformation on solutions.

Flatness

Flatness: Σ is flat if this holds with Σ' trivial:

$$\Sigma \quad \begin{pmatrix} x \\ u \\ \dot{u} \\ \vdots \end{pmatrix} \quad \begin{array}{c} v = \phi(x, u, \dots, u^{(K)}) \\ \longrightarrow \\ \longleftarrow \\ (x, u) = \psi(v, \dots, v^{(K')}) \end{array} \quad \begin{pmatrix} v \\ \dot{v} \\ \ddot{v} \\ \vdots \end{pmatrix} \quad \Sigma' \quad \text{no relation}$$

Initially, [Fliess & al] stated this in terms of differential fields extensions. Obviously more restrictive (ϕ, ψ should be algebraic).

Equivalence can however be translated into isomorphism of differential algebras or conjugation of a vector field on a (infinite) jet manifold.

Further characterizations

Differential algebra: $\mathcal{A}_\Sigma = \{(\text{germs of}) C^\omega \text{ functions of a finite number of variables among } x, u, \dot{u}, \dots\}$,

dérivation: $F = f(x, u) \frac{\partial}{\partial x} + \dot{u} \frac{\partial}{\partial u} + \ddot{u} \frac{\partial}{\partial \dot{u}} + \dots$

[Jakubczyk 1993]: Σ is equivalent to Σ' if and only \mathcal{A}_Σ and $\mathcal{A}_{\Sigma'}$ are isomorphic diff. algebras.

[Fliess & al.], [JBP] 1992: it also translates into a diffeomorphism between “manifolds” where coordinates are x, u, \dot{u}, \dots and z, v, \dot{v}, \dots , that conjugates F to G .

Similar to Lie-Bäcklund transformations.

Checkable conditions ?

How to decide flatness of a system Σ ?

or equivalence of Σ to Σ' ?

Invariants ?

- ① In principle, if K and K' are fixed, one may decide upon existence of ϕ and ψ , that depend on a fixed number of variables. (see static feedback)
- ② ... **but no known a priori bound on K, K' !!**
- ③ However, many physical systems are flat and this is useful.
- ④ A lot of current work on all possible choices of flat outputs (they are far from unique), respecting symmetries, etc... see [Respondek, Nicolau et al], [Murray et al.], [Rouchon, Martin et al.] ...

Checkable conditions ?

Difficult point is: necessary conditions.

If no ϕ, ψ for some K , why not for $K + 1$?

Invariants:

- ① m (number of inputs) is an invariant of dynamic equivalence.
- ② If $m = 1$, dynamic equivalence is static equivalence. [Charlet et al., 1991], [JBP 1993].
- ③ Singular curves are an obstruction to flatness: if $t \mapsto (x(t), u(t))$ is singular, flatness fails at any truncated jet $(x(0), u(0), \dot{u}(0), \dots, u^{(j)}(0))$ [common knowledge ?]
- ④ Ruled manifold criterium

Ruled systems

Regularity assumption

Σ and Σ' define smooth (C^ω) sub-bundles of $T\mathbb{R}^n$ and $T\mathbb{R}^{n'}$.

E.g. $\text{Rank} \frac{\partial f}{\partial u} = m$, $\text{Rank} \frac{\partial g}{\partial v} = m'$

- $\Sigma \rightarrow \mathbb{R}^n$ sub-bundle of $T\mathbb{R}^n \rightarrow \mathbb{R}^n$.
 $\Sigma_x = \{f(x, u), u \in \mathbb{R}^m\}$ sub-manifold of the linear space $T_x\mathbb{R}^n$.
- A submanifold of an affine space is called (locally) **ruled** if it is a union of straight lines (locally of open segments).

Definition (ruled system)

Σ is ruled iff each Σ_x is a ruled submanifold of $T_x\mathbb{R}^n (\approx \mathbb{R}^n)$.

Note: This property is preserved by static equivalence.

Necessary condition for flatness

$$(\Sigma) \quad \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad (\Sigma') \text{ "trivial" .}$$

Theorem ([Rouchon], [Sluis], 1992)

A flat system must be ruled

What if

$$\begin{array}{ll} (\Sigma) & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \\ (\Sigma') & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, \quad v \in \mathbb{R}^{m'} \end{array} \quad \text{dynamic equivalent?}$$

Necessary condition for equivalence

$$\begin{aligned}(\Sigma) \quad & \dot{x} = f(x, u), \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m \\(\Sigma') \quad & \dot{z} = g(z, v), \quad z \in \mathbb{R}^{n'}, v \in \mathbb{R}^{m'}\end{aligned}$$

Theorem (JBP, 2009)

If Σ and Σ' are dynamic equivalent and $n = n'$, then, locally,

- ① either they are **static equivalent**,
- ② or they are **both ruled**.

If $n < n'$, Σ' must be ruled.

In the C^∞ case, ① and ② may both occur for the same Σ, Σ' .

Note: the condition for flatness is a consequence (trivial system is ruled).

Flatness with 2 controls and 3 states

$$\dot{z} = f(x, y, z, \dot{x}, \dot{y})$$

[Avanessof, JBP 2007].

We look for a parameterization x, y, z function of $(u, \dot{u}, \dots, u^{(k)}, v, \dots, v^{(\ell)})$, $k \leq \ell$, such that all solutions are covered. one has

- Either no parameterization because of ruled criterion of non-controllability,
- or parameterization with $(k, \ell) = (1, 2)$,
- or there could be a parameterization, but k and ℓ have to be no smaller than 3 and the largest one no smaller than 4.

Examples

Example 1: $\dot{z} = y + (\dot{y} - z\dot{x})\dot{x}$. Parameterization of order (1, 2) around jets such that $\ddot{x} + \dot{x}^3 \neq 1$, given by :

$$x = v, \quad y = \frac{\dot{v}^2 u + \dot{v}}{\ddot{v} + \dot{v}^3 - 1}, \quad z = \frac{(1 - \ddot{v})u + \dot{v}\dot{v}}{\ddot{v} + \dot{v}^3 - 1}. \quad (*)$$

Formulas can be “inverted” by $u = -z + y\dot{x}$, $v = x$, hence flatness. Note on singularity: $\ddot{x} + \dot{x}^3 = 1$ is the equation of the singular curves. There is a parameterization of higher order at jets of order 3 such that $x^{(3)} + 3\dot{x}^2\ddot{x} \neq 0$.

Example 2: $\dot{z} = y + (\dot{y} - z\dot{x})^2\dot{x}$: if there is a parameterization, it has order at least (3, 4). Conjecture: no parameterization.

Open questions

- ① Given Σ , give an a priori bound on the number K of derivatives.
- ② Prove that at least one system for which above mentioned necessary conditions do not work is not flat, or does not admit a parameterization.
- ③ Does existence of a parameter imply flatness ? (exogenous implies exogenous, according to [Fliess et al.])

Thank you all for attention, and
thank you for your action in the community, Bronek !