

FULLER SINGULARITIES FOR GENERIC CONTROL-AFFINE SYSTEMS WITH AN EVEN NUMBER OF CONTROLS

FRANCESCO BOAROTTO, YACINE CHITOUR, AND MARIO SIGALOTTI

ABSTRACT. In this article we study how bad can be the singularities of a time-optimal trajectory of a generic control affine system. Under the assumption that the control has an even number of scalar components and belongs to a closed ball we prove that singularities cannot be, generically, worse than finite order accumulations of Fuller points, with order of accumulation lower than a bound depending only on the dimension of the manifold where the system is set.

1. INTRODUCTION

1.1. Time-optimal trajectories of control-affine systems. Let M be a smooth and connected n -dimensional manifold. Given $k + 1$ smooth vector fields f_0, \dots, f_k on M , we study control systems of the form

$$(1.1) \quad \dot{q} = f_0(q) + \sum_{i=1}^k u_i f_i(q), \quad q \in M, \quad u \in \overline{B}_1^k,$$

where $B_1^k = \{u \in \mathbb{R}^k \mid \|u\| < 1\}$ is the (open) unit ball contained in \mathbb{R}^k , and \overline{B}_1^k denotes its closure. Systems of the form (1.1) are called *control-affine systems*, and the geometric aspects of their evolution has attracted a lot of interest in the mathematical control community (see e.g. [4, 10, 17]).

An *admissible trajectory* of (1.1) is a Lipschitz continuous curve $q : [0, T] \rightarrow M$, $T > 0$, for which there exists $u \in L^\infty([0, T], \overline{B}_1^k)$ such that

$$\dot{q}(t) = f_0(q(t)) + \sum_{i=1}^k u_i(t) f_i(q(t))$$

holds almost everywhere on $[0, T]$.

Definition 1. The time-optimal control problem associated with (1.1) consists into finding the admissible trajectories $q : [0, T] \rightarrow M$ of the system that minimize the time needed to join $q(0)$ and $q(T)$, among all the admissible curves. Admissible trajectories that solve the time-optimal control problem associated with (1.1) are called *time-optimal trajectories*.

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Candidate time-optimal trajectories are characterized by the Pontryagin maximum principle [20] (PMP, in short). Every admissible time-optimal trajectory can be lifted to a Lipschitz continuous trajectory $\lambda : [0, T] \rightarrow T^*M$ of an associated time-dependent Hamiltonian system (see Section 2.1 for details). Moreover, $\lambda(t) \neq 0$ for every $t \in [0, T]$, and for almost every $t \in [0, T]$ the triple $(q(t), \lambda(t), u(t))$ has the property that

$$(1.2) \quad \langle \lambda(t), \sum_{i=1}^k u_i(t) f_i(q(t)) \rangle = \max_{v \in \overline{B}_1^k} \langle \lambda(t), \sum_{i=1}^k v_i f_i(q(t)) \rangle.$$

The triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ is said to be an *extremal triple*, and the PMP reduces the study of time-optimal trajectories to the study of extremal triples. We call *extremal trajectory* any admissible trajectory which is part of an extremal triple, so that any time-optimal trajectory is an extremal trajectory, but the converse does not hold in general.

1.2. Regularity of extremal trajectories. Our goal is to establish regularity results for time-optimal trajectories of control-affine systems. Our methods, however, apply to the broader class of extremal ones.

Given an extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$, the control u can be smoothly reconstructed from the maximality condition (1.2) whenever $\lambda(t)$ is not simultaneously orthogonal to $f_1(q(t)), \dots, f_k(q(t))$. However, smoothness may stop at times where $\lambda(t)$ annihilates $f_1(q(t)), \dots, f_k(q(t))$ and, actually, for any given measurable control $t \mapsto u(t)$, there exist a dynamical system of the form (1.1) and an initial datum $q_0 \in M$ for which the admissible trajectory driven by u and starting at q_0 is time-optimal. This has been noticed in [24] for the single-input case, i.e., when $k = 1$, but can be easily extended to the general case. It makes anyhow sense to investigate regularity properties of extremal trajectories for generic systems or, more generally, for systems satisfying low-codimension non-degeneracy conditions. The single-input case, in particular, gave rise to a vast literature (see, e.g., [5, 7, 8, 21, 22, 23, 25] and the references therein).

Recently, the same questions about the regularity of time-optimal trajectories have been posed in the multi-dimensional input case, but only few results are available [3, 11, 12, 14, 19, 26].

Definition 2. Given an admissible trajectory $q : [0, T] \rightarrow M$, we denote by O_q the maximal open subset of $[0, T]$ such that there exists a control $u \in L^\infty([0, T], \overline{B}_1^k)$, associated with $q(\cdot)$, which is smooth on O_q . We also define Σ_q (or Σ , if no ambiguity is possible) as

$$\Sigma_q = [0, T] \setminus O_q.$$

An isolated point of Σ is usually called a *switching time*. The accumulation of switching times is referred to in the literature as *Fuller phenomenon* (after the pathbreaking work [15]), or also *chattering* or *Zeno behavior*.

Definition 3 (Fuller times). Let us define Σ_0 to be the set of isolated points of Σ . Inductively, we set Σ_j to be the set of isolated points of $\Sigma \setminus (\bigcup_{i=0}^{j-1} \Sigma_i)$. A time $t \in \Sigma_j$ is said to be a *Fuller time of order j* . Finally, we declare points of

$$\Sigma_\infty = \Sigma \setminus \left(\bigcup_{j \geq 0} \Sigma_j \right)$$

to be Fuller times of infinite order.

Remark 4. For every $j \in \mathbb{N}$, the set Σ_j consists of isolated points only, hence it is countable.

We measure the worst stable behavior of “generic” systems of the form (1.1) in terms of the maximal order of their Fuller times. The more an instant t is nested among Fuller times of high order, the greater is the number of relations satisfied by the vectors $f_0(q(t)), \dots, f_k(q(t))$.

Transversality theory is then used to guarantee that generically not too many of these conditions can hold at the same point. As opposed to the analysis in [6], we restrict ourselves to the case of global frames of everywhere linearly independent vector fields, and the word generic must be intended with respect to this property.

Definition 5. For every open set $U \subset M$, we denote by

- $\text{Vec}(U)$ the set of smooth vector fields f on U , endowed with the C^∞ -Whitney topology.
- $\text{Vec}(U)^{k+1}$ the set of all $(k+1)$ -tuples $\mathbf{f} = (f_0, \dots, f_k)$ in $\text{Vec}(U)$ with the corresponding product topology.
- $\text{Vec}(U)_0^{k+1}$ the set of everywhere linearly independent $(k+1)$ -tuples of vector fields on U , that is,

$$\text{Vec}(U)_0^{k+1} = \{ \mathbf{f} \in \text{Vec}(U)^{k+1} \mid f_0(q) \wedge \dots \wedge f_k(q) \neq 0 \text{ for every } q \in U \}.$$

We equip $\text{Vec}(U)_0^{k+1}$ with the topology inherited from $\text{Vec}(U)^{k+1}$.

The next statement contains the precise formulation of our main result, which is obtained under the condition $k = 2m$, that is, assuming that the number of controlled vector fields is even.

Theorem 6. *Let $m, n \in \mathbb{N}$ be such that $2m + 1 \leq n$. Let M be a n -dimensional smooth manifold. There exist a positive integer K depending only on n and an open and dense set $\mathcal{U} \subset \text{Vec}(M)_0^{2m+1}$ such that, if the $(2m+1)$ -tuple $\mathbf{f} = (f_0, \dots, f_{2m})$ is in \mathcal{U} , then every extremal trajectory $q(\cdot)$ of the time-optimal control problem*

$$\dot{q} = f_0(q) + \sum_{i=1}^{2m} u_i f_i(q), \quad q \in M, \quad u \in \overline{B}_1^{2m},$$

has at most Fuller times of order K , i.e.,

$$\Sigma = \Sigma_0 \cup \dots \cup \Sigma_K,$$

where Σ and Σ_j are as in Definitions 2 and 3.

Combining Theorem 6 and Remark 4, we deduce that any extremal trajectory $q(\cdot)$ of a generic control-affine system of the form (1.1) with $k = 2m$ is smooth out of a countable set.

1.3. Remarks on the main result and open problems. We conclude this introduction proposing two lines of investigation related to our study. The first one consists into extending our analysis to the case of linearly dependent frames, as the first and the third author have done in [6, §4.1] for the single-input case. Even though we expect that similar arguments work also in the multi-input case, the differential structure of the singular locus where the fields f_0, \dots, f_{2m} become dependent is more complicated, and needs to be properly investigated.

A different, and possibly more substantial line of research consists into establishing Theorem 6 for systems of the form (1.1) and an *odd number* (greater than one) of controls. The fact that an extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ crosses the singular locus $\{ \lambda \in T^*M \mid \langle \lambda, f_i(q) \rangle = 0, i = 1, \dots, 2m, q = \pi(\lambda) \}$ imposes in the even case a differential condition that we can exploit to begin our iterative arguments (Proposition 20). This condition is based on the results in [3] where the switching behavior in time-optimal trajectories for multi-input control-affine systems is characterized (see also [11] for a study in the same spirit for a class of control-affine systems issuing from the circular restricted three-body problem). In the odd case, it is not clear how to derive such a first additional relation at times at which an extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ crosses the singular locus. In the single-input case, this difficulty has been overcome with a suitable analysis of extremal trajectories around Fuller times [6, Theorem 18], but the arguments there

depend decisively on the fact that the control is scalar. For the general odd case, the problem is open, and new ideas are required.

1.4. Structure of the paper. In Section 2 we present the Pontryagin maximum principle (PMP) to recast the time-optimal problem into its proper geometric framework. Based on the Hamiltonian formalism of the PMP, we establish a differentiation lemma that we will use intensively in the paper (Lemma 10). Section 2 also contains some general observation on the maximal order of the Fuller times in a set (Section 2.3) and classical definitions about jet spaces and transversality theory (Section 2.4). Section 3 collects additional algebraic material on skew-symmetric matrices that we need in subsequent arguments. Sections 4 and 5 are devoted to the recursive characterization of dependence conditions holding at accumulations of Fuller times, when the Goh matrix is, respectively, invertible and singular. Finally, in Section 6, we conclude the proof of the main result, Theorem 6.

2. MAIN TECHNICAL TOOLS

2.1. The Pontryagin maximum principle. Let us introduce some technical notations that we will employ extensively throughout the rest of the paper. Let $\pi : T^*M \rightarrow M$ be the cotangent bundle, and $s \in \Lambda^1(T^*M)$ be the tautological Liouville one-form on T^*M . The non-degenerate skew-symmetric form $\sigma = ds \in \Lambda^2(T^*M)$ endows T^*M with a canonical symplectic structure.

With any C^1 function $p : T^*M \rightarrow \mathbb{R}$ let us associate its Hamiltonian lift $\vec{p} \in C(T^*M, TT^*M)$ by the condition

$$\sigma_\lambda(\cdot, \vec{p}) = d_\lambda p.$$

Fix $\mathbf{f} = (f_0, \dots, f_{2m}) \in \text{Vec}(M)^{2m+1}$. The Pontryagin Maximum Principle (PMP, for short) [20] gives then a necessary condition satisfied by candidate time-optimal trajectories of

$$(2.1) \quad \dot{q} = f_0(q) + \sum_{i=1}^{2m} u_i f_i(q), \quad q \in M, \quad u \in \overline{B}_1^{2m},$$

recalled in the theorem below. Introducing the control-dependent Hamiltonian function $\mathcal{H} : T^*M \times \mathbb{R}^{2m} \rightarrow \mathbb{R}$ by

$$(2.2) \quad \mathcal{H}(\lambda, v) = \langle \lambda, f_0(q) + \sum_{i=1}^{2m} v_i f_i(q) \rangle, \quad q = \pi(\lambda),$$

the precise statement is the following.

Theorem 7 (PMP). *Let $q : [0, T] \rightarrow M$ be a time-optimal trajectory of (2.1), associated with a control $u(\cdot)$. Then there exists an absolutely continuous curve $\lambda : [0, T] \rightarrow T^*M$ such that $(q(\cdot), \lambda(\cdot), u(\cdot))$ is an extremal triple, i.e., in terms of the control-dependent Hamiltonian \mathcal{H} introduced in (2.2), one has*

$$(2.3) \quad \begin{aligned} \lambda(t) &\in T_{q(t)}^*M \setminus \{0\}, \quad \forall t \in [0, T], \\ \mathcal{H}(\lambda(t), u(t)) &= \max\{\mathcal{H}(\lambda(t), v) \mid v \in \overline{B}_1^{2m}\} \quad \text{for a.e. } t \in [0, T], \\ (2.4) \quad \dot{\lambda}(t) &= \vec{\mathcal{H}}(\lambda(t), u(t)), \quad \text{for a.e. } t \in [0, T]. \end{aligned}$$

Definition 8. For any extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$, we call the corresponding trajectory $t \mapsto q(t)$ a time-extremal trajectory, and the curve $t \mapsto \lambda(t)$ its associated time-extremal lift.

For every $i = 0, \dots, 2m$, let us define the smooth functions $h_i : T^*M \rightarrow \mathbb{R}$ by

$$h_i(\lambda) := \langle \lambda, f_i(q) \rangle, \quad q = \pi(\lambda).$$

More generally, let k be an integer and $D = i_1 \cdots i_k$ a multi-index of $\{0, 1, \dots, 2m\}$, and let $|D| := k$ be the length of D . A multi-index $D = i \cdots i$ with k consecutive occurrences of the index i is denoted as $D = i^k$. We use f_D to denote the vector field defined by

$$f_D = [f_{i_1}, [\cdots, [f_{i_{k-1}}, f_{i_k}] \cdots]],$$

and h_D to denote the smooth function on T^*M given by $\langle \lambda, f_D \rangle$ for $\lambda \in T^*M$.

By a slight abuse of notations, given a time-extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ defined on $[0, T]$, we define $h_i(t) := h_i(\lambda(t))$ for every $i = 1, \dots, 2m$ and $t \in [0, T]$. Throughout the rest of the paper, we further extend this convention in the following way: whenever $\varphi : T^*M \rightarrow \mathbb{R}$ is a scalar function defined on T^*M and $t \mapsto \lambda(t)$ is an integral curve of $\vec{\mathcal{H}}$, we denote by $\varphi(t)$ the evaluation of φ at $\lambda(t)$ if no ambiguity is possible.

Denote by I the set $\{1, \dots, 2m\}$ and by h_I the map $h_I : T^*M \rightarrow \mathbb{R}^{2m}$ defined by

$$h_I(\lambda) = (h_1(\lambda), \dots, h_{2m}(\lambda)).$$

Let us first recall that the time-extremal control u is smooth (up to modification on a set of measure zero) on the open set $R_q := \{t \in [0, T] \mid h_I(t) \neq 0\}$, i.e., in terms of the set Σ_q introduced in Definition 2,

$$(2.5) \quad \Sigma_q \subset \{t \in [0, T] \mid h_I(t) = 0\}.$$

Indeed, the maximality condition (2.3) provided by the PMP yields the explicit characterization

$$u(t) = \frac{h_I(t)}{\|h_I(t)\|}, \quad t \in R_q.$$

Therefore an extremal trajectory on R_q is an integral curve of the vector field

$$\lambda \mapsto \vec{\mathcal{H}} \left(\lambda, \frac{h_I(\lambda)}{\|h_I(\lambda)\|} \right),$$

which is well-defined and smooth on $T^*M \setminus \{\lambda \in T^*M \mid h_I(\lambda) = 0\}$. In particular, its integral curves are smooth as well.

We also recall the following differentiation formula along a time-extremal lift $t \mapsto \lambda(t)$, which follows as a consequence of the symplectic structure on T^*M (see [1, Section 3.3]).

Proposition 9. *Let $\varphi : T^*M \rightarrow \mathbb{R}$ be a C^1 function, and let $\lambda : [0, T] \rightarrow T^*M$ be a solution of (2.4) corresponding to a control $u : [0, T] \rightarrow \overline{B}_1^{2m}$. Then*

$$(2.6) \quad \frac{d}{dt} \varphi(\lambda(t)) = \{h_0, \varphi\}(\lambda(t)) + \sum_{i=1}^{2m} u_i(t) \{h_i, \varphi\}(\lambda(t)) \quad \text{a.e. on } [0, T].$$

In particular, Proposition 9 implies that for every $X \in \text{Vec}(M)$ and every extremal triple associated with (2.1) the identity

$$\frac{d}{dt} \langle \lambda(t), X(q(t)) \rangle = \langle \lambda(t), [f_0 + \sum_{i=1}^{2m} u_i(t) f_i, X](q(t)) \rangle$$

holds true for a.e. t (here we apply the proposition to $\varphi(\lambda) = \langle \lambda, X(\pi(\lambda)) \rangle$).

Denote by $M_{j,k}(\mathbb{R})$ the set of $j \times k$ matrices with real entries and let $M_j(\mathbb{R}) = M_{j,j}(\mathbb{R})$. We introduce the map

$$(2.7) \quad \begin{aligned} H_{II} : T^*M &\rightarrow M_{2m}(\mathbb{R}), \\ \lambda &\mapsto (\{h_i, h_j\}(\lambda))_{i,j=1}^{2m}. \end{aligned}$$

For every $\lambda \in T^*M$, the skew symmetric matrix $H_{II}(\lambda)$ is called the *Goh matrix*. Defining $h_{0I} : T^*M \rightarrow M_{2m,1}(\mathbb{R})$ to be the vector-valued function $(h_{0i}(\lambda))_{i=1}^{2m}$ and differentiating h_I along a time-extremal triple, we find by the previous considerations that

$$\dot{h}_I(t) = h_{0I}(t) - H_{II}(t)u(t)$$

for a.e. t (notice that the minus sign is a consequence of considering the transposition in (2.6)). In particular, within the set R , the dynamics of h_I are described by

$$\dot{h}_I(t) = h_{0I}(t) - H_{II}(t) \frac{h_I(t)}{\|h_I(t)\|}.$$

2.2. A differentiation lemma. We present in this section a result that we will extensively use in the paper. It concerns the differentiation along an extremal curve of a smooth function on T^*M that vanishes at a converging sequence of times.

Lemma 10. *Let $(q(\cdot), \lambda(\cdot), u(\cdot))$ be an extremal triple on $[0, T]$ associated with (2.1). Assume that there exists a sequence of times $(t_l)_{l \in \mathbb{N}}$ in $[0, T]$ such that $t_l \rightarrow t^* \in [0, T]$ and $t_l \neq t^*$ for every $l \in \mathbb{N}$. Then there exists $u^* \in \overline{B}_1^{2m}$ such that, for every smooth function $\varphi : T^*M \rightarrow \mathbb{R}$ satisfying $\varphi(\lambda(t_l)) = 0$ for every $l \in \mathbb{N}$,*

$$\{h_0, \varphi\}(\lambda(t^*)) + \sum_{i=1}^{2m} u_i^* \{h_i, \varphi\}(\lambda(t^*)) = 0.$$

Proof. Since $u(\cdot) \in L^\infty([0, T], \overline{B}_1^{2m})$, there exists a subsequence $(t_{l_w})_{w \in \mathbb{N}}$ such that the limit

$$u^* := \lim_{w \rightarrow \infty} \frac{1}{t^* - t_{l_w}} \int_{t_{l_w}}^{t^*} u(t) dt$$

exists and belongs to \overline{B}_1^{2m} .

Consider a smooth function $\varphi : T^*M \rightarrow \mathbb{R}$ such that $\varphi(\lambda(t_l)) = 0$ for every $l \in \mathbb{N}$. By continuity we have $\varphi(\lambda(t^*)) = 0$, so that by Proposition 9 for every $l \in \mathbb{N}$ we can write

$$(2.8) \quad \begin{aligned} 0 &= \frac{\varphi(\lambda(t^*)) - \varphi(\lambda(t_l))}{t^* - t_l} = \frac{1}{t^* - t_l} \int_{t_l}^{t^*} \frac{d}{dt} \varphi(\lambda(t)) dt \\ &= \frac{1}{t^* - t_l} \int_{t_l}^{t^*} (\{h_0, \varphi\}(\lambda(t)) + \sum_{i=1}^{2m} u_i(t) \{h_i, \varphi\}(\lambda(t))) dt. \end{aligned}$$

Rewriting (2.8) along the subsequence t_{l_w} and taking the limit as $w \rightarrow \infty$ permits then to conclude, since $t \mapsto \{h_i, \varphi\}(\lambda(t))$ is absolutely continuous for every $i = 0, \dots, 2m$. \square

2.3. Fuller order of a set. For a subset Ξ of \mathbb{R} we denote by Ξ_0 its subset made of isolated points and, inductively, by Ξ_j the set of isolated points of $\Xi \setminus (\bigcup_{i=0}^{j-1} \Xi_i)$, $j \geq 1$.

Definition 11. We say that Ξ has *Fuller order* $k \in \mathbb{N}$ if $\Xi = \Xi_0 \cup \dots \cup \Xi_k$ and $\Xi_k \neq \emptyset$. We say that \emptyset has *Fuller order* -1 and that Ξ has *Fuller order* ∞ if $\Xi \setminus (\bigcup_{i=0}^k \Xi_i) \neq \emptyset$ for every $k \in \mathbb{N}$.

Remark 12. The notion of Fuller order is strictly related to the one of Cantor-Bendixson rank: if X is a topological space (in particular, a subset of \mathbb{R} with the induced topology) the *Cantor-Bendixson rank* of X is the least ordinal such that $X^{(\alpha)} = X^{(\alpha+1)}$, where $X^{(1)} = \{x \in X \mid x \in \overline{X \setminus \{x\}}\}$ is the *derived subset* of X , $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$, and $X^{(\beta)} = \bigcap_{\alpha < \beta} X^{(\alpha)}$. For *scattered* sets, i.e., sets such that $X^{(k)} = \emptyset$ for some $k \in \mathbb{N}$, the Cantor-Bendixson rank is equal to the Fuller order plus 1. For perfect sets, on the contrary, the Fuller order is infinite and the Cantor-Bendixson rank is zero.

The properties of the Fuller order described in the following two results have been probably already observed in the context of Cantor-Bendixson rank but we were not able to find a precise reference for them.

Lemma 13. *Let Ξ, \mathfrak{S} be two subsets of \mathbb{R} . If Ξ has Fuller order at least k and \mathfrak{S} has Fuller order at most j , with $k > j \geq 0$, then $\Xi \setminus \mathfrak{S}$ has Fuller order at least $k - j - 1$.*

Proof. Without loss of generality Ξ has order k and \mathfrak{S} has order j . Notice that it is enough to prove the lemma in the case $j = 0$, since every set \mathfrak{S}_i , $i = 0, \dots, h$, is of Fuller order 0 and

$$\Xi \setminus \mathfrak{S} = (\dots((\Xi \setminus \mathfrak{S}_0) \setminus \mathfrak{S}_1) \dots) \setminus \mathfrak{S}_j).$$

Let us prove the property by induction on k , assuming that $\mathfrak{S} = \mathfrak{S}_0$. In the case $k = 1$, we just need to notice that $\Xi \setminus \mathfrak{S}$ is nonempty and hence has nonnegative Fuller order. Assume now that the property holds for $k - 1$ and let us prove it for k . Consider a point $x \in \Xi_k$. If x is in \mathfrak{S} , then there exists a neighborhood of x which does not contain any point of \mathfrak{S} except x . Since x is a density point for Ξ_{k-1} , we deduce that there exist points in Ξ_{k-1} at positive distance from \mathfrak{S} . Hence $\Xi \setminus \mathfrak{S}$ has Fuller order at least $k - 1$. Assume now that x is in $\Xi \setminus \mathfrak{S}$. Notice that, by the induction hypothesis, for every neighborhood U of x , the set $U \cap ((\Xi_0 \cup \dots \cup \Xi_{k-1}) \setminus \mathfrak{S})$ has Fuller order at least $k - 2$. We can then extract a sequence in $((\Xi_0 \cup \dots \cup \Xi_{k-1}) \setminus \mathfrak{S})_{k-2}$ converging to x . We deduce that $\Xi \setminus \mathfrak{S}$ has Fuller order at least $k - 1$. \square

As an immediate consequence, we get the following result.

Corollary 14. *Let $k \geq 1$, $j \geq 0$, and $\Xi \subset \mathbb{R}$ be the union of Ξ^1, \dots, Ξ^k . If Ξ^i has Fuller order at most j for every $i \in \{1, \dots, k\}$, then Ξ has Fuller order at most $k(j + 1)$.*

2.4. Jet spaces and transversality. Following [13], for any nonempty open subset U of M we introduce:

- JTU : the jet space of the smooth vector fields on U ,
- $J^N TU$, $N \in \mathbb{N}$: the jet space of order N ,
- $J_{2m+1}^N TU$: the fiber product $J^N TU \times_U \dots \times_U J^N TU$ of $2m + 1$ copies of $J^N TU$,
- $J_q^N TU$: the fiber of $J^N TU$ at $q \in U$,
- $J_{2m+1,q}^N TU$: the fiber of $J_{2m+1}^N TU$ at $q \in U$,
- $T_{2m+1,N}$ the typical fiber of $J_{2m+1}^N TU$.

The spaces JTU , $J^N TU$ and $J_{2m+1}^N TU$ are endowed with the Whitney C^∞ topology.

If N is a positive integer and $f \in \text{Vec}(U)$ (respectively, $\mathbf{f} \in \text{Vec}(U)^{2m+1}$), we use $j^N(f)$ and $j_q^N(f)$ (respectively, $j^N(\mathbf{f})$ and $j_q^N(\mathbf{f})$) to denote respectively the jet of order N associated with f (respectively, the $(2m + 1)$ -tuple of jets of order N associated with \mathbf{f}) and its evaluation at $q \in U$ (respectively, the evaluation of $j^N(\mathbf{f})$ at $q \in U$).

Fix $N \in \mathbb{N}$ and let $P(n, N)$ be the set of all polynomial mappings

$$G := (G^1, \dots, G^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n, \quad \deg(G^i) \leq N, \quad \text{for every } 1 \leq i \leq n.$$

Similarly, we call $P(n, N)^{2m+1}$ the set of all $(2m + 1)$ -tuples of elements in $P(n, N)$, that is,

$$P(n, N)^{2m+1} = \{(Q_1, \dots, Q_{2m+1}) \mid Q_i \in P(n, N), 1 \leq i \leq 2m + 1\}.$$

Assume from now on that U is the domain of a coordinate chart (x, U) centered at some $q \in U$. This allows one to identify the typical fiber $T_{2m+1, N}$ of $J_{2m+1}^N TU$ with $P(n, N)^{2m+1}$ as explained below. There is a standard way [7] of introducing coordinates on the semi-algebraic set

$$\Omega := \{(Q_1, \dots, Q_{2m+1}) \in P(n, N)^{2m+1} \mid Q_1(0) \wedge \dots \wedge Q_{2m+1}(0) \neq 0\} \subset P(n, N)^{2m+1},$$

which we briefly recall.

Let $\mathcal{K}_0 = \{0\}$, and \mathcal{K}_k be the set of k -tuples of ordered integers in $\{1, \dots, n\}$. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a homogeneous polynomial of degree k , and $\xi = (\xi_1, \dots, \xi_k) \in (\mathbb{R}^n)^k$, the polarization of f along ξ is the real number

$$Pf(\xi) := D_{\xi_1} \dots D_{\xi_k} f,$$

where, for every $\eta \in \mathbb{R}^n$, $D_\eta f$ denotes the directional derivative of f along η .

Given $\widehat{Q} \in \Omega$, we complete $(\widehat{Q}_1(0), \dots, \widehat{Q}_{2m+1}(0))$ to a basis of \mathbb{R}^n with $n - 2m - 1$ vectors $v_{2m+2}, \dots, v_n \in \mathbb{R}^n$. There exists a neighborhood $V \subset \Omega$ of \widehat{Q} such that the map

$$\begin{aligned} \text{ev} : V &\rightarrow (\mathbb{R}^n)^n \\ Q &\mapsto (Q_1(0), \dots, Q_{2m+1}(0), v_{2m+2}, \dots, v_n) \end{aligned}$$

associates with any element $Q \in V$ a basis of \mathbb{R}^n . For $1 \leq i \leq n$ and $Q \in V$, we also employ the notation $\text{ev}(Q)_i$ to refer to the i -th component of $\text{ev}(Q)$. In particular $\text{ev}(Q)_i \in \mathbb{R}^n$. This allows to introduce a coordinate chart X_V on V , in such a way that every $Q = (Q_1, \dots, Q_{2m+1}) \in V$ can be written with coordinates

$$\left\{ X_{i, \sigma}^j \mid 1 \leq i \leq 2m + 1, 1 \leq j \leq n, \sigma \in \mathcal{K}_k, 0 \leq k \leq N \right\},$$

where the element $X_{i, \sigma}^j$ denotes the polarization of the j -th coordinate of the homogeneous part of degree $k = |\sigma|$ of Q_i along the element $(\text{ev}(Q)_{\sigma_1}, \dots, \text{ev}(Q)_{\sigma_k})$.

Consider the now the chart (X_V, x) on the domain $V \times U \subset \Omega \times M$. If $\sigma \in \mathcal{K}_k$, define $\sigma! = \sigma_1! \dots \sigma_k!$ and $x^\sigma = x_1^{\sigma_1} \dots x_k^{\sigma_k}$. In local coordinates, Q_i is represented by

$$Q_i = \frac{\partial}{\partial x_i} + \sum_{\substack{1 \leq k \leq N \\ \sigma \in \mathcal{K}_k}} \frac{x^\sigma}{\sigma!} X_{i, \sigma}, \quad X_{i, \sigma} = \sum_{j=1}^n X_{i, \sigma}^j \frac{\partial}{\partial x_j},$$

and $X_{i, \sigma}$ is a constant vector field.

If $1 \leq i \neq k \leq 2m + 1$, in these local coordinates we see that $[Q_i, Q_k](0) = Q_{ik}(0) = X_{k, i} - X_{i, k}$ and similarly, if $Q_{i^l k}$ denotes the l -fold iterated bracket $\text{ad}_{Q_i}^l(Q_k)$, we deduce inductively that $Q_{i^l k}(0) = X_{k, i^l} + R_{i, k, l}$, where $R_{i, k, l}$ is a polynomial in the coordinates $X_{s, \sigma}^a$, with $1 \leq a \leq n$, $1 \leq s \leq 2m + 1$, $|\sigma| \leq l$ and $\sigma \neq j^l$. Similar computations can be carried out for all iterated brackets.

Remark 15. Let $((x, \psi), \pi^{-1}(U))$ be the induced chart on T^*U , where $\psi = (\psi_r)_{r=1}^n$. In particular, we use λ_ψ to denote the elements of T_0^*M given in coordinates by $(0, \psi)$. The typical fiber $\widehat{T}_{2m+1, N}$ of the vector bundle $J_{2m+1}^N TU \times_U T^*U$ is isomorphic to $P(n, N)^{2m+1} \times \mathbb{R}^n$. Clearly, $h_{ik}(\lambda_\psi) = \langle \psi, X_{k, i} \rangle - \langle \psi, X_{i, k} \rangle$ and, for $l \geq 1$,

$$h_{i^l k}(\lambda_\psi) = \langle \psi, Q_{i^l k}(0) \rangle = \langle \psi, X_{k, i^l} \rangle + \langle \psi, R'_{i, k, l} \rangle,$$

where $R'_{i,k,l}$ is a polynomial in the coordinates $\psi_r, X_{s,\sigma}^a$ with $1 \leq a, r \leq n$, $1 \leq s \leq 2m+1$, $|\sigma| \leq l$ and $\sigma \neq j^l$. By an induction argument, $h_D(\lambda_\psi)$, with D a multi-index, can be expressed as a polynomial function in terms of the coordinates $\psi_r, X_{s,\sigma}^a$. Therefore, this choice of the chart (X_V, x) allows one to see every h_D and $h_D \circ \text{ev}$ as a real-valued function on $J_{2m+1}^N TU \times_U T^*U$ and on its typical fiber $\widehat{T}_{2m+1,N}$, respectively, where N is large enough. This will also be the case for any polynomial function in the h_D 's.

The following result follows by standard transversality arguments (see, e.g., [2, 16]).

Lemma 16 (Transversality Lemma). *Let $N \in \mathbb{N}$. Let \mathcal{B} be a closed subset of $J_{2m+1}^N TM$ and assume that for every $q \in M$ there exists a coordinate chart (x, U) centered at q such that $\mathcal{B} \cap J_{2m+1}^N TU$ is semi-algebraic in the coordinates (X_V, x) introduced above. For every $q \in M$ let $\mathcal{B}_q := \mathcal{B} \cap J_{2m+1,q}^N TM$. Let \mathcal{V} be the open subset of $\text{Vec}(M)_0^{2m+1}$ made of the $(2m+1)$ -tuples $\mathbf{f} = (f_0, \dots, f_{2m})$ such that, for every $q \in M$, $j_q^N(\mathbf{f}) \notin \mathcal{B}_q$. Assume that \mathcal{B}_q has codimension larger than or equal to $n+1$ in $J_{2m+1,q}^N TM$ for every $q \in M$. Then \mathcal{V} is also dense in $\text{Vec}(M)_0^{2m+1}$.*

3. ALGEBRAIC CONSIDERATIONS

3.1. Decomposition of skew-symmetric matrices. We collect in this section some general facts regarding the algebraic structure of skew-symmetric matrices. For any $l \in \mathbb{N}$, we recall that the notation $\mathfrak{so}(l)$ stands for the linear space of $l \times l$ skew-symmetric real matrices. We begin by [recalling some useful properties concerning the Pfaffian of a skew-symmetric matrix](#).

Lemma 17. *Let $A \in \mathfrak{so}(2m)$. Then the following properties hold true.*

- i) $\det(A) = \text{Pf}(A)^2$, where $\text{Pf}(A)$, called the Pfaffian of A , is a homogeneous polynomial in the entries of A of degree m .
- ii) There exists a $2m \times 2m$ skew-symmetric matrix $\text{adj}^{\text{Pf}}(A)$, called the adjoint Pfaffian of A , such that its entries are homogeneous polynomial of degree $m-1$ in the entries of A and

$$\text{adj}^{\text{Pf}}(A)A = \text{Pf}(A)\text{Id}_{2m}.$$

Proof. Item i) is classical, and we refer the reader to [18] for a proof. [Concerning Item ii\), it can be found, for instance, in \[9, Equation \(3.2\)\].](#) \square

The next proposition collects a list of useful properties valid for general skew-symmetric matrices of size k .

Proposition 18. *Let $k \in \mathbb{N}$ and $A \in \mathfrak{so}(k)$ be nonzero. Then the following holds true.*

- i) The rank of A is an even integer $1 \leq 2m_0 \leq k$ and there exists a nonzero principal minor of order $2m_0$. As a consequence, there exists a permutation matrix P such that

$$(3.1) \quad P^T A P = \begin{pmatrix} A_1 & A_2 \\ -A_2^T & A_3 \end{pmatrix},$$

where $A_1 \in \mathfrak{so}(2m_0)$ is invertible, $A_2 \in M_{2m_0, k-2m_0}(\mathbb{R})$, and $A_3 \in \mathfrak{so}(k-2m_0)$.

- ii) With $P^T A P$ presented as in (3.1) one has

$$\ker(P^T A P) = \text{span}\{(-A_1^{-1} A_2 x_2, x_2) \mid x_2 \in \mathbb{R}^{k-2m_0}\}.$$

In particular, A_1 , A_2 , and A_3 satisfy the relation

$$A_2^T A_1^{-1} A_2 + A_3 = 0.$$

iii) Let e_1, \dots, e_{k-2m_0} be the canonical basis of \mathbb{R}^{k-2m_0} . Define

$$v_i = \left(-\text{adj}^{\text{Pf}}(A_1)A_2e_i, \text{Pf}(A_1)e_i \right), \quad 1 \leq i \leq k - 2m_0,$$

where $\text{adj}^{\text{Pf}}(A_1)$ denotes the adjoint Pfaffian of A_1 introduced in Lemma 17. Then the family v_1, \dots, v_{k-2m_0} is a basis of $\ker(P^T AP)$, and the coordinates of each v_i , for $i = 1, \dots, k - 2m_0$, are homogeneous polynomials of degree m_0 in the entries of A .

Proof. We begin by i). First note that the conclusion is equivalent to prove that A admits a $2m_0 \times 2m_0$ nonzero principal minor, i.e., the determinant of an $2m_0 \times 2m_0$ principal submatrix. Recall that, for $1 \leq l \leq k$, the coefficient of $(-1)^l x^{k-l}$ of the characteristic polynomial of any $k \times k$ matrix is equal to the sum of its $l \times l$ principal minors. If A is a $k \times k$ skew-symmetric matrix, notice that its principal submatrices are themselves skew-symmetric. One deduces that the coefficients of $(-1)^l x^{k-l}$ in the characteristic polynomial P_A of A are zero if l is odd and sums of squares if l is even, according to i) of Lemma 17. Moreover, if the rank of A is equal to $2m_0$, then $P_A(x) = x^{k-2m_0}Q(x)$ with $Q(0) \neq 0$ since A is diagonalizable over \mathbb{C} . Hence the coefficient of x^{k-2m_0} of P_A is nonzero, yielding the existence of a $2m_0 \times 2m_0$ nonzero principal minor.

We pass now to Point ii). Let us consider any element $w = (w_1, w_2) \in \ker(P^T AP)$. Computing the product $P^T APw = 0$, and recalling that A_1 is invertible, we obtain the relations

$$w_1 = -A_1^{-1}A_2w_2, \quad (A_2^T A_1^{-1}A_2 + A_3)w_2 = 0.$$

By assumption, $\ker(P^T AP)$ has dimension $k - 2m_0$, therefore there exists a basis $w_2^1, \dots, w_2^{k-2m_0}$ of \mathbb{R}^{k-2m_0} , such that the elements

$$(A_1^{-1}A_2w_2^i, w_2^i), \quad i = 1, \dots, k - 2m_0,$$

belong to $\ker(P^T AP)$ and are linearly independent. In particular the $(k - 2m_0) \times (k - 2m_0)$ skew-symmetric matrix $(A_2^T A_1^{-1}A_2 + A_3)$ has a $(k - 2m_0)$ -dimensional kernel, and therefore it is the zero matrix.

As for Point iii), it is sufficient to notice that the elements

$$v_i := \text{Pf}(A_1)(-A_1^{-1}A_2e_i, e_i), \quad i = 1, \dots, k - 2m_0,$$

form a basis of $\ker(P^T AP)$ and that, by Lemma 17,

$$v_i = (-\text{adj}^{\text{Pf}}(A_1)A_2e_i, \text{Pf}(A_1)e_i), \quad i = 1, \dots, k - 2m_0,$$

and, in particular, the coordinates of v_i are homogeneous polynomials of degree m_0 in the entries of A . \square

3.2. Consequences on the structure of the Goh matrix. We apply here below Proposition 18 to the skew-symmetric Goh matrix H_{II} defined in (2.7).

Let $(q(\cdot), \lambda(\cdot), u(\cdot))$ be a time-extremal triple of (2.1), and assume that $t^* \in [0, T]$ is such that $1 \leq \text{rank}(H_{II}(t^*)) = 2m_0 \leq 2m$. Then, up to a permutation of the basis of \mathbb{R}^{2m} we can present $H_{II}(t^*)$ in the block form

$$H_{II}(t^*) = \begin{pmatrix} H_{II}^{2m_0}(t^*) & E(t^*) \\ -E(t^*)^T & F(t^*) \end{pmatrix},$$

where $H_{II}^{2m_0}(t^*) \in M_{2m_0}(\mathbb{R})$ and $F(t^*) \in M_{2(m-m_0)}(\mathbb{R})$ are skew-symmetric matrices, $H_{II}^{2m_0}(t^*)$ is invertible and $E(t^*) \in M_{2m_0, 2(m-m_0)}(\mathbb{R})$. Then the following holds true.

Proposition 19. *There exist a relatively open interval $\mathcal{J} \subset [0, T]$ containing t^* , and smooth functions $v_1, \dots, v_{2(m-m_0)} : [0, T] \rightarrow \mathbb{R}^{2m}$ such that:*

- i) for every $i = 1, \dots, 2(m - m_0)$ and every $t \in \mathcal{J}$, letting e_i be the i -th element of the canonical basis of $\mathbb{R}^{2(m-m_0)}$,

$$v_i(t) = \begin{pmatrix} -\text{adj}^{\text{Pf}}(H_{II}^{2m_0}(t))E(t)e_i \\ \text{Pf}(H_{II}^{2m_0}(t))e_i \end{pmatrix}$$

is a $2m$ -dimensional vector whose components are homogeneous polynomials of degree m_0 in the entries $h_{ij}(t)$ of the Goh matrix;

- ii) if $t \in \mathcal{J}$ is such that $\text{rank}(H_{II}(t)) = 2m_0$, then

$$\ker(H_{II}(t)) = \text{span}\{v_1(t), \dots, v_{2(m-m_0)}(t)\};$$

- iii) if $t \in \mathcal{J}$ is such that $\text{rank}(H_{II}(t)) = 2m_0$, the non-trivial relations expressed by the matrix equality

$$E(t)^T \text{adj}^{\text{Pf}}(H_{II}^{2m_0}(t))E(t) + \text{Pf}(H_{II}^{2m_0}(t))F(t) = 0$$

are homogeneous polynomial relations of degree $m_0 + 1$ in the entries $h_{ij}(t)$ of the Goh matrix.

4. ITERATED ACCUMULATIONS OF POINTS IN Σ WITH INVERTIBLE GOH MATRIX

Let $(q(\cdot), \lambda(\cdot), u(\cdot))$ be an extremal triple of (2.1). Consider the set

$$(4.1) \quad \Sigma^{2m} := \Sigma \cap \{t \in [0, T] \mid \det H_{II}(t) \neq 0\},$$

where Σ is the set constructed in Definition 2. In analogy with Definition 3, we define Σ_0^{2m} to be the set of isolated points of Σ^{2m} and, inductively, we set Σ_j^{2m} to be the set of isolated points of $\Sigma^{2m} \setminus (\bigcup_{i=0}^{j-1} \Sigma_i^{2m})$.

The starting point of the study of accumulations of singularities in Σ^{2m} is the following result.

Proposition 20. *Let $t^* \in \Sigma^{2m} \setminus \Sigma_0^{2m}$. Then*

$$\|H_{II}(t^*)^{-1}h_{0I}(t^*)\| = 1.$$

Proof. Since $t^* \in \Sigma^{2m} \subset \Sigma$, we have that $\det(H_{II}(t^*)) \neq 0$ and we deduce from (2.5) that $h_I(t^*) = 0$. Moreover, since $t^* \notin \Sigma_0^{2m}$, there exists a nontrivial sequence $(t_l)_{l \in \mathbb{N}} \subset \Sigma^{2m}$ converging to t^* such that $h_I(t_l) = 0$ for every $l \in \mathbb{N}$. Applying Lemma 10 to $\varphi = h_i$, $i \in I$, we infer the existence of $u^* \in \overline{B}_1^{2m}$ such that

$$h_{0I}(t^*) - H_{II}(t^*)u^* = 0,$$

that is, we deduce that $h_{0I}(t^*) \in H_{II}(t^*)\overline{B}_1^{2m}$.

Assume by contradiction that $h_{0I}(t^*) \in H_{II}(t^*)B_1^{2m}$. Then we deduce from [3, Theorem 3.4] that h_I vanishes identically in a relative neighborhood $\mathcal{J} \subset [0, T]$ of t^* . Note that [3, Theorem 3.4] is stated for time-optimal trajectories, but it actually holds true for extremal trajectories, since its proof only relies on the properties of the extremal flow characterized by the PMP.

Upon shrinking \mathcal{J} , we can assume that $\det(H_{II}(t)) \neq 0$ for every $t \in \mathcal{J}$. Differentiating the relation $h_I|_{\mathcal{J}} \equiv 0$, we find that $u(t) = H_{II}(t)^{-1}h_{0I}(t)$ holds true a.e. on \mathcal{J} . The differential system generated by the Hamiltonian function

$$H^0(p) = \langle p, f_0(q) \rangle + \sum_{i=1}^{2m} (H_{II}(p)^{-1}h_{0I}(p))_i \langle p, f_i(q) \rangle, \quad p \in T^*M, \quad q = \pi(p),$$

where $(H_{II}(p)^{-1}h_{0I}(p))_i$ is the i -th component of $H_{II}(p)^{-1}h_{0I}(p)$, is well-defined on the set $\{p \in T^*M \mid \text{rank}(H_{II}(p)) = 2m\}$. Moreover, the time-extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ satisfies

$$\dot{\lambda}(t) = \vec{H}^0(\lambda(t)),$$

almost everywhere on J , that is, it is an integral curve of \vec{H}^0 on J . But this forces $u(\cdot)$ to be smooth on J , contradicting the assumption that t^* is an element of Σ^{2m} . The contradiction argument yields

$$\|H_{II}(t^*)^{-1}h_{0I}(t^*)\| = 1,$$

and the statement follows. \square

As a direct consequence of Lemma 17 and Proposition 20, we deduce the following.

Corollary 21. *Let $t^* \in \Sigma^{2m} \setminus \Sigma_0^{2m}$. Then, defining the symmetric $2m \times 2m$ matrix $S_H(t^*) := \text{adj}^{\text{Pf}}(H_{II})^2(t^*)$, one has*

$$\langle S_H(t^*)h_{0I}(t^*), h_{0I}(t^*) \rangle + \det(H_{II}(t^*)) = 0.$$

In particular, $\langle S_H(t^*)h_{0I}(t^*), h_{0I}(t^*) \rangle \neq 0$.

Definition 22. Define the smooth functions $(\phi_\ell)_{\ell \in \mathbb{N}}$ and the matrix-valued functions $(\Phi_\ell)_{\ell \in \mathbb{N}}$ on T^*M by

$$(4.2) \quad \begin{aligned} \phi_0(\lambda) &= \langle S_H(\lambda)h_{0I}(\lambda), h_{0I}(\lambda) \rangle + \det(H_{II}(\lambda)), \\ \Phi_0(\lambda) &= \begin{pmatrix} h_{0I}(\lambda) & -H_{II}(\lambda) \\ \{h_0, \phi_0\}(\lambda) & \{h_I, \phi_0\}(\lambda)^T \end{pmatrix} \in M_{2m+1}(\mathbb{R}), \end{aligned}$$

and, inductively with respect to $\ell \geq 0$,

$$(4.3) \quad \phi_{\ell+1}(\lambda) = \det(\Phi_\ell(\lambda)), \quad \Phi_{\ell+1}(\lambda) = \begin{pmatrix} h_{0I}(\lambda) & -H_{II}(\lambda) \\ \{h_0, \phi_{\ell+1}\}(\lambda) & \{h_I, \phi_{\ell+1}\}(\lambda)^T \end{pmatrix} \in M_{2m+1}(\mathbb{R}).$$

Remark 23. By Point ii) of Lemma 17, we see that ϕ_0 in (4.2) is a polynomial function in the elements h_{ik} for $i \in \{0, \dots, 2m\}$ and $k \in I$. Moreover, we deduce inductively that all the functions $(\phi_\ell)_{\ell \in \mathbb{N}}$ are polynomial functions in the elements $\text{ad}_{h_{i_1}} \circ \dots \circ \text{ad}_{h_{i_\nu}}(h_{jk})(\lambda)$ for $\nu \in \mathbb{N}$ and $i_1, \dots, i_\nu, j, k \in \{0, \dots, 2m\}$.

It is useful to make the following observation on the structure of the constraint $\phi_\ell(\lambda) = 0$. Its proof can be obtained by an easy inductive argument.

Lemma 24. *Let $\ell \in \mathbb{N}$ and $\lambda \in T^*M$. Then*

$$\phi_\ell(\lambda) = \text{ad}_{h_0}^\ell(\phi_0)(\lambda) \det(H_{II}(\lambda))^\ell + B_\ell(\lambda),$$

where $B_\ell(\lambda)$ is the evaluation of a polynomial depending only on ℓ at a point whose coordinates are $h_{ik}(\lambda)$ for $i \in \{0, \dots, 2m\}$ and $k \in I$, and $\text{ad}_{h_{i_1}} \circ \dots \circ \text{ad}_{h_{i_\nu}}(\phi_0)(\lambda)$ for $0 \leq \nu \leq \ell$ and $i_1, \dots, i_\nu \in \{0, \dots, 2m\}$, with the property that if $\nu = \ell$ then $(i_1, \dots, i_\nu) \neq (0, \dots, 0)$.

The following result illustrates the relation between the functions ϕ_ℓ and the Fuller order of the set Σ^{2m} .

Proposition 25. *Let $\ell \in \mathbb{N}$ and $t^* \in \Sigma^{2m} \setminus \bigcup_{j=0}^\ell \Sigma_j^{2m}$. Then $\phi_j(\lambda(t^*)) = 0$ for every $j = 0, \dots, \ell$.*

Proof. First notice that, since Σ^{2m} is relatively open in Σ , one has $\Sigma_j^{2m} = \Sigma^{2m} \cap \Sigma_j$ for every $j \geq 0$.

We proceed by induction, observing that the case $\ell = 0$ follows from Corollary 21.

Assume the conclusion to be true for some integer $\ell \geq 0$, and let us establish it for $\ell + 1$. Pick $t^* \in \Sigma^{2m} \setminus \bigcup_{j=0}^{\ell+1} \Sigma_j^{2m}$ and a sequence $(t_w)_{w \in \mathbb{N}} \subset \Sigma^{2m} \setminus \bigcup_{j=0}^{\ell} \Sigma_j^{2m}$ converging to t^* . The inductive step yields that $\phi_j(t_w) = 0$ for $j = 0, \dots, \ell$ and $w \in \mathbb{N}$. The equalities $\phi_j(t^*) = 0$, $j = 0, \dots, \ell$, follow by continuity, and we are left to prove that $\phi_{\ell+1}(t^*) = 0$. Lemma 10 applies both to $\varphi = \phi_\ell$ and $\varphi = h_j$, $j \in I$, and allows to conclude that there exists $u^* \in \overline{B}_1^{2m}$ such that

$$\Phi_{\ell+1}(\lambda(t^*)) \begin{pmatrix} 1 \\ u^* \end{pmatrix} = 0,$$

where $\Phi_{\ell+1}$ is defined as in (4.3). Hence, $\phi_{\ell+1}(\lambda(t^*)) = \det(\Phi_{\ell+1}(\lambda(t^*))) = 0$. \square

In the next lemma, using the fact that the conditions $\phi_\ell = 0$ define independent constraints on the jets, we deduce from Proposition 25 and Lemma 16 that the set Σ^{2m} has Fuller order at most $2n - 1$.

Lemma 26. *There exists an open and dense set $\mathcal{V}_{2m} \subset \text{Vec}(M)_0^{2m+1}$ such that, for every $\mathbf{f} = (f_0, \dots, f_{2m}) \in \mathcal{V}_{2m}$ and every extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ of (2.1),*

$$(4.4) \quad \Sigma^{2m} = \bigcup_{j=0}^{2n-1} \Sigma_j^{2m}.$$

Proof. The proof of the lemma follows a classical strategy found, e.g., in [7]. Let us construct the set $\widehat{\mathcal{B}} \subset J_{2m+1}^{2n+1}TM \times_M T^*M$ by

$$\widehat{\mathcal{B}} = \left\{ (j_q^{2n+1}(\mathbf{f}), \lambda) \mid (q, \lambda) \in T^*M, \mathbf{f} = (f_0, \dots, f_{2m}) \in \text{Vec}(M)_0^{2m+1}, \right. \\ \left. \det(H_{II}(\lambda)) \neq 0, \phi_0(\lambda) = \dots = \phi_{2n-1}(\lambda) = 0 \right\},$$

where $\phi_0, \dots, \phi_{2n-1}$ are defined in (4.2) and (4.3). We denote then by \mathcal{B} the canonical projection of $\widehat{\mathcal{B}}$ onto $J_{2m+1}^{2n+1}TM$. Similarly, for $q \in M$, we define $\widehat{\mathcal{B}}_q \subset J_{2m+1,q}^{2n+1}TM \times T_q^*M$ by

$$\widehat{\mathcal{B}}_q := \widehat{\mathcal{B}} \cap J_{2m+1,q}^{2n+1}TM \times T_q^*M,$$

and by \mathcal{B}_q the canonical projection of $\widehat{\mathcal{B}}_q$ onto $J_{2m+1,q}^{2n+1}TM$.

Notice that, for every coordinate chart (x, U) , $\widehat{\mathcal{B}} \cap J_{2m+1}^{2n+1}TU \times T^*U$ is an algebraic subset of $J_{2m+1}^{2n+1}TU \times T^*U$ for the coordinates (X_V, x, ψ) introduced in Section 2.4. Hence, $\mathcal{B} \cap J_{2m+1}^{2n+1}TU$ is a semi-algebraic subset of $J_{2m+1}^{2n+1}TU$.

We now consider the set \mathcal{V}_{2m} of vector fields $\mathbf{f} \in \text{Vec}(M)_0^{2m+1}$ verifying the following: for every $q \in M$, $j_q^{2n+1}(\mathbf{f}) \notin \mathcal{B}_q$. We claim that (4.4) holds true if $\mathbf{f} \in \mathcal{V}_{2m}$. In fact, arguing by contradiction, assume that for such an \mathbf{f} and an extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ of (2.1), there exists $t^* \in \Sigma^{2m} \setminus \bigcup_{j=0}^{2n-1} \Sigma_j^{2m}$. Then, Proposition 25 implies that

$$(j_{q(t^*)}^{2n+1}(\mathbf{f}), \lambda(t^*)) \in \widehat{\mathcal{B}},$$

yielding that $j_{q(t^*)}^{2n+2}(\mathbf{f}) \in \mathcal{B}_{q(t^*)}$ and contradicting the fact that $\mathbf{f} \in \mathcal{V}_{2m}$. The claim follows.

We conclude the proof of Lemma 26 thanks to Lemma 16, by showing that for every $q \in M$, the set \mathcal{B}_q defined above has codimension larger than or equal to $n + 1$ in $J_{2m+1,q}^{2n+1}TM$.

Let $q \in M$, and consider a local coordinate chart (x, U) on M centered at q . Lift this chart to a coordinate chart $((x, \psi), \pi^{-1}(U))$ on T^*U as in Remark 15, and recall that $J_{2m+1,q}^{2n+1}TM \times T_q^*M$

is isomorphic to $P(n, 2n + 1)^{2m+1} \times \mathbb{R}^n$. By taking into account Remark 23, the map

$$E_\phi^{2n} : P(n, 2n + 1)^{2m+1} \times \mathbb{R}^n \rightarrow \mathbb{R}^{2n},$$

$$(Q, \psi) \mapsto (\phi_0(\lambda_\psi), \dots, \phi_{2n-1}(\lambda_\psi)),$$

is well defined. Then, up to the identification of $J_{2m+1, q}^{2n+1} TU \times T_q^* U$ and $P(n, 2n + 1)^{2m+1} \times \mathbb{R}^n$, $\widehat{\mathcal{B}}_q = \{(Q, \psi) \in (E_\phi^{2n})^{-1}(0) \mid \det(H_{II}(\lambda_\psi)) \neq 0\}$.

In order to prove that \mathcal{B}_q has codimension larger than or equal to $n + 1$ we first show that $\widehat{\mathcal{B}}_q$ has codimension $2n$ by proving that E_ϕ^{2n} is a submersion at every point of $\widehat{\mathcal{B}}_q$. To that purpose, we compute in local coordinates the maps $\phi_i(\lambda_\psi)$ for $0 \leq i \leq 2n - 1$.

Following (4.2) and recalling that $S_H(\lambda) \in M_{2m}(\mathbb{R})$ is symmetric, we have

$$(4.5) \quad \phi_0(\lambda) = \sum_{i,j=1}^{2m} P_{i,j}(\lambda) h_{0i}(\lambda) h_{0j}(\lambda) + R_0(\lambda),$$

where the $P_{i,j}(\lambda)$ and $R_0(\lambda)$ are polynomial functions in the variables $h_{st}(\lambda)$, with $1 \leq s, t \leq 2m$, and not all the $P_{i,j}(\lambda)$ are zero. In local coordinates this gives

$$(4.6) \quad \phi_0(\lambda_\psi) = \sum_{i,j=1}^{2m} P_{i,j}(\psi) \langle \psi, X_{0,i} \rangle \langle \psi, X_{0,j} \rangle + R_0(\psi),$$

where the $P_{i,j}(\psi)$ and $R_0(\psi)$ are now polynomial functions in the variables $\langle \psi, X_{s,t} \rangle$, with $1 \leq s, t \leq 2m$, and not all the $P_{i,j}(\psi)$ are zero.

From Lemma 24, (4.5) and an easy inductive argument, one deduces that, for $0 \leq l \leq 2n - 1$,

$$\phi_l(\lambda) = \det(H_{II}(\lambda))^l \sum_{i,j=1}^{2m} P_{i,j,l}(\lambda) (h_{0^{l+1}i}(\lambda) h_{0j}(\lambda) + h_{0i}(\lambda) h_{0^{l+1}j}(\lambda)) + R_{0,l}(\lambda),$$

where the $P_{i,j,l}(\lambda)$ are (not all zero) polynomial functions in the variables $h_{st}(\lambda)$, $1 \leq s, t \leq 2m$ and $R_{0,l}(\lambda)$ is a polynomial function in the variables $\text{ad}_{h_{i_1}} \circ \dots \circ \text{ad}_{h_{i_\nu}}(\phi_0)(\lambda)$ for $0 \leq \nu \leq l$ and $i_1, \dots, i_\nu \in \{0, \dots, 2m\}$, with the property that if $\nu = l$ then $(i_1, \dots, i_\nu) \neq (0, \dots, 0)$. In local coordinates one deduces that, for $0 \leq l \leq 2n - 1$,

$$(4.7) \quad \phi_l(\lambda_\psi) = \det(H_{II}(\lambda_\psi))^l \sum_{i,j=1}^{2m} P_{i,j,l}(\psi) \left(\langle \psi, X_{0^{l+1}i} \rangle \langle \psi, X_{0,j} \rangle + \langle \psi, X_{0,i} \rangle \langle \psi, X_{0^{l+1}j} \rangle \right) + R_{0,l}(\psi),$$

where the $P_{i,j,l}(\psi)$ are polynomial functions in the variables $\langle \psi, X_{s,t} \rangle$, $1 \leq s, t \leq 2m$ and $R_{0,l}(\psi)$ is a polynomial function in the variables $\langle \psi, X_{i_1 \dots i_\nu} \rangle$, for $0 \leq \nu \leq l$ and $i_1, \dots, i_\nu \in \{0, \dots, 2m\}$, with the property that if $\nu = l$ then $(i_1, \dots, i_\nu) \neq (0, \dots, 0)$. From (4.6) and (4.7), one deduces that the map E_ϕ^{2n} is a submersion at every point of $\widehat{\mathcal{B}}_q$, since the polynomials $P_{i,j,l}$ are not all zero.

We proved that $\widehat{\mathcal{B}}_q$ has codimension $2n$, from which it follows readily that the codimension of \mathcal{B}_q is larger than or equal to $2n - n + 1 = n + 1$ by projection, where the extra term $+1$ is due to the homogeneity of each of the relations $\phi_l(\lambda_\psi) = 0$ with respect to λ_ψ . This concludes the proof of Lemma 26. \square

5. ITERATED ACCUMULATIONS OF POINTS IN Σ WITH SINGULAR GOH MATRIX

We consider in this section the complementary case in which the Goh matrix H_{II} does not have full rank.

Let us fix $1 \leq a \leq m$, and consider the sets

$$\begin{aligned}\Sigma^{2(m-a)} &= \Sigma \cap \{t \in [0, T] \mid \text{rank } H_{II}(t) = 2(m-a)\}, \\ (T^*M)^{2(m-a)} &= T^*M \cap \{\lambda \in T^*M \mid \text{rank } H_{II}(\lambda) = 2(m-a)\}.\end{aligned}$$

Observe that the notation is consistent with the notation Σ^{2m} introduced in (4.1), which effectively corresponds to the case $a = 0$.

By point i) of Proposition 18, for every $\lambda \in (T^*M)^{2(m-a)}$ there exists a permutation matrix $P_\lambda \in M_{2m}(\mathbb{R})$ such that

$$(5.1) \quad P_\lambda^T H_{II}(\xi) P_\lambda = \begin{pmatrix} H_{II}^{2(m-a), \lambda}(\xi) & E^\lambda(\xi) \\ -E^\lambda(\xi)^T & F^\lambda(\xi) \end{pmatrix} \quad \text{for every } \xi \in T^*M,$$

where $H_{II}^{2(m-a), \lambda} : T^*M \rightarrow M_{2(m-a)}(\mathbb{R})$, $E^\lambda : T^*M \rightarrow M_{2(m-a), 2a}(\mathbb{R})$ and $F^\lambda : T^*M \rightarrow M_{2a}(\mathbb{R})$ are matrix-valued functions, with the property that $H_{II}^{2(m-a), \lambda}(\lambda)$ is of maximal rank (equal to $2(m-a)$).

Remark 27. We assume the permutation matrix P_λ to be chosen according to the following algorithmic rule: pick the subset J_0^λ of I of cardinality $2(m-a)$ such that the matrix extracted from $H_{II}(\lambda)$ with row and column indices in J_0^λ is invertible and which is minimal for the lexicographic order among all the subsets of I with the same property. (Subsets of I of cardinality $2(m-a)$ are here identified with strings of indices of length $2(m-a)$.) Then if $J_0^\lambda = \{j_1, \dots, j_{2(m-a)}\}$ and $I \setminus J_0^\lambda = \{\ell_1, \dots, \ell_{2a}\}$ with $j_1 < \dots < j_{2(m-a)}$ and $\ell_1 < \dots < \ell_{2a}$, pick as permutation the reordering of $1, \dots, 2m$ into $j_1, \dots, j_{2(m-a)}, \ell_1, \dots, \ell_{2a}$.

Consider the smooth vector-valued functions

$$v_i^\lambda : T^*M \rightarrow \mathbb{R}^{2m}, \quad \xi \mapsto \begin{pmatrix} -\text{adj}^{\text{Pf}}(H_{II}^{2(m-a), \lambda}(\xi)) E^\lambda(\xi) e_i \\ \text{Pf}(H_{II}^{2(m-a), \lambda}(\xi)) e_i \end{pmatrix}, \quad i = 1, \dots, 2a,$$

where e_1, \dots, e_{2a} denotes the canonical basis of \mathbb{R}^{2a} , with the convention that $v_i^\lambda(\xi) = e_i$ when $a = m$. By point iii) of Proposition 18, there exists a neighborhood $O_\lambda \subset T^*M$ of λ such that the collection $\{v_i^\lambda(\xi) \mid 1 \leq i \leq 2a\}$ parametrizes the kernel of $P_\lambda^T H_{II}(\xi) P_\lambda$ for every $\xi \in O_\lambda \cap (T^*M)^{2(m-a)}$. We also define for $1 \leq i \leq 2a$, the functions

$$(5.2) \quad \begin{aligned}\kappa_i^\lambda : T^*M &\rightarrow \mathbb{R}, \\ \xi &\mapsto \langle P_\lambda^T h_{0I}(\xi), v_i^\lambda(\xi) \rangle,\end{aligned}$$

and, finally, letting

$$(5.3) \quad \begin{aligned}G^\lambda : T^*M &\rightarrow \mathfrak{so}(2a), \\ \xi &\mapsto E^\lambda(\xi)^T \text{adj}^{\text{Pf}}(H_{II}^{2(m-a), \lambda}(\xi)) E^\lambda(\xi) + \text{Pf}(H_{II}^{2(m-a), \lambda}(\xi)) F^\lambda(\xi),\end{aligned}$$

we list all of the $a(2a-1)$ independent entries of G^λ as a collection of functions $g_l^\lambda : T^*M \rightarrow \mathbb{R}$, for $1 \leq l \leq a(2a-1)$. Notice that $G^\lambda(\xi) = F^\lambda(\xi) = H_{II}(\xi)$ if $a = m$.

Proposition 28. *Let $1 \leq a \leq m$ and consider, for $1 \leq i \leq 2a$ and $1 \leq l \leq a(2a-1)$, the functions κ_i^λ and g_l^λ defined in (5.2) and (5.3), respectively. Consider an extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$. Then the following holds true:*

- (i) if $t \in \Sigma^{2(m-a)}$, then $g_l^{\lambda(t)}(t) = 0$, $l = 1, \dots, a(2a-1)$;
- (ii) if moreover $t \in \Sigma^{2(m-a)} \setminus \Sigma_0$, we also have $\kappa_i^{\lambda(t)}(t) = 0$ for every $i = 1, \dots, 2a$.

Proof. Our considerations being local, it is not restrictive to work with the Goh matrix H_{II} in the block form (5.1). The fact that for $t \in \Sigma^{2(m-a)}$ and $1 \leq l \leq a(2a-1)$, $g_l^{\lambda(t)}(t) = 0$ is the content of Point iii) of Proposition 19. If, in addition, t is in $\Sigma^{2(m-a)} \setminus \Sigma_0$, then by definition there exists a nontrivial sequence $(t_l)_{l \in \mathbb{N}} \subset \Sigma_0$ that converges to t and yielding by (2.5) and Lemma 10 the existence of some $u^* \in \overline{B}_1^{2m}$ such that

$$h_{0I}(t) - H_{II}(t)u^* = 0.$$

Since $H_{II}(t)$ is a skew-symmetric matrix, the above relation implies that

$$h_{0I}(t) \in \ker(H_{II}(t))^\perp,$$

whence $\kappa_i^{\lambda(t)}(t) = 0$ for every $1 \leq i \leq 2a$. □

The following rather long and technical definition aims at identifying sufficiently many independent functions that vanish at high order density points of Σ .

Definition 29. Let $\lambda \in (T^*M)^{2(m-a)}$ with $1 \leq a \leq m$ and consider $\kappa_1^\lambda, \dots, \kappa_{2a}^\lambda : T^*M \rightarrow \mathbb{R}$ and $g_1^\lambda, \dots, g_{a(2a-1)}^\lambda : T^*M \rightarrow \mathbb{R}$ defined as in (5.2) and (5.3), respectively. For every $r \in \mathbb{N}$ consider $\rho_r^\lambda \in \{2(m-a), \dots, 2m\}$, $J_r^\lambda \subset \{1, \dots, 2m\}$, $\mu_r^\lambda : T^*M \rightarrow \mathbb{R}$, $S_r^\lambda : T^*M \rightarrow M_{\rho_r^\lambda, 2m}(\mathbb{R})$, $T_r^\lambda : T^*M \rightarrow M_{\rho_r^\lambda+1, 2m}(\mathbb{R})$, and $V_r^\lambda : T^*M \rightarrow M_{\rho_r^\lambda, 1}(\mathbb{R})$ defined inductively as follows:

- $\rho_0^\lambda = 2(m-a)$, $\mu_0^\lambda = g_1^\lambda$, J_0^λ is the set defined in Remark 27, and

$$S_0^\lambda(\xi) = \begin{pmatrix} H_{II}^{2(m-a), \lambda}(\xi) & E^\lambda(\xi) \end{pmatrix}, \quad V_0^\lambda(\xi) = \begin{pmatrix} h_{01}(\xi) \\ \vdots \\ h_{0, 2(m-a)}(\xi) \end{pmatrix}.$$

(Here and in the following, $\{1, \dots, 2(m-a)\}$ is identified with J_0^λ by the permutation described in Remark 27.) Notice that $S_0^\lambda(\xi)$ is the $2(m-a) \times 2m$ matrix obtained by selecting only rows of the Goh matrix $H_{II}(\xi)$ with indices in J_0^λ ;

- for $r \geq 1$, define ρ_r^λ to be the rank of $S_r^\lambda(\lambda)$ and J_r^λ to be the subset of $\{1, \dots, 2m\}$ of cardinality ρ_r^λ such that the matrix extracted from $S_r^\lambda(\lambda)$, with column indices in J_r^λ is invertible, and which is minimal for the lexicographic order among all subsets of $\{1, \dots, 2m\}$ with the same property.

Let, moreover, for $r \geq 0$,

$$T_r^\lambda(\xi) = \begin{pmatrix} S_r^\lambda(\xi) \\ \{h_I, \mu_r^\lambda\}(\xi) \end{pmatrix}$$

and notice that the rank of $T_r^\lambda(\lambda)$ is either equal to ρ_r^λ or to $\rho_r^\lambda + 1$;

- if $\text{rank}(T_r^\lambda(\lambda)) = \rho_r^\lambda + 1$, set

$$S_{r+1}^\lambda(\xi) = T_r^\lambda(\xi), \quad V_{r+1}^\lambda(\xi) = \begin{pmatrix} V_r^\lambda(\xi) \\ \{h_0, \mu_r^\lambda\}(\xi) \end{pmatrix}.$$

Then $\rho_{r+1}^\lambda = \rho_r^\lambda + 1$ and set $\mu_{r+1}^\lambda = \kappa_{\rho_{r+1}^\lambda - \rho_0^\lambda}^\lambda$;

- if $\text{rank}(T_r^\lambda(\lambda)) = \rho_r^\lambda$ set

$$S_{r+1}^\lambda(\xi) = S_r^\lambda(\xi), \quad V_{r+1}^\lambda(\xi) = V_r^\lambda(\xi).$$

Then $\rho_{r+1}^\lambda = \rho_r^\lambda$. Let, moreover, $Z_r^\lambda(\cdot)$ be the matrix extracted from $S_r^\lambda(\cdot)$ with column indices in J_r^λ , and define

$$\begin{aligned} \tilde{S}_r^\lambda : T^*M &\rightarrow M_{\rho_r^\lambda+1}(\mathbb{R}) \\ \xi &\mapsto \begin{pmatrix} V_r^\lambda(\xi) & Z_r^\lambda(\xi) \\ \{h_0, \mu_r^\lambda\}(\xi) & \{h_{J_r^\lambda}, \mu_r^\lambda\}(\xi) \end{pmatrix}. \end{aligned}$$

Set then $\mu_{r+1}^\lambda(\xi) = \det(\tilde{S}_r^\lambda(\xi))$ for every $\xi \in T^*M$.

Notice once again that, by Proposition 19, the functions $\kappa_1^\lambda, \dots, \kappa_{2a}^\lambda$ and $g_1^\lambda, \dots, g_{a(2a-1)}^\lambda$ are polynomials in the elements h_{jk} for $j, k \in \{0, \dots, 2m\}$. Inductively, the construction of Definition 29 implies that all the functions $(\mu_r^\lambda)_{r \in \mathbb{N}}$, and the entries of the matrix-valued functions $(S_r^\lambda)_{r \in \mathbb{N}}$, $(T_r^\lambda)_{r \in \mathbb{N}}$ and $(V_r^\lambda)_{r \in \mathbb{N}}$ are polynomials in the elements $\text{ad}_{h_{i_1}} \circ \dots \circ \text{ad}_{h_{i_\nu}}(h_{jk})$ for $\nu \in \mathbb{N}$ and $i_1, \dots, i_\nu, j, k \in \{0, \dots, 2m\}$.

For every $\lambda \in \cup_{a=1}^m (T^*M)^{2(m-a)}$ the sequence $(\rho_r^\lambda)_{r \in \mathbb{N}}$ is nondecreasing and takes values in $\{0, \dots, 2m\}$. Hence, given any $N \in \mathbb{N}$, the pigeonhole principle implies that for every λ there exists $r \leq 2mN$ such that

$$(5.4) \quad \rho_r^\lambda = \rho_{r+1}^\lambda = \dots = \rho_{r+N}^\lambda.$$

Given $N \in \mathbb{N}$ and $\lambda \in \cup_{a=1}^m (T^*M)^{2(m-a)}$, we define

$$R_N(\lambda) = (\rho_0^\lambda, \dots, \rho_{(2m+1)N}^\lambda, J_0^\lambda, \dots, J_{(2m+1)N}^\lambda).$$

We denote by Υ_N the range of R_N and we notice that it is of finite cardinality.

The main property justifying the above definition is the following.

Proposition 30. *Fix $N \geq 1$ and $\bar{R} \in \Upsilon_N$. For $k = 0, \dots, 2(m+1)N$, denote by μ_k the function such that $\mu_k^\lambda = \mu_k$ for every λ such that $R_N(\lambda) = \bar{R}$. Let $(q(\cdot), \lambda(\cdot), u(\cdot))$ be an extremal triple of (2.1) and define*

$$\mathfrak{S}^{\bar{R}} = \{t \in \Sigma \mid R_N(\lambda(t)) = \bar{R}\}.$$

Denote by $\mathfrak{S}_0^{\bar{R}}$ the set of isolated points of $\mathfrak{S}^{\bar{R}}$ and, inductively, by $\mathfrak{S}_j^{\bar{R}}$ the set of isolated points of $\mathfrak{S}^{\bar{R}} \setminus (\cup_{i=0}^{j-1} \mathfrak{S}_i^{\bar{R}})$. Then, for every $k \in \{0, \dots, 2(m+1)N\}$ and every

$$t \in \mathfrak{S}^{\bar{R}} \setminus \left(\bigcup_{j=0}^k \mathfrak{S}_j^{\bar{R}} \right),$$

we have

$$\mu_0(t) = \dots = \mu_k(t) = 0.$$

Proof. Let us first notice that $\rho_k^\lambda, J_k^\lambda, V_k^\lambda$ and the other matrices introduced in Definition 29 do not depend on λ provided that $R_N(\lambda) = \bar{R}$. To simplify the notations we then drop the index λ .

Let us prove the proposition by induction on k . For $k = 0$ recall that $\mu_0 = g_1$ and the conclusion follows from Proposition 28. The same argument works in the inductive step from $k - 1$ to k whenever $\rho_{k-1} < \rho_k$, since in this case $\mu_k = \kappa_{\rho_k - \rho_0}$. When, instead, $\rho_{k-1} = \rho_k$, notice that by the inductive assumption and by Lemma 10 there exists $u^* \in \bar{B}_1^{2m}$ such that $\{h_0, \mu_j\} + \sum_{i=1}^{2m} u_i^* \{h_i, \mu_j\}$ and $\{h_0, h_\ell\} + \sum_{i=1}^{2m} u_i^* \{h_i, h_\ell\}$ vanish at $\lambda(t)$ for every $j = 1, \dots, k-1$ and every $\ell = 1, \dots, 2m$. In particular,

$$\begin{pmatrix} 1 & u^* \end{pmatrix} \in \ker \begin{pmatrix} V_{k-1}(t) & S_{k-1}(t) \\ \{h_0, \mu_{k-1}\}(t) & \{h_I, \mu_{k-1}\}(t) \end{pmatrix}.$$

Since, moreover, the ranks of $\begin{pmatrix} S_{k-1}(t) \\ \{h_I, \mu_{k-1}\}(t) \end{pmatrix}$ and of its extracted matrix $\begin{pmatrix} Z_{k-1}(t) \\ \{h_{J_{k-1}}, \mu_{k-1}\}(t) \end{pmatrix}$ are equal, we deduce that there exists $v^* \in \mathbb{R}^{\rho_k}$ such that

$$\begin{pmatrix} 1 & v^* \end{pmatrix} \in \ker \begin{pmatrix} V_{k-1}(t) & Z_{k-1}(t) \\ \{h_0, \mu_{k-1}\}(t) & \{h_{J_{k-1}}, \mu_{k-1}\}(t) \end{pmatrix}.$$

Thus $\det(\tilde{S}_k)(t) = \mu_k(t) = 0$, proving the claim. \square

In order to study the independence of the constraints $\mu_j(\lambda) = 0$ we investigate in the next lemma their expression.

Lemma 31. *Fix $N \geq 1$ and $\bar{R} \in \Upsilon_N$. For $k = 0, \dots, 2(m+1)N$, denote by ρ_k the integer such that $\rho_k^\lambda = \rho_k$ for every λ such that $R_N(\lambda) = \bar{R}$, and define similarly μ_k, J_k, Z_k and the other matrices introduced in Definition 29. Let $r, k \geq 0$ be such that $r + k \leq (2m+1)N$,*

$$\rho_r = \dots = \rho_{r+k},$$

and either $r = 0$ or $\rho_{r-1} < \rho_r$. Then

$$(5.5) \quad \mu_{r+j}(\xi) = \text{ad}_{h_0}^j(\kappa_{\rho_r - \rho_0})(\xi) \det(Z_r(\xi))^j + P_j(\xi), \quad \forall j \in \{0, \dots, k\}, \xi \in T^*M,$$

where $P_j(\xi)$ is the evaluation of a polynomial depending only on j at variables of the form $h_{i_\ell}(\xi)$ with $i \in \{0, \dots, 2m\}$ and $\ell \in J_r$, or $\text{ad}_{h_{i_1}} \circ \dots \circ \text{ad}_{h_{i_\nu}}(\mu_\ell)(\xi)$ with $1 \leq \nu \leq j$, $i_1, \dots, i_\nu \in \{0, \dots, 2m\}$, and $\ell \in \{0, \dots, r\}$, with the property that if $\ell = r$ then $(i_1, \dots, i_\nu) \neq (0, \dots, 0)$.

Proof. Let us prove Equation (5.5) by induction on j . In the case $j = 0$, by the assumption made on r , $\mu_r = \kappa_{\rho_r - \rho_0}$ and the conclusion follows. For $j = 1, \dots, k$, $\mu_{r+j} = \det(\tilde{S}_{r+j-1})$, $V_{r+j} = V_r$, $Z_{r+j} = Z_r$, and a simple recursive argument allows to conclude. \square

Using the properties of the functions μ_j obtained in the last two results, we are able to prove the following lemma on the Fuller order of the set $\mathfrak{S}^{\bar{R}}$ introduced in the statement of Proposition 30.

Lemma 32. *Let $N \in \mathbb{N}$ and $\bar{R} \in \Upsilon_N$. Assume that $N \geq 2n$. Then there exists an open and dense set $\mathcal{V}_{\bar{R}} \subset \text{Vec}(M)_0^{2m+1}$ such that, for every $(f_0, \dots, f_{2m}) \in \mathcal{V}_{\bar{R}}$, for every extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ of (2.1), $\mathfrak{S}^{\bar{R}}$ is of Fuller order at most $2(m+1)N$.*

Proof. Let us use the same notational convention for μ_j, ρ_j and the other objects introduced in Definition 29 as in the statement of Lemma 31. Let $r \in \{0, \dots, 2mN\}$ be minimal such that

$$\rho_r = \dots = \rho_{r+N}.$$

(compare with formula (5.4).)

Reasoning as in Lemma 26, define $\mathcal{B} \subset J_{2m+1}^{(2m+1)N+2}TM$ by projecting on $J_{2m+1}^{(2m+1)N+2}TM$ the set $\widehat{\mathcal{B}} \subset J_{2m+1}^{(2m+1)N+2}TM \times_M T^*M$ defined by

$$\widehat{\mathcal{B}} = \left\{ \left(j_q^{(2m+1)N+2}(\mathbf{f}), \lambda \right) \mid (q, \lambda) \in T^*M, \mathbf{f} = (f_0, \dots, f_{2m}) \in \text{Vec}(M)_0^{2m+1}, \right. \\ \left. \det(Z_r(\lambda)) \neq 0, \mu_r(\lambda) = \dots = \mu_{r+N}(\lambda) = 0 \right\}.$$

Moreover, for $q \in M$, we set $\widehat{\mathcal{B}}_q = \widehat{\mathcal{B}} \cap J_{2m+1, q}^{(2m+1)N+2}TM \times T_q^*M$ and $\mathcal{B}_q = \mathcal{B} \cap J_{2m+1, q}^{(2m+1)N+2}TM$.

We define the open set $\mathcal{V}_{\bar{R}}$ as the set of $\mathbf{f} \in \text{Vec}(M)_0^{2m+1}$ with the property that, for every $q \in M$, $j_q^{(2m+1)N+2}(\mathbf{f}) \notin \mathcal{B}_q$. We claim that $\mathfrak{S}^{\bar{R}}$ is of Fuller order at most $2(m+1)N$ if $\mathbf{f} \in \mathcal{V}_{\bar{R}}$. Indeed, assume by contradiction that for $\mathbf{f} \in \mathcal{V}_{\bar{R}}$ and an extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$

of (2.1) there exists $t^* \in \mathfrak{S}^{\bar{R}} \setminus \left(\bigcup_{k=0}^{2(m+1)N} \mathfrak{S}_k^{\bar{R}} \right)$. We deduce that $j_{q(t^*)}^{(2m+1)N+2}(\mathbf{f}) \in \mathcal{B}_{q(t^*)}$ by Proposition 30, from which the contradiction follows.

To conclude as in Lemma 26 and deduce from Lemma 16 that $\mathcal{V}_{\bar{R}}$ is dense in $\text{Vec}(M)_0^{2m+1}$, it suffices to show that for every $q \in M$ the codimension of \mathcal{B}_q in $J_{2m+1,q}^{(2m+1)N+2}TM$ is larger than or equal to $n+1$.

Let $q \in M$, and consider a local coordinate chart (x, U) on M centered at q . Lift this chart to a coordinate chart $((x, \psi), \pi^{-1}(U))$ on T^*U as in Section 2.4. By construction, $\mathcal{B} \cap J_{2m+1,q}^{2n+1}TU$ is a semi-algebraic subset of $J_{2m+1,q}^{(2m+1)N+2}TU$.

Recall that $J_{2m+1,q}^{(2m+1)N+2}TM \times T^*M$ is isomorphic to $P(n, (2m+1)N+2)^{2m+1} \times \mathbb{R}^n$. Owing again to Remark 15, the map

$$\begin{aligned} \mu^N : P(n, (2m+1)N+2)^{2m+1} \times \mathbb{R}^n &\rightarrow \mathbb{R}^N, \\ (Q, \psi) &\mapsto (\mu_r(\lambda_\psi), \dots, \mu_{r+N}(\lambda_\psi)) \end{aligned}$$

is well defined, and $\widehat{\mathcal{B}}_q = \{(Q, \psi) \in (\mu^N)^{-1}(0) \mid \det(Z_r(\lambda_\psi)) \neq 0\}$. From here, we conclude as in Lemma 26. By Proposition 30 we have

$$(5.6) \quad \mu_{r+l}(\lambda) = \text{ad}_{h_0}^l(\kappa_{\rho_r - \rho_0})(\lambda) \det(Z_r(\lambda))^l + R_l(\lambda),$$

where $R_l(\lambda)$ is the evaluation of a polynomial depending only on l at variables of the form $h_{i_\ell}(\lambda)$ with $i \in \{0, \dots, 2m\}$ and $\ell \in J_r$, or $\text{ad}_{h_{i_1}} \circ \dots \circ \text{ad}_{h_{i_\nu}}(\mu_\ell)(\lambda)$ with $1 \leq \nu \leq l$, $i_1, \dots, i_\nu \in \{0, \dots, 2m\}$, and $\ell \in \{0, \dots, r\}$, with the property that if $\ell = r$ then $(i_1, \dots, i_\nu) \neq (0, \dots, 0)$. A routine computation of (5.6) in local coordinates $((X_{i,j})_{i,j=0}^{2m}, (\psi_r)_{r=1}^n)$ allows to conclude that the map μ^N is a submersion at every point of $\widehat{\mathcal{B}}_q$, whence we conclude that the codimension of \mathcal{B}_q is greater than or equal to $N - n + 1 \geq 2n - n + 1 = n + 1$, where again the $+1$ follows by the homogeneity of the relations $\mu_r(\lambda_\psi) = \dots = \mu_{r+N}(\lambda_\psi) = 0$ with respect to λ_ψ . The conclusion follows. \square

6. PROOF OF THEOREM 6

Let $N \geq 2n$ and define $\mathcal{U} = \mathcal{V}_{2m} \cap (\bigcap_{\bar{R} \in \Upsilon_N} \mathcal{V}_{\bar{R}})$, where \mathcal{V}_{2m} is as in Lemma 26 and the sets $\mathcal{V}_{\bar{R}}$ as in Lemma 32.

In particular, \mathcal{U} is open and dense in $\text{Vec}(M)_0^{2m+1}$, and has the property that for every $(f_0, \dots, f_{2m}) \in \mathcal{U}$, every extremal triple $(q(\cdot), \lambda(\cdot), u(\cdot))$ of (2.1), Σ^{2m} is of Fuller order at most $2n-1$ and, for every $\bar{R} \in \Upsilon_N$, $\mathfrak{S}^{\bar{R}}$ is of Fuller order at most $2(m+1)N$.

Denote by N^* the cardinality of Υ_N . Notice that N^* only depends on n and m . Since $\Sigma = \Sigma^{2m} \cup \left(\bigcup_{\bar{R} \in \Upsilon_N} \mathfrak{S}^{\bar{R}} \right)$, we deduce from Corollary 14 that Σ has Fuller order at most $(2(m+1)N+1)N^* + 2n$. Finally, since $m \leq (n-1)/2$, we conclude the proof of Theorem 6 by taking $K = \max\{(2(m+1)N+1)N^* + 2n \mid m = 1, \dots, \lfloor (n-1)/2 \rfloor\}$.

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DIPARTIMENTO DI MATEMATICA TULLIO LEVI-CIVITA, UNIVERSITÀ DEGLI STUDI DI PADOVA, ITALY
E-mail address: francesco.boarotto@math.unipd.it

UNIVERSITÉ PARIS-SUD, L2S, CENTRALESUPÉLEC, UNIVERSITÉ PARIS-SACLAY, GIF-SUR-YVETTE, FRANCE
E-mail address: yacine.chitour@l2s.centralesupelec.fr

INRIA & LABORATOIRE JACQUES-LOUIS LIONS, CNRS, SORBONNE UNIVERSITÉ, UNIVERSITÉ DE PARIS, FRANCE
E-mail address: Mario.Sigalotti@inria.fr