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To cite this version:

Miguel Couceiro, Jimmy Devillet, Jean-Luc Marichal, Pierre Mathonet. Reducibity of n-ary semigroups: from quasitriviality towards idempotency. 2019. hal-02293908

HAL Id: hal-02293908

https://inria.hal.science/hal-02293908v1

Preprint submitted on 22 Sep 2019

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REDUCIBILITY OF n-ARY SEMIGROUPS: FROM QUASITRIVIALITY TOWARDS IDEMPOTENCY

MIGUEL COUCEIRO, JIMMY DEVILLET, JEAN-LUC MARICHAL, AND PIERRE MATHONET

ABSTRACT. Let X be a nonempty set. Denote by $\mathcal{F}_k^n$ the class of associative operations $F\colon X^n \to X$ satisfying the condition $F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$ whenever at least k of the elements $x_1,\dots,x_n$ are equal to each other. The elements of $\mathcal{F}_1^n$ are said to be quasitrivial and those of $\mathcal{F}_n^n$ are said to be idempotent. We show that $\mathcal{F}_1^n = \cdots = \mathcal{F}_{n-2}^n \subsetneq \mathcal{F}_{n-1}^n \subsetneq \mathcal{F}_n^n$. The class $\mathcal{F}_1^n$ was recently characterized by Couceiro and Devillet [2], who showed that its elements are reducible to binary associative operations. However, some elements of $\mathcal{F}_n^n$ are not reducible. In this paper, we characterize the class $\mathcal{F}_{n-1}^n \smallsetminus \mathcal{F}_1^n$ and show that its elements are reducible. In particular, we show that each of these elements is an extension of an n-ary Abelian group operation whose exponent divides $n-1$.

1. INTRODUCTION

Let X be a nonempty set, let $|X|$ be its cardinality, and let $n \geq 2$ be an integer. An n-ary operation $F\colon X^n \to X$ is said to be *associative* if

$$F(x_1,\dots,x_{i-1},F(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{2n-1}) \\ = F(x_1,\dots,x_i,F(x_{i+1},\dots,x_{i+n}),x_{i+n+1},\dots,x_{2n-1}),$$

for all $x_1,\dots,x_{2n-1} \in X$ and all $1 \leq i \leq n-1$. The pair (X,F) is then called an *n-ary semigroup*. This notion actually stems back to Dörnte [6] and has led to the concept of n-ary group, which was first studied by Post [11].

In [5] the authors investigated associative n-ary operations that are determined by binary associative operations. An n-ary operation $F\colon X^n \to X$ is said to be *reducible to* an associative binary operation $G\colon X^2 \to X$ if there are $G^m\colon X^{m+1} \to X$ $(m = 1,\dots,n-1)$ such that $G^{n-1} = F$, $G^1 = G$, and

$$G^m(x_1,\dots,x_{m+1}) = G^{m-1}(x_1,\dots,x_{m-1},G(x_m,x_{m+1})), \qquad m \geq 2.$$

The pair (X,F) is then said to be the *n-ary extension* of (X,G).

Also, an n-ary operation $F\colon X^n \to X$ is said to be

- *idempotent* if $F(x,\dots,x) = x$ for all $x \in X$,
- *quasitrivial* if $F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$ for all $x_1,\dots,x_n \in X$.

We observe that any quasitrivial n-ary operation is idempotent. However, the converse is not true. For instance, the ternary operation $F\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x,y,z) = x - y + z$ is idempotent but not quasitrivial.

Recall that an element $e \in X$ is said to be *neutral* for $F\colon X^n \to X$ if

$$F((k-1)\cdot e, x, (n-k)\cdot e) = x, \qquad x \in X,\ k \in \{1,\dots,n\}.$$

Date: September 22, 2019.

2010 *Mathematics Subject Classification.* Primary 20M10, 20N15; Secondary 16B99, 20K25.

Key words and phrases. Semigroup, polyadic semigroup, Abelian group, reducibility, quasitriviality, idempotency.

Here and throughout, for any $k \in \{1,\dots,n\}$ and any $x \in X$, the notation $k \cdot x$ stands for the k-tuple $x,\dots,x$. For instance, we have

$$F(3\cdot x, 0\cdot y, 2\cdot z) \;=\; F(x,x,x,z,z).$$

We also denote the set of neutral elements for an operation $F\colon X^n \to X$ by E_F.

The quest for conditions under which an associative n-ary operation is reducible to an associative binary operation gained an increasing interest since the pioneering work of Post [11] (see, e.g., [1,2,4,5,7,8]). For instance, Dudek and Mukhin [5, Theorem 1] proved that an associative operation $F\colon X^n \to X$ is reducible to an associative binary operation if and only if one can *adjoin to X a neutral element e for F*; that is, there is an n-ary associative operation $\tilde{F}\colon (X\cup\{e\})^n \to X\cup\{e\}$ such that e is a neutral element for $\tilde{F}$ and $\tilde{F}|_{X^n} = F$. In this case, a binary reduction G_e of F can be defined by

$$G_e(x,y) \;=\; \tilde{F}(x,(n-2)\cdot e, y) \quad x,y \in X.$$

Also, it was recently observed [2, Corollary 2.3] that all the quasitrivial associative n-ary operations are reducible to associative binary operations. On the other hand, the associative idempotent ternary operation $F\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x,y,z) = x - y + z$ is neither quasitrivial nor reducible (see, e.g., [13] or more recently [10]).

In this paper, we are interested in studying conditions under which an idempotent n-ary semigroup is reducible to a semigroup. The observations above lead us to investigate certain subclasses of idempotent n-ary semigroups containing the quasitrivial ones, for instance by requiring the condition

$$F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}$$

to hold on at least some subsets of X^n. More precisely, for every set $S \subseteq \{1,\dots,n\}$, we define

$$D_S^n \;=\; \{(x_1,\dots,x_n) \in X^n : x_i = x_j,\ \forall i,j \in S\}.$$

Also, for every $k \in \{1,\dots,n\}$, we set

$$D_k^n \;=\; \bigcup_{\substack{S\subseteq\{1,\dots,n\}\\ |S|\geq k}} D_S^n \;=\; \bigcup_{\substack{S\subseteq\{1,\dots,n\}\\ |S|=k}} D_S^n.$$

Thus, the set D_k^n consists of those tuples of X^n for which at least k components are equal to each other. In particular, $D_1^n = X^n$ and $D_n^n = \{(x,\dots,x) : x \in X\}$.

For every $k \in \{1,\dots,n\}$, denote by $\mathcal{F}_k^n$ the class of those associative n-ary operations $F\colon X^n \to X$ that satisfy

$$F(x_1,\dots,x_n) \in \{x_1,\dots,x_n\}, \qquad (x_1,\dots,x_n) \in D_k^n.$$

We say that these operations are *quasitrivial on D_k^n*.

Thus defined, $\mathcal{F}_1^n$ is the class of quasitrivial associative n-ary operations and $\mathcal{F}_n^n$ is the class of idempotent associative n-ary operations. Since the sets D_k^n are nested in the sense that $D_{k+1}^n \subsetneq D_k^n$ for $1 \leq k \leq n-1$, the classes $\mathcal{F}_k^n$ clearly form a filtration, that is,

$$\mathcal{F}_1^n \subseteq \mathcal{F}_2^n \subseteq \cdots \subseteq \mathcal{F}_n^n.$$

It is easy to see that $\mathcal{F}_1^n = \mathcal{F}_2^n = \cdots = \mathcal{F}_n^n$ if $|X| \leq 2$.

Quite surprisingly, we have the following result, which shows that this filtration actually reduces to three nested classes only. The proof is given in the next section.

Proposition 1.1. *For every $n \geq 3$, we have $\mathcal{F}_1^n = \mathcal{F}_{n-2}^n$.*

We observe that the class $\mathcal{F}_1^n = \mathcal{F}_2^n = \cdots = \mathcal{F}_{n-2}^n$ was characterized by Couceiro and Devillet [2] who showed that all its elements are reducible. In particular, we have the following result.

Proposition 1.2 (see [2, Corollary 3.11])**.** *An operation $F\colon X^n \to X$ is associative, quasitrivial, and has at most one neutral element if and only if it is reducible to an associative and quasitrivial binary operation $G\colon X^2 \to X$. In that case, we have $G(x,y) = F(x,y,\ldots,y)$ for any $x,y \in X$.*

Remark 1. We observe that the class of associative and quasitrivial operations $G\colon X^2 \to X$ was characterized in [9, Theorem 1].

In this paper, we provide a characterization of the class $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ and show that all its elements are also reducible to binary associative operations. This result is stated in Theorem 1.5 and Corollary 1.6. The proofs of these results are given in the next section.

Recall that an *n-ary groupoid* is a set equipped with an n-ary operation. Moreover, two n-ary groupoids (X,F) and (Y,F') are said to be *isomorphic* if there exists a bijection $\phi\colon X \to Y$ such that

$$\phi(F(x_1,\ldots,x_n)) = F'(\phi(x_1),\ldots,\phi(x_n)),$$

for every $x_1,\ldots,x_n \in X$. In that case, the operations F and F' are said to be *conjugate to each other*.

The *exponent* of a group (X,G) with neutral element e is the smallest integer $m \geq 1$ such that $G^{m-1}(m\cdot x) = e$ for any $x \in X$. Otherwise, (X,G) is said to have *zero exponent*.

Remark 2. A group (X,G) has an exponent dividing n if and only if $G^n((n+1)\cdot x) = x$ for any $x \in X$.

Let us recall the following important result, due to Prüfer and Baer (see, e.g., [12, Corollary 10.37]).

Proposition 1.3 (see [12, Corollary 10.37])**.** *If (X,G) is an Abelian group of bounded exponent, then it is isomorphic to a direct sum of cyclic groups.*

The following immediate corollary provides a characterization of the class of Abelian groups with nonzero exponents.

Corollary 1.4. *Let $m \geq 1$ be an integer. An Abelian group (X,G) is of exponent m if and only if it is isomorphic to a direct sum of cyclic groups whose exponent is m.*

Recall that an element $a \in X$ is said to be an *annihilator* for $F\colon X^n \to X$ if $F(x_1,\ldots,x_n) = a$ whenever $a \in \{x_1,\ldots,x_n\}$.

Theorem 1.5. *Suppose that $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \varnothing$. An associative operation F is in $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ if and only if there exists a unique subset $Y \subseteq X$, with $|Y| \geq 3$, such that the following assertions hold.*

(a) *$(Y, F|_{Y^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$.*
(b) *$F|_{(X\setminus Y)^n}$ is quasitrivial and has at most one neutral element.*
(c) *Any $x \in X \setminus Y$ is an annihilator for $F|_{(\{x\}\cup Y)^n}$.*

Moreover, we have that $Y = E_F$.

Remark 3. The associativity assumption can be removed in Theorem 1.5 if we replace condition (c) by the following two conditions.

(c') For any $y \in Y$ and any $x_1, \ldots, x_{n-1} \in X$ we have

$$F(x_1, \ldots, x_{n-1}, y) \;=\; F(x_1, \ldots, y, x_{n-1}) \;=\; \cdots \;=\; F(y, x_1, \ldots, x_{n-1}).$$

(c") For any $k \in \{1, \ldots, n-1\}$, any $y_1, \ldots, y_k \in Y$, and any $x_1, \ldots, x_{n-k} \in X \smallsetminus Y$ we have

$$\begin{aligned} &F(y_1, \ldots, y_k, x_1, \ldots, x_{n-k}) \\ &\quad = \quad F(x_1, \ldots, x_{i-1}, (k+1) \cdot x_i, x_{i+1}, \ldots, x_{n-k}), \qquad i \in \{1, \ldots, n-k\}. \end{aligned}$$

The proof is a little tedious but straightforward.

Corollary 1.6. *Every operation in $\mathcal{F}^n_{n-1}$ is reducible to a binary associative operation.*

A reformulation of Theorem 1.5 based on binary reductions is given in the following corollary.

Corollary 1.7. *Suppose that $\mathcal{F}^n_{n-1} \smallsetminus \mathcal{F}^n_1 \neq \varnothing$. An operation F is in $\mathcal{F}^n_{n-1} \smallsetminus \mathcal{F}^n_1$ if and only if it is reducible to an associative operation $G\colon X^2 \to X$ and there exists a unique subset $Y \subseteq X$, with $|Y| \geq 3$, such that the following conditions hold.*

(a) *$(Y, G|_Y)$ is an Abelian group whose exponent divides $n-1$.*
(b) *$G|_{(X \smallsetminus Y)^2}$ is quasitrivial.*
(c) *Any $x \in X \smallsetminus Y$ is an annihilator for $G|_{(\{x\} \cup Y)^2}$.*

Moreover, we have that $Y = E_F$.

Corollary 1.7 is of particular interest as it allows us to easily construct n-ary operations in $\mathcal{F}^n_{n-1} \smallsetminus \mathcal{F}^n_1$. For instance, for any integers $n \geq 3$ and $p \geq 1$, the operation associated with any $(np+1)$-ary extension of $(\mathbb{Z}_n, +)$ is in $\mathcal{F}^{np+1}_{np} \smallsetminus \mathcal{F}^{np+1}_1$ by Corollary 1.7. To give another example, consider the chain $(X, \leq) = (\{1,2,3,4,5\}, \leq)$ together with the operation $G\colon X^2 \to X$ defined by the following conditions:

- $(\{1,2,3\}, G|_{\{1,2,3\}^2})$ is isomorphic to $(\mathbb{Z}_3, +)$,
- $G|_{\{4,5\}^2} = \vee|_{\{4,5\}^2}$, where $\vee\colon X^2 \to X$ is the maximum operation for $\leq$,
- for any $x \in \{1,2,3\}$, $G(x,4) = G(4,x) = 4$ and $G(x,5) = G(5,x) = 5$.

Then for any integer $p \geq 1$ we have that the operation associated with any $(3p+1)$-ary extension of $(\{1,2,3,4,5\}, G)$ is in $\mathcal{F}^{3p+1}_{3p} \smallsetminus \mathcal{F}^{3p+1}_1$, again by Corollary 1.7.

We also have the following corollary which follows from Theorem 1.5.

Corollary 1.8. *Let $n \geq 3$ and assume that any of the two following conditions holds.*

- *$n = 3$ and $|X| \geq 4$.*
- *$n \geq 4$ and $|X| \geq n-1$.*

Then $\mathcal{F}^n_1 \subsetneq \mathcal{F}^n_{n-1}$.

For instance, consider the operation $F\colon X^3 \to X$ defined on $X = \mathbb{Z}_2^2$ by

$$F(x_1, x_2, x_3) \;=\; x_1 + x_2 + x_3 \pmod 2,$$

where $(\bmod 2)$ is understood componentwise. Then, for any $x, y \in \mathbb{Z}_2^2$, we have $F(x,x,y) = y$, and hence $F \in \mathcal{F}^3_2$. But we have

$$F\Big(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\Big) \;=\; \begin{bmatrix}1\\1\end{bmatrix},$$

which shows that $F \notin \mathcal{F}^3_1$. However, the converse of Corollary 1.8 does not hold in general. For instance, the operation associated with the 7-ary extension of $(\mathbb{Z}_3, +)$ is in $\mathcal{F}^7_6 \smallsetminus \mathcal{F}^7_1$. However, we have that $|\mathbb{Z}_3| = 3$.

Moreover, we observe that $\mathcal{F}_{n-1}^n \subsetneq \mathcal{F}_n^n$. For instance, if $(X, \leq)$ is a lattice that is not a chain, then the ternary operation $F\colon X^3 \to X$ defined by $F(x, y, z) = x \vee y \vee z$ is in $\mathcal{F}_3^3$. However, it is not in $\mathcal{F}_2^3$ since $F(x, x, y) \notin \{x, y\}$ whenever x and y are not comparable, i.e, $x \vee y \notin \{x, y\}$.

In proving Theorem 1.5 we will make use of the following characterization of the class of associative operations $F\colon X^n \to X$ for which the elements of X are all neutral.

Theorem 1.9. *Let $F\colon X^n \to X$ ($n \geq 3$) be an associative operation. Then $E_F = X$ if and only if (X, F) is the n-ary extension of an Abelian group whose exponent divides $n - 1$.*

Recall from [5, Lemma 1] that if an associative operation $F\colon X^n \to X$ has a neutral element e, then it is reducible to the associative operation $G_e\colon X^2 \to X$ defined by

$$G_e(x, y) = F(x, (n-2) \cdot e, y), \qquad x, y \in X. \tag{1}$$

We observe that e is the unique neutral element of G_e.

Using this result, we can also show that all reductions of an associative operation $F\colon X^n \to X$ obtained from neutral elements are conjugate to each other. For instance, the ternary sum on $\mathbb{Z}_2$ has two neutral elements, namely 0 and 1. By [5, Lemma 1] it is reducible to the operations $G_0, G_1\colon \mathbb{Z}_2^2 \to \mathbb{Z}_2^2$ defined by $G_0(x, y) = x + y \pmod 2$ and $G_1(x, y) = x + y + 1 \pmod 2$. It is easy to see that the semigroups $(\mathbb{Z}_2, G_0)$ and $(\mathbb{Z}_2, G_1)$ are isomorphic.

Proposition 1.10. *Let $F\colon X^n \to X$ ($n \geq 3$) be an associative operation and let $e_1, e_2 \in E_F$. Then (X, G_{e_1}) and (X, G_{e_2}) are isomorphic.*

The proofs of the results above are given in Section 2. In Section 3, we discuss an alternative hierarchy, which provides variants of Theorem 1.5 and Corollary 1.7.

2. Technicalities and proofs of the results

Let us start by providing the proofs of Propositions 1.1 and 1.10.

Proof of Proposition 1.1. We only need to prove that $\mathcal{F}_{n-2}^n \subseteq \mathcal{F}_1^n$, and so we can assume that $n \geq 4$. Let $F \in \mathcal{F}_{n-2}^n$ and let us show by induction that for every $k \in \{1, \ldots, n\}$ we have

$$F(k \cdot x_1, x_{k+1}, \ldots, x_n) \in \{x_1, x_{k+1}, \ldots, x_n\}, \qquad x_1, x_{k+1}, \ldots, x_n \in X. \tag{2}$$

By hypothesis, condition (2) holds for any $k \in \{n-2, n-1, n\}$. Let us now assume that it holds for some $k \in \{2, \ldots, n\}$ and let us show that it still holds for $k-1$. Using associativity and idempotency, we have

$$\begin{aligned} F((k-1) \cdot x_1, x_k, \ldots, x_n) &= F(F(n \cdot x_1), (k-2) \cdot x_1, x_k, \ldots, x_n) \\ &= F(k \cdot x_1, F((n-2) \cdot x_1, x_k, x_{k+1}), \ldots, x_n). \end{aligned}$$

By induction assumption, the latter expression lies in $\{x_1, x_k, \ldots, x_n\}$. This completes the proof. □

Proof of Proposition 1.10. The map $\psi\colon X \to X$ defined by

$$\psi(x) = F(e_1, e_2, x, (n-3) \cdot e_1)$$

is a bijection and we have $\psi^{-1}(x) = F((n-2)\cdot e_2, x, e_1)$. We then have

$$\begin{aligned}
&G^{e_2}(\psi(x),\psi(y))\\
&\quad= F(F(e_1,e_2,x,(n-3)\cdot e_1),(n-2)\cdot e_2,F(e_1,e_2,y,(n-3)\cdot e_1))\\
&\quad= F(F(e_1,e_2,x,(n-3)\cdot e_1),F((n-2)\cdot e_2,e_1,e_2),y,(n-3)\cdot e_1)\\
&\quad= F(F(e_1,e_2,x,(n-3)\cdot e_1),e_1,y,(n-3)\cdot e_1)\\
&\quad= F(e_1,e_2,F(x,(n-2)\cdot e_1,y),(n-3)\cdot e_1)\\
&\quad= \psi(G^{e_1}(x,y)),
\end{aligned}$$

which completes the proof. □

Let us now prove Theorem 1.9. To this extent, we first state and prove some intermediate results.

Lemma 2.1. *Let $F\colon X^n \to X$ be an associative operation and let $e \in E_F$. Then, for any $x_1,\dots,x_{n-1} \in X$ we have*

$$F(x_1,\dots,x_{n-1},e) = F(x_1,\dots,e,x_{n-1}) = \cdots = F(e,x_1,\dots,x_{n-1}).$$

Moreover, for any $x \in X$ the restriction $F|_{(\{x\}\cup E_F)^n}$ is symmetric.

Proof. Let $x_1,\dots,x_{n-1} \in X$ and let G_e be the reduction of F defined by (1). Since $E_{G_e} = \{e\}$, we have

$$\begin{aligned}
F(x_1,\dots,x_{n-1},e) &= G_e^{n-1}(x_1,\dots,x_{n-1},e) = G_e^{n-1}(x_1,\dots,e,x_{n-1}) = \cdots\\
&= G_e^{n-1}(e,x_1,\dots,x_{n-1}),
\end{aligned}$$

which proves the first part of the statement. The second part is a direct consequence of the first part. □

Lemma 2.2. *Let $F\colon X^n \to X$ be an associative operation such that $E_F \neq \varnothing$. Then F preserves E_F, i.e., $F(E_F^n) \subseteq E_F$.*

Proof. Let $e_1,\dots,e_n \in E_F$ and let us show that $F(e_1,\dots,e_n) \in E_F$. By Lemma 2.1 and associativity of F, for any $x \in X$ we have

$$\begin{aligned}
&F((n-1)\cdot F(e_1,\dots,e_n),x)\\
&\quad= F(F(e_1,(n-1)\cdot e_2),F(e_1,(n-1)\cdot e_3),\dots,F(e_1,(n-1)\cdot e_n),x)\\
&\qquad= F((n-1)\cdot e_1,x) = x.
\end{aligned}$$

Similarly, for any $x \in X$ we can show that

$$F(i\cdot F(e_1,\dots,e_n),x,(n-i-1)\cdot F(e_1,\dots,e_n)) = x, \qquad i \in \{1,\dots,n-2\},$$

and the proof is now complete. □

Combining Lemmas 2.1 and 2.2, we immediately derive the following result.

Corollary 2.3. *If (X,F) is an n-ary monoid, then $(E_F, F|_{E_F^n})$ is a symmetric n-ary monoid.*

Proof of Theorem 1.9. (Sufficiency) Obvious.

(Necessity) Suppose that $X = E_F$. Let $e \in E_F$ and $G_e\colon X^2 \to X$ be the corresponding reduction of F defined by (1). By Corollary 2.3, we have that F is symmetric. Thus, we have that G_e also is symmetric. Moreover, since G_e is a binary reduction of F and $E_F = X$, it follows that

$$G_e(G_e^{n-2}((n-1)\cdot x),y) = y = G_e(y,G_e^{n-2}((n-1)\cdot x)), \qquad x,y \in X,$$

which shows that $G_e^{n-2}((n-1)\cdot x) \in E_{G_e}$ for any $x \in X$. However, since $E_{G_e} = \{e\}$, we have that $G_e^{n-2}((n-1)\cdot x) = e$ for any $x \in X$. Thus, (X, G_e) is an Abelian group whose exponent divides $n-1$. □

The following result follows immediately from Theorem 1.9.

Corollary 2.4. *If (X, F) is an n-ary monoid, then $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$.*

Let us now prove Theorem 1.5. To this extent, we first state and prove some intermediate results. We have the following remarkable result, which characterizes the existence of a pair of neutral elements for $F \in \mathcal{F}_{n-1}^n$ by means of two identities.

Lemma 2.5. *Let $F \in \mathcal{F}_{n-1}^n$ and let $a, b \in X$ such that $a \neq b$. Then $a, b \in E_F$ if and only if $F((n-1)\cdot a, b) = b$ and $F(a, (n-1)\cdot b) = a$.*

Proof. (Necessity) Obvious.

(Sufficiency) For any $x \in X$, we have

$$\begin{aligned} F((n-1)\cdot a, x) &= F((n-2)\cdot a, F(a, (n-1)\cdot b), x) \\ &= F(F((n-1)\cdot a, b), (n-2)\cdot b, x) \;=\; F((n-1)\cdot b, x), \end{aligned}$$

which implies that $F((n-1)\cdot a, x) = x = F((n-1)\cdot b, x)$ for any $x \in X$. Similarly, we can show that $F(x, (n-1)\cdot a) = x = F(x, (n-1)\cdot b)$ for any $x \in X$. Also, we observe that for any $k \in \{1, \ldots, n-2\}$, the maps $\psi_k, \xi_k \colon X \to X$ defined by

$$\begin{aligned} \psi_k(x) &= F(k\cdot a, x, (n-k-1)\cdot a) \\ \xi_k(x) &= F(k\cdot b, x, (n-k-1)\cdot b) \end{aligned}$$

are bijections with inverse maps ψ_{n-k-1} and ξ_{n-k-1}. It then follows that, for any $k \in \{1, \ldots, n-2\}$, we have

$$F(k\cdot a, x, (n-k-1)\cdot a) \;=\; x \;=\; F(k\cdot b, x, (n-k-1)\cdot b), \qquad x \in X,$$

which shows that $a, b \in E_F$. □

Given an associative operation $F \colon X^n \to X$, we can define the sequence $(F^q)_{q\geq 1}$ of $(qn-q+1)$-ary associative operations inductively by the rules $F^1 = F$ and

$$F^q(x_1, \ldots, x_{qn-q+1}) \;=\; F^{q-1}(x_1, \ldots, x_{(q-1)n-q+1}, F(x_{(q-1)n-q+2}, \ldots, x_{qn-q+1})),$$

for any integer $q \geq 2$ and any $x_1, \ldots, x_{qn-q+1} \in X$.

Proposition 2.6. *Let $F \in \mathcal{F}_{n-1}^n$. For any $a_1, \ldots, a_n \in X$ such that $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$, we have that $a_1, \ldots, a_n, F(a_1, \ldots, a_n) \in E_F$. Moreover, $F|_{(X\setminus E_F)^n}$ is quasitrivial.*

Proof. Let us prove by induction on $k \in \{1, \ldots, n-1\}$ that the condition

$$F((n-k)\cdot a_1, a_2, \ldots, a_{k+1}) \notin \{a_1, \ldots, a_{k+1}\}$$

implies that $a_1, \ldots, a_{k+1} \in E_F$. For $k = 1$, there is nothing to prove. We thus assume that the result holds true for a given $k \in \{1, \ldots, n-2\}$ and we show that it still holds for $k+1$. Now, consider elements $a_1, \ldots, a_{k+2}$ such that

$$F((n-k-1)\cdot a_1, a_2, \ldots, a_{k+2}) \notin \{a_1, \ldots, a_{k+2}\}. \tag{3}$$

We first prove that $a_1, a_2 \in E_F$.

If $a_1 = a_2$, then $a_1, \ldots, a_{k+2} \in E_F$ by the induction hypothesis.

If $a_1 \neq a_2$, then we prove that $F((n-1)\cdot a_1, a_2) = a_2$ and $F(a_1, (n-1)\cdot a_2) = a_1$, which show that $a_1, a_2 \in E_F$ by Lemma 2.5.

- For the sake of a contradiction, assume first that $F((n-1)\cdot a_1, a_2) = a_1$. Then, for $\ell \in \{1,\dots,n-2\}$ we have

$$\begin{aligned} &F((n-k-1)\cdot a_1, a_2, \dots, a_{k+2}) \\ &\quad = F^{\ell+1}(((n-k-1)+\ell(n-2))\cdot a_1, (\ell+1)\cdot a_2, \dots, a_{k+2}). \end{aligned} \tag{4}$$

  Choosing $\ell = n-k-1$ and using idempotency of F, we obtain

$$F((n-k-1)\cdot a_1, a_2, \dots, a_{k+2}) = F^2((n-1)\cdot a_1, (n-k)\cdot a_2, a_3, \dots, a_{k+2}).$$

  Since the left-hand side of this equation does not lie in $\{a_1,\dots,a_{k+2}\}$, by (3) we obtain

$$F((n-k)\cdot a_2, a_3, \dots, a_{k+2}) \notin \{a_1,\dots,a_{k+2}\}.$$

  By the induction hypothesis, we have $a_2,\dots,a_{k+2} \in E_F$. Then choosing $\ell = n-2$ in (4) and using idempotency and the fact that $a_2 \in E_F$, we obtain

$$\begin{aligned} &F((n-k-1)\cdot a_1, a_2, \dots, a_{k+2}) \\ &\quad = F^{n-1}(((n-k-1)+(n-2)^2)\cdot a_1, (n-1)\cdot a_2, \dots, a_{k+2}) \\ &\quad = F^2((n-k)\cdot a_1, (n-1)\cdot a_2, a_3, \dots, a_{k+2}) \\ &\quad = F((n-k)\cdot a_1, a_3, \dots, a_{k+2}). \end{aligned}$$

  By the induction hypothesis, we have $a_1 \in E_F$. We then have $F((n-1)\cdot a_1, a_2) = a_2 \neq a_1$, a contradiction.
- Assume now that $F(a_1, (n-1)\cdot a_2) = a_2$. Then, for $\ell \in \{1,\dots,n-2\}$ we have

$$\begin{aligned} &F((n-k-1)\cdot a_1, a_2, \dots, a_{k+2}) \\ &\quad = F^{\ell+1}((n-k-1+\ell)\cdot a_1, (\ell(n-2)+1)\cdot a_2, \dots, a_{k+2}). \end{aligned}$$

  For $\ell = k$, using idempotency and the fact that $k(n-2)+1 = n-k+(k-1)(n-1)$, we obtain

$$\begin{aligned} &F((n-k-1)\cdot a_1, a_2, \dots, a_{k+2}) \\ &\quad = F^2((n-1)\cdot a_1, (n-k)\cdot a_2, a_3, \dots, a_{k+2}). \end{aligned}$$

  Thus, $F((n-k)\cdot a_2, a_3, \dots, a_{k+2}) \notin \{a_1,\dots,a_{k+2}\}$. By the induction hypothesis, we have $a_2,\dots,a_{k+2} \in E_F$. It follows that $F(a_1, (n-1)\cdot a_2) = a_1 \neq a_2$, a contradiction.

Now, since $a_2 \in E_F$, it commutes with all other arguments of F by Lemma 2.1. Also, by (3) we have

$$F((n-k-1)\cdot a_1, a_3, \dots, a_{k+2}, a_2) \notin \{a_1,\dots,a_{k+2}\},$$

and thus $a_3 \in E_F$. Repeating this argument, we have that $a_1,\dots,a_{k+2} \in E_F$. Furthermore, $F(a_1,\dots,a_n) \in E_F$ by Lemma 2.2. The second part is straightforward. □

Proof of Corollary 1.6. This follows from [2, Corollary 2.3], Proposition 2.6, and [5, Lemma 1]. □

Remark 4. In the proof of Corollary 1.6 we used [2, Corollary 2.3] which is based on results obtained by Ackerman [1]. In the appendix we provide an alternative proof of Corollary 1.6 that does not make use of [2, Corollary 2.3].

In [2, Proposition 3.7], it was shown that a quasitrivial n-ary semigroup cannot have more than two neutral elements. The next result shows that an operation in $\mathcal{F}^n_{n-1}$ is quasitrivial whenever it has at most two neutral elements.

Corollary 2.7. *An operation $F \in \mathcal{F}^n_{n-1}$ is quasitrivial if and only if $|E_F| \leq 2$.*

Proof. (Necessity) This follows from [2, Proposition 3.7].

(Sufficiency) Suppose to the contrary that $|E_F| \leq 2$ and that F is not quasitrivial, i.e., there exist $a_1, \ldots, a_n \in X$ such that $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$. By Proposition 2.6, we have that $a_1, \ldots, a_n, F(a_1, \ldots, a_n) \in E_F$. Thus, we must have

$$|\{a_1, \ldots, a_n, F(a_1, \ldots, a_n)\}| \leq 2,$$

which contradicts the idempotency of F and the preservation of E_F by F (see Lemma 2.2). □

Proposition 2.8. *Let $F \in \mathcal{F}^n_{n-1}$ and suppose that $|E_F| \geq 3$. Then, any element $x \in X \smallsetminus E_F$ is an annihilator of $F|_{(\{x\} \cup E_F)^n}$. Moreover, $F|_{(X \smallsetminus E_F)^n}$ is quasitrivial and has at most one neutral element.*

Proof. Let $x \in X \smallsetminus E_F$ and $e \in E_F$ and let us show that $F(k \cdot x, (n-k) \cdot e) = x$ for any $k \in \{1, \ldots, n-1\}$. If $k = 1$, then the latter identity follows from the definition of a neutral element. Now, suppose that there exists $k \in \{2, \ldots, n-1\}$ such that $F(k \cdot x, (n-k) \cdot e) \neq x$. Since $x \in X \smallsetminus E_F$, by Proposition 2.6 we must have $F(k \cdot x, (n-k) \cdot e) = e$. But then, using the associativity of F, we get

$$\begin{aligned} F((n-1) \cdot x, e) &= F((n-1) \cdot x, F(k \cdot x, (n-k) \cdot e)) \\ &= F(k \cdot x, (n-k) \cdot e) = e, \end{aligned}$$

and we conclude by Lemma 2.5 that $x \in E_F$, which contradicts our assumption. Thus, we have

$$F(k \cdot x, (n-k) \cdot e) = x, \qquad k \in \{1, \ldots, n-1\}. \tag{5}$$

Now, let us show that $F(k \cdot x, e_{k+1}, \ldots, e_n) = x$ for any $k \in \{1, \ldots, n-1\}$ and any $e_{k+1}, \ldots, e_n \in E_F$. To this extent, we only need to show that

$$F(k \cdot x, e_{k+1}, \ldots, e_n) = F((k+1) \cdot x, e_{k+2}, \ldots, e_n),$$

for any $k \in \{1, \ldots, n-1\}$ and any $e_{k+1}, \ldots, e_n \in E_F$. So, let $k \in \{1, \ldots, n-1\}$ and $e_{k+1}, \ldots, e_n \in E_F$. Using (5) and the associativity of F we get

$$\begin{aligned} F(k \cdot x, e_{k+1}, \ldots, e_n) &= F((k-1) \cdot x, F(2 \cdot x, (n-2) \cdot e_{k+1}), e_{k+1}, \ldots, e_n) \\ &= F(k \cdot x, F(x, (n-1) \cdot e_{k+1}), e_{k+2}, \ldots, e_n) \\ &= F((k+1) \cdot x, e_{k+2}, \ldots, e_n), \end{aligned}$$

which completes the proof by idempotency of F and Lemma 2.1. For the second part of the proposition, we observe that $F|_{(X \smallsetminus E_F)^n}$ is quasitrivial by Proposition 2.6. Also, using (5) and the associativity of F, for any $x, y \in X \smallsetminus E_F$ and any $e \in E_F$ we obtain

$$\begin{aligned} F((n-1) \cdot x, y) &= F((n-1) \cdot x, F(e, (n-1) \cdot y)) \\ &= F(F((n-1) \cdot x, e), (n-1) \cdot y) = F(x, (n-1) \cdot y), \end{aligned}$$

which shows that $F|_{(X \smallsetminus E_F)^n}$ cannot have more than one neutral element. □

Proof of Theorem 1.5. (Necessity) This follows from Corollaries 2.4 and 2.7 and Proposition 2.8.

(Sufficiency) It is not difficult to see that $Y \subseteq E_F$. By Corollary 2.7, we also have that $|E_F| \geq 3$. By Corollary 2.4, the pair $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$. Thus, since Y is unique, it follows that $Y = E_F$. Finally, the quasitriviality of F on D^n_{n-1} is straightforward to check. □

Proof of Corollary 1.7. This follows from Theorem 1.5, Corollary 1.6, and Proposition 1.2. □

3. An alternative hierarchy

For any integer $k \geq 1$, let S_k^n be the set of n-tuples $(x_1, \ldots, x_n) \in X^n$ such that $|\{x_1, \ldots, x_n\}| \leq k$. Of course, we have $D_k^n \subseteq S_{n-k+1}^n$ for $k \in \{1, \ldots, n\}$. Now, denote by $\mathcal{G}_k^n$ the class of those associative n-ary operations $F\colon X^n \to X$ satisfying

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}, \qquad (x_1, \ldots, x_n) \in S_k^n.$$

We say that these operations are *quasitrivial on* S_k^n.

It is not difficult to see that if $F \in \mathcal{G}_k^n$, then $F \in \mathcal{F}_{n-k+1}^n$. Due to Proposition 1.1, we have $\mathcal{G}_n^n = \cdots = \mathcal{G}_3^n$, and hence we only need to consider operations in $\mathcal{G}_2^n$. The analog of Theorem 1.5 can then be stated as follows.

Theorem 3.1. *Suppose that $\mathcal{G}_2^n \smallsetminus \mathcal{G}_n^n \neq \varnothing$. An associative operation F is in $\mathcal{G}_2^n \smallsetminus \mathcal{G}_n^n$ if and only if n is odd and there exists a unique subset $Y \subseteq X$, with $|Y| \geq 3$, such that the following conditions hold.*

(a) *$(Y, F|_{Y^n})$ is the n-ary extension of an Abelian group of exponent 2.*
(b) *$F|_{(X \smallsetminus Y)^n}$ is quasitrivial and has at most one neutral element.*
(c) *Any $x \in X \smallsetminus Y$ is an annihilator for $F|_{(\{x\} \cup Y)^n}$.*

Moreover, we have that $Y = E_F$.

Proof. (Necessity) Due to Theorem 1.5, we only need to show that condition (a) holds. By Corollary 2.7, we have that $|E_F| \geq 3$. Also, by Corollary 2.4, we have that $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$. In particular, for any $e \in E_F$, we have that (E_F, G_e) is an Abelian group whose exponent divides $n-1$. However, since the neutral element is the only idempotent element of a group and since $G_e(e', e') \in \{e, e'\}$ for any $e, e' \in E_F$, it follows that $G_e(e', e') = e$ for any $e, e' \in E_F$, i.e., for any $e \in E_F$ we have that (E_F, G_e) is a group of exponent 2. (Recall that an element $x \in X$ is said to be *idempotent* for an operation $F\colon X^n \to X$ if $F(n \cdot x) = x$.) Therefore, we conclude that $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group of exponent 2. Also, since 2 divides $n-1$ we conclude that n is odd.

(Sufficiency) It is not difficult to see that $Y \subseteq E_F$. Now, let $e \in E_F$ and let us show that $e \in Y$. Suppose to the contrary that $e \notin Y$, i.e., $e \in X \smallsetminus Y$. But then by (c) we have

$$F((n-1) \cdot e, y) = e, \qquad y \in Y,$$

which contradicts the definition of a neutral element. Thus, we have $Y = E_F$. Also, since (Y, F) is the n-ary extension of an Abelian group of exponent 2 and n is odd, it follows that $F \in \mathcal{G}_2^n$. The remaining properties follow from Theorem 1.5. □

The following corollary follows from Theorem 3.1, Proposition 1.2, and Corollary 1.6.

Corollary 3.2. *Suppose that $\mathcal{G}_2^n \smallsetminus \mathcal{G}_n^n \neq \varnothing$. An operation F is in $\mathcal{G}_2^n \smallsetminus \mathcal{G}_n^n$ if and only if it is reducible to an associative operation $G\colon X^2 \to X$, n is odd, and there exists a unique subset $Y \subseteq X$, with $|Y| \geq 3$, such that the following conditions hold.*

(a) *(Y, G) is an Abelian group of exponent 2.*
(b) *$G|_{(X \smallsetminus Y)^2}$ is quasitrivial.*
(c) *Any $x \in X \smallsetminus Y$ is an annihilator for $G|_{(\{x\} \cup Y)^2}$.*

Moreover, we have that $Y = E_F$.

Corollary 3.2 is particularly interesting as it allows to construct easily n-ary operations in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$. For instance, consider the set $X = \{1,2,3,4,5,6\}$ together with the operation $G\colon X^2 \to X$ defined by the following conditions:

- $(\{1,2,3,4\}, G|_{\{1,2,3,4\}^2})$ is isomorphic to $(\mathbb{Z}_2^2, +)$,
- $G|_{\{5,6\}^2} = \pi_1|_{\{5,6\}^2}$, where $\pi_1\colon X^2 \to X$ is defined by $\pi_1(x,y) = x$ for any $x, y \in X$,
- for any $x \in \{1,2,3,4\}$, $G(x,5) = G(5,x) = 5$ and $G(x,6) = G(6,x) = 6$.

Then for any integer $p \geq 1$, we have that the operation associated with any $(2p+1)$-ary extension of $(\{1,2,3,4,5,6\}, G)$ is in $\mathcal{G}_2^{2p+1} \setminus \mathcal{G}_{2p+1}^{2p+1}$ by Corollary 3.2.

CONCLUDING REMARKS

In this paper we characterized the class $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$, i.e., the class of those associative operations $F\colon X^n \to X$ that are not quasitrivial but satisfy the condition $F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$ whenever at least $n-1$ of the elements $x_1,\ldots,x_n$ are equal to each other (Theorem 1.5). Moreover, we proved that any such operation is reducible to a binary operation (Corollary 1.6). This led to an alternative characterization of the class $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ based on binary reductions (Corollary 1.7). Finally, we characterized the class $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$, i.e., the class of those associative operations $F\colon X^n \to X$ that are not quasitrivial but satisfy the condition $F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$ whenever $|\{x_1,\ldots,x_n\}| \leq 2$ (Theorem 3.1 and Corollary 3.2).

The main results of this paper thus characterize several relevant subclasses of associative and idempotent n-ary operations. However, the characterization of the class $\mathcal{F}_n^n$ of associative and idempotent n-ary operations still eludes us. This and related enumeration results [2, 3] constitute a topic of current research.

ACKNOWLEDGMENTS

The second author is supported by the Luxembourg National Research Fund under the project PRIDE 15/10949314/GSM.

APPENDIX: ALTERNATIVE PROOF OF COROLLARY 1.6

We provide an alternative proof of Corollary 1.6 that does not use [2, Corollary 2.3]. To this extent, we first prove the following general result.

Proposition 3.3. *Let $F \in \mathcal{F}_n^n$. The following assertions are equivalent.*

(i) *F is reducible to an associative and idempotent operation $G\colon X^2 \to X$.*
(ii) *$F((n-1)\cdot x, y) = F(x, (n-1)\cdot y)$ for any $x, y \in X$.*

Proof. The implications $(i) \Rightarrow (ii)$ is straightforward. Now, let us show that $(ii) \Rightarrow (i)$. So, suppose that

$$F((n-1)\cdot x, y) \;=\; F(x, (n-1)\cdot y) \qquad x, y \in X, \tag{6}$$

and consider the operation $G\colon X^2 \to X$ defined by $G(x,y) = F((n-1)\cdot x, y)$ for any $x, y \in X$. It is not difficult to see that G is associative and idempotent. Now, let $x_1,\ldots,x_n \in X$ and let us show that $G^{n-1}(x_1,\ldots,x_n) = F(x_1,\ldots,x_n)$. Using repeatedly (6) and the

idempotency of F we obtain

$$\begin{aligned} G^{n-1}(x_1,\ldots,x_n) &= F^{n-1}((n-1)\cdot x_1,(n-1)\cdot x_2,\ldots,(n-1)\cdot x_{n-1},x_n) \\ &= F^{n-1}((2n-3)\cdot x_1,x_2,(n-1)\cdot x_3,\ldots,(n-1)\cdot x_{n-1},x_n) \\ &= \cdots \\ &= F^{n-1}((n(n-1)-2(n-1)+1)\cdot x_1,x_2,x_3,\ldots,x_{n-1},x_n) \\ &= F(x_1,\ldots,x_n), \end{aligned}$$

which shows that F is reducible to G. □

Remark 5. Let $\leq$ be a total ordering on X. An operation $F\colon X^n \to X$ is said to be $\leq$-*preserving* if $F(x_1,\ldots,x_n) \leq F(x'_1,\ldots,x'_n)$ whenever $x_i \leq x'_i$ for any $i \in \{1,\ldots,n\}$. One of the main results of Kiss and Somlai [7, Theorem 4.8] is that every $\leq$-preserving operation $F \in \mathcal{F}_n^n$ is reducible to an associative, idempotent, and $\leq$-preserving binary operation. The reader can easily verify that the latter is an immediate consequence of Proposition 3.3 above.

The following result is the key for the alternative proof of Corollary 1.6.

Proposition 3.4. *Let $F \in \mathcal{F}_{n-1}^n$. The following assertions are equivalent.*

(i) *F is reducible to an associative and quasitrivial operation $G\colon X^2 \to X$.*
(ii) *F is reducible to an associative and idempotent operation $G\colon X^2 \to X$.*
(iii) *$F((n-1)\cdot x,y) = F(x,(n-1)\cdot y)$ for any $x,y \in X$.*
(iv) $|E_F| \leq 1$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ and the implication $(iii) \Rightarrow (iv)$ are straightforward. Also, the equivalence $(ii) \Leftrightarrow (iii)$ follows from Proposition 3.3. Now, let us show that $(iv) \Rightarrow (iii)$. So, suppose that $|E_F| \leq 1$ and suppose to the contrary that there exist $x,y \in X$ with $x \neq y$ such that $F((n-1)\cdot x,y) \neq F(x,(n-1)\cdot y)$. We have two cases to consider. If $F((n-1)\cdot x,y) = y$ and $F(x,(n-1)\cdot y) = x$, then by Lemma 2.5 we have that $x,y \in E_F$, which contradicts our assumption on E_F. Otherwise, if $F((n-1)\cdot x,y) = x$ and $F(x,(n-1)\cdot y) = y$, then we have

$$\begin{aligned} x = F((n-1)\cdot x,y) &= F((n-1)\cdot x,F(n\cdot y)) \\ &= F(F((n-1)\cdot x,y),(n-1)\cdot y) = F(x,(n-1)\cdot y) = y, \end{aligned}$$

which contradicts the fact that $x \neq y$. □

Proof of Corollary 1.6. This follows from Proposition 3.4 and [5, Lemma 1]. □

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