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1 **ON THE APPROXIMATION OF ELECTROMAGNETIC FIELDS BY**  
2 **EDGE FINITE ELEMENTS. PART 3: SENSITIVITY TO**  
3 **COEFFICIENTS**

4 PATRICK CIARLET JR\*

5 **Abstract.** In bounded domains, the regularity of the solutions to boundary value problems  
6 depends on the geometry, and on the coefficients that enter into the definition of the model. This is in  
7 particular the case for the time-harmonic Maxwell equations, whose solutions are the electromagnetic  
8 fields. In this paper, emphasis is put on the electric field. We study the regularity in terms of the  
9 fractional order Sobolev spaces  $H^s$ ,  $s \in [0, 1]$ . Precisely, our first goal is to determine the regularity  
10 of the electric field and of its curl, that is to find some regularity exponent  $\tau \in (0, 1)$ , such that they  
11 both belong to  $H^s$ , for all  $s \in [0, \tau)$ . After that, one can derive error estimates. Here, the error is  
12 defined as the difference between the exact field and its approximation, where the latter is built with  
13 Nédélec's first family of finite elements. In addition to the regularity exponent, one needs to derive  
14 a stability constant that relates the norm of the error to the norm of the data: this is our second  
15 goal. We provide explicit expressions for both the regularity exponent and the stability constant  
16 with respect to the coefficients. We also discuss the accuracy of these expressions, and we provide  
17 some numerical illustrations.

18 **Key words.** Maxwell's equations, interface problem, edge elements, sensitivity to coefficients,  
19 error estimates

20 **AMS subject classifications.** 78A48, 35B65, 65N30

21 **1. Introduction.** We study the numerical approximation of electromagnetic  
22 fields governed by Maxwell's equations. More precisely, our goal is to characterize  
23 the dependence of the error between the exact and computed fields, with respect to  
24 the coefficients that define the model (PDEs, supplemented with boundary condi-  
25 tions). This paper is the third part of the series entitled "On the Approximation of  
26 Electromagnetic Fields by Edge Finite Elements" [12, 13].

27 For Maxwell's equations, the coefficients are the electric permittivity, the magnetic  
28 permeability and the conductivity. Classically, the model is recast as an equivalent  
29 variational formulation. The first goal is to determine the value of the constants that  
30 appear in the analysis of the variational formulation, which are the continuity mod-  
31 ulus of the forms, and the coercivity or inf-sup constants. Then, one performs the  
32 numerical analysis of the model. In addition to the above-mentioned constants, one  
33 has to estimate the order of convergence, which depends on the (extra-)regularity of  
34 the fields ; this (extra-)regularity depends itself on the behavior of the coefficients,  
35 and on the geometry of the model. In particular, it is crucial to use ad hoc norms  
36 to measure the fields and the data, and particular care is devoted to the definition  
37 of those norms. We observe that if the coefficients belong to a set not reduced to a  
38 singleton (eg. random coefficients), then the (extra-)regularity may vanish in some  
39 limit cases.

40  
41 The outline of the paper is as follows.  
42 In the next section, we introduce the model problem (see eg. [2]), set in a bounded  
43 region of  $\mathbb{R}^3$ , with volume sources. We prescribe some a priori conditions on the co-  
44 efficients, and on the source terms ; the coefficients are only supposed to be piecewise  
45 smooth, hence they may be discontinuous. The variational formulation is introduced.

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46 Then, in section 3, we recall the main results regarding the discretisation of elec-  
 47 tromagnetic fields by Nédélec’s finite elements [30, 29, 12, 20]. We also define the  
 48 function spaces that will allow us to perform the analysis of the model. Particular  
 49 attention is paid to the fractional order Sobolev spaces, which play a crucial role in  
 50 the analysis. These are defined using either the real interpolation method [28, 33, 9],  
 51 or with the help of the Sobolev-Slobodeckij norms [24, 25, 11]: this yields two ways to  
 52 measure their elements. In section 4, we provide the (well-known) estimates for the  
 53 continuity modulus and the coercivity constant related to the variational formulation,  
 54 with respect to the coefficients of the model.

55 There remains to estimate the so-called regularity exponent and the stability constant,  
 56 which relate the norm of the electromagnetic fields in *ad hoc* fractional order Sobolev  
 57 spaces norms to the norm of the source terms. Estimating these last two quantities  
 58 with respect to the coefficients of the model is less classical. Hence, most novelties in  
 59 the paper are contained in the approach developed in sections 5 and 6. In section 5, we  
 60 recall how one can split the electromagnetic fields into a regular part and a gradient  
 61 part [4], so the (lack of) regularity of the fields rests on the regularity of the gradients.  
 62 This is the subject of section 6. We use a perturbation argument *à la* Jochmann [27]  
 63 or Bonito *et al* [6] to estimate this regularity with respect to the coefficients of the  
 64 model. We call it the *global approach*. When the coefficient are piecewise constant,  
 65 one may also use the *local approach*, see Appendix B. The main novelties are threefold:  
 66 the extension of existing results to problems with complex-valued coefficients, set in  
 67 a non-topologically trivial domain; the use of the two measures for elements of the  
 68 Sobolev spaces, and their interplay; the design of estimates for the numerical error  
 69 that depend only on the coefficients of the model (see Theorem 6.15). To conclude,  
 70 we illustrate our results by two examples in section 7.

71  
 72 We denote constant fields by the symbol  *cst* . Vector-valued (respectively tensor-  
 73 valued) function spaces are written in boldface character (resp. blackboard bold  
 74 characters). Given a non-empty open set  $\mathcal{O}$  of  $\mathbb{R}^3$ , we use the notation  $(\cdot|\cdot)_{0,\mathcal{O}}$  (re-  
 75 spectively  $\|\cdot\|_{0,\mathcal{O}}$ ) for the  $L^2(\mathcal{O})$  and the  $\mathbf{L}^2(\mathcal{O}) := (L^2(\mathcal{O}))^3$  hermitian scalar prod-  
 76 ucts (resp. norms). More generally,  $(\cdot|\cdot)_{s,\mathcal{O}}$  and  $\|\cdot\|_{s,\mathcal{O}}$  (respectively  $|\cdot|_{s,\mathcal{O}}$ ) denote  
 77 the hermitian scalar product and the norm (resp. semi-norm) of the Sobolev spaces  
 78  $H^s(\mathcal{O})$  and  $\mathbf{H}^s(\mathcal{O}) := (H^s(\mathcal{O}))^3$  for  $s \in \mathbb{R}$  (resp. for  $s > 0$ ). *The index  $zmv$  indi-*  
 79 *cates zero-mean-value fields.* If moreover the boundary  $\partial\mathcal{O}$  is Lipschitz,  $\mathbf{n}$  denotes  
 80 the unit outward normal vector field to  $\partial\mathcal{O}$ . Finally, it is assumed that the reader  
 81 is familiar with function spaces related to Maxwell’s equations, such as  $\mathbf{H}(\mathbf{curl}; \mathcal{O})$ ,  
 82  $\mathbf{H}_0(\mathbf{curl}; \mathcal{O})$ ,  $\mathbf{H}(\text{div}; \mathcal{O})$ ,  $\mathbf{H}_0(\text{div}; \mathcal{O})$  etc. A priori,  $\mathbf{H}(\mathbf{curl}; \mathcal{O})$  is endowed with the  
 83 norm  $\mathbf{v} \mapsto (\|\mathbf{v}\|_{0,\mathcal{O}}^2 + \|\mathbf{curl} \mathbf{v}\|_{0,\mathcal{O}}^2)^{1/2}$ , etc. We refer to the monographs of Monk [29]  
 84 and Assous *et al* [4] for details. We will define more specialized function spaces later  
 85 on.

86 **2. The model problem.** Let  $\Omega$  be a *domain* in  $\mathbb{R}^3$ , that is an open, connected  
 87 and bounded subset of  $\mathbb{R}^3$  with a Lipschitz-continuous boundary  $\partial\Omega$ . We are interested  
 88 in solving the time-harmonic Maxwell’s equations (with time-dependence  $\exp(-i\omega t)$ ,  
 89 for a given pulsation  $\omega > 0$ ),

90 (2.1) 
$$i\omega \mathbf{d} + \mathbf{curl} \mathbf{h} = \mathbf{j} \text{ in } \Omega,$$

91 (2.2) 
$$-i\omega \mathbf{b} + \mathbf{curl} \mathbf{e} = 0 \text{ in } \Omega,$$

92 (2.3) 
$$\text{div} \mathbf{d} = \varrho \text{ in } \Omega,$$

93 (2.4) 
$$\text{div} \mathbf{b} = 0 \text{ in } \Omega.$$

94 Above,  $(\mathbf{e}, \mathbf{d}, \mathbf{h}, \mathbf{b})$  are the electromagnetic fields. We suppose that  $\mathbf{d}$  and  $\mathbf{b}$  are related  
 95 to  $\mathbf{e}$  and  $\mathbf{h}$  by the constitutive relations

$$96 \quad (2.5) \quad \mathbf{d} = \varepsilon \mathbf{e}, \quad \mathbf{b} = \mu \mathbf{h} \text{ in } \Omega,$$

97 where the real-valued coefficient  $\varepsilon$  is the electric permittivity and the real-valued  
 98 coefficient  $\mu$  is the magnetic permeability.

99 The source terms  $\mathbf{j}$  and  $\varrho$  are respectively the current density and the charge density,  
 100 and they are related by the charge conservation equation

$$101 \quad (2.6) \quad -\omega \varrho + \operatorname{div} \mathbf{j} = 0.$$

102 We suppose that the current density may be written as

$$103 \quad (2.7) \quad \mathbf{j} = \mathbf{j}_{ext} + \sigma \mathbf{e} \text{ in } \Omega,$$

104 where  $\mathbf{j}_{ext}$  is an externally imposed current, and the real-valued coefficient  $\sigma$  is the  
 105 conductivity.

106 **2.1. A priori assumptions.** Classically, the electromagnetic fields all belong  
 107 to  $\mathbf{L}^2(\Omega)$  and the coefficients  $\varepsilon$ ,  $\mu$  and  $\sigma$  have a fixed-sign (positive): we make these  
 108 assumptions from now on. We also assume throughout this work that these coefficients  
 109 together with their inverses belong to  $L^\infty(\Omega)$ , and we use the notations  $\varepsilon_{max} =$   
 110  $\|\varepsilon\|_{L^\infty(\Omega)}$ ,  $\varepsilon_{min} = (\|\varepsilon^{-1}\|_{L^\infty(\Omega)})^{-1}$ , etc.

111 We choose the data  $(\mathbf{j}_{ext}, \varrho)$  in  $\mathbf{H}(\operatorname{div}; \Omega) \times H^{-1}(\Omega)$ . It is also possible to choose  
 112  $\mathbf{j}_{ext} \in \mathbf{L}^2(\Omega)$  with  $\operatorname{div} \mathbf{j}_{ext} \in H^{-t}(\Omega)$  for some  $t \in (0, 1)$ , but we assume for simplicity  
 113 that  $\operatorname{div} \mathbf{j}_{ext} \in L^2(\Omega)$ . We refer to §6.5 for the study of the more general case.

114 Finally, we assume that the medium  $\Omega$  is surrounded by a perfect conductor, so that  
 115 the boundary condition below holds:

$$116 \quad (2.8) \quad \mathbf{e} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

117 Eqs. (2.1)-(2.8) together with the assumptions on the coefficients and on the source  
 118 terms define our *model problem*. When we focus on the discretization, see §3.3 and  
 119 afterwards, we assume that  $\Omega$  is a *Lipschitz polyhedron*, that  $\varepsilon, \sigma$  are *piecewise smooth*  
 120 *on*  $\Omega$ , and that  $\mu$  is *constant on*  $\Omega$ . We call this setting the *polyhedral model problem*.  
 121 Let us mention that once the field  $\mathbf{e}$  is known, then all other electromagnetic fields  $\mathbf{d}$ ,  
 122  $\mathbf{b}$  and  $\mathbf{h}$  are known too. As a consequence, we focus on the study of the field  $\mathbf{e}$ . In  
 123 particular, we note that  $\mathbf{e}$  belongs to the function space  $\mathbf{H}_0(\mathbf{curl}; \Omega)$ .

124 **2.2. Variational formulation.** In the spirit of the charge conservation equation  
 125 tion, let us introduce  $\varrho_{ext} = -i/\omega \operatorname{div} \mathbf{j}_{ext} \in L^2(\Omega)$ . Our model problem can be  
 126 formulated in the electric field  $\mathbf{e}$  only, namely

$$127 \quad (2.9) \quad \begin{cases} \text{Find } \mathbf{e} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \text{ such that} \\ -\omega^2 \varepsilon_\sigma \mathbf{e} + \mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{e}) = \omega \mathbf{j}_{ext} \text{ in } \Omega \\ \operatorname{div} \varepsilon_\sigma \mathbf{e} = \varrho_{ext} \text{ in } \Omega. \end{cases}$$

128 Above, the complex-valued coefficient  $\varepsilon_\sigma$  is defined by  $\varepsilon_\sigma = \varepsilon + i\sigma/\omega$ . Note that in  
 129 (2.9), the equation  $\operatorname{div} \varepsilon_\sigma \mathbf{e} = \varrho_{ext}$  is implied by the second-order equation  $-\omega^2 \varepsilon_\sigma \mathbf{e} +$   
 130  $\mathbf{curl}(\mu^{-1} \mathbf{curl} \mathbf{e}) = \omega \mathbf{j}_{ext}$ , together with the charge conservation equation (2.6) and  
 131 the splitting of the current (2.7), so it can be omitted. Moreover, one can check that  
 132 the equivalent variational formulation in  $\mathbf{H}_0(\mathbf{curl}; \Omega)$  writes

$$133 \quad (2.10) \quad \begin{cases} \text{Find } \mathbf{e} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \text{ such that} \\ (\mu^{-1} \mathbf{curl} \mathbf{e} | \mathbf{curl} \mathbf{v})_{0,\Omega} - \omega^2 (\varepsilon_\sigma \mathbf{e} | \mathbf{v})_{0,\Omega} = \omega (\mathbf{j}_{ext} | \mathbf{v})_{0,\Omega}, \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega). \end{cases}$$

134 Under the assumptions on the coefficients, this variational formulation is well-posed  
 135 (see for instance [4, §8.3.2]). In other words,

$$136 \quad (2.11) \quad \begin{aligned} & \exists C_{(\varepsilon, \mu, \sigma)} > 0, \text{ such that } \forall \mathbf{j}_{ext} \in \mathbf{L}^2(\Omega), \exists! \mathbf{e} \text{ solution to (2.10), and} \\ & \|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl}; \Omega)} \leq C_{(\varepsilon, \mu, \sigma)} \|\mathbf{j}_{ext}\|_{0, \Omega}. \end{aligned}$$

137 **3. Discretisation of electromagnetic fields.** Here, we define finite element  
 138 approximations of the electric field  $\mathbf{e}$ . We also recall how one can build an a pri-  
 139 ori error estimate between  $\mathbf{e}$  and its approximation. When we study the numerical  
 140 approximations and for the ease of exposition, we assume that  $\Omega$  is a Lipschitz poly-  
 141 hedron (polyhedral model problem). To define finite dimensional subspaces  $(\mathbf{V}_h)_h$   
 142 of  $\mathbf{H}_0(\mathbf{curl}; \Omega)$ , we choose the so-called Nédélec's first family of edge finite elements,  
 143 defined on simplicial meshes of  $\Omega$ . We follow here [12, §2.4]. It is sufficient to use  
 144 first-order finite elements because we focus on electromagnetic fields with low regular-  
 145 ity.  $\bar{\Omega}$  is triangulated by a shape regular family of meshes  $(\mathcal{T}_h)_h$ , made up of (closed)  
 146 simplices, generically denoted by  $K$ . A mesh is indexed by  $h := \max_K h_K$  (the mesh-  
 147 size), where  $h_K$  is the diameter of  $K$ . Denoting by  $\rho_K$  the diameter of the largest ball  
 148 inscribed in  $K$ , we assume that there exists a shape regularity parameter  $\varsigma > 0$  such  
 149 that for all  $h$ , for all  $K \in \mathcal{T}_h$ , it holds  $h_K \leq \varsigma \rho_K$ . Nédélec's  $\mathbf{H}(\mathbf{curl}; \Omega)$ -conforming  
 150 (first family, first-order) finite element spaces are defined as

$$151 \quad \mathbf{V}_h := \{\mathbf{v}_h \in \mathbf{H}_0(\mathbf{curl}; \Omega) : \mathbf{v}_h|_K \in \mathcal{R}_1(K), \forall K \in \mathcal{T}_h\},$$

152 where  $\mathcal{R}_1(K)$  is the six-dimensional vector space of polynomials on  $K$

$$153 \quad \mathcal{R}_1(K) := \{\mathbf{v} \in \mathbf{P}_1(K) : \mathbf{v}(\mathbf{x}) = \mathbf{a} + \mathbf{b} \times \mathbf{x}, \mathbf{a}, \mathbf{b} \in \mathbb{R}^3\}.$$

154 According to [30, Theorem 1], any element  $\mathbf{v}$  in  $\mathcal{R}_1(K)$  is uniquely determined by the  
 155 degrees of freedom in the moment set  $M_E(\mathbf{v})$ :

$$156 \quad M_E(\mathbf{v}) := \left( \int_e \mathbf{v} \cdot \mathbf{t} \, dl \right)_{e \in A_K}.$$

157 Above,  $A_K$  is the set of edges of  $K$ , and  $\mathbf{t}$  is a unit vector along the edge  $e$ . The global  
 158 set of moments on  $\mathbf{V}_h$  is then obtained by taking one degree of freedom as above per  
 159 interior edge of  $\mathcal{T}_h$ . We recall that the basic approximability properties writes (cf.  
 160 [29, Lemma 7.10])

$$161 \quad (3.1) \quad \lim_{h \rightarrow 0} \left( \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{H}(\mathbf{curl}; \Omega)} \right) = 0, \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega).$$

162 Assuming for simplicity that the integrals are computed exactly, the discrete electric  
 163 problem writes

$$164 \quad (3.2) \quad \begin{cases} \text{Find } \mathbf{e}_h \in \mathbf{V}_h \text{ such that} \\ (\mu^{-1} \mathbf{curl} \mathbf{e}_h | \mathbf{curl} \mathbf{v}_h)_{0, \Omega} - \omega^2 (\varepsilon_\sigma \mathbf{e}_h | \mathbf{v}_h)_{0, \Omega} = \omega (\mathbf{j}_{ext} | \mathbf{v}_h)_{0, \Omega}, \forall \mathbf{v}_h \in \mathbf{V}_h. \end{cases}$$

165 Because the exact problem is well-posed, cf. (2.11), one may apply Céa's lemma to  
 166 find

$$167 \quad (3.3) \quad \exists C_{(\varepsilon, \mu, \sigma)}^\sharp > 0, \forall h, \|\mathbf{e} - \mathbf{e}_h\|_{\mathbf{H}(\mathbf{curl}; \Omega)} \leq C_{(\varepsilon, \mu, \sigma)}^\sharp \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{e} - \mathbf{v}_h\|_{\mathbf{H}(\mathbf{curl}; \Omega)}.$$

168 Classically, the constant  $C_{(\varepsilon, \mu, \sigma)}^\sharp$  depends on the coercivity constant and on the norm  
 169 of the sesquilinear form in the left-hand side of (2.10) and (3.2). This constant is  
 170 investigated in detail in section 4. It follows from (3.1) that

$$171 \quad \lim_{h \rightarrow 0} \|e - e_h\|_{\mathbf{H}(\mathbf{curl}; \Omega)} = 0.$$

172 In order to obtain a result which is more accurate, typically a convergence rate in the  
 173 order of  $h^s$  for some  $s > 0$ , one has to use information on the (extra-)regularity of the  
 174 electric field. Let us recall how this can be achieved.

175 **3.1. A few reminders about Sobolev spaces.** Let  $\mathcal{O} \subset \Omega$  be a non-empty  
 176 connected open subset of  $\mathbb{R}^3$  with Lipschitz boundary. To give a precise meaning to  
 177 the regularity of a scalar or vector field on  $\mathcal{O}$ , we use the well-known Sobolev scale  
 178  $(H^s(\mathcal{O}))_s$ .

179 (0) For  $s \in \mathbb{N}$ , one uses the standard definition:

$$180 \quad H^s(\mathcal{O}) := \{v \in L^2(\mathcal{O}) \text{ s.t. } \forall \alpha \in \mathbb{N}^3, |\alpha| \leq s, \partial_\alpha v \in L^2(\mathcal{O})\},$$

181 equipped with the norm  $\|v\|_{s, \mathcal{O}} := (\sum_{\alpha \in \mathbb{N}^3, |\alpha| \leq s} \|\partial_\alpha v\|_{L^2(\mathcal{O})}^2)^{1/2}$ . Obviously,  $H^0(\mathcal{O}) =$   
 182  $L^2(\mathcal{O})$ .

183 (1) To define those spaces for  $s > 0$ ,  $s \notin \mathbb{N}$ , several possibilities exist. Let us begin  
 184 with the real interpolation method [28] (see also Appendix A), which allows us to  
 185 define those Hilbert spaces for non-integer indices  $s = m + \sigma$ ,  $m \in \mathbb{N}$ ,  $\sigma \in (0, 1)$ , as

$$186 \quad H^s(\mathcal{O}) := (H^m(\mathcal{O}), H^{m+1}(\mathcal{O}))_{\sigma, 2}.$$

187 The corresponding norm is denoted by  $\|\cdot\|_{s, \mathcal{O}}$ . In particular, for all  $0 \leq s \leq t$ , it  
 188 holds that  $H^t(\mathcal{O}) \subset H^s(\mathcal{O})$  with continuous embedding [9, §14]:

$$189 \quad \exists C_{(s,t)} > 0, \forall v \in H^t(\mathcal{O}), \quad \|v\|_{s, \mathcal{O}} \leq C_{(s,t)} \|v\|_{t, \mathcal{O}}.$$

190 Given  $0 < s_0 \leq s_1 < t < 1$ ,  $s \mapsto C_{(s,t)}$  is continuous on  $[s_0, s_1]$ .

191 A well-known alternative is to define, for  $\sigma \in (0, 1)$ :

$$192 \quad \underline{H}^\sigma(\mathcal{O}) := \{v \in L^2(\mathcal{O}) \text{ s.t. } |v|_{\underline{H}^\sigma(\mathcal{O})} < \infty\},$$

193 where

$$194 \quad |v|_{\underline{H}^\sigma(\mathcal{O})} := \left( \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|v(\mathbf{x}) - v(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{3+2\sigma}} d\mathbf{x} d\mathbf{y} \right)^{1/2} \text{ is the Sobolev-Slobodeckij semi-norm,}$$

195 and  $\underline{H}^\sigma(\mathcal{O})$  is endowed with the Sobolev-Slobodeckij norm

$$196 \quad \|v\|_{\underline{H}^\sigma(\mathcal{O})} := \left( \|v\|_{0, \mathcal{O}}^2 + |v|_{\underline{H}^\sigma(\mathcal{O})}^2 \right)^{1/2}.$$

197 And then, for  $s = m + \sigma$ ,  $m \in \mathbb{N}$ ,  $\sigma \in (0, 1)$ :

$$198 \quad \underline{H}^s(\mathcal{O}) := \{v \in H^m(\mathcal{O}) \text{ s.t. } \forall \alpha \in \mathbb{N}^3 \text{ with } |\alpha| = m, \partial_\alpha v \in \underline{H}^\sigma(\mathcal{O})\},$$

199 endowed with the Sobolev-Slobodeckij norm

$$200 \quad \|v\|_{\underline{H}^s(\mathcal{O})} := \left( \|v\|_{m, \mathcal{O}}^2 + \sum_{\alpha \in \mathbb{N}^3, |\alpha|=m} |\partial_\alpha v|_{\underline{H}^\sigma(\mathcal{O})}^2 \right)^{1/2}.$$

201 The Sobolev-Slobodeckij semi-norm is

$$202 \quad |v|_{\underline{H}^s(\mathcal{O})} := \left( \sum_{\alpha \in \mathbb{N}^3, |\alpha|=m} |\partial_\alpha v|_{\underline{H}^\sigma(\mathcal{O})}^2 \right)^{1/2}.$$

203 For all  $s \in \mathbb{R}^+ \setminus \mathbb{N}$ , it holds that  $H^s(\mathcal{O}) = \underline{H}^s(\mathcal{O})$  algebraically and topologically:

$$204 \quad \exists m_{(s)}, M_{(s)} > 0, \forall v \in H^s(\mathcal{O}), \quad m_{(s)} \|v\|_{\underline{H}^s(\mathcal{O})} \leq \|v\|_{s,\mathcal{O}} \leq M_{(s)} \|v\|_{\underline{H}^s(\mathcal{O})}.$$

205 However, in a bounded set  $\mathcal{O}$ , there are no results on the uniform equivalence of  
 206 Sobolev-Slobodeckij norms and real interpolation norms when  $s$  spans  $(0, 1)$ , ie. on  
 207 bounding one norm with the other times a constant that is independent of  $s \in (0, 1)$ .  
 208 We refer to [25, 11] for illuminating discussions on this topic. On the other hand (see  
 209 [24] or [9, §14]), if  $s$  spans  $[s_0, s_1]$  with  $0 < s_0 \leq s_1 < 1$ , there is a uniform equivalence  
 210 of norms: in other words,  $m, m^{-1}, M, M^{-1}$  are continuous on  $[s_0, s_1]$ .

211 (2) For  $s \geq 0$ ,  $H_0^s(\mathcal{O})$  is the closure of  $\mathcal{D}(\mathcal{O})$  in  $H^s(\mathcal{O})$ . For  $s \in [0, \frac{1}{2}]$ , it holds that  
 212  $H_0^s(\mathcal{O}) = H^s(\mathcal{O})$  algebraically and topologically, see for instance [21, Theorem 1.4.2.4]  
 213 ; while for  $s > \frac{1}{2}$ , it holds that  $H_0^s(\mathcal{O}) \subsetneq H^s(\mathcal{O})$ .

214 (3) For  $s < 0$ ,  $\tilde{H}^s(\mathcal{O})$  is the topological dual of  $H_0^{-s}(\mathcal{O})$ .

215 (4) For  $s \geq 0$ ,  $\tilde{H}^s(\mathcal{O})$  (also denoted in the literature by  $H_{00}^s(\mathcal{O})$ ) is composed of  
 216 elements of  $H^s(\mathcal{O})$  such that the continuation by zero outside  $\mathcal{O}$  belongs to  $H^s(\mathbb{R}^3)$  ;  
 217 for  $s \notin \frac{1}{2} + \mathbb{N}$ , it holds that  $\tilde{H}^s(\mathcal{O}) = H_0^s(\mathcal{O})$ , while for  $s \in \frac{1}{2} + \mathbb{N}$ , it holds that  
 218  $\tilde{H}^s(\mathcal{O}) \subsetneq H_0^s(\mathcal{O})$ . Going back to the real interpolation method, for non-integer in-  
 219 dices  $s = m + \sigma$ ,  $m \in \mathbb{N}$ ,  $\sigma \in (0, 1)$ , one has  $\tilde{H}^s(\mathcal{O}) = (H_0^m(\mathcal{O}), H_0^{m+1}(\mathcal{O}))_{\sigma, 2}$ .

220

221 For the *regularity studies*, we choose the real interpolation method, while we  
 222 use the double-integral Sobolev-Slobodeckij norms and semi-norms to perform the  
 223 *numerical analysis*, and derive convergence rates.

224 **3.2. Piecewise smooth fields.** The set  $\mathcal{P} := \{\Omega_j\}_{j=1, \dots, J}$  is called a *parti-*  
 225 *tion* of  $\Omega$  if  $(\Omega_j)_{j=1, \dots, J}$  are disjoint domains, and it holds  $\Omega = \cup_{j=1}^J \overline{\Omega_j}$ . When the  
 226  $(\Omega_j)_{j=1, \dots, J}$  are Lipschitz polyhedra, we use the term *polyhedral partition*. Given a  
 227 partition, we introduce the corresponding *interface*  $\Sigma := \cup_{1 \leq j \neq j' \leq J} (\partial\Omega_j \cap \partial\Omega_{j'})$ . For  
 228 a field  $v$  defined on  $\Omega$ , we denote by  $v_j$  its restriction to  $\Omega_j$ , for all  $j$ . In relation to  
 229 the partition  $\mathcal{P}$  and for  $s \geq 0$ , we define

$$230 \quad PH^s(\Omega) := \{v \in L^2(\Omega) : v_j \in H^s(\Omega_j), 1 \leq j \leq J\}, \text{ endowed with}$$

$$231 \quad \|v\|_{PH^s(\Omega)} := \left( \sum_{1 \leq j \leq J} \|v_j\|_{s, \Omega_j}^2 \right)^{1/2} \quad \text{or} \quad \|v\|_{\underline{PH}^s(\Omega)} := \left( \sum_{1 \leq j \leq J} \|v_j\|_{\underline{H}^s(\Omega_j)}^2 \right)^{1/2}.$$

232 To simplify the notations, the reference to  $\mathcal{P}$  is usually omitted. Let us recall the  
 233 technical result (Theorem 4.1 of [1], or Lemma 2.1 of [6]).

234 **PROPOSITION 3.1.** *For all  $s \in [0, 1]$ , it holds that*

$$235 \quad \|v\|_{PH^s(\Omega)} \leq \|v\|_{s, \Omega}, \quad \forall v \in H^s(\Omega).$$

236 Note that one has  $PH^s(\Omega) = H^s(\Omega)$  algebraically and topologically for all partitions  
 237 and for all  $s \in [0, \frac{1}{2})$ .

238 Finally, we introduce

$$239 \quad \mathbf{PH}^s(\mathbf{curl}; \Omega) := \{\mathbf{v} \in \mathbf{PH}^s(\Omega) : \mathbf{curl} \mathbf{v} \in \mathbf{PH}^s(\Omega)\} \text{ for } s > 0;$$

$$240 \quad PW^{1,\infty}(\Omega) := \{\zeta \in L^\infty(\Omega) : \zeta_j \in W^{1,\infty}(\Omega_j), 1 \leq j \leq J\}.$$

241  $\mathbf{PH}^s(\mathbf{curl}; \Omega)$  is endowed with the graph norm.

242 And we endow  $PW^{1,\infty}(\Omega)$  with the norm  $\|\zeta\|_{PW^{1,\infty}(\Omega)} := \|\zeta\|_{L^\infty(\Omega)} + |\zeta|_{PW^{1,\infty}(\Omega)}$ ,  
 243 and the semi-norm  $|\zeta|_{PW^{1,\infty}(\Omega)} := \max_{1 \leq j \leq J} \|\nabla \zeta_j\|_{L^\infty(\Omega_j)}$ . For a piecewise constant  
 244 coefficient  $\zeta$ , it holds that  $\|\zeta\|_{PW^{1,\infty}(\Omega)} = \|\zeta\|_{L^\infty(\Omega)} = \max_{1 \leq j \leq J} |\zeta_j|$ .  
 245 When the partition is trivial, that is  $\mathcal{P} = \{\Omega\}$ , we omit the  $P$  or  $\mathbf{P}$  in the name of  
 246 the function space.

247 We note that, for the polyhedral model, the assumption on the coefficients writes  
 248  $\varepsilon, \sigma \in PW^{1,\infty}(\Omega)$ , and the interface  $\Sigma$  can be viewed as the locus of the discontinuities  
 249 of at least one the two coefficients. More generally, if  $\varepsilon, \sigma, \mu \in PW^{1,\infty}(\Omega)$ ,  $\Sigma$  is the  
 250 locus of the discontinuities of at least one the three coefficients.

251 **3.3. Finite element interpolation or quasi-interpolation operators.** In a  
 252 Lipschitz polyhedron  $\Omega$ , one can build finite element interpolation, or quasi-interpolation,  
 253 operators that act on piecewise smooth fields, with range in  $\mathbf{V}_h$ . For a polyhedral  
 254 partition  $\mathcal{P} := \{\Omega_j\}_{j=1,\dots,J}$ , the family of meshes  $(\mathcal{T}_h)_h$  is said to be conforming if, for  
 255 all  $h$ , for all  $K \in \mathcal{T}_h$ , there exists  $j_0$  such that  $K \subset \Omega_{j_0}$ . Let us recall briefly the theory  
 256 of finite element interpolation. Classically, those results are obtained by studying the  
 257 properties of the mappings to the reference element, using Sobolev-Slobodeckij semi-  
 258 norms. It holds, for conforming meshes,

$$259 \quad (3.4) \quad \forall s \in (0, 1], \exists \underline{C}_{(\varsigma,s)}^{interp} > 0, \forall \mathbf{v} \in \mathbf{PH}^s(\mathbf{curl}; \Omega), \forall h,$$

$$\|\mathbf{v} - \Pi_h^{interp} \mathbf{v}\|_{\mathbf{H}(\mathbf{curl}; \Omega)} \leq \underline{C}_{(\varsigma,s)}^{interp} h^s \{\|\mathbf{v}\|_{\underline{\mathbf{PH}}^s(\Omega)} + \|\mathbf{curl} \mathbf{v}\|_{\underline{\mathbf{PH}}^s(\Omega)}\}.$$

260 In (3.4), the interpolation operator  $\Pi_h^{interp}$  is defined in [29, §5.5] for  $s > \frac{1}{2}$ , respec-  
 261 tively is the so-called combined interpolation operator of [12, §4.2] for  $s \leq \frac{1}{2}$ . Regard-  
 262 ing the theory of finite element quasi-interpolation, a similar result can be derived.  
 263 Namely, that (3.4) holds, where  $\Pi_h^{interp}$  now stands for the quasi-interpolation opera-  
 264 tor defined in [20, §3.5]. The two finite element interpolation and quasi-interpolation  
 265 bounds are identical, bearing in mind that  $\Pi_h^{interp}$  is either the interpolation, or the  
 266 quasi-interpolation, operator. In addition, we note that  $\underline{C}_{(\varsigma,s)}^{interp}$  is not proven to be  
 267 independent of  $s$  in the above mentioned papers. On the other hand, one can check  
 268 that  $\underline{C}_{(\varsigma,s)}^{interp}$  depends continuously on  $s$  in  $(0, 1)$  with the help of the tools proposed  
 269 in those papers. For the derivation of those continuous dependence results, we refer  
 270 precisely to: the proof of Theorem 3.3 in [20] using abstract estimates from [19, §5]  
 271 for the quasi-interpolation; resp. [12, §4.2] using estimates for the Scott-Zhang inter-  
 272 polation, for the combined interpolation. Both proofs rely on [9, §14.3].

273 With the help of the results on the equivalence of norms of §3.1(1), we conclude that  
 274 one can write (3.4) with the real interpolation norms

$$275 \quad (3.5) \quad \forall s \in (0, 1], \exists C_{(\varsigma,s)}^{interp} > 0, \forall \mathbf{v} \in \mathbf{PH}^s(\mathbf{curl}; \Omega), \forall h,$$

$$\|\mathbf{v} - \Pi_h^{interp} \mathbf{v}\|_{\mathbf{H}(\mathbf{curl}; \Omega)} \leq C_{(\varsigma,s)}^{interp} h^s \|\mathbf{v}\|_{\mathbf{PH}^s(\mathbf{curl}; \Omega)}.$$

276 Furthermore, one can choose  $s \mapsto C_{(\varsigma,s)}^{interp}$  that is continuous on  $[s_0, s_1]$ , for all  $0 <$   
 277  $s_0 \leq s_1 < 1$ .



278 **3.4. Extra-regularity of the electric field and convergence rate.** Since  
 279  $\mathbf{j}_{ext} \in \mathbf{H}(\operatorname{div}; \Omega)$ , one may prove that the electric field that solves (2.9)-(2.10) enjoys  
 280 extra smoothness. More precisely, the aim is to prove that

$$281 \quad (3.6) \quad \begin{aligned} & \exists \tau_{(\varepsilon, \mu, \sigma)} > 0, \forall t \in [0, \tau_{(\varepsilon, \mu, \sigma)}), \exists C_{(\varepsilon, \mu, \sigma, t)}^*, \forall \mathbf{j}_{ext} \in \mathbf{H}(\operatorname{div}; \Omega), \\ & \mathbf{e} \in \mathbf{PH}^t(\operatorname{curl}; \Omega) \text{ and } \|\mathbf{e}\|_{\mathbf{PH}^t(\operatorname{curl}; \Omega)} \leq C_{(\varepsilon, \mu, \sigma, t)}^* \|\mathbf{j}_{ext}\|_{\mathbf{H}(\operatorname{div}; \Omega)}. \end{aligned}$$

282 Above,  $\tau_{(\varepsilon, \mu, \sigma)}$  plays the role of a *regularity exponent*, while  $C_{(\varepsilon, \mu, \sigma, s)}^*$  can be seen as  
 283 a *stability constant*.

284 Let  $\Theta$  a set of coefficients  $(\varepsilon, \mu, \sigma)$  whose elements are all piecewise smooth on the same  
 285 partition, and  $\mathbf{j}_{ext}$  the data, be given, and assume that  $\underline{\tau} = \inf_{(\varepsilon, \mu, \sigma) \in \Theta} \tau_{(\varepsilon, \mu, \sigma)} > 0$ ,  
 286 where  $\tau_{(\varepsilon, \mu, \sigma)}$  is defined in (3.6). Regrouping all the previous results, one concludes  
 287 first that for all  $(\varepsilon, \mu, \sigma) \in \Theta$ , the solution  $\mathbf{e}$  to (2.9)-(2.10) is in  $\bigcap_{s \in [0, \underline{\tau})} \mathbf{PH}^s(\operatorname{curl}; \Omega)$ ;  
 288 and second that one has the error estimates

$$289 \quad (3.7) \quad \begin{aligned} & \forall s \in [0, \underline{\tau}), \\ & \|\mathbf{e} - \mathbf{e}_h\|_{\mathbf{H}(\operatorname{curl}; \Omega)} \leq C_{(\varepsilon, \mu, \sigma)}^\sharp C_{(s, s)}^{interp} C_{(\varepsilon, \mu, \sigma, s)}^* h^s \|\mathbf{j}_{ext}\|_{\mathbf{H}(\operatorname{div}; \Omega)}. \end{aligned}$$

290 In the error estimates (3.7), only  $C_{(\varepsilon, \mu, \sigma)}^\sharp$  and  $C_{(\varepsilon, \mu, \sigma, s)}^*$  depend on the coefficients  
 291  $(\varepsilon, \mu, \sigma)$ . Also, for a given  $\epsilon \in (0, \underline{\tau})$ , since  $s \mapsto C_{(s, s)}^{interp}$  is continuous for  $s > 0$ , one  
 292 may replace  $C_{(s, s)}^{interp}$  by the  $s$ -independent  $\max_{s \in [\epsilon, \underline{\tau})} C_{(s, s)}^{interp}$ , for all  $s \in [\epsilon, \underline{\tau})$ .  
 293

294 Our purpose is now to estimate more precisely the constants that appear in (3.3),  
 295 (3.5), (3.6) and (3.7). The dependency of  $C_{(\varepsilon, \mu, \sigma)}^\sharp$  on  $(\varepsilon, \mu, \sigma)$  is addressed in sec-  
 296 tion 4. For  $\tau_{(\varepsilon, \mu, \sigma)}$  and  $C_{(\varepsilon, \mu, \sigma, s)}^*$ , this dependency can be studied via the *global*  
 297 *approach*, which relies on a decomposition of the electric field, and of its curl, into  
 298 a regular part, and a gradient part. To obtain this splitting, we adapt [4, Chapter  
 299 6] to the case of complex-valued coefficients, and in the process we generalize the  
 300 results of [15] to the case of a non-topologically trivial domain: this is the subject  
 301 of section 5. In section 6, one studies the regularity of the gradient part, where the  
 302 scalar potential is governed by a second order elliptic PDEs complemented either with  
 303 Dirichlet boundary conditions (for the electric field) or with Neumann boundary con-  
 304 ditions (for its curl). The global approach, in the spirit of [27, 6], uses a perturbation  
 305 argument, where the regularity of the gradient part, ie. of its scalar potential, is  
 306 derived in comparison to the regularity of the solution to the Laplace equation with  
 307 the same boundary condition. Indeed, in a domain  $\Omega$  and for  $L^2(\Omega)$  volume data, it  
 308 is known from [26] that the gradient of the solution to the Laplace equation belongs  
 309 to  $\mathbf{H}^{\frac{1}{2}}(\Omega)$ . Using interpolation theory, one can find a regularity exponent  $\tau_{(\varepsilon, \mu, \sigma)}$  in  
 310 (3.6) for our problem. Up to our knowledge, this analysis has only been carried out  
 311 for PDEs with real-valued coefficients. Here, we check in particular that the analysis  
 312 proposed in [6] can be extended to the case of complex-valued coefficients. The main  
 313 results are: the derivation of a regularity exponent  $\tau_{(\varepsilon, \mu, \sigma)}$  that depends polynomially  
 314 on the coefficients, and the computation of an upper bound for the stability constant  
 315  $C_{(\varepsilon, \mu, \sigma, s)}^*$  when  $s$  spans  $[0, \tau_{(\varepsilon, \mu, \sigma)})$ . Finally, in section 7 we illustrate the theory on  
 316 two examples for which the singular behavior can be determined explicitly.

317 *Remark 3.2.* For piecewise constant coefficients, there exists an alternative, which  
 318 focuses on the singular behavior of the gradient part of the solution by finding directly  
 319 the “best” regularity exponent  $\tau_{(\varepsilon, \mu, \sigma)}^{opt}$  attached to this part of the solution. We call  
 320 it the *local approach*, see Appendix B. On the other hand, providing an upper bound  
 321 with the local approach for  $C_{(\varepsilon, \mu, \sigma, s)}^*$  when  $s$  spans  $[0, \tau_{(\varepsilon, \mu, \sigma)}^{opt})$ , is an open question.

322 **4. Estimating the constant**  $C_{(\varepsilon, \mu, \sigma)}^\sharp$ . Let  $\mathbf{V} := \mathbf{H}_0(\mathbf{curl}; \Omega)$  be endowed with  
 323  $\|\mathbf{v}\|_{\mathbf{V}} := (\|\mathbf{v}\|_{0, \Omega}^2 + \|\mathbf{curl} \mathbf{v}\|_{0, \Omega}^2)^{1/2}$ , and  $a(\cdot, \cdot)$  be the sesquilinear form on  $\mathbf{V}$  defined  
 324 by:

$$325 \quad (\mathbf{v}, \mathbf{w}) \mapsto (\mu^{-1} \mathbf{curl} \mathbf{v} | \mathbf{curl} \mathbf{w})_{0, \Omega} - \omega^2 (\varepsilon_\sigma \mathbf{v} | \mathbf{w})_{0, \Omega}.$$

326 Then, let  $C_{(\varepsilon, \mu, \sigma)}^{cont}$  be the best continuity constant, or continuity modulus, of  $a(\cdot, \cdot)$ :

$$327 \quad C_{(\varepsilon, \mu, \sigma)}^{cont} = \sup_{\mathbf{v}, \mathbf{w} \in \mathbf{V} \setminus \{0\}} \frac{|a(\mathbf{v}, \mathbf{w})|}{\|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}}},$$

328 resp.  $C_{(\varepsilon, \mu, \sigma)}^{coer}$  be the best coercivity constant of  $a(\cdot, \cdot)$ :

$$329 \quad C_{(\varepsilon, \mu, \sigma)}^{coer} = \inf_{\mathbf{v} \in \mathbf{V} \setminus \{0\}} \frac{|a(\mathbf{v}, \mathbf{v})|}{\|\mathbf{v}\|_{\mathbf{V}}^2}.$$

330

331 **PROPOSITION 4.1.** *Let the coefficients  $\varepsilon$ ,  $\mu$  and  $\sigma$  be as in section 2.1. Then the*  
 332 *sesquilinear form  $a(\cdot, \cdot)$  is continuous, with  $C_{(\varepsilon, \mu, \sigma)}^{cont} \leq \max(\omega(\omega^2 \varepsilon_{max}^2 + \sigma_{max}^2)^{1/2}, \mu^{-1})$ ,*  
 333 *and it is coercive with  $C_{(\varepsilon, \mu, \sigma)}^{coer} \geq \frac{1}{2} \sigma_{min} \min(\omega \varepsilon_{min} \varepsilon_{max}^{-1}, \mu^{-1}(\omega^2 \varepsilon_{max}^2 + \frac{1}{2} \sigma_{min}^2)^{-1/2})$ .*

334 *Proof.* (We omit the subscript  $0, \Omega$  for the  $L^2(\Omega)$ -scalar product and norm). Re-  
 335 garding continuity, given  $\mathbf{v}, \mathbf{w} \in \mathbf{V}$ , one finds

$$336 \quad |a(\mathbf{v}, \mathbf{w})| \leq \omega^2 \|\varepsilon_\sigma\|_{L^\infty(\Omega)} \|\mathbf{v}\| \|\mathbf{w}\| + \mu^{-1} \|\mathbf{curl} \mathbf{v}\| \|\mathbf{curl} \mathbf{w}\|$$

$$337 \quad \leq \max(\omega^2 \|\varepsilon_\sigma\|_{L^\infty(\Omega)}, \mu^{-1}) (\|\mathbf{v}\| \|\mathbf{w}\| + \|\mathbf{curl} \mathbf{v}\| \|\mathbf{curl} \mathbf{w}\|)$$

$$338 \quad \leq \max(\omega^2 \|\varepsilon_\sigma\|_{L^\infty(\Omega)}, \mu^{-1}) \|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}}$$

$$339 \quad \leq \max(\omega(\omega^2 \varepsilon_{max}^2 + \sigma_{max}^2)^{1/2}, \mu^{-1}) \|\mathbf{v}\|_{\mathbf{V}} \|\mathbf{w}\|_{\mathbf{V}}$$

340 because  $\|\varepsilon_\sigma\|_{L^\infty(\Omega)} \leq (\varepsilon_{max}^2 + \omega^{-2} \sigma_{max}^2)^{1/2}$ .

341 Regarding coercivity, given  $\mathbf{v} \in \mathbf{V}$ , if we let  $\mathbf{c} = \mathbf{curl} \mathbf{v}$ , one finds

$$342 \quad |a(\mathbf{v}, \mathbf{v})|^2 = (-\omega^2 (\varepsilon \mathbf{v} | \mathbf{v}) + \mu^{-1} \|\mathbf{c}\|^2)^2 + \omega^2 (\sigma \mathbf{v} | \mathbf{v})^2$$

$$343 \quad = \omega^4 (\varepsilon \mathbf{v} | \mathbf{v})^2 + \mu^{-2} \|\mathbf{c}\|^4 - 2\omega^2 \mu^{-1} (\varepsilon \mathbf{v} | \mathbf{v}) \|\mathbf{c}\|^2 + \omega^2 (\sigma \mathbf{v} | \mathbf{v})^2$$

$$344 \quad \geq (\omega^4 - \omega^2 \eta) (\varepsilon \mathbf{v} | \mathbf{v})^2 + \mu^{-2} (1 - \omega^2 \eta^{-1}) \|\mathbf{c}\|^4 + \omega^2 (\sigma \mathbf{v} | \mathbf{v})^2,$$

345 for all  $\eta > 0$  (Young's inequality). Then,

$$346 \quad |a(\mathbf{v}, \mathbf{v})|^2 \geq (\omega^4 - \omega^2 \eta) (\varepsilon \mathbf{v} | \mathbf{v})^2 + \mu^{-2} (1 - \omega^2 \eta^{-1}) \|\mathbf{c}\|^4 + \omega^2 \sigma_{min}^2 \|\mathbf{v}\|^4$$

$$347 \quad \geq \omega^2 \varepsilon_{min}^2 (\omega^2 + \sigma_{min}^2 \varepsilon_{max}^{-2} - \eta) \|\mathbf{v}\|^4 + \mu^{-2} (1 - \omega^2 \eta^{-1}) \|\mathbf{c}\|^4.$$

348 As a consequence, choosing  $\eta \in (\omega^2, \omega^2 + \sigma_{min}^2 \varepsilon_{max}^{-2})$ , one derives coercivity. For  
 349 instance, let  $\eta = \omega^2 + \frac{1}{2} \sigma_{min}^2 \varepsilon_{max}^{-2}$ . It follows that

$$350 \quad |a(\mathbf{v}, \mathbf{v})|^2 \geq \frac{1}{2} \omega^2 \sigma_{min}^2 \frac{\varepsilon_{min}^2}{\varepsilon_{max}^2} \|\mathbf{v}\|^4 + \frac{\sigma_{min}^2}{2\mu^2 (\omega^2 \varepsilon_{max}^2 + \frac{1}{2} \sigma_{min}^2)} \|\mathbf{c}\|^4$$

$$351 \quad \geq \frac{\sigma_{min}^2}{2} \min \left( \omega^2 \frac{\varepsilon_{min}^2}{\varepsilon_{max}^2}, \frac{1}{\mu^2 (\omega^2 \varepsilon_{max}^2 + \frac{1}{2} \sigma_{min}^2)} \right) (\|\mathbf{v}\|^4 + \|\mathbf{c}\|^4)$$

$$352 \quad \geq \frac{\sigma_{min}^2}{4} \min \left( \omega^2 \frac{\varepsilon_{min}^2}{\varepsilon_{max}^2}, \frac{1}{\mu^2 (\omega^2 \varepsilon_{max}^2 + \frac{1}{2} \sigma_{min}^2)} \right) \|\mathbf{v}\|_{\mathbf{V}}^4.$$

353 Hence,  $|a(\mathbf{v}, \mathbf{v})| \geq \frac{1}{2} \sigma_{min} \min(\omega \varepsilon_{min} \varepsilon_{max}^{-1}, \mu^{-1} (\omega^2 \varepsilon_{max}^2 + \frac{1}{2} \sigma_{min}^2)^{-1/2}) \|\mathbf{v}\|_{\mathbf{V}}^2$ .  $\square$

354 COROLLARY 4.2. *Let the coefficients  $\varepsilon$ ,  $\mu$  and  $\sigma$  be as in section 2.1. Then the*  
 355 *error estimate (3.3) holds with*

$$356 \quad C_{(\varepsilon, \mu, \sigma)}^\sharp \leq \frac{2 \max(\omega^2 \varepsilon_{max}^2 + \sigma_{max}^2)^{1/2}, \mu^{-1}}{\sigma_{min} \min(\omega \varepsilon_{min} \varepsilon_{max}^{-1}, \mu^{-1} (\omega^2 \varepsilon_{max}^2 + \frac{1}{2} \sigma_{min}^2)^{-1/2})}.$$

357 *Proof.* This is an obvious consequence of the fact that one can choose

$$358 \quad C_{(\varepsilon, \mu, \sigma)}^\sharp = \frac{C_{(\varepsilon, \mu, \sigma)}^{cont}}{C_{(\varepsilon, \mu, \sigma)}^{coer}}$$

359 in the error estimate (3.3). □

360 **5. Splitting into a regular part and a gradient part.** Below, we recall  
 361 some results of [4], and we adapt them to the case of complex-valued coefficients if  
 362 necessary. Let  $\xi$  be a coefficient defined on  $\Omega$ , we assume in the current section that  
 363  $\xi$  fulfills:

$$364 \quad (5.1) \quad \begin{cases} \xi \text{ is a complex-valued measurable scalar field on } \Omega, \xi, \xi^{-1} \in L^\infty(\Omega), \\ \exists \xi_- > 0, \theta^* \in [0, 2\pi), \Re(\exp(-i\theta^*)\xi) \geq \xi_- \text{ a.e. in } \Omega. \end{cases}$$

365

366 LEMMA 5.1. *Let the coefficients  $\varepsilon$  and  $\sigma$  be as in section 2.1. Then  $\xi = \varepsilon_\sigma$  fulfills*  
 367 *(5.1), where  $\theta^*$  can be any element of  $[0, \pi/2]$ .*

368 *Remark 5.2.* In other words,  $\xi = \varepsilon_\sigma$  belongs to a subclass of those coefficients that  
 369 are defined by (5.1). In the case where  $\sigma \geq 0$  (in particular, in the non-conducting  
 370 case, that is when it holds that  $\sigma = 0$  on some region of  $\Omega$ ), the above result still  
 371 holds for all  $\theta^* \in [0, \pi/2)$ . On the other hand, a real-valued, sign-changing coefficient  
 372  $\xi$  does not fulfill (5.1). We refer to [8, 7, 18, 10] for those more “exotic” configurations  
 373 of Maxwell’s equations, in which  $\varepsilon$  or  $\mu$  are real-valued and exhibit a sign-change.

374 *Proof.* One has  $\varepsilon_\sigma, \varepsilon_\sigma^{-1} \in L^\infty(\Omega)$ . The result follows from  $\Re(\exp(-i\theta^*)\varepsilon_\sigma) \geq$   
 375  $\cos \theta^* \varepsilon_{min} + \sin \theta^* \sigma_{min}/\omega > 0$  a.e. in  $\Omega$ . □

376 Define

$$377 \quad \mathbf{X}_{Dir}(\Omega, \xi) := \{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega) : \xi \mathbf{v} \in \mathbf{H}(\text{div}; \Omega)\},$$

$$378 \quad \mathbf{X}_{Neu}(\Omega, \xi) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega) : \xi \mathbf{v} \in \mathbf{H}_0(\text{div}; \Omega)\}.$$

379 The function spaces  $\mathbf{X}_{Dir}(\Omega, \xi)$  and  $\mathbf{X}_{Neu}(\Omega, \xi)$  are endowed with the graph norm  
 380  $\mathbf{v} \mapsto (\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl}; \Omega)}^2 + \|\xi \mathbf{v}\|_{\mathbf{H}(\text{div}; \Omega)}^2)^{1/2}$ . In the particular case where  $\xi$  is equal to 1, one  
 381 writes  $\mathbf{X}_B(\Omega)$  instead of  $\mathbf{X}_B(\Omega, 1)$ , for  $B \in \{Dir, Neu\}$ . We also define the subspaces  
 382 of *regular* fields, resp. the *null* subspaces:

$$383 \quad \mathbf{H}_B(\Omega) := \mathbf{X}_B(\Omega) \cap \mathbf{H}^1(\Omega), \quad B \in \{Dir, Neu\},$$

$$384 \quad \mathbf{Z}_B(\Omega) := \{\mathbf{v} \in \mathbf{X}_B(\Omega) : \mathbf{curl} \mathbf{v} = 0, \text{div} \mathbf{v} = 0 \text{ in } \Omega\}, \quad B \in \{Dir, Neu\}.$$

385 In our case, both  $\varepsilon_\sigma$  and  $\mu$  fulfill (5.1) and moreover, since  $\mathbf{j}_{ext} \in \mathbf{H}(\text{div}; \Omega)$ , we  
 386 note that the solution  $\mathbf{e}$  to (2.9) is such that  $\mathbf{e} \in \mathbf{X}_{Dir}(\Omega, \varepsilon_\sigma)$ , and  $\mu^{-1} \mathbf{curl} \mathbf{e} \in$   
 387  $\mathbf{X}_{Neu}(\Omega, \mu)$ .

388 **5.1. Geometric framework.** The domain  $\Omega$  can be topologically non-trivial,  
389 or with a non-connected boundary. Regarding the first item, we assume that:  
390 either **(Top)** $_{I=0}$  'given any curl-free vector field  $\mathbf{v} \in \mathbf{C}^1(\Omega)$ , there exists  $p \in C^0(\Omega)$   
391 such that  $\mathbf{v} = \nabla p$  in  $\Omega$ ';  
392 or **(Top)** $_{I>0}$  'there exist  $I$  non-intersecting manifolds,  $\Sigma_1, \dots, \Sigma_I$ , with bound-  
393 aries  $\partial\Sigma_i \subset \partial\Omega$ , such that, if we let  $\dot{\Omega} = \Omega \setminus \bigcup_{i=1}^I \Sigma_i$ , given any curl-free vector field  
394  $\mathbf{v}$ , there exists  $\dot{p} \in C^0(\dot{\Omega})$  such that  $\mathbf{v} = \nabla \dot{p}$  in  $\dot{\Omega}$ '.  
395 When  $I = 0$ ,  $\dot{\Omega} = \Omega$ . For short, we write **(Top)** $_I$  to cover both instances. One can  
396 build cuts that are piecewise plane, see [23, Chapter 6]. Finally, we assume that  $\dot{\Omega}$  is a  
397 connected set. For the polyhedral model problem, we assume that **(Top)** $_I$  is fulfilled.  
398 The domain  $\Omega$  is said to be topologically trivial when  $I = 0$ . When  $I > 0$ , the set  $\dot{\Omega}$   
399 has pseudo-Lipschitz boundary in the sense of [3].

400  
401 The a priori regularity of elements of  $\mathbf{X}_{Dir}(\Omega)$  and  $\mathbf{X}_{Neu}(\Omega)$  is described in [3,  
402 Remark 2.16 and Proposition 3.7]. Below,  $\subset$  refers to an algebraical and topological  
403 embedding.

404 **PROPOSITION 5.3.** *Let  $\Omega$  be a Lipschitz polyhedron: there exists  $\sigma_{Dir} \in (\frac{1}{2}, 1]$  such*  
405 *that it holds that  $\mathbf{X}_{Dir}(\Omega) \subset \mathbf{H}^{\sigma_{Dir}}(\Omega)$ . Assume in addition that **(Top)** $_I$  is fulfilled:*  
406 *there exists  $\sigma_{Neu} \in (\frac{1}{2}, 1]$  such that it holds that  $\mathbf{X}_{Neu}(\Omega) \subset \mathbf{H}^{\sigma_{Neu}}(\Omega)$ .*  
407 *Let  $\Omega$  be a domain: the embeddings hold with  $\sigma_{Dir} = \sigma_{Neu} = \frac{1}{2}$ .*

408 **COROLLARY 5.4.** *With the same assumptions as in Proposition 5.3, it holds that*  
409  *$\mathbf{Z}_{Dir}(\Omega) \subset \mathbf{H}^{\sigma_{Dir}}(\Omega)$  and  $\mathbf{Z}_{Neu}(\Omega) \subset \mathbf{H}^{\sigma_{Neu}}(\Omega)$ .*

410 Finally, one can prove that the null spaces  $\mathbf{Z}_{Dir}(\Omega)$  and  $\mathbf{Z}_{Neu}(\Omega)$  are finite dimen-  
411 sional vector spaces.

412 **5.2. Splittings of fields.** We provide now splittings into a regular part, and  
413 a gradient part, of elements of  $\mathbf{X}_{Dir}(\Omega, \xi)$  ("electric case"), resp. of elements of  
414  $\mathbf{X}_{Neu}(\Omega, \xi)$  ("magnetic case"), called *regular/gradient splittings*. The proofs can be  
415 found in §6.1.6 and §6.2.6 of [4]. We provide some comments on these splittings below.  
416

417 **THEOREM 5.5.** *Let  $\Omega$  be a domain such that **(Top)** $_I$  is fulfilled, and assume that  $\xi$*   
418 *fulfills (5.1). Then, there exists a continuous splitting operator acting from  $\mathbf{X}_{Dir}(\Omega, \xi)$*   
419 *to  $\mathbf{H}_{Dir}(\Omega) \times \mathbf{Z}_{Dir}(\Omega) \times H_0^1(\Omega)$ .*  
420 *More precisely, given  $\mathbf{v} \in \mathbf{X}_{Dir}(\Omega, \xi)$ ,*

$$421 \quad (5.2) \quad \exists (\mathbf{v}_{reg}, \mathbf{z}, p_0) \in \mathbf{H}_{Dir}(\Omega) \times \mathbf{Z}_{Dir}(\Omega) \times H_0^1(\Omega), \quad \mathbf{v} = \mathbf{v}_{reg} + \mathbf{z} + \nabla p_0 \text{ in } \Omega.$$

422 *One has*

$$423 \quad (5.3) \quad \|\mathbf{v}_{reg}\|_{1,\Omega} + \|\mathbf{v}_{reg}\|_{\mathbf{X}_{Dir}(\Omega)} + \|\mathbf{z}\|_{\sigma_{Dir},\Omega} + \|\mathbf{v}_{reg} + \mathbf{z}\|_{1/2,\Omega} \leq C_{\mathbf{X}}^{Dir} \|\mathbf{v}\|_{\mathbf{H}(\text{curl};\Omega)}.$$

424 *The scalar field  $p_0$  is governed by the variational formulation:*

$$425 \quad (5.4) \quad \begin{cases} \text{Find } p_0 \in H_0^1(\Omega) \text{ such that} \\ (\xi \nabla p_0 | \nabla \psi)_{0,\Omega} = -(\xi \mathbf{z} | \nabla \psi)_{0,\Omega} - (\xi \mathbf{v}_{reg} | \nabla \psi)_{0,\Omega} \\ \quad \quad \quad - (\text{div } \xi \mathbf{v} | \psi)_{0,\Omega}, \quad \forall \psi \in H_0^1(\Omega). \end{cases}$$

426 **THEOREM 5.6.** *Let  $\Omega$  be a domain such that **(Top)** $_I$  is fulfilled, and assume*  
427 *that  $\xi$  fulfills (5.1). Then, there exists a continuous splitting operator acting from*

428  $\mathbf{X}_{Neu}(\Omega, \xi)$  to  $\mathbf{H}_{zmv}^1(\Omega) \times \mathbf{Z}_{Neu}(\Omega) \times H_{zmv}^1(\Omega)$ .  
 429 More precisely, given  $\mathbf{v} \in \mathbf{X}_{Neu}(\Omega, \xi)$ ,

430 (5.5)  $\exists(\mathbf{w}_{reg}, \mathbf{z}, q_0) \in \mathbf{H}_{zmv}^1(\Omega) \times \mathbf{Z}_{Neu}(\Omega) \times H_{zmv}^1(\Omega)$ ,  $\mathbf{v} = \mathbf{w}_{reg} + \mathbf{z} + \nabla q_0$  in  $\Omega$ .

431 One has

432 (5.6)  $\|\mathbf{w}_{reg}\|_{1,\Omega} + \|\mathbf{w}_{reg}\|_{\mathbf{X}_{Neu}(\Omega)} + \|\mathbf{z}\|_{\sigma_{Neu},\Omega} + \|\mathbf{w}_{reg} + \mathbf{z}\|_{1/2,\Omega} \leq C_{\mathbf{X}}^{Neu} \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl};\Omega)}$ .

433 The scalar field  $q_0$  is governed by the variational formulation:

434 (5.7) 
$$\left\{ \begin{array}{l} \text{Find } q_0 \in H_{zmv}^1(\Omega) \text{ such that} \\ (\xi \nabla q_0 | \nabla \psi)_{0,\Omega} = -(\xi \mathbf{z} | \nabla \psi)_{0,\Omega} - (\xi \mathbf{w}_{reg} | \nabla \psi)_{0,\Omega} \\ -(\operatorname{div} \xi \mathbf{v} | \psi)_{0,\Omega}, \quad \forall \psi \in H_{zmv}^1(\Omega). \end{array} \right.$$

435 In the splitting (5.2) of  $\mathbf{v} \in \mathbf{X}_{Dir}(\Omega, \xi)$ , all three terms  $\mathbf{v}_{reg}, \mathbf{z}, \nabla p_0$  have vanishing  
 436 tangential components on the boundary  $\partial\Omega$ , whereas in the splitting (5.5) of  $\mathbf{v} \in$   
 437  $\mathbf{X}_{Neu}(\Omega, \xi)$ ,  $\mathbf{w}_{reg}$  does not verify a homogeneous boundary condition in general. Since  
 438  $\xi$  fulfills (5.1), both variational formulations (5.4) and (5.7) are well-posed. Finally,  
 439 we note that regarding the a priori regularity in (5.2), one has  $\mathbf{v}_{reg} \in \mathbf{H}^1(\Omega)$  and  $\mathbf{z} \in$   
 440  $\mathbf{H}^{\sigma_{Dir}}(\Omega)$ . Likewise, regarding the a priori regularity in (5.5), one has  $\mathbf{w}_{reg} \in \mathbf{H}^1(\Omega)$   
 441 and  $\mathbf{z} \in \mathbf{H}^{\sigma_{Neu}}(\Omega)$ .

442 **5.3. Comments.** One may easily generalize the splitting theory to the case  
 443 where  $\xi$  is a complex-valued, measurable, *tensor* field. As a matter of fact, it is  
 444 straightforward to check that if  $\xi$  fulfills:

445 
$$\left\{ \begin{array}{l} \xi \text{ is a complex-valued measurable tensor field on } \Omega, \quad \xi, \xi^{-1} \in \mathbb{L}^\infty(\Omega), \\ \exists \xi_- > 0, \theta^* \in [0, 2\pi), \quad \forall \mathbf{z} \in \mathbb{C}^3, \quad \Re(\exp(-i\theta^*) \xi \mathbf{z} \cdot \bar{\mathbf{z}}) \geq \xi_- |\mathbf{z}|^2 \text{ a.e. in } \Omega, \end{array} \right.$$

446 then the conclusions of Theorems 5.5 and 5.6 still apply. Obviously, (5.4) and (5.7)  
 447 are well-posed.

448 In the special case where  $\xi$  is a *normal tensor* field ( $\xi^* \xi = \xi \xi^*$  a.e. in  $\Omega$ ), or equiv-  
 449 alently there exists a unitary tensor field  $\mathbb{U}$  and a diagonal tensor field  $\mathbb{D}$  such that  
 450  $\xi = \mathbb{U}^{-1} \mathbb{D} \mathbb{U}$  a.e. in  $\Omega$ , one can reformulate the second line of the above condition as

451 (5.8)  $\exists \xi_- > 0, \theta^* \in [0, 2\pi), \quad \min_{k=1,2,3} \Re(\exp(-i\theta^*) \mathbb{D}_{kk}) \geq \xi_- \text{ a.e. in } \Omega$ .

452 **6. The global approach for finding a regularity exponent  $\tau_{(\varepsilon,\mu,\sigma)}$  and a**  
 453 **stability constant  $C_{(\varepsilon,\mu,\sigma,s)}^*$ .** To estimate the regularity exponent, we adapt some  
 454 results of [6] to the case of complex-valued coefficients. Let  $\xi$  be a coefficient defined  
 455 on  $\Omega$ , we assume in the current section that  $\xi$  fulfills (5.1). This assumption prescribes  
 456 that

457 (6.1) 
$$\xi \in \{z = \rho \exp(i\theta), \rho \in [\xi_-, \xi_{max}], \theta \in [\theta_{min}, \theta_{max}]\} \text{ a.e. in } \Omega, \text{ where}$$

$$\xi_{max} := \|\xi\|_{L^\infty(\Omega)}, \text{ and } 0 \leq \theta_{max} - \theta_{min} \leq 2 \arccos\left(\frac{\xi_-}{\xi_{max}}\right).$$

458 In other words, since  $\arccos(\xi_-/\xi_{max}) < \pi/2$  the coefficient  $\xi$  takes its values in some  
 459 open, *half plane* in  $\mathbb{C}$ . If the coefficients  $\varepsilon$  and  $\sigma$  are as in section 2.1, then  $\xi = \varepsilon_\sigma$   
 460 takes its values in some open, *quarter plane* in  $\mathbb{C}$ .

461

462 We recall that  $\mathbf{e} \in \mathbf{X}_{Dir}(\Omega, \varepsilon_\sigma)$  and  $\mu^{-1} \mathbf{curl} \mathbf{e} \in \mathbf{X}_{Neu}(\Omega, \mu)$ . Hence, according  
463 to Theorems 5.5 and 5.6, we may write:

$$464 \quad (6.2) \quad \mathbf{e} = \mathbf{e}_{reg} + \mathbf{z}_e + \nabla p_0 \text{ in } \Omega, \quad \mathbf{e}_{reg} \in \mathbf{H}^1(\Omega), \quad \mathbf{z}_e \in \mathbf{H}^{\sigma_{Dir}}(\Omega);$$

$$465 \quad (6.3) \quad \mu^{-1} \mathbf{curl} \mathbf{e} = \mathbf{c}_{reg} + \mathbf{z}_c + \nabla q_0 \text{ in } \Omega, \quad \mathbf{c}_{reg} \in \mathbf{H}^1(\Omega), \quad \mathbf{z}_c \in \mathbf{H}^{\sigma_{Neu}}(\Omega).$$

466 In addition, it holds that

$$467 \quad \|\mathbf{e}_{reg}\|_{1,\Omega} + \|\mathbf{e}_{reg}\|_{\mathbf{X}_{Dir}(\Omega)} + \|\mathbf{z}_e\|_{\sigma_{Dir},\Omega} + \|\mathbf{e}_{reg} + \mathbf{z}_e\|_{1/2,\Omega} \leq C_{\mathbf{X}}^{Dir} \|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\Omega)},$$

$$468 \quad \|\mathbf{c}_{reg}\|_{1,\Omega} + \|\mathbf{c}_{reg}\|_{\mathbf{X}_{Neu}(\Omega)} + \|\mathbf{z}_c\|_{\sigma_{Neu},\Omega} + \|\mathbf{c}_{reg} + \mathbf{z}_c\|_{1/2,\Omega} \leq C_{\mathbf{X}}^{Neu} \|\mu^{-1} \mathbf{curl} \mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\Omega)}.$$

469 It now remains to evaluate the regularity and norm of the gradient parts  $\nabla p_0$  and  
470  $\nabla q_0$ . Note that  $p_0$  is governed by the second-order scalar PDE (5.4) with *Dirichlet*  
471 *boundary condition*, while  $q_0$  is governed by the second-order scalar PDE (5.7) with  
472 *Neumann boundary condition*. We will use this vocabulary in the following to address  
473 both cases.

474 **6.1. Preliminary results.** To start with, given  $\mathcal{O} \subset \Omega$  a non-empty connected  
475 open subset of  $\mathbb{R}^3$  with Lipschitz boundary, let  $\mathcal{H}^0(\mathcal{O})$  be equal to  $L^2(\mathcal{O})$  in the  
476 Dirichlet case, resp.  $L^2_{zmv}(\mathcal{O})$  in the Neumann case, and  $\mathcal{H}^1(\mathcal{O})$  be equal to  $H^1_0(\mathcal{O})$   
477 in the Dirichlet case, resp.  $H^1_{zmv}(\mathcal{O})$  in the Neumann case. We equip  $\mathcal{H}^1(\mathcal{O})$  with the  
478 norm  $\|v\|_{\mathcal{H}^1(\mathcal{O})} := \|\nabla v\|_{0,\mathcal{O}}$ .

479 Then, for  $s \in (0, 1)$ , we introduce  $\mathcal{H}^s(\mathcal{O})$ , the Sobolev space obtained by the real  
480 interpolation method between  $\mathcal{H}^1(\mathcal{O})$  and  $\mathcal{H}^0(\mathcal{O})$ : if needed, we distinguish the two  
481 cases by writing  $\mathcal{H}^s_{Dir}(\mathcal{O})$ , resp.  $\mathcal{H}^s_{Neu}(\mathcal{O})$ . In particular, by definition (cf. §3.1), it  
482 holds that  $\mathcal{H}^s_{Dir}(\mathcal{O}) = \tilde{H}^s(\mathcal{O})$  for all  $s \in [0, 1]$ , and we recall that  $H^s(\mathcal{O}) = \tilde{H}^s(\mathcal{O})$   
483 for all  $s \in [0, \frac{1}{2})$ .

484 We denote by  $\mathcal{H}^{-s}(\mathcal{O})$  the dual space of  $\mathcal{H}^s(\mathcal{O})$ , for  $s \in [0, 1]$ . Finally, for  $s \in [0, 1]$ ,  
485 we define  $\mathcal{H}^{1+s}(\mathcal{O}) := \{v \in \mathcal{H}^1(\mathcal{O}) \text{ s.t. } \nabla v \in \mathbf{H}^s(\mathcal{O})\}$ , equipped with the norm  
486  $\|v\|_{\mathcal{H}^{1+s}(\mathcal{O})} := \|\nabla v\|_{s,\mathcal{O}}$ .

487 **LEMMA 6.1.** *Given  $s \in [0, 1]$ , there exists  $C_{(s)}^P > 0$  such that*

$$488 \quad \forall v \in \mathcal{H}^{1+s}(\mathcal{O}), \quad \|v\|_{\mathcal{H}^{1+s}(\mathcal{O})} \leq \|v\|_{1+s,\mathcal{O}} \leq C_{(s)}^P \|v\|_{\mathcal{H}^{1+s}(\mathcal{O})}.$$

489 *Proof.* The result is obvious for  $s \in \{0, 1\}$ , according to the Poincaré inequality.  
490 We let now  $s \in (0, 1)$ .

491 For the left inequality, notice that

$$492 \quad \forall v \in H^1(\mathcal{O}), \quad \|\nabla v\|_{0,\Omega} \leq \|v\|_{1,\Omega}; \quad \forall v \in H^2(\mathcal{O}), \quad \|\nabla v\|_{1,\Omega} \leq \|v\|_{2,\Omega}.$$

493 As a consequence, the left inequality follows. This is the so-called *exact sequence*  
494 *property*. Following Appendix A, if we let  $v \in \mathcal{H}^{1+s}(\mathcal{O})$ :

$$\begin{aligned} 495 \quad & \|v\|_{\mathcal{H}^{1+s}(\mathcal{O})} := \|\nabla v\|_{s,\mathcal{O}} \\ 496 \quad & = \|t^{-s} \inf_{\substack{\nabla v = \mathbf{v}_0 + \mathbf{v}_1 \\ \mathbf{v}_0 \in \mathbf{L}^2(\mathcal{O}), \mathbf{v}_1 \in \mathbf{H}^1(\mathcal{O})}} (\|\mathbf{v}_0\|_{0,\mathcal{O}}^2 + t^2 \|\mathbf{v}_1\|_{1,\mathcal{O}}^2)^{1/2} \|_{L^2(0,\infty; \frac{dt}{t})} \\ 497 \quad & \leq \|t^{-s} \inf_{\substack{v = v_0 + v_1 \\ v_0 \in H^1(\mathcal{O}), v_1 \in H^2(\mathcal{O})}} (\|\nabla v_0\|_{0,\mathcal{O}}^2 + t^2 \|\nabla v_1\|_{1,\mathcal{O}}^2)^{1/2} \|_{L^2(0,\infty; \frac{dt}{t})} \\ 498 \quad & \text{(cf. above)} \leq \|t^{-s} \inf_{\substack{v = v_0 + v_1 \\ v_0 \in H^1(\mathcal{O}), v_1 \in H^2(\mathcal{O})}} (\|v_0\|_{1,\mathcal{O}}^2 + t^2 \|v_1\|_{2,\mathcal{O}}^2)^{1/2} \|_{L^2(0,\infty; \frac{dt}{t})} \\ 499 \quad & =: \|v\|_{1+s,\mathcal{O}}. \end{aligned}$$

500 For the right inequality, let us introduce the Poincaré constant:

$$501 \quad \mathcal{C} := \sup_{v \in \mathcal{H}^1(\mathcal{O}) \setminus \{0\}} \frac{\|v\|_{0,\mathcal{O}}}{\|\nabla v\|_{0,\mathcal{O}}}.$$

502 Using the equivalence of norms, the definition of  $\|\cdot\|_{\underline{H}^{1+s}(\mathcal{O})}$ , the Poincaré inequality,  
503 and finally the equivalence of norms again, we find:

$$\begin{aligned} 504 \quad \|v\|_{1+s,\mathcal{O}} &\leq M_{(1+s)} \|v\|_{\underline{H}^{1+s}(\mathcal{O})} := M_{(1+s)} \left( \|v\|_{1,\mathcal{O}}^2 + |\nabla v|_{\underline{H}^s(\mathcal{O})}^2 \right)^{1/2} \\ 505 \quad &\leq M_{(1+s)} \left( (1 + \mathcal{C}^2) \|\nabla v\|_{0,\mathcal{O}}^2 + |\nabla v|_{\underline{H}^s(\mathcal{O})}^2 \right)^{1/2} \\ 506 \quad &\leq M_{(1+s)} (1 + \mathcal{C}^2)^{1/2} \|\nabla v\|_{\underline{H}^s(\mathcal{O})} \\ 507 \quad &\leq M_{(1+s)} (1 + \mathcal{C}^2)^{1/2} m_{(s)}^{-1} \|\nabla v\|_{s,\mathcal{O}} =: M_{(1+s)} (1 + \mathcal{C}^2)^{1/2} m_{(s)}^{-1} \|v\|_{\mathcal{H}^{1+s}(\mathcal{O})}. \end{aligned}$$

508 Hence, one may choose  $C_{(s)}^P = M_{(1+s)} (1 + \mathcal{C}^2)^{1/2} m_{(s)}^{-1}$ . □

509 If we let  $\underline{s} \in [0, 1]$ , we want to find the a priori regularity of the solution to

$$510 \quad (6.4) \quad \begin{cases} \text{Find } u \in \mathcal{H}^1(\Omega) \text{ such that} \\ (\xi \nabla u | \nabla v)_{0,\Omega} = \langle f, v \rangle_{\mathcal{H}^1(\Omega)}, \quad \forall v \in \mathcal{H}^1(\Omega), \end{cases}$$

511 and  $f$  is some data in  $\mathcal{H}^{-\underline{s}}(\Omega)$ .

512 If  $\xi$  is constant on  $\Omega$ , that is if one considers the Laplace operator with Dirichlet  
513 boundary condition, or with Neumann boundary condition, then one may apply the  
514 classical results of [26] or [32] (see [6, p. 504]). See also Proposition 6.7 below. In the  
515 statement of the next Theorem, the constant  $c^{Lap}(s)$  depends on  $\Omega$ . For the sake of  
516 conciseness, we omit this dependence.

517 **THEOREM 6.2.** *Let  $\xi \neq 0$  be constant on  $\Omega$ . Then, for all  $s \in [0, \frac{1}{2})$ , there exists*  
518  *$c(s) := c^{Lap}(s) > 0$  such that for all  $f \in \mathcal{H}^{s-1}(\Omega)$ , the solution  $u \in \mathcal{H}^1(\Omega)$  to (6.4)*  
519 *belongs to  $\mathcal{H}^{s+1}(\Omega)$ , and*

$$520 \quad \|u\|_{\mathcal{H}^{s+1}(\Omega)} \leq \frac{c(s)}{\xi_{max}} \|f\|_{\mathcal{H}^{s-1}(\Omega)}.$$

521 **DEFINITION 6.3.** *Let the coefficient  $\xi$  fulfill (5.1). We say that  $\xi$  fulfills the coef-*  
522 *ficient assumption if there exists a partition  $\mathcal{P}$  of  $\Omega$  such that  $\xi \in PW^{1,\infty}(\Omega)$ .*

523 If  $\xi$  fulfills the coefficient assumption on a partition, then  $\xi^{-1}$  fulfills the coefficient  
524 assumption on the same partition.

525

526 From now on in the current section, we consider the case where  $\xi \neq 0$  is a scalar,  
527 non-constant, *complex-valued* coefficient that fulfills the coefficient assumption on a  
528 partition  $\mathcal{P} := \{\Omega_j\}_{j=1,\dots,J}$ . In [6], the authors study the case of a symmetric-tensor,  
529 real-valued coefficient  $\xi$ . There are similarities between the two cases, and also some  
530 differences, that are highlighted below. We refer to §6.5 for a generalization to the  
531 case of a normal-tensor, complex-valued coefficient  $\xi$ . Let

$$532 \quad \Lambda_\xi := \frac{|\xi|_{PW^{1,\infty}(\Omega)}}{\xi_{max}}.$$

533 By definition, it holds that  $\|\xi\|_{PW^{1,\infty}(\Omega)} = \xi_{max}(1 + \Lambda_\xi)$ . For a piecewise constant  
534 coefficient  $\xi$ , one has  $\Lambda_\xi = 0$ . Otherwise,  $\Lambda_\xi > 0$ .

535

536 For  $s \in [0, \frac{1}{2})$ , choosing  $\mathcal{O} \in \{\Omega_j, 1 \leq j \leq J\}$ , we denote by  $D_j^s$  the norm of the  
 537 natural imbedding of  $\mathbf{H}^s(\Omega_j)$  into  $\widetilde{\mathbf{H}}^s(\Omega_j)$ :

$$538 \quad (6.5) \quad \begin{aligned} D_j^s &:= \sup_{\mathbf{v}_j \in \mathbf{H}^s(\Omega_j) \setminus \{0\}} \frac{\|\mathbf{v}_j\|_{\widetilde{\mathbf{H}}^s(\Omega_j)}}{\|\mathbf{v}_j\|_{s, \Omega_j}}, \quad 1 \leq j \leq J; \\ D_s &:= \max(1, \max_{1 \leq j \leq J} D_j^s) \geq 1. \end{aligned}$$

539

540 *Remark 6.4.* It holds that  $\lim_{s \rightarrow \frac{1}{2}} D_j^s = +\infty$ , because constant, non-vanishing  
 541 fields defined on  $\Omega_j$  belong to  $\mathbf{H}^s(\Omega_j)$ , but not to  $\widetilde{\mathbf{H}}^s(\Omega_j)$ .

542 Also, we denote the Poincaré constants by:

$$543 \quad (6.6) \quad C_j := \sup_{\mathbf{v}_j \in \mathbf{H}_0^1(\Omega_j) \setminus \{0\}} \frac{\|\mathbf{v}_j\|_{0, \Omega_j}}{\|\nabla \mathbf{v}_j\|_{0, \Omega_j}}, \quad 1 \leq j \leq J; \quad C := \max_{1 \leq j \leq J} C_j > 0.$$

544 We note that, obviously, the constants  $\Lambda_\xi$ ,  $(D_j^s)_j$ ,  $D_s$ ,  $(C_j)_j$  and  $C$  all depend on  $\Omega$ ,  
 545 and on the partition  $\mathcal{P}$ . These dependences are omitted.

546 Then, we define the multiplicative operator  $m_\xi \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{L}^2(\Omega))$  by:  $m_\xi \mathbf{v}(\mathbf{x}) =$   
 547  $\xi(\mathbf{x}) \mathbf{v}(\mathbf{x})$ , for all  $\mathbf{v} \in \mathbf{L}^2(\Omega)$ , a.e.  $\mathbf{x} \in \Omega$ . One may now adapt the proof of Proposition  
 548 2.1 of [6] to the complex-valued case, to find...

549 **PROPOSITION 6.5.** *Let  $\xi$  fulfill the coefficient assumption. Then, for all  $s \in [0, \frac{1}{2})$ ,*  
 550 *it holds that  $m_\xi \in \mathcal{L}(\mathbf{H}^s(\Omega), \widetilde{\mathbf{H}}^s(\Omega))$  and in addition,*

$$551 \quad (6.7) \quad \|m_\xi\|_{\mathcal{L}(\mathbf{H}^s(\Omega), \widetilde{\mathbf{H}}^s(\Omega))} \leq \xi_{\max} N_\xi^s, \quad \text{where } N_\xi^s := D_s(2(1 + C^2 \Lambda_\xi^2))^{s/2}.$$

552 *Furthermore, for all  $r \in [0, \frac{1}{2})$ , it holds that*

$$553 \quad (6.8) \quad \|m_\xi\|_{\mathcal{L}(\mathbf{H}^s(\Omega), \widetilde{\mathbf{H}}^s(\Omega))} \leq \xi_{\max} (N_\xi^r)^{s/r}, \quad \forall s \in [0, r].$$

554 We then recall the technical Lemmas 3.1 and 3.2 of [6], which are independent of the  
 555 coefficient  $\xi$ . Introduce the operator  $D \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathcal{H}^{-1}(\Omega))$  defined by

$$556 \quad \langle D\mathbf{v}, q \rangle_{\mathcal{H}^1(\Omega)} = (\mathbf{v} | \nabla q)_{0, \Omega}, \quad \forall \mathbf{v} \in \mathbf{L}^2(\Omega), \quad \forall q \in \mathcal{H}^1(\Omega).$$

557

558 **PROPOSITION 6.6.** *For all  $s \in [0, 1]$ , one has*

$$559 \quad (6.9) \quad D \in \mathcal{L}(\widetilde{\mathbf{H}}^s(\Omega), \mathcal{H}^{s-1}(\Omega)) \quad \text{and} \quad \|D\|_{\mathcal{L}(\widetilde{\mathbf{H}}^s(\Omega), \mathcal{H}^{s-1}(\Omega))} \leq 1.$$

560 Introduce the operator  $L \in \mathcal{L}(\mathcal{H}^{-1}(\Omega), \mathcal{H}^1(\Omega))$  defined by

$$561 \quad (\nabla(Lv) | \nabla q)_{0, \Omega} = \langle v, q \rangle_{\mathcal{H}^1(\Omega)}, \quad \forall v \in \mathcal{H}^{-1}(\Omega), \quad \forall q \in \mathcal{H}^1(\Omega).$$

562 One may rephrase Theorem 6.2 (with  $\xi = 1$ ) as follows.

563 **PROPOSITION 6.7.** *For all  $r \in [0, \frac{1}{2})$ , one has  $L \in \mathcal{L}(\mathcal{H}^{r-1}(\Omega), \mathcal{H}^{r+1}(\Omega))$  and*  
 564 *there exists  $K_r \geq 1$  such that it holds that:*

$$565 \quad (6.10) \quad \|L\|_{\mathcal{L}(\mathcal{H}^{r-1}(\Omega), \mathcal{H}^{r+1}(\Omega))} \leq K_r;$$

$$566 \quad (6.11) \quad \text{for all } s \in [0, r], \quad \|L\|_{\mathcal{L}(\mathcal{H}^{s-1}(\Omega), \mathcal{H}^{s+1}(\Omega))} \leq (K_r)^{s/r}.$$

567 Obviously,  $K_r$  depends on  $\Omega$ . This dependence is omitted.



568 **6.2. Regularity of scalar fields.** We transpose Theorem 3.1 of [6] to the  
569 complex-valued case. Since the second half of the proof (choice of the parameter  
570  $k$ , here complex-valued; explicit dependence of the regularity exponent  $\tau$  and stabil-  
571 ity constant  $c$  on  $\mathcal{P}$ ,  $\xi$ ...) is quite different in this case, we provide it for the sake of  
572 completeness.

573 **THEOREM 6.8.** *Let  $\xi$  be a scalar, non-constant, complex-valued coefficient that*  
574 *fulfills the coefficient assumption. Then, there exists a regularity exponent  $\tau_\xi :=$*   
575  *$\tau(\mathcal{P}, \xi_-/\xi_{max}, \Lambda_\xi) \in (0, \frac{1}{2})$  such that, for all  $s \in [0, \tau_\xi)$ , there exists a constant*  
576  *$c(s, \xi) := c(\mathcal{P}, s, \xi_-/\xi_{max}, \Lambda_\xi)$  such that for all  $f \in \mathcal{H}^{s-1}(\Omega)$ , the solution  $u \in \mathcal{H}^1(\Omega)$*   
577 *to (6.4) belongs to  $\mathcal{H}^{s+1}(\Omega)$ , and*

$$578 \quad \|u\|_{\mathcal{H}^{s+1}(\Omega)} \leq \frac{c(s, \xi)}{\xi_{max}} \|f\|_{\mathcal{H}^{s-1}(\Omega)}.$$

579 *Remark 6.9.* By introducing the scaling factor  $(\xi_{max})^{-1}$ , one is able to refine the  
580 dependence of the constant  $c$  on the coefficient, namely that  $c$  depends only on the  
581 amplitude  $\xi_-/\xi_{max}$  and the local, scaled, variations  $\Lambda_\xi$ . And because the coefficient  
582  $\xi$  is non-constant, condition (5.1) yields  $\xi_-/\xi_{max} \in (0, 1)$ .

583 *Proof.* Let  $k \in \mathbb{C} \setminus \{0\}$ . Using the operator  $D$ , we note that given  $f^* \in \mathcal{H}^{-1}(\Omega)$   
584 and denoting by  $u^*$  the solution to (6.4) with data  $f^*$ , it holds that

$$585 \quad f^* = D(\xi \nabla u^*) = D(k \nabla u^*) - D((k - \xi) \nabla u^*) = D(\nabla(ku^*)) - D((1 - \frac{\xi}{k}) \nabla(ku^*)).$$

586 Introducing  $\bar{\xi} = (1 - \xi/k) \in PW^{1,\infty}(\Omega)$  and  $v^* = ku^* \in \mathcal{H}^1(\Omega)$ , we get  $f^* = D(\nabla v^*) -$   
587  $D(\bar{\xi} \nabla v^*)$ . Because  $LD\nabla$  is equal to the identity operator in  $\mathcal{L}(\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega))$ , it  
588 follows that

$$589 \quad Lf^* = v^* - L(D(\bar{\xi} \nabla v^*)) = v^* - L(D(m_{\bar{\xi}}(\nabla v^*))).$$

590 Let us now denote  $Q := LDm_{\bar{\xi}}\nabla \in \mathcal{L}(\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega))$ . If moreover  $Q$  belongs to  
591  $\mathcal{L}(\mathcal{H}^{s+1}(\Omega), \mathcal{H}^{s+1}(\Omega))$  for some  $s \in (0, \frac{1}{2})$ , then we derive from the above that if we  
592 consider some data  $f \in \mathcal{H}^{s-1}(\Omega)$  in (6.4), one has

$$593 \quad (6.12) \quad |k| \|u\|_{\mathcal{H}^{s+1}(\Omega)} \leq \frac{\|L\|_{s-1, s+1}}{1 - \|Q\|_{s+1, s+1}} \|f\|_{\mathcal{H}^{s-1}(\Omega)},$$

594 *under the condition  $\|Q\|_{s+1, s+1} < 1$ , where  $\|L\|_{s-1, s+1} := \|L\|_{\mathcal{L}(\mathcal{H}^{s-1}(\Omega), \mathcal{H}^{s+1}(\Omega))}$  and*  
595  $\|Q\|_{s+1, s+1} := \|Q\|_{\mathcal{L}(\mathcal{H}^{s+1}(\Omega), \mathcal{H}^{s+1}(\Omega))}$ .

596 Given  $s \in (0, \frac{1}{2})$ , let  $v \in \mathcal{H}^{s+1}(\Omega)$ . One checks successively that:  $\nabla v \in \mathbf{H}^s(\Omega)$ ;  
597  $m_{\bar{\xi}}(\nabla v) \in \widetilde{\mathbf{H}}^s(\Omega)$  (cf. Proposition 6.5);  $D(m_{\bar{\xi}}\nabla v) \in \mathcal{H}^{s-1}(\Omega)$  (cf. Proposition 6.6);  
598  $L(Dm_{\bar{\xi}}\nabla v) \in \mathcal{H}^{s+1}(\Omega)$  (cf. Proposition 6.7). In addition, all those results are  
599 accompanied by a continuous dependence. Hence, one has  $Q \in \mathcal{L}(\mathcal{H}^{s+1}(\Omega), \mathcal{H}^{s+1}(\Omega))$ ,  
600 with

$$601 \quad (6.13) \quad \|Q\|_{s+1, s+1} \leq \|L\|_{s-1, s+1} \|D\|_{\mathcal{L}(\widetilde{\mathbf{H}}^s(\Omega), \mathcal{H}^{s-1}(\Omega))} \|m_{\bar{\xi}}\|_{\mathcal{L}(\mathbf{H}^s(\Omega), \widetilde{\mathbf{H}}^s(\Omega))}.$$

602 So it remains to prove that there exists  $\tau \in (0, \frac{1}{2})$  such that, for all  $s \in [0, \tau)$ ,  
603  $\|Q\|_{s+1, s+1} < 1$ , where the bound on the norm is derived by an appropriate choice of  
604  $k \in \mathbb{C} \setminus \{0\}$ .

605 To that aim, let  $\tau_0 \in (0, \frac{1}{2})$  be given, and let  $s \in [0, \tau_0)$ . Using the bound (6.13), we  
606 obtain first that

$$607 \quad (6.14) \quad \|Q\|_{s+1, s+1} \leq (K_{\tau_0})^{s/\tau_0} \bar{\xi}_{max} (N_{\bar{\xi}}^{\tau_0})^{s/\tau_0},$$

608 where  $\bar{\xi}_{max} := \|\bar{\xi}\|_{L^\infty(\Omega)}$ , according to (6.8), (6.11) and Proposition 6.6. We know  
 609 from Proposition 6.7 that  $K_{\tau_0} \geq 1$ , and that it is independent of  $k$  and  $\xi$ . Let us study  
 610 now the behavior of  $\bar{\xi}_{max}$  and  $N_{\bar{\xi}}^{\tau_0}$  with respect to  $k$  and  $\xi$ , so that for an appropriate  
 611 choice of  $k$  and of  $\tau \in (0, \tau_0]$ , one can guarantee that

$$612 \quad (6.15) \quad (K_{\tau_0})^{s/\tau_0} \bar{\xi}_{max} (N_{\bar{\xi}}^{\tau_0})^{s/\tau_0} < 1, \quad \forall s \in [0, \tau).$$

613 To achieve (6.15), one needs that  $\bar{\xi}_{max} < 1$ . Indeed, once  $k$  is given, it holds that  
 614  $\lim_{s \rightarrow 0^+} (K_{\tau_0})^{s/\tau_0} (N_{\bar{\xi}}^{\tau_0})^{s/\tau_0} = 1$ .

615 Let  $k = \bar{\rho} \exp(i\tilde{\theta})$ ,  $\bar{\rho} > 0$ ,  $\tilde{\theta} \in [0, 2\pi)$ . Keeping the notations of (6.1), one has

$$616 \quad |\bar{\xi}(\mathbf{x})|^2 = |1 - \xi(\mathbf{x})/k|^2 = 1 + \frac{\rho(\mathbf{x})}{\bar{\rho}} \left( \frac{\rho(\mathbf{x})}{\bar{\rho}} - 2 \cos(\theta(\mathbf{x}) - \tilde{\theta}) \right), \text{ a.e. } \mathbf{x} \in \Omega.$$

617 It follows that  $|\bar{\xi}(\mathbf{x})| < 1$  a.e.  $\mathbf{x} \in \Omega$  if, and only if,

$$618 \quad \frac{1}{2} \frac{\rho(\mathbf{x})}{\bar{\rho}} < \cos(\theta(\mathbf{x}) - \tilde{\theta}), \text{ a.e. } \mathbf{x} \in \Omega.$$

619 Choosing the angular part  $\tilde{\theta} = \frac{1}{2}(\theta_{min} + \theta_{max})$  and recalling that  $\xi_{max} = \|\rho\|_{L^\infty(\Omega)}$   
 620 (cf. (6.1)), this condition is implied by

$$621 \quad \frac{1}{2} \frac{\xi_{max}}{\bar{\rho}} < \cos\left(\frac{1}{2}(\theta_{max} - \theta_{min})\right).$$

622 Since one has  $\theta_{max} - \theta_{min} \leq 2 \arccos(\xi_- / \xi_{max})$ , a sufficient condition is

$$623 \quad \frac{1}{2} \frac{\xi_{max}}{\bar{\rho}} < \frac{\xi_-}{\xi_{max}} \iff \frac{1}{2} \frac{(\xi_{max})^2}{\xi_-} < \bar{\rho}.$$

624 So, let us choose  $k = \gamma (\xi_{max})^2 / \xi_- \exp(\frac{i}{2}(\theta_{min} + \theta_{max}))$  for some  $\gamma \in (\frac{1}{2}, \infty)$  to be  
 625 determined. With this value of  $k = k(\gamma)$ , one can find an upper bound for  $\bar{\xi}_{max}$ :

$$626 \quad |\bar{\xi}(\mathbf{x})|^2 = 1 + \frac{\rho(\mathbf{x})(\xi_-)^2}{\gamma^2 (\xi_{max})^3} \left( \frac{\rho(\mathbf{x})}{\xi_{max}} - 2\gamma \frac{\xi_{max}}{\xi_-} \cos(\theta(\mathbf{x}) - \tilde{\theta}) \right), \text{ a.e. } \mathbf{x} \in \Omega.$$

627 But  $\rho(\mathbf{x}) \leq \xi_{max}$  and  $\cos(\theta(\mathbf{x}) - \tilde{\theta}) \geq \cos(\frac{1}{2}(\theta_{max} - \theta_{min}))$  a.e.  $\mathbf{x} \in \Omega$ , so

$$628 \quad \frac{\rho(\mathbf{x})}{\xi_{max}} - 2\gamma \frac{\xi_{max}}{\xi_-} \cos(\theta(\mathbf{x}) - \tilde{\theta}) \leq 1 - 2\gamma \frac{\xi_{max}}{\xi_-} \cos\left(\frac{1}{2}(\theta_{max} - \theta_{min})\right)$$

$$629 \quad \leq 1 - 2\gamma, \text{ a.e. } \mathbf{x} \in \Omega.$$

630 Since  $1 - 2\gamma < 0$ , we now observe that  $\rho(\mathbf{x}) \geq \xi_-$  a.e.  $\mathbf{x} \in \Omega$  leads to

$$631 \quad |\bar{\xi}(\mathbf{x})|^2 \leq 1 + \frac{1 - 2\gamma}{\gamma^2} \left( \frac{\xi_-}{\xi_{max}} \right)^3, \text{ a.e. } \mathbf{x} \in \Omega.$$

632 The minimum of  $\gamma \mapsto (1 - 2\gamma)/\gamma^2$  on  $(\frac{1}{2}, \infty)$  is obtained for  $\gamma = 1$  and is equal to  $-1$ .  
 633 So we finally choose  $k = k(1)$ , i.e.

$$634 \quad (6.16) \quad k = \frac{(\xi_{max})^2}{\xi_-} \exp\left(\frac{i}{2}(\theta_{min} + \theta_{max})\right),$$

635 and conclude that, for this choice of  $k$ , it holds that

$$636 \quad (6.17) \quad \bar{\xi}_{max} \leq \left(1 - \left(\frac{\xi_-}{\xi_{max}}\right)^3\right)^{1/2}.$$

637 With this choice for  $k$ , one can also infer an upper bound for  $N_{\bar{\xi}}^{\tau_0} = D_{\tau_0}(2(1 +$   
638  $C^2\Lambda_{\bar{\xi}}^2))^{\tau_0/2}$ , where  $\Lambda_{\bar{\xi}} = |\bar{\xi}|_{PW^{1,\infty}(\Omega)}/\bar{\xi}_{max} = \max_{1 \leq j \leq J} \|\nabla \bar{\xi}_j\|_{L^\infty(\Omega_j)}/\bar{\xi}_{max}$ .

639 Let us bound  $\bar{\xi}_{max}$  from below:

$$\begin{aligned} 640 \quad 0 < \left(1 - \frac{\xi_-}{\xi_{max}}\right)^2 &\leq \left(1 - \frac{\rho(\mathbf{x})\xi_-}{(\xi_{max})^2}\right)^2, \quad \text{a.e. } \mathbf{x} \in \Omega, \\ 641 \quad &= 1 + \frac{\rho(\mathbf{x})(\xi_-)^2}{(\xi_{max})^3} \left(\frac{\rho(\mathbf{x})}{\xi_{max}} - 2\frac{\xi_{max}}{\xi_-}\right), \quad \text{a.e. } \mathbf{x} \in \Omega, \\ 642 \quad &\leq 1 + \frac{\rho(\mathbf{x})(\xi_-)^2}{(\xi_{max})^3} \left(\frac{\rho(\mathbf{x})}{\xi_{max}} - 2\frac{\xi_{max}}{\xi_-} \cos(\theta(\mathbf{x}) - \tilde{\theta})\right), \quad \text{a.e. } \mathbf{x} \in \Omega, \\ 643 \quad &= |\bar{\xi}(\mathbf{x})|^2, \quad \text{a.e. } \mathbf{x} \in \Omega \\ 644 \quad \text{so } |\bar{\xi}(\mathbf{x})| &\geq 1 - \frac{\xi_-}{\xi_{max}} > 0, \quad \text{a.e. } \mathbf{x} \in \Omega. \end{aligned}$$

645 Hence,  $\bar{\xi}_{max} \geq 1 - \xi_-/\xi_{max}$ .

646 Next, for  $1 \leq j \leq J$ :

$$647 \quad \|\nabla \bar{\xi}_j\|_{L^\infty(\Omega_j)} = \frac{1}{|k|} \|\nabla \xi_j\|_{L^\infty(\Omega_j)} = \frac{\xi_-}{(\xi_{max})^2} \|\nabla \xi_j\|_{L^\infty(\Omega_j)}.$$

648 It follows that<sup>1</sup>:

$$\begin{aligned} 649 \quad \Lambda_{\bar{\xi}} &\leq \frac{1}{\left(1 - \frac{\xi_-}{\xi_{max}}\right)} \max_{1 \leq j \leq J} \|\nabla \bar{\xi}_j\|_{L^\infty(\Omega_j)}, \\ 650 \quad &= \frac{1}{\left(1 - \frac{\xi_-}{\xi_{max}}\right)} \frac{\xi_-}{\xi_{max}} \frac{\max_{1 \leq j \leq J} \|\nabla \xi_j\|_{L^\infty(\Omega_j)}}{\xi_{max}}, \\ 651 \quad \text{so } \Lambda_{\bar{\xi}} &\leq \frac{1}{\left(\frac{\xi_{max}}{\xi_-} - 1\right)} \Lambda_{\xi}. \end{aligned}$$

652 We conclude that

$$653 \quad N_{\bar{\xi}}^{\tau_0} \leq D_{\tau_0} \left(2\left(1 + \frac{C^2}{\left(\frac{\xi_{max}}{\xi_-} - 1\right)^2} \Lambda_{\xi}^2\right)\right)^{\tau_0/2}.$$

654 From this point on, one can choose  $\tau \leq \tau_0$  to ensure that (6.15), and so (6.14), hold.

655 Due to the upper bounds on  $\bar{\xi}_{max}$  and  $N_{\bar{\xi}}^{\tau_0}$ , it is sufficient that

$$656 \quad \left(1 - \left(\frac{\xi_-}{\xi_{max}}\right)^3\right)^{1/2} \left(2(K_{\tau_0} D_{\tau_0})^{2/\tau_0} \left(1 + \frac{C^2}{\left(\frac{\xi_{max}}{\xi_-} - 1\right)^2} \Lambda_{\xi}^2\right)\right)^{s/2} < 1, \quad \forall s \in [0, \tau).$$

<sup>1</sup>Notice that in the particular case where  $\xi$  is piecewise constant, we have that  $\Lambda_{\bar{\xi}} = \Lambda_{\xi} = 0$ .

657 Since  $2(K_{\tau_0}D_{\tau_0})^{2/\tau_0} \geq 2$  (see Proposition 6.7 and (6.5)) and  $(1 + C^2 \dots) \geq 1$ , this  
 658 last condition is equivalent to

$$659 \quad \left(1 - \left(\frac{\xi_-}{\xi_{max}}\right)^3\right) \left(2(K_{\tau_0}D_{\tau_0})^{2/\tau_0} \left(1 + \frac{C^2}{\left(\frac{\xi_{max}}{\xi_-} - 1\right)^2} \Lambda_\xi^2\right)\right)^\tau < 1.$$

660 This leads to choosing  
 (6.18)

$$661 \quad \tau_\xi = \min \left( \tau_0, -\frac{\log \left(1 - \left(\frac{\xi_-}{\xi_{max}}\right)^3\right)}{\log 2 + \frac{2}{\tau_0} \log(K_{\tau_0}D_{\tau_0}) + \log \left(1 + \frac{C^2}{\left(\frac{\xi_{max}}{\xi_-} - 1\right)^2} \Lambda_\xi^2\right)} \right) \in (0, \tau_0].$$

662 Finally, we also conclude from (6.12), (6.14), (6.16) and the bounds above that, for all  
 663  $s \in [0, \tau)$ , for all  $f \in \mathcal{H}^{s-1}(\Omega)$ , the solution  $u \in \mathcal{H}^1(\Omega)$  to (6.4) belongs to  $\mathcal{H}^{s+1}(\Omega)$ ,  
 664 and

$$665 \quad \|u\|_{\mathcal{H}^{s+1}(\Omega)} \leq \frac{\xi_-}{(\xi_{max})^2} \frac{(K_{\tau_0})^{s/\tau_0}}{1 - (K_{\tau_0})^{s/\tau_0} \bar{\xi}_{max} (N_{\bar{\xi}}^{\tau_0})^{s/\tau_0}} \|f\|_{\mathcal{H}^{s-1}(\Omega)}$$

$$666 \quad \leq \frac{c(s, \xi)}{\xi_{max}} \|f\|_{\mathcal{H}^{s-1}(\Omega)}, \text{ where}$$

$$667 \quad (6.19) \quad c(s, \xi) = \frac{\frac{\xi_-}{\xi_{max}} (K_{\tau_0})^{s/\tau_0}}{1 - \left(2(K_{\tau_0}D_{\tau_0})^{2/\tau_0} \left(1 + \frac{C^2}{\left(\frac{\xi_{max}}{\xi_-} - 1\right)^2} \Lambda_\xi^2\right)\right)^{s/2} \left(1 - \left(\frac{\xi_-}{\xi_{max}}\right)^3\right)^{1/2}}.$$

668 This proves the claim.  $\square$

669 According to (6.17), one finds that  $\bar{\xi}_{max} = 0$  only in the particular case where  $\xi_- =$   
 670  $\xi_{max}$ . As a matter of fact, in this case, we know from (5.1) that  $\xi(\mathbf{x}) = \xi_- \exp(i\theta^*)$   
 671 a.e. in  $\Omega$ . Hence, the operator  $D m_\xi \nabla$  is proportional to the Laplacian  $D \nabla$  and  
 672 the result is trivial: one can even pick any regularity exponent  $\tau$  lower than  $\frac{1}{2}$ , cf.  
 673 Theorem 6.2.

674 On the other hand, for a *piecewise constant* coefficient  $\xi$ , one has  $\Lambda_\xi = 0$ , so that once  
 675  $\tau_0 \in (0, \frac{1}{2})$  is chosen, (6.18) and (6.19) respectively simplify to:

$$676 \quad (6.20) \quad \tau_\xi = \min \left( \tau_0, -\frac{\log \left(1 - \left(\frac{\xi_-}{\xi_{max}}\right)^3\right)}{\log 2 + \frac{2}{\tau_0} \log(K_{\tau_0}D_{\tau_0})} \right) \in (0, \tau_0];$$

$$677 \quad (6.21) \quad c(s, \xi) = \frac{\frac{\xi_-}{\xi_{max}} (K_{\tau_0})^{s/\tau_0}}{1 - (2(K_{\tau_0}D_{\tau_0})^{2/\tau_0})^{s/2} \left(1 - \left(\frac{\xi_-}{\xi_{max}}\right)^3\right)^{1/2}}.$$

678 For our model problem, of special interest is the electric case, with the coefficient  
 679  $\xi = \varepsilon_\sigma$ : in this case,  $(\varepsilon_\sigma)_-$  can be chosen among  $(\cos \theta^* \varepsilon_{min} + \sin \theta^* \sigma_{min}/\omega)_{\theta^* \in [0, \pi/2]}$ .  
 680 In particular

$$681 \quad (\varepsilon_\sigma)_- \geq \max_{\theta^* \in [0, \pi/2]} (\cos \theta^* \varepsilon_{min} + \sin \theta^* \sigma_{min}/\omega) = ((\varepsilon_{min})^2 + (\sigma_{min}/\omega)^2)^{1/2}.$$

682 On the other hand,

$$683 \quad (\varepsilon_\sigma)_{max} \leq ((\varepsilon_{max})^2 + (\sigma_{max}/\omega)^2)^{1/2}.$$

684 When both  $\varepsilon$  and  $\sigma$  are constant on  $\Omega$ , the operator  $-\operatorname{div} \varepsilon_\sigma \nabla$  (with Dirichlet bound-  
 685 ary condition) is proportional to the Laplace operator (with Dirichlet boundary condi-  
 686 tion), and one has  $(\varepsilon_{min})^2 + (\sigma_{min}/\omega)^2 = (\varepsilon_{max})^2 + (\sigma_{max}/\omega)^2$ , so the bounds on  
 687  $(\varepsilon_\sigma)_-$  and  $(\varepsilon_\sigma)_{max}$  are sharp. According again to Theorem 6.2, any  $\tau_{\varepsilon_\sigma} < \frac{1}{2}$  with  
 688  $c_{\varepsilon_\sigma} = c^{Lap}(\tau_{\varepsilon_\sigma})$  is admissible in this case.

689 In the other configurations (non-constant  $\varepsilon_\sigma$ ), let  $\tau_0 \in (0, \frac{1}{2})$  be fixed, and introduce

$$690 \quad R_{\varepsilon_\sigma} := (\varepsilon_\sigma)_{max}/(\varepsilon_\sigma)_- > 1;$$

691 then the regularity exponent (6.18), respectively the stability constant (6.19) of The-  
 692 orem 6.8, are given by

$$693 \quad (6.22) \quad \tau_{\varepsilon_\sigma} = \min \left( \tau_0, -\frac{\log(1 - (R_{\varepsilon_\sigma})^{-3})}{\log 2 + \frac{2}{\tau_0} \log(K_{\tau_0} D_{\tau_0}) + \log \left( 1 + \frac{C^2}{(R_{\varepsilon_\sigma} - 1)^2} \Lambda_{\varepsilon_\sigma}^2 \right)} \right) \in (0, \tau_0];$$

$$694 \quad (6.23) \quad c(s, \varepsilon_\sigma) = \frac{(R_{\varepsilon_\sigma})^{-1} (K_{\tau_0})^{s/\tau_0}}{1 - \left( 2(K_{\tau_0} D_{\tau_0})^{2/\tau_0} \left( 1 + \frac{C^2}{(R_{\varepsilon_\sigma} - 1)^2} \Lambda_{\varepsilon_\sigma}^2 \right) \right)^{s/2} (1 - (R_{\varepsilon_\sigma})^{-3})^{1/2}}.$$

695 **6.3. Bounding the norm of scalar fields.** We now bound the norm of the  
 696 right-hand sides of the variational formulations (5.4) and (5.7), governing resp.  $p_0$   
 697 and  $q_0$ .

698 **LEMMA 6.10.** *Let  $\Omega$  be a domain such that  $(\mathbf{Top})_I$  is fulfilled, and assume that*  
 699  *$\xi$  fulfills the coefficient assumption. Let  $\mathbf{v} \in \mathbf{X}_B(\Omega, \xi)$  be given, for  $B \in \{Dir, Neu\}$ .*  
 700 *Let  $s \in [0, \frac{1}{2})$ : if  $B = Dir$ , the right-hand sides  $f$  defined by (5.4) belong to  $\mathcal{H}_{Dir}^{s-1}(\Omega)$ ;*  
 701 *resp. if  $B = Neu$  the right-hand sides  $f$  defined by (5.7) belong to  $\mathcal{H}_{Neu}^{s-1}(\Omega)$ . In*  
 702 *addition, for all  $s \in [0, \frac{1}{2})$ , it holds that*

$$703 \quad (6.24) \quad \|f\|_{\mathcal{H}^{s-1}(\Omega)} \leq c^B s^{1/2} \|\operatorname{div} \xi \mathbf{v}\|_{0,\Omega} + C_{(s, \frac{1}{2})} \xi_{max} N_\xi^s \|\mathbf{v}\|_{\mathbf{H}(\operatorname{curl}; \Omega)}.$$

704 *Proof.* Let us focus on the case  $B = Dir$  (proof is similar in the case  $B = Neu$ ).  
 705 We use the same notations as in Theorem 5.5. Introduce  $f \in \mathcal{H}^{-1}(\Omega)$ :

$$706 \quad f : \psi \mapsto (\xi(\mathbf{z} + \mathbf{v}_{reg})|\nabla \psi)_{0,\Omega} + (\operatorname{div} \xi \mathbf{v}|\psi)_{0,\Omega}, \quad \forall \psi \in \mathcal{H}^1(\Omega).$$

707 Let  $s \in [0, \frac{1}{2})$  be given.

708 First, it is obvious that  $f_0 : \psi \mapsto (\operatorname{div} \xi \mathbf{v}|\psi)_{0,\Omega}$  belongs to  $\mathcal{H}^{s-1}(\Omega)$ . Indeed, according  
 709 to Appendix A:

$$710 \quad \forall \psi \in \mathcal{H}^{1-s}(\Omega), |(\operatorname{div} \xi \mathbf{v}|\psi)_{0,\Omega}| \leq \|\operatorname{div} \xi \mathbf{v}\|_{0,\Omega} \|\psi\|_{0,\Omega} \leq c^{Dir} s^{1/2} \|\operatorname{div} \xi \mathbf{v}\|_{0,\Omega} \|\psi\|_{\mathcal{H}^{1-s}(\Omega)}.$$

711 Hence,  $f_0 \in \mathcal{H}^{s-1}(\Omega)$  and  $\|f_0\|_{\mathcal{H}^{s-1}(\Omega)} \leq c^{Dir} s^{1/2} \|\operatorname{div} \xi \mathbf{v}\|_{0,\Omega}$ .

712 Then we recall that  $\mathbf{z} + \mathbf{v}_{reg}$  belongs to  $\mathbf{H}^{\frac{1}{2}}(\Omega) \subset \mathbf{H}^s(\Omega)$ , so that  $m_\xi(\mathbf{z} + \mathbf{v}_{reg}) =$   
 713  $\xi(\mathbf{z} + \mathbf{v}_{reg}) \in \widehat{\mathbf{H}}^s(\Omega)$  according to Proposition 6.5. Then it follows from Proposition 6.6  
 714 that  $D(m_\xi(\mathbf{z} + \mathbf{v}_{reg})) \in \mathcal{H}^{s-1}(\Omega)$ . In other words,  $\psi \mapsto (\xi(\mathbf{z} + \mathbf{v}_{reg})|\nabla \psi)_{0,\Omega}$  also  
 715 belongs to  $\mathcal{H}^{s-1}(\Omega)$ .

716 Regarding the norm estimate (6.24), we simply use the bounds (6.7) and (6.9) together  
 717 with (5.3) to conclude the proof: the dependence in  $s$  (the constant  $C_{(s, \frac{1}{2})}$ ) comes from

718 the continuous embedding  $\mathbf{H}^{\frac{1}{2}}(\Omega) \subset \mathbf{H}^s(\Omega)$ .  $\square$

719 Given  $r \in [0, \frac{1}{2})$ , one may use (6.8) for  $s \in [0, r]$ , and thus replace (6.24) for all  
 720  $s \in [0, r]$  by

$$721 \quad (6.25) \quad \|f\|_{\mathcal{H}^{s-1}(\Omega)} \leq c^B s^{1/2} \|\operatorname{div} \xi \mathbf{v}\|_{0,\Omega} + C_{(s, \frac{1}{2})} \xi_{max} (N_{\xi}^r)^{s/r} \|\mathbf{v}\|_{\mathbf{H}(\operatorname{curl}; \Omega)}.$$

722 **6.4. Application to the polyhedral model problem.** We recall that, un-  
 723 der the assumptions defining the polyhedral model problem, the coefficients  $(\varepsilon, \mu, \sigma)$   
 724 are such that  $\varepsilon, \sigma$  fulfill the coefficient assumption, and  $\mu$  is constant on  $\Omega$ . Accord-  
 725 ing to the definition of the model problem (2.9), it always hold that  $\mu^{-1} \operatorname{curl} \mathbf{e} \in$   
 726  $\mathbf{X}_{Neu}(\Omega, \mu)$ . Here, because  $\mu$  is a constant,  $\mathbf{X}_{Neu}(\Omega, \mu) = \mathbf{X}_{Neu}(\Omega)$ . It follows that  
 727  $\mu^{-1} \operatorname{curl} \mathbf{e} \in \mathbf{H}^{\sigma_{Neu}}(\Omega)$  with  $\sigma_{Neu} > \frac{1}{2}$ , cf. Proposition 5.3. Then, regarding the  
 728 choice of a regularity exponent for the electric field  $\mathbf{e}$  itself, because  $\varepsilon_{\sigma} \in PW^{1,\infty}(\Omega)$ ,  
 729 we note that either any  $\tau_{\varepsilon_{\sigma}} < \frac{1}{2}$  is admissible (constant  $\varepsilon_{\sigma}$ ), or that it is given by  
 730 (6.22) (non-constant  $\varepsilon_{\sigma}$ ). Indeed, one has the regular/gradient splitting (6.2):

$$731 \quad \mathbf{e} = \mathbf{e}_{reg} + \mathbf{z}_e + \nabla p_0 \text{ in } \Omega, \text{ where } \mathbf{e}_{reg} \in \mathbf{H}^1(\Omega), \mathbf{z}_e \in \mathbf{H}^{\sigma_{Dir}}(\Omega).$$

732 The regularity of the gradient part, namely  $\nabla p_0 \in \mathbf{H}^s(\Omega)$  is a straightforward con-  
 733 sequence of Theorems 6.2 (constant  $\varepsilon_{\sigma}$ ) and 6.8 (non-constant  $\varepsilon_{\sigma}$ ), provided that the  
 734 right-hand side  $f$  given there belongs to  $\mathcal{H}^{s-1}(\Omega)$ , for all values  $s \in [0, \tau_{\varepsilon_{\sigma}})$ . But, since  
 735 this regularity result on  $f$  was proven in Lemma 6.10, one has indeed  $\nabla p_0 \in \mathbf{H}^s(\Omega)$ ,  
 736 for all  $s \in [0, \tau_{\varepsilon_{\sigma}})$ .

737

738 It follows that we can provide values for  $\tau_{(\varepsilon, \mu, \sigma)}$  (regularity exponent), resp.  
 739  $C_{(\varepsilon, \mu, \sigma, s)}^*$  (stability constant), in (3.6). Because the limiting value of a regularity  
 740 exponent is constrained by  $\tau_{\varepsilon_{\sigma}}$  ( $\tau_{\varepsilon_{\sigma}} < \frac{1}{2} < \sigma_{Neu}$ ), we choose

$$741 \quad \tau_{(\varepsilon, \mu, \sigma)} := \tau_{\varepsilon_{\sigma}} \in (0, \frac{1}{2}).$$

742 From now on, we assume that we are given a set  $\Theta$  of coefficients  $(\varepsilon, \mu, \sigma)$ , such  
 743 that  $\varepsilon, \sigma$  fulfill the coefficient assumption, and  $\mu$  is constant on  $\Omega$ . Moreover, we  
 744 consider the case where  $\underline{\tau} := \inf_{(\varepsilon, \mu, \sigma) \in \Theta} \tau_{(\varepsilon, \mu, \sigma)} > 0$ . This covers in particular the  
 745 case where  $\Theta$  is a singleton  $\{(\varepsilon^0, \mu^0, \sigma^0)\}$ , and  $\underline{\tau} = \tau_{(\varepsilon^0, \mu^0, \sigma^0)}$ .

746 **LEMMA 6.11.** *Let  $p_0$  be defined as in (5.4). For all  $(\varepsilon, \mu, \sigma) \in \Theta$ , for all  $s \in [0, \underline{\tau})$ ,*  
 747 *the norm in  $\mathcal{H}^{s-1}(\Omega)$  of the right-hand side  $f$  in (5.4) is bounded by*

$$748 \quad \|f\|_{\mathcal{H}^{s-1}(\Omega)} \leq c_0^0 \omega \|\mathbf{j}_{ext}\|_{0,\Omega} + c_{\operatorname{div}}^0 \omega^{-1} \|\operatorname{div} \mathbf{j}_{ext}\|_{0,\Omega}, \text{ where}$$

$$749 \quad c_0^0 := C_{(s, \frac{1}{2})} (\varepsilon_{\sigma})_{max} (N_{\varepsilon_{\sigma}}^{\underline{\tau}})^{s/\underline{\tau}} (C_{(\varepsilon, \mu, \sigma)}^{coer})^{-1}, \text{ and } c_{\operatorname{div}}^0 := c^{Dir} s^{1/2}.$$

750 **Remark 6.12.** We recall that

$$751 \quad (N_{\varepsilon_{\sigma}}^{\underline{\tau}})^{s/\underline{\tau}} := (D_{\underline{\tau}})^{s/\underline{\tau}} (2(1 + C^2 \Lambda_{\varepsilon_{\sigma}}^2))^{s/2},$$

752 where  $D_{\underline{\tau}}$  is defined at (6.5), resp.  $C$  is the Poincaré constant defined at (6.6), and  
 753  $\Lambda_{\varepsilon_{\sigma}} := ((\varepsilon_{\sigma})_{max})^{-1} |\varepsilon_{\sigma}|_{PW^{1,\infty}(\Omega)}$ . Note that only  $D_{\underline{\tau}}$  and  $C$ , and hence  $N_{\varepsilon_{\sigma}}^{\underline{\tau}}$ , depend  
 754 on the partition induced by  $(\varepsilon, \mu, \sigma)$ .

755 *Proof.* According to (6.25), we know that the right-hand side that defines  $\nabla p_0$  is  
 756 bounded by

$$757 \quad \|f\|_{\mathcal{H}^{s-1}(\Omega)} \leq c^{Dir} s^{1/2} \|\operatorname{div} \varepsilon_{\sigma} \mathbf{e}\|_{0,\Omega} + C_{(s, \frac{1}{2})} (\varepsilon_{\sigma})_{max} (N_{\varepsilon_{\sigma}}^{\underline{\tau}})^{s/\underline{\tau}} \|\mathbf{e}\|_{\mathbf{H}(\operatorname{curl}; \Omega)}.$$

758 Going back to the model problem (2.9), the left part of the upper bound is readily  
 759 replaced by  $c^{Dir} s^{1/2} \omega^{-1} \|\operatorname{div} \mathbf{j}_{ext}\|_{0,\Omega}$ . Then, using the coercivity bound of Propo-  
 760 sition 4.1:

$$761 \quad C_{(\varepsilon,\mu,\sigma)}^{coer} \|\mathbf{e}\|_{\mathbf{H}(\operatorname{curl};\Omega)}^2 \leq \omega |(\mathbf{j}_{ext}|\mathbf{e})_{0,\Omega}| \leq \omega \|\mathbf{j}_{ext}\|_{0,\Omega} \|\mathbf{e}\|_{\mathbf{H}(\operatorname{curl};\Omega)},$$

762 we find for the right part the bound

$$763 \quad C_{(s,\frac{1}{2})} (\varepsilon_\sigma)_{max} (N_{\varepsilon_\sigma}^\tau)^{s/\tau} (C_{(\varepsilon,\mu,\sigma)}^{coer})^{-1} \omega \|\mathbf{j}_{ext}\|_{0,\Omega},$$

764 which proves the claim.  $\square$

765 The bound on  $\|\nabla p_0\|_{s,\Omega} = \|p_0\|_{\mathcal{H}^{1+s}(\Omega)}$  follows: for all  $s \in [0, \tau]$ ,

$$766 \quad \|\nabla p_0\|_{s,\Omega} \leq \frac{c_{Dir}(s, \varepsilon_\sigma)}{(\varepsilon_\sigma)_{max}} (c_0^0 \omega \|\mathbf{j}_{ext}\|_{0,\Omega} + c_{div}^0 \omega^{-1} \|\operatorname{div} \mathbf{j}_{ext}\|_{0,\Omega}),$$

767 where  $c_{Dir}(s, \varepsilon_\sigma)$  is given in Theorem 6.2 or Theorem 6.8;  $c_{Dir}(s, \varepsilon_\sigma)$  depends on the  
 768 partition induced by  $(\varepsilon, \mu, \sigma)$ .

769 We are now in a position to estimate the norm of  $\mathbf{e}$  in  $\mathbf{PH}^s(\operatorname{curl};\Omega)$ , for all values  
 770  $s \in [0, \tau]$ , which then leads to the desired convergence rate.

771 LEMMA 6.13. For all  $(\varepsilon, \mu, \sigma) \in \Theta$ , for all  $s \in [0, \tau]$ , one has the estimate

$$772 \quad \|\mathbf{e}\|_{\mathbf{PH}^s(\operatorname{curl};\Omega)} \leq C_{0(\varepsilon,\mu,\sigma,s)}^* \|\mathbf{j}_{ext}\|_{0,\Omega} + C_{div(\varepsilon,\mu,\sigma,s)}^* \|\operatorname{div} \mathbf{j}_{ext}\|_{0,\Omega}, \text{ where}$$

$$773 \quad C_{0(\varepsilon,\mu,\sigma,s)}^* := \left( \frac{C_{(s,\frac{1}{2})}}{C_{(\varepsilon,\mu,\sigma)}^{coer}} \left( c_{Dir}(s, \varepsilon_\sigma) (N_{\varepsilon_\sigma}^\tau)^{s/\tau} + C_{\mathbf{X}}^{Dir} \right) \right. \\ 774 \quad \left. + C_{(s,\sigma_{Neu})} I_{\mathbf{X}_{Neu}} \left( \frac{(1 + \mu \omega^2 (\varepsilon_\sigma)_{max})}{C_{(\varepsilon,\mu,\sigma)}^{coer}} + \mu \right) \right) \omega,$$

$$775 \quad C_{div(\varepsilon,\mu,\sigma,s)}^* := \frac{c_{Dir}(s, \varepsilon_\sigma) c^{Dir} s^{1/2}}{(\varepsilon_\sigma)_{max}} \omega^{-1},$$

776 and  $I_{\mathbf{X}_{Neu}}$  denotes the norm of the embedding  $\mathbf{X}_{Neu}(\Omega) \subset \mathbf{H}^{\sigma_{Neu}}(\Omega)$ .

777 Remark 6.14. The above is slightly different from (3.6), where both contributions  
 778 of the norm  $\|\mathbf{j}_{ext}\|_{\mathbf{H}(\operatorname{div};\Omega)}$  are merged. Also, only  $c_{Dir}(s, \varepsilon_\sigma)$  and  $N_{\varepsilon_\sigma}^\tau$  depend on the  
 779 partition induced by  $(\varepsilon, \mu, \sigma)$ .

780 Proof. One has  $\|\mathbf{e}\|_{s,\Omega} \leq \|\mathbf{e}_{reg} + \mathbf{z}_e\|_{s,\Omega} + \|\nabla p_0\|_{s,\Omega}$ , and the bound on  $\|\nabla p_0\|_{s,\Omega}$   
 781 is given right above. On the other hand,

$$782 \quad \|\mathbf{e}_{reg} + \mathbf{z}_e\|_{s,\Omega} \leq C_{(s,\frac{1}{2})} \|\mathbf{e}_{reg} + \mathbf{z}_e\|_{1/2,\Omega} \\ 783 \quad \leq C_{(s,\frac{1}{2})} C_{\mathbf{X}}^{Dir} \|\mathbf{e}\|_{\mathbf{H}(\operatorname{curl};\Omega)} \leq \frac{C_{(s,\frac{1}{2})} C_{\mathbf{X}}^{Dir}}{C_{(\varepsilon,\mu,\sigma)}^{coer}} \omega \|\mathbf{j}_{ext}\|_{0,\Omega}.$$

784 We conclude that: for all  $s \in [0, \tau]$ ,

$$785 \quad \|\mathbf{e}\|_{s,\Omega} \leq \left( \frac{c_{Dir}(s, \varepsilon_\sigma) c_0^0}{(\varepsilon_\sigma)_{max}} + \frac{C_{(s,\frac{1}{2})} C_{\mathbf{X}}^{Dir}}{C_{(\varepsilon,\mu,\sigma)}^{coer}} \right) \omega \|\mathbf{j}_{ext}\|_{0,\Omega} + \frac{c_{Dir}(s, \varepsilon_\sigma) c_{div}^0 \omega^{-1}}{(\varepsilon_\sigma)_{max}} \|\operatorname{div} \mathbf{j}_{ext}\|_{0,\Omega},$$

786 where  $c_0^0$  and  $c_{div}^0$  are defined in Lemma 6.11.

787 Regarding the norm of  $\operatorname{curl} \mathbf{e}$ , we recall that  $\operatorname{curl} \mathbf{e} \in \mathbf{X}_{Neu}(\Omega)$ . But  $\mathbf{X}_{Neu}(\Omega)$  is

788 continuously embedded in  $\mathbf{H}^{\sigma_{Neu}}(\Omega)$  (Proposition 5.3) so we find that for all  $s \in [0, \underline{\tau}]$ ,

$$\begin{aligned}
789 \quad & \|\mathbf{curl} \mathbf{e}\|_{s,\Omega} \leq C_{(s,\sigma_{Neu})} \|\mathbf{curl} \mathbf{e}\|_{\sigma_{Neu},\Omega} \\
790 \quad & \leq C_{(s,\sigma_{Neu})} I_{\mathbf{X}_{Neu}} \|\mathbf{curl} \mathbf{e}\|_{\mathbf{X}_{Neu}(\Omega)} \\
791 \quad & = C_{(s,\sigma_{Neu})} I_{\mathbf{X}_{Neu}} \|\mathbf{curl} \mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\Omega)}.
\end{aligned}$$

792 Next, we have, using the model problem (2.9):

$$\begin{aligned}
793 \quad & \|\mathbf{curl} \mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq \|\mathbf{curl} \mathbf{e}\|_{0,\Omega} + \|\mathbf{curl} \mathbf{curl} \mathbf{e}\|_{0,\Omega} \\
794 \quad & = \|\mathbf{curl} \mathbf{e}\|_{0,\Omega} + \mu \|\mathbf{curl} \mu^{-1} \mathbf{curl} \mathbf{e}\|_{0,\Omega} \\
795 \quad & = \|\mathbf{curl} \mathbf{e}\|_{0,\Omega} + \mu \|\omega^2 \varepsilon_\sigma \mathbf{e} + \omega \mathbf{j}_{ext}\|_{0,\Omega} \\
796 \quad & \leq (1 + \mu \omega^2 (\varepsilon_\sigma)_{max}) \|\mathbf{e}\|_{\mathbf{H}(\mathbf{curl};\Omega)} + \mu \omega \|\mathbf{j}_{ext}\|_{0,\Omega} \\
797 \quad & \leq \left( \frac{(1 + \mu \omega^2 (\varepsilon_\sigma)_{max})}{C_{(\varepsilon,\mu,\sigma)}^{coer}} + \mu \right) \omega \|\mathbf{j}_{ext}\|_{0,\Omega}.
\end{aligned}$$

798 Hence, for all  $s \in [0, \underline{\tau}]$ ,

$$799 \quad \|\mathbf{curl} \mathbf{e}\|_{s,\Omega} \leq C_{(s,\sigma_{Neu})} I_{\mathbf{X}_{Neu}} \left( \frac{(1 + \mu \omega^2 (\varepsilon_\sigma)_{max})}{C_{(\varepsilon,\mu,\sigma)}^{coer}} + \mu \right) \omega \|\mathbf{j}_{ext}\|_{0,\Omega}.$$

800 Then, using Proposition 3.1, we find, for all  $s \in [0, \underline{\tau}]$ :

$$801 \quad \|\mathbf{e}\|_{\mathbf{PH}^s(\mathbf{curl};\Omega)} \leq \|\mathbf{e}\|_{\mathbf{PH}^s(\Omega)} + \|\mathbf{curl} \mathbf{e}\|_{\mathbf{PH}^s(\Omega)} \leq \|\mathbf{e}\|_{s,\Omega} + \|\mathbf{curl} \mathbf{e}\|_{s,\Omega},$$

802 and the conclusion follows.  $\square$

803 We recall that  $C_{(\varepsilon,\mu,\sigma)}^\sharp = C_{(\varepsilon,\mu,\sigma)}^{cont} (C_{(\varepsilon,\mu,\sigma)}^{coer})^{-1}$  is the constant appearing in C ea's lemma  
804 (and bounded in  4) and that  $\varsigma > 0$  is the shape regularity parameter of the family  
805 of meshes. Then one has the following convergence rate for the polyhedral model  
806 problem.

807 **THEOREM 6.15.** *Let  $(\varepsilon, \mu, \sigma)$  be such that  $\varepsilon, \sigma$  fulfill the coefficient assumption,*  
808 *and  $\mu$  is constant on  $\Omega$ . For all  $s \in [0, \tau_{(\varepsilon,\mu,\sigma)})$ , there exist constants  $C_{(\varsigma,s)}^{interp}$ ,*  
809  *$C_{0(\varepsilon,\mu,\sigma,s)}^*$ , and  $C_{div(\varepsilon,\mu,\sigma,s)}^*$  such that for all  $\mathbf{j}_{ext} \in \mathbf{H}(\text{div};\Omega)$  and all  $h$ , the error*  
810 *estimate hold:*

$$\begin{aligned}
811 \quad & \|\mathbf{e} - \mathbf{e}_h\|_{\mathbf{H}(\mathbf{curl};\Omega)} \leq C_{(\varepsilon,\mu,\sigma)}^\sharp C_{(\varsigma,s)}^{interp} h^s \left( C_{0(\varepsilon,\mu,\sigma,s)}^* \|\mathbf{j}_{ext}\|_{0,\Omega} \right. \\
812 \quad (6.26) \quad & \left. + C_{div(\varepsilon,\mu,\sigma,s)}^* \|\text{div} \mathbf{j}_{ext}\|_{0,\Omega} \right).
\end{aligned}$$

813 Let  $\Theta$  be a set of coefficients  $(\varepsilon, \mu, \sigma)$  whose elements are all piecewise smooth on the  
814 same polyhedral partition, and assume that  $\underline{\tau} := \inf_{(\varepsilon,\mu,\sigma) \in \Theta} \tau_{(\varepsilon,\mu,\sigma)} > 0$ . Then (6.26)  
815 holds true for all  $s \in [0, \underline{\tau}]$ .

816 **Remark 6.16.** The above is slightly different from (3.7), where both contributions  
817 of the norm  $\|\mathbf{j}_{ext}\|_{\mathbf{H}(\text{div};\Omega)}$  are merged.

818 *Proof.* It is straightforward to derive the result (6.26) by using successively (3.3)  
819 and (3.5), and finally the estimate of Lemma 6.13 for  $\underline{\tau} = \tau_{(\varepsilon,\mu,\sigma)}$ .  $\square$



820 **6.5. A few possible generalizations.** Let us mention two cases we have ex-  
821 cluded so far: first when  $\mu$  fulfills the coefficient assumption, but  $\mu$  is not constant on  
822  $\Omega$ ; second when  $\mathbf{j}_{ext} \notin \mathbf{H}(\operatorname{div}; \Omega)$ , but  $\mathbf{j}_{ext} \in \mathbf{L}^2(\Omega)$  with  $\operatorname{div} \mathbf{j}_{ext} \in H^{-t}(\Omega)$  for some  
823  $t \in (0, 1)$ .

824

825 In the first situation, all the previous analyses apply, except when one addresses  
826 the regularity of  $\mu^{-1} \operatorname{curl} \mathbf{e}$  with respect to the scale  $(\mathbf{H}^s(\Omega))_s$ . Although it still  
827 holds that  $\mu^{-1} \operatorname{curl} \mathbf{e} \in \mathbf{X}_{Neu}(\Omega, \mu)$ , one has  $\mathbf{X}_{Neu}(\Omega, \mu) \neq \mathbf{X}_{Neu}(\Omega)$ . To find a  
828 regularity exponent, one uses now (6.3), where the regularity is determined by the  
829 gradient part  $\nabla q_0$ : see Theorem 6.8 (Neumann case), which yields the value of the  
830 regularity exponent  $\tau_\mu$ . One then chooses

$$831 \quad \tau_{(\varepsilon, \mu, \sigma)} := \min(\tau_{\varepsilon, \sigma}, \tau_\mu) \in (0, \frac{1}{2}).$$

832 Next, one derives an estimate on  $\|\operatorname{curl} \mathbf{e}\|_{s, \Omega}$ , for all  $s \in (0, \tau_{(\varepsilon, \mu, \sigma)})$ . Noting that the  
833 multiplicative operator  $m_\mu$  belongs to  $\mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{H}^s(\Omega))$ , it follows that

$$834 \quad \|\operatorname{curl} \mathbf{e}\|_{s, \Omega} \leq \|m_\mu\|_{\mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{H}^s(\Omega))} \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{s, \Omega}.$$

835 The first quantity,  $\|m_\mu\|_{\mathcal{L}(\mathbf{H}^s(\Omega), \mathbf{H}^s(\Omega))}$ , is easily bounded from above, thanks to  
836 Proposition 6.5 and (6.5).

837 Then, using (6.3), one writes

$$838 \quad \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{s, \Omega} \leq \|\mathbf{c}_{reg} + \mathbf{z}_c\|_{s, \Omega} + \|\nabla q_0\|_{s, \Omega}.$$

839 Thanks to (5.6), one has

$$840 \quad \|\mathbf{c}_{reg} + \mathbf{z}_c\|_{s, \Omega} \leq C_{(s, \frac{1}{2})} \|\mathbf{c}_{reg} + \mathbf{z}_c\|_{1/2, \Omega} \leq C_{(s, \frac{1}{2})} C_{\mathbf{X}}^{Neu} \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{\mathbf{H}(\operatorname{curl}; \Omega)}.$$

841 On the other hand, according to Theorem 6.8 it holds that

$$842 \quad \|\nabla q_0\|_{s, \Omega} \leq \frac{c_{Neu}(s, \mu)}{\mu_{max}} \|f\|_{\mathcal{H}^{s-1}(\Omega)},$$

843 where  $f$  is the right-hand side of (5.7). Using Lemma 6.10 yields

$$844 \quad \|\nabla q_0\|_{s, \Omega} \leq c_{Neu}(s, \mu) C_{(s, \frac{1}{2})} (N_\mu^\tau)^{s/\tau} \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{\mathbf{H}(\operatorname{curl}; \Omega)}.$$

845 Aggregating the two estimates, one finds now

$$846 \quad \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{s, \Omega} \leq C_{(s, \frac{1}{2})} \left( C_{\mathbf{X}}^{Neu} + c_{Neu}(s, \mu) (N_\mu^\tau)^{s/\tau} \right) \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{\mathbf{H}(\operatorname{curl}; \Omega)}.$$

847 Writing finally

$$\begin{aligned} 848 \quad \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{\mathbf{H}(\operatorname{curl}; \Omega)} &\leq \|\mu^{-1} \operatorname{curl} \mathbf{e}\|_{0, \Omega} + \|\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{e}\|_{0, \Omega} \\ 849 &\leq \frac{1}{\mu_{min}} \|\operatorname{curl} \mathbf{e}\|_{0, \Omega} + \|\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{e}\|_{0, \Omega} \\ 850 \quad (\text{cf. (2.9)}) &\leq \left( \frac{1}{\mu_{min}} + \omega^2 (\varepsilon_\sigma)_{max} \right) \|\mathbf{e}\|_{\mathbf{H}(\operatorname{curl}; \Omega)} + \omega \|\mathbf{j}_{ext}\|_{0, \Omega} \end{aligned}$$

851 and using the coercivity, the rest of the estimates follow easily.

852

853 In the second situation, namely when  $\operatorname{div} \mathbf{j}_{ext} \in H^{-t}(\Omega)$  for some  $t \in (0, 1)$ , one  
 854 must use a generalized regular/gradient splitting. Precisely, one introduces (see [4,  
 855 Theorem 6.1.15]):

$$856 \quad \mathbf{X}_{Dir}(\Omega, \xi, -t) := \{\mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega) : \operatorname{div} \xi \mathbf{v} \in H^{-t}(\Omega)\}.$$

857 Then, one may generalize Theorem 5.5 to elements of  $\mathbf{X}_{Dir}(\Omega, \xi, -t)$ . the only dif-  
 858 ference is that  $\nabla p_0$  is now governed by

$$859 \quad \begin{cases} \text{Find } p_0 \in H_0^1(\Omega) \text{ such that} \\ (\xi \nabla p_0 | \nabla \psi)_{0,\Omega} = -(\xi \mathbf{z} | \nabla \psi)_{0,\Omega} - (\xi \mathbf{v}_{reg} | \nabla \psi)_{0,\Omega} - \langle \operatorname{div} \xi \mathbf{v}, \psi \rangle_{H_0^1(\Omega)}, \quad \forall \psi \in H_0^1(\Omega). \end{cases}$$

860 With this result at hand, one may proceed as before, replacing the occurrences of  
 861  $\|\operatorname{div} \mathbf{j}_{ext}\|_{0,\Omega}$  by  $\|\operatorname{div} \mathbf{j}_{ext}\|_{-t,\Omega}$ : one simply notices that when  $\operatorname{div} \mathbf{j}_{ext} \in H^{-t}(\Omega)$ , one  
 862 may still apply Lemma 6.10, but only for all  $s \in [0, \min(\frac{1}{2}, 1-t))$ . Hence, one chooses

$$863 \quad \tau_{(\varepsilon, \mu, \sigma, t)} := \min(\tau_{\varepsilon, \sigma}, \tau_{\mu}, 1-t) \in (0, \frac{1}{2}).$$

864 Computations can then be carried out.

865

866 Then, what happens when  $\Theta$  is not reduced to a singleton? For simplicity<sup>2</sup>, let  
 867 us consider that all its elements  $(\varepsilon, \mu, \sigma)$  are such that  $\varepsilon, \sigma, \mu$  are piecewise-constant  
 868 on a *fixed partition* and that  $\mathbf{j}_{ext} \in \mathbf{H}(\operatorname{div}; \Omega)$ . Then the regularity exponent writes

$$869 \quad \underline{\tau} := \inf_{(\varepsilon, \mu, \sigma) \in \Theta} \min(\tau_{\varepsilon, \sigma}, \tau_{\mu}).$$

870 According to (6.20), we see that if condition

$$871 \quad (6.27) \quad \sup_{(\varepsilon, \mu, \sigma) \in \Theta} \frac{(\varepsilon_{\sigma})_{max}}{(\varepsilon_{\sigma})_{-}} + \sup_{(\varepsilon, \mu, \sigma) \in \Theta} \frac{\mu_{max}}{\mu_{min}} < \infty$$

872 holds, then  $\underline{\tau} > 0$  and for any  $(\varepsilon, \mu, \sigma) \in \Theta$  one may apply the previous results for  
 873 all  $s \in [0, \underline{\tau}]$ . Note that this condition is comparable to the one found using the  
 874 local approach, see (B.8), because one has  $(\varepsilon_{\sigma})_{max} \leq \omega^{-1}(\varepsilon_{max}^2 \omega^2 + \sigma_{max}^2)^{1/2}$ , and  
 875  $(\varepsilon_{\sigma})_{-} \geq \omega^{-1}(\varepsilon_{min}^2 \omega^2 + \sigma_{min}^2)^{1/2}$ . On the other hand, there exist configurations such  
 876 that if, eg.,  $\sup_{(\varepsilon, \mu, \sigma) \in \Theta} \mu_{max}/\mu_{min} = \infty$ , it holds that  $\inf_{(\varepsilon, \mu, \sigma) \in \Theta} \tau_{\mu} = 0$ . In other  
 877 words, there is no (extra-)regularity in this limit case. We refer to §7.2 for an illus-  
 878 tration.

879

880 Also, what can be said in the context of §5.3, that is when  $\xi$  is a complex-valued,  
 881 measurable, *tensor* field that fulfills the coefficient assumption? It turns out that one  
 882 may address the case of a *normal tensor* field, ie.  $\xi = \mathbb{U}^{-1} \mathbb{D} \mathbb{U}$  a.e. in  $\Omega$ , where  $\mathbb{U}$   
 883 is a unitary tensor field, resp.  $\mathbb{D}$  is a diagonal tensor field, defined in  $\Omega$ . Let us briefly  
 884 explain why.

885 First, Proposition 6.5 still applies. Then, (5.1) is now replaced by (5.8): but the latter  
 886 can be seen as the equivalent of the former, imposed on  $\mathbb{D}_{11}$ ,  $\mathbb{D}_{22}$  and  $\mathbb{D}_{33}$ . Hence one  
 887 may use the reformulated (6.1), namely

$$888 \quad \begin{aligned} & \mathbb{D}_{kk} \in \{z = \rho \exp(i\theta), \rho \in [\xi_{-}, \xi_{max}], \theta \in [\theta_{min}, \theta_{max}]\} \text{ a.e. in } \Omega, \text{ for } k = 1, 2, 3, \\ & \text{where } \xi_{max} := \sup_{\mathbf{z} \in \mathbb{C}^3 \setminus \{0\}} \operatorname{ess\,sup}_{\Omega} \frac{|\xi \mathbf{z}|}{|\mathbf{z}|}, \text{ and } 0 \leq \theta_{max} - \theta_{min} \leq 2 \arccos \left( \frac{\xi_{-}}{\xi_{max}} \right), \end{aligned}$$

<sup>2</sup> If more generally the coefficients are piecewise smooth, or if the partition depends on the element  $(\varepsilon, \mu, \sigma)$ , the condition is more involved than (6.27) proposed below.

889 in the proof of Theorem 6.8.

890 The proof then proceeds as before, and one can conclude that Theorem 6.8 still holds.

891 The rest of the proofs are unchanged.

892

893 Finally, let us mention that the non-conductive case ( $\sigma = 0$ ) can be handled  
894 similarly, under the assumption that the model problem is well-posed (ie.  $\omega^2$  is not  
895 an eigenvalue of the corresponding eigenproblem). In this case the coercivity constant  
896 is *frequency dependent*, in the sense that it is inversely proportional to the distance of  
897  $\omega^2$  to the closest eigenvalue, see for instance [4, §8.3]. On the other hand, the estimates  
898 on the regularity exponent and on the stability constant can still be recovered in this  
899 context.

900 **7. Evaluating the regularity exponent.** Below, we evaluate the "sharpness"  
901 of the bounds on  $\tau_{(\varepsilon,\mu,\sigma)}$  on two examples.

902 **7.1. The coplanar waveguide.** First, let us consider the coplanar waveguide  
903 case, as provided by the MORwiki Community [34, 5]. Precisely, the geometry of  
904 interest is a parallelepiped, see Figure 1. The upper part of the domain is made of  
905 air, while the bottom part is made of a substrate (in yellow) in which three perfectly  
906 conducting striplines (in blue) are embedded. The electric permittivity  $\varepsilon$  and the  
907 conductivity  $\sigma$  are piecewise constant (with different values in the air and in the sub-  
908 strate), while the magnetic permeability  $\mu$  takes the same value in the air and in the  
909 substrate.

910 The resulting  $\Omega$  is thus equal to the parallelepiped minus the three striplines, and the  
911 interface  $\Sigma$  separates the two materials (air, substrate): it is flat, see Figure 1. Impor-  
912 tantly, all angles, either at the boundary, or at the interface between to two materials,  
913 are equal to multiples of  $\pi/2$ . One solves the model problem (2.9) in this configuration  
914 for a given  $\mathbf{j}_{ext}$ .

915

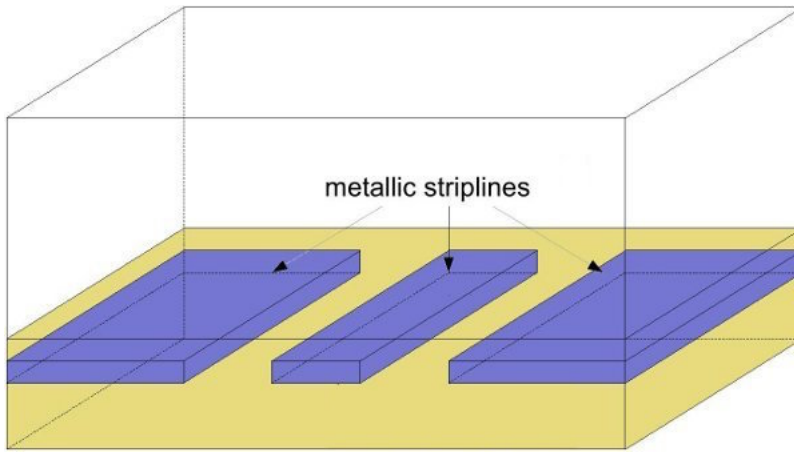


FIG. 1. A coplanar waveguide [34].

916 Since the coefficients are piecewise constant, one may use the local approach (cf.  
917 Appendix B). According to the framework developed there, one has to study the local  
918 problems on interior domains (no intersection with  $\partial\Omega$ , cf. §B.1) resp. on bound-  
919 ary domains (cf. §B.2), to determine the "best regularity" exponent  $\tau_{(\varepsilon,\mu,\sigma)}^{opt}$ . In the

920 present case, studying local problems on interior domains, see (B.7), one finds that  
 921 there is no constraint on  $\tau_{(\varepsilon,\mu,\sigma)}^{opt}$  because the interface is flat. On the other hand, the  
 922 study of local problems on boundary domains is more involved a priori; there are edge  
 923 problems with interface (at the intersection  $\partial\Omega \cap \Sigma$ ), edge problems without interface  
 924 (on  $\partial\Omega$ ), and finally corner problems without interface (on  $\partial\Omega$ ).

925 Interestingly, for the edge problems with interface and because the diedric angles are  
 926 equal to  $\pi/2$  in both materials, one notices that, using a symmetry argument (odd  
 927 reflection for the Dirichlet boundary condition, resp. even reflection for the Neumann  
 928 boundary condition, see [22, p. 41]), one can recast the problem as an interior prob-  
 929 lem, with a flat interface. Hence,  $\tau_{\mathbf{e},\Sigma}(\xi) = 1$  for  $\xi \in \{\mu, \varepsilon_\sigma\}$ .

930 Also, making the same observation on the value of the angles, and using again a  
 931 symmetry argument, one can recast the corner problem without interface as an edge  
 932 problem without interface: it follows that  $\tau_{\mathbf{c},\partial\Omega}(\xi) = \tau_{\mathbf{e},\partial\Omega}(\xi)$ , where  $\tau_{\mathbf{e},\partial\Omega}(\xi)$  is char-  
 933 acterized next, for  $\xi \in \{\mu, \varepsilon_\sigma\}$ , and one has  $\tau_{(\varepsilon,\mu,\sigma)}^{opt} = \min(1, \tau_{\mathbf{e},\partial\Omega}(\mu), \tau_{\mathbf{e},\partial\Omega}(\varepsilon_\sigma))$ .

934 So, in the end, there remains to study the edge problem without interface to determine  
 935 the value of  $\tau_{(\varepsilon,\mu,\sigma)}^{opt}$ . But this is a standard problem: one looks for the regularity ex-  
 936 ponent in an L-shape (local) domain  $\mathcal{O}$  for the Laplace operator (there is no interface,  
 937 hence no jump of the coefficient) with either homogeneous Dirichlet or, homogeneous  
 938 Neumann, boundary condition: it is well-known that  $\tau_{\mathbf{e},\partial\Omega}(\mu) = \tau_{\mathbf{e},\partial\Omega}(\varepsilon_\sigma) = 2/3$ .

939 Aggregating all results, one concludes that  $\tau_{(\varepsilon,\mu,\sigma)}^{opt} = 2/3$  for the coplanar waveguide,  
 940 *independently of the values of the coefficients*  $(\varepsilon, \mu, \sigma)$ . In other words the lower  
 941 bounds provided by (6.27) and (B.8) are not “sharp”.

942 **7.2. The checkerboard.** We study now a simple example (cf. for instance  
 943 Dauge’s benchmark [16]), to illustrate the fact that the conditions (6.27) and (B.8)  
 944 can be “sharp”: let us consider the domain  $\Omega := (-1, 1) \times (-1, 1) \times (0, 1)$ , made of  
 945 four cubes, stacked together:

$$946 \quad \begin{aligned} \Omega_1 &:= (-1, 0) \times (-1, 0) \times (0, 1), & \Omega_2 &:= (0, 1) \times (-1, 0) \times (0, 1), \\ \Omega_3 &:= (0, 1) \times (0, 1) \times (0, 1), & \Omega_4 &:= (-1, 0) \times (0, 1) \times (0, 1). \end{aligned}$$

947 We assume that  $\varepsilon$  and  $\sigma$  are constant, while  $\mu$  is piecewise constant, and equal to 1  
 948 in  $\Omega_1 \cup \Omega_3$ , resp. to  $\delta \in (0, 1)$  in  $\Omega_2 \cup \Omega_4$ .

949 Again, all angles are multiples of  $\pi/2$ . Proceeding as in §7.1, one observes first that  
 950  $\tau_{(\varepsilon,\mu,\sigma)}^{opt} = \min(1, \tau_{\mathbf{e},\Sigma}(\mu))$ . Then, solving (B.7) and looking for the smallest non-zero  
 951 eigenvalue  $\nu_0$ , one may check that it is governed by

$$952 \quad \cos\left(\sqrt{\nu_0} \frac{\pi}{2}\right) = \frac{1 - \delta}{1 + \delta}, \text{ or equivalently } \sqrt{\nu_0} = \frac{2}{\pi} \arccos\left(\frac{1 - \delta}{1 + \delta}\right).$$

953 Performing the expansion in the limit  $\delta \rightarrow 0$ , one finds that  $\sqrt{\nu_0} \sim \frac{4}{\pi} \sqrt{\delta}$ . Since  
 954  $\tau_{\mathbf{e},\Sigma(\mu)} = \sqrt{\nu_0}$ , we find  $\tau_{(\varepsilon,\mu,\sigma)}^{opt} \sim \frac{4}{\pi} \sqrt{\delta}$ , so there is no (extra-)regularity in the limit  
 955 case: in this sense the conditions (6.27) and (B.8) are “sharp” for the checkerboard.  
 956 To conclude, we provide some excerpts from Dauge’s benchmark [16], see Table 1.  
 957 These numerical values corroborate the asymptotic formula when  $\delta$  goes to 0.

958 **7.3. Comments.** When applicable, the local approach allows one to compute  
 959 the “best” regularity exponent  $\tau_{(\varepsilon,\mu,\sigma)}^{opt}$ , up to numerical precision. As a matter of fact,  
 960 one has to solve numerically a series of eigenproblems, see §§B.1-B.2. Or, in more  
 961 favorable cases (cf. §7.1), it is even known exactly. However, the theory we recalled in  
 962 Appendix B is limited to the case of piecewise constant coefficients, on a polyhedral

$\delta$	$\frac{4}{\pi}\sqrt{\delta}$	$(\sqrt{\nu_0})_{computed}$
$10^{-1}$	$4.0263 \cdot 10^{-1}$	$3.8996 \cdot 10^{-1}$
$10^{-2}$	$1.2732 \cdot 10^{-1}$	$1.2690 \cdot 10^{-1}$
$10^{-8}$	$1.2732 \cdot 10^{-4}$	$1.2732 \cdot 10^{-4}$

TABLE 1  
Asymptotic and computed values of  $\sqrt{\nu_0}$ .

963 partition. Or, at least, to coefficients that are locally (piecewise) constant near the  
964 interface and locally smooth near the boundary. And, near the boundary and for  
965 smooth coefficients, one may use the so-called *frozen coefficients* technique, cf. [21,  
966 §5.2] or [17, §5]. In principle, the value of  $\tau_{(\varepsilon,\mu,\sigma)}^{opt}$  can still be computed. But when  
967 the coefficients are only piecewise smooth, the technique no longer applies.  
968 Also, there is no obvious way to compute the constant  $C_{(\varepsilon,\mu,\sigma,t)}^*$  appearing in (3.6)  
969 when  $t$  spans  $[0, \tau_{(\varepsilon,\mu,\sigma)}^{opt})$ , or to provide bounds of such a constant with respect to the  
970 coefficients, with the help of the local approach.  
971 On the other hand, the global approach allows one to address all of the above, on a  
972 partition made of (possibly) non-polyhedral domains. It is only when the discretiza-  
973 tion is concerned that one assumes the partition to be made of polyhedra.

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979 and measures of electromagnetic fields than the ones presented here.

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1051           **Appendix A. Real interpolation method.** We follow here [33, §§22-23]  
1052 and [9, §14]. Let  $H_0, H_1$  be two Hilbert spaces, continuously embedded into a third  
1053 Hilbert space  $H$ :

- 1054           •  $H_0 + H_1$  is equipped with the norm  $\|v\|_{H_0 + H_1} = \inf_{v=v_0+v_1} (\|v_0\|_{H_0} + \|v_1\|_{H_1})$ ;
- 1055           • resp.,  $H_0 \cap H_1$  is equipped with the norm  $\|v\|_{H_0 \cap H_1} = \max(\|v\|_{H_0}, \|v\|_{H_1})$ .

1056 One introduces, for  $v \in H_0 + H_1$ ,  $K(t; v) := \inf_{v=v_0+v_1} (\|v_0\|_{H_0}^2 + t^2\|v_1\|_{H_1}^2)^{1/2}$ . For  
1057  $s \in (0, 1)$ , one defines the interpolated space

$$1058 \quad (H_0, H_1)_{s,2} := \left\{ v \in H_0 + H_1 \text{ s.t. } t^{-s}K(t; v) \in L^2(0, \infty; \frac{dt}{t}) \right\},$$

1059 equipped with the norm  $\|v\|_{(H_0, H_1)_{s,2}} := \|t^{-s}K(t; v)\|_{L^2(0, \infty; \frac{dt}{t})}$ .

1060

1061 In the case where  $H_1 \subset H_0$ , we use the notation  $H_s = (H_0, H_1)_{s,2}$ . In this case  
 1062 and after elementary computations, one finds that :

1063 • For all  $s, s' \in (0, 1)$ ,  $s \leq s'$ :

1064 
$$\exists C_{(s,s')}, \forall v \in H_{s'}, \quad \|v\|_{H_s} \leq C_{(s,s')} \|v\|_{H_{s'}}.$$

1065 • For all  $s \in (0, 1)$ ,  $H_s \subset H_0$ :

1066 
$$\exists c_0 > 0, \forall s \in (0, 1), \forall v \in H_s, \quad \|v\|_{H_0} \leq c_0 s^{1/2} (1-s)^{1/2} \|v\|_{H_s}.$$

1067 • For all  $s \in (0, 1)$ ,  $H_1 \subset H_s$ :

1068 
$$\exists c_1 > 0, \forall s \in (0, 1), \forall v \in H_1, \quad s^{1/2} (1-s)^{1/2} \|v\|_{H_s} \leq c_1 \|v\|_{H_1}.$$

1069 If  $H_1 = H_0$ , the above holds for  $c_0 = 2$ , resp.  $c_1 = \frac{1}{\sqrt{2}}$ .

1070 **Appendix B. The local approach for finding a regularity exponent**

1071  $\tau_{(\varepsilon,\mu,\sigma)}$ . We know that  $e$  and  $\mu^{-1} \mathbf{curl} e$  may be split into regular and gradient  
 1072 parts (6.2)-(6.3). The regularity of the regular parts is known to be independent of  
 1073 the coefficients (see Proposition 5.3). On the other hand, the regularity of the gradient  
 1074 parts,  $\nabla p_0$  governed by (5.4), resp.  $\nabla q_0$  governed by (5.7), depends a priori on the co-  
 1075 efficients: it is now determined by the local approach. Based on the global approach,  
 1076 we have obtained a regularity exponent  $\tau_{(\varepsilon,\mu,\sigma)}$  which is strictly lower than  $\frac{1}{2}$ . So it  
 1077 may not be equal to the “best” regularity exponent  $\tau_{(\varepsilon,\mu,\sigma)}^{opt}$ . Indeed, one knows that  
 1078 the best regularity exponent is always lower than or equal to 1, and that there exist  
 1079 configurations for which it is equal to 1, see eg. [3]. To summarize,  $\tau_{(\varepsilon,\mu,\sigma)}^{opt} \in (0, 1]$   
 1080 and

- 1081 • either  $p_0, q_0 \in PH^2(\Omega)$  always holds, in which case  $\tau_{(\varepsilon,\mu,\sigma)}^{opt} = 1$  ;  
 1082 • or  $p_0, q_0 \in \bigcap_{s \in [0, \tau_{(\varepsilon,\mu,\sigma)}^{opt})} PH^{1+s}(\Omega)$ , and possibly  $p_0, q_0 \notin PH^{1+\tau_{(\varepsilon,\mu,\sigma)}^{opt}}(\Omega)$ ,  
 1083 always holds, in which case  $\tau_{(\varepsilon,\mu,\sigma)}^{opt} \in (0, 1)$ .

1084 Below, we recall how one can characterize the best regularity exponent  $\tau_{(\varepsilon,\mu,\sigma)}^{opt}$  by the  
 1085 local approach, in the case where the *coefficients*  $(\varepsilon, \mu, \sigma)$  are as in section 2.1, and  
 1086 moreover  $(\varepsilon, \mu, \sigma)$  are *piecewise constant over a polyhedral partition*. Below, we focus  
 1087 on the influence of the interface  $\Sigma$ , that is to say on the influence of the coefficients.  
 1088 Obviously, the influence of the boundary  $\partial\Omega$  must be taken into account. But unless  
 1089 the interface intersects with it, the coefficients do not play a role in the regularity of  
 1090 the solution there, only the geometry of the boundary does. Along the same lines, if  
 1091 the coefficient is constant on  $\Omega$ , the local approach allows one to determine the “best”  
 1092 (largest) value of  $\sigma_{Dir}$  and  $\sigma_{Neu}$ . We follow here [14, 15] and Refs. therein: according  
 1093 to Kondratiev’s theory, one studies the second order elliptic PDEs locally all over  $\Omega$ ,  
 1094 and in particular the (local) regularity of its solution [21, §8.2].

1095 **B.1. Interior domain.** Let  $\mathcal{O}$  be the domain on which a local problem is de-  
 1096 fined. We consider here that  $\mathcal{O}$  is an *interior domain*, that is  $\partial\mathcal{O} \cap \partial\Omega = \emptyset$ . The case  
 1097 of a boundary domain ( $\partial\mathcal{O} \cap \partial\Omega \neq \emptyset$ ) is sketched in §B.2.

1098 The first type of local problems occurs when  $\bar{\mathcal{O}}$  does not intersect with the interface,  
 1099 and then the solution belongs to  $H^2(\mathcal{O})$ .

1100 There are three other types of local problems, namely:

- 1101 • problems where the interface is a smooth manifold, in this case the solution  
 1102 is piecewise smooth, ie. it belongs to  $PH^2(\mathcal{O})$  ;

- 1103 • edge problems where the interface is a smooth manifold, *except for one edge*  $\mathbf{e}$ ,  
 1104 that allow to determine the so-called edge singularities: there exists  $\tau_{\mathbf{e}} \in (0, 1)$   
 1105 such that the solution always belongs to  $\bigcap_{s \in [0, \tau_{\mathbf{e}})} PH^{1+s}(\mathcal{O})$ , but may not  
 1106 belong to  $PH^{1+\tau_{\mathbf{e}}}(\mathcal{O})$ ;
- 1107 • corner problems, where the interface is smooth, *except for several edges that*  
 1108 *intersect at a corner*  $\mathbf{c}$ , that allow to determine the so-called corner singu-  
 1109 larities: there exists  $\tau_{\mathbf{c}} \in (0, 1)$  such that the solution always belongs to  
 1110  $\bigcap_{s \in [0, \tau_{\mathbf{c}})} PH^{1+s}(\mathcal{O})$ , but may not belong to  $PH^{1+\tau_{\mathbf{c}}}(\mathcal{O})$ .

1111 We focus first on *corner problems*. The case of the edge problem is treated next.  
 1112 For a corner  $\mathbf{c}$ , the singularities are obtained as non-zero quasi-homogeneous functions  
 1113 which solve the same problem in  $\Gamma$  with zero right-hand side, where  $\Gamma$  is the *infinite*  
 1114 *cone* that coincides with the domain  $\mathcal{O}$  at  $\mathbf{c}$ . Introducing  $\mathbb{S}^2$  the unit sphere, resp.  
 1115  $(\rho, \theta, \varphi) \in \mathbb{R}^+ \times [0, \pi) \times [0, 2\pi)$  the spherical coordinates, centered at  $\mathbf{c}$ , and defining  
 1116  $G := \Gamma \cap \mathbb{S}^2$ , one can choose a priori those functions in the sets

$$1117 \quad S_{\mathbf{c}}^{\lambda}(\Gamma) := \{ \Psi = \rho^{\lambda} \psi(\theta, \varphi) \text{ s.t. } \psi \in H^1(G) \}, \text{ where } \lambda \in \mathbb{C}.$$

1118 More precisely (see [15, p. 818]), one should look for quasi-homogeneous func-  
 1119 tions of the type  $\rho^{\lambda} \sum_{q=0, Q} (\log \rho)^q \psi_q(\theta, \varphi)$  with  $Q \in \mathbb{N}$ . However it is sufficient  
 1120 for our purposes – determining the exponent – to focus on homogeneous functions.  
 1121 In spherical coordinates, we recall that the volume element writes  $\rho^2 d\rho d\zeta$ , where  
 1122  $d\zeta := \sin \theta d\theta d\varphi$ , whereas the gradient writes  $\nabla v = \partial_{\rho} v e_{\rho} + \rho^{-1} \nabla_{\zeta} v$ , with  $\nabla_{\zeta} v :=$   
 1123  $\partial_{\theta} v e_{\theta} + (\sin \theta)^{-1} \partial_{\varphi} v e_{\varphi}$ .

1124 Because  $\mathcal{O}$  is an interior domain, observe that one has  $G = \mathbb{S}^2$  and  $\partial\Gamma = \emptyset$ .  
 1125 For the local corner problem at hand, since one is looking for  $\Psi_{\mathbf{c}} = \rho^{\lambda_{\mathbf{c}}} \psi(\theta, \varphi)$  in  
 1126  $H_{loc}^1(\Gamma)$ , one finds that a necessary and sufficient condition on the exponent  $\lambda_{\mathbf{c}}$  is that  
 1127  $\Re(\lambda_{\mathbf{c}}) > -\frac{1}{2}$ ; and moreover that

$$1128 \quad \tau_{\mathbf{c}}(\xi) := \min_{\lambda_{\mathbf{c}} \text{ st. } \Re(\lambda_{\mathbf{c}}) > -\frac{1}{2}} \left( \lambda_{\mathbf{c}} + \frac{1}{2} \right),$$

1129 where  $\Psi_{\mathbf{c}} \neq 0$  is a (non-smooth) function governed by  $\text{div}(\xi \nabla \Psi_{\mathbf{c}}) = 0$  in  $\Gamma$ . The  
 1130 coefficient  $\xi$  being independent of  $\rho$ , one easily checks that it is equivalent to finding  
 1131 solutions to the eigenproblem

$$1132 \quad (\text{B.1}) \quad \begin{cases} \text{Find } \psi \in H^1(\mathbb{S}^2) \setminus \{0\}, \nu \in \mathbb{C} \text{ such that} \\ \int_{\mathbb{S}^2} \xi \nabla_{\zeta} \psi \cdot \nabla_{\zeta} \bar{\psi}' d\zeta = \nu \int_{\mathbb{S}^2} \xi \psi \bar{\psi}' d\zeta, \forall \psi' \in H^1(\mathbb{S}^2), \end{cases}$$

1133 with the relation  $\nu = \lambda_{\mathbf{c}}(\lambda_{\mathbf{c}} + 1)$ . Note that  $\psi = 1$  and  $\nu = 0$  is an eigenpair of  
 1134 (B.1), which yields the values  $\lambda_{\mathbf{c}} = 0$  or  $\lambda_{\mathbf{c}} = -1$ . The latter is excluded, because one  
 1135 has necessarily  $\Re(\lambda_{\mathbf{c}}) > -\frac{1}{2}$ . Whereas the former yields  $\Psi_{\mathbf{c}} = 1$ , which is a smooth  
 1136 function, and thus one concludes that no singular behavior is associated with  $\nu = 0$ .  
 1137 Then, choosing  $\psi' = \psi$  in (B.1) one finds that

$$1138 \quad \int_{\mathbb{S}^2} \xi |\nabla_{\zeta} \psi|^2 d\zeta = \nu \int_{\mathbb{S}^2} \xi |\psi|^2 d\zeta.$$

1139 Using the notation  $v_R := \Re(v)$  and  $v_I := \Im(v)$  for complex-valued fields, and taking  
 1140 respectively the real and imaginary part of the previous equation, one derives the



1141 relations

$$1142 \quad (\text{B.2}) \quad \nu_R := \frac{(\int_{\mathbb{S}^2} \xi_R |\nabla_\zeta \psi|^2 d\zeta) (\int_{\mathbb{S}^2} \xi_R |\psi|^2 d\zeta) + (\int_{\mathbb{S}^2} \xi_I |\nabla_\zeta \psi|^2 d\zeta) (\int_{\mathbb{S}^2} \xi_I |\psi|^2 d\zeta)}{(\int_{\mathbb{S}^2} \xi_R |\psi|^2 d\zeta)^2 + (\int_{\mathbb{S}^2} \xi_I |\psi|^2 d\zeta)^2},$$

$$1143 \quad (\text{B.3}) \quad \nu_I := \frac{(\int_{\mathbb{S}^2} \xi_I |\nabla_\zeta \psi|^2 d\zeta) (\int_{\mathbb{S}^2} \xi_R |\psi|^2 d\zeta) - (\int_{\mathbb{S}^2} \xi_R |\nabla_\zeta \psi|^2 d\zeta) (\int_{\mathbb{S}^2} \xi_I |\psi|^2 d\zeta)}{(\int_{\mathbb{S}^2} \xi_R |\psi|^2 d\zeta)^2 + (\int_{\mathbb{S}^2} \xi_I |\psi|^2 d\zeta)^2}.$$

1144 Considering both cases  $\xi = \mu$  and  $\xi = \varepsilon_\sigma$ , one notices that, under the assumptions of  
 1145 section 2.1, the spectral theorem can be applied to characterize the solutions to the  
 1146 eigenproblem (B.1), see for instance [31, §2.1]. In particular, the eigenfunctions can be  
 1147 chosen as the elements of a Hilbert basis of  $L^2(\mathbb{S}^2)$ . We already observed that  $\psi = 1$   
 1148 is an eigenfunction (with related eigenvalue  $\nu = 0$ ). Hence, as a consequence of the  
 1149 Poincaré inequality in  $H_{zmv}^1(\mathbb{S}^2)$ , there exists  $c_P > 0$  such that, for all eigenfunctions  
 1150  $\psi$  related to a non-zero eigenvalue, it holds that

$$1151 \quad (\text{B.4}) \quad \frac{\int_{\mathbb{S}^2} |\nabla_\zeta \psi|^2 d\zeta}{\int_{\mathbb{S}^2} |\psi|^2 d\zeta} \geq c_P^2.$$

1152 For the two cases of interest:

1153 • If  $\xi = \mu$  (real-valued coefficient case), one has according to (B.2)-(B.3) that

$$1154 \quad (\text{B.5}) \quad \nu = \frac{\int_{\mathbb{S}^2} \mu |\nabla_\zeta \psi|^2 d\zeta}{\int_{\mathbb{S}^2} \mu |\psi|^2 d\zeta} \geq \frac{\mu_{min}}{\mu_{max}} c_P^2 > 0.$$

1155 Recall that  $\nu = \lambda_c(\lambda_c + 1)$ , ie.  $\lambda_c = -\frac{1}{2} \pm \sqrt{\nu + \frac{1}{4}}$ . Due to the condition  
 1156  $\Re(\lambda_c) > -\frac{1}{2}$  and because one has  $\nu > 0$ , the only admissible relation is  
 1157  $\lambda_c = -\frac{1}{2} + \sqrt{\nu + \frac{1}{4}}$ . Hence  $\lambda_c > 0$  for all non-zero eigenvalues  $\nu$ , which yields  
 1158  $\tau_c(\mu) > 1/2$  *independently of the values of  $\mu_{min}$  and  $\mu_{max}$ .*

1159 • If  $\xi = \varepsilon_\sigma$  (complex-valued coefficient case), one has according to (B.2) that

$$1160 \quad (\text{B.6}) \quad \nu_R \geq \frac{(\xi_R)_{min}^2 + (\xi_I)_{min}^2}{(\xi_R)_{max}^2 + (\xi_I)_{max}^2} \frac{\int_{\mathbb{S}^2} |\nabla_\zeta \psi|^2 d\zeta}{\int_{\mathbb{S}^2} |\psi|^2 d\zeta} \geq \frac{\varepsilon_{min}^2 \omega^2 + \sigma_{min}^2}{\varepsilon_{max}^2 \omega^2 + \sigma_{max}^2} c_P^2 > 0,$$

1161 while, according to (B.3),  $\nu_I$  can take positive or negative values. Due to  
 1162 the condition  $\Re(\lambda_c) > -\frac{1}{2}$  and because one has now  $\nu_R > 0$ , see (B.6), the  
 1163 only admissible relation is again  $\lambda_c = -\frac{1}{2} + \sqrt{\nu + \frac{1}{4}}$ . Let  $\nu + \frac{1}{4} = \rho_\nu \exp(i\theta_\nu)$ ,  
 1164  $\rho_\nu > 0$ ,  $\theta_\nu \in [0, 2\pi)$ . Then  $\Re(\sqrt{\nu + \frac{1}{4}}) = (\rho_\nu)^{1/2} \cos(\theta_\nu/2)$ , with  $\rho_\nu \geq (\nu_R + \frac{1}{4})$   
 1165 and  $\cos^2(\theta_\nu/2) = \frac{1}{2}(1 + \cos \theta_\nu) = \frac{1}{2}(1 + (\rho_\nu)^{-1}(\nu_R + \frac{1}{4}))$ . Hence,

$$1166 \quad \Re(\sqrt{\nu + \frac{1}{4}}) = \left( \frac{1}{2} \left( \rho_\nu + \nu_R + \frac{1}{4} \right) \right)^{1/2} \geq \left( \nu_R + \frac{1}{4} \right)^{1/2} > \frac{1}{2}.$$

1167 This yields  $\tau_c(\varepsilon_\sigma) > 1/2$  *independently of the values of  $\varepsilon_{max}$ ,  $\varepsilon_{min}$ ,  $\sigma_{max}$  and*  
 1168  $\sigma_{min}$ .

1169 We focus now on *edge problems*. For an edge  $\mathbf{e}$ , the singularities are obtained as  
 1170 non-zero quasi-homogeneous functions which solve the same problem in  $\Gamma$  with zero  
 1171 right-hand side, where  $\Gamma \times \mathbb{R}$  is the *infinite sector* that coincides with the domain  $\mathcal{O}$   
 1172 at  $\mathbf{e}$ . Introducing  $\mathbb{S}^1$  the unit circle, resp.  $(\rho, \theta, z) \in \mathbb{R}^+ \times [0, 2\pi) \times \mathbb{R}$  the cylindrical  
 1173 coordinates with  $\mathbf{e} \subset \{z = 0\}$ , and defining  $G := \Gamma \cap \mathbb{S}^1$ , one can choose a priori those  
 1174 functions in the sets

$$1175 \quad S_{\mathbf{e}}^\lambda(\Gamma) := \{\Psi = \rho^\lambda \psi(\theta) \text{ s.t. } \psi \in H^1(G)\}, \text{ where } \lambda \in \mathbb{C}.$$

1176 One should look for quasi-homogeneous functions, however restricting to homogeneous  
 1177 is again sufficient to determine the exponent. In the polar coordinates  $(\rho, \theta)$ , we recall  
 1178 that the surface element writes  $\rho d\rho d\theta$ , whereas the gradient writes  $\nabla v = \partial_\rho v \mathbf{e}_\rho +$   
 1179  $\rho^{-1} \partial_\theta v \mathbf{e}_\theta$ . Because  $\mathcal{O}$  is an interior domain, observe that one has  $G = \mathbb{S}^1$  and  $\partial\Gamma = \emptyset$ .  
 1180 For the local edge problem at hand, since one is looking for  $\Psi_{\mathbf{e}} = \rho^{\lambda_{\mathbf{e}}} \psi(\theta)$  in  $H_{loc}^1(\Gamma)$ ,  
 1181 one finds that a necessary and sufficient condition on the exponent  $\lambda_{\mathbf{e}}$  is that  $\Re(\lambda_{\mathbf{e}}) >$   
 1182  $0$ ; and moreover that

$$1183 \quad \tau_{\mathbf{e}}(\xi) := \min_{\lambda_{\mathbf{e}} \text{ st. } \Re(\lambda_{\mathbf{e}}) > 0} \lambda_{\mathbf{e}},$$

1184 where  $\Psi_{\mathbf{e}} \neq 0$  is a (non-smooth) function governed by  $\text{div}(\xi \nabla \Psi_{\mathbf{e}}) = 0$  in  $\Gamma$ . The  
 1185 coefficient  $\xi$  being independent of  $\rho$ , it is equivalent to finding solutions to the eigen-  
 1186 problem

$$1187 \quad (\text{B.7}) \quad \begin{cases} \text{Find } \psi \in H^1(\mathbb{S}^1) \setminus \{0\}, \nu \in \mathbb{C} \text{ such that} \\ \int_{\mathbb{S}^1} \xi \partial_\theta \psi \partial_\theta \overline{\psi'} d\theta = \nu \int_{\mathbb{S}^1} \xi \psi \overline{\psi'} d\theta, \forall \psi' \in H^1(\mathbb{S}^1), \end{cases}$$

1188 with the relation  $\nu = \lambda_{\mathbf{e}}^2$ . The spectral theorem can be applied under the assumptions  
 1189 of section 2.1 for  $\xi = \mu$  and  $\xi = \varepsilon_\sigma$ . As previously,  $\psi = 1$  and  $\nu = 0$  is an eigenpair  
 1190 of (B.7), leading to  $\Psi_{\mathbf{e}} = 1$ , and one concludes again that no singular behavior is  
 1191 associated with  $\nu = 0$ . So, using the Poincaré inequality in  $H_{zmv}^1(\mathbb{S}^1)$ , there exists  
 1192  $c_P > 0$  such that, for all eigenfunctions  $\psi$  related to a non-zero eigenvalue, the bound  
 1193 (B.4) holds, and one also recovers (B.2)-(B.3), with  $\nabla_\zeta$  replaced by  $\partial_\theta \psi$ , resp.  $d\zeta$  by  
 1194  $d\theta$ . Then:

- 1195 • If  $\xi = \mu$  (real-valued coefficient case), one derives again the lower bound  
 1196 (B.5) on  $\nu$ . Due to the condition  $\Re(\lambda_{\mathbf{e}}) > 0$ , the only admissible relation is  
 1197  $\lambda_{\mathbf{e}} = \sqrt{\nu}$ . Hence  $\lambda_{\mathbf{e}} > 0$  for all non-zero eigenvalues  $\nu$ , which yields  $\tau_{\mathbf{e}}(\mu) > 0$ .  
 1198 More precisely, one gets the lower bound

$$1199 \quad \tau_{\mathbf{e}}(\mu) \geq c_P \left( \frac{\mu_{min}}{\mu_{max}} \right)^{1/2}.$$

- 1200 • If  $\xi = \varepsilon_\sigma$  (complex-valued coefficient case), one has the lower bound (B.6)  
 1201 for  $\nu_R$ . And, according to (B.3),  $\nu_I$  can take positive or negative values.  
 1202 Due to the condition  $\Re(\lambda_{\mathbf{e}}) > 0$  and because  $\nu_R > 0$ , one has  $\lambda_{\mathbf{e}} = \sqrt{\nu}$ .  
 1203 Writing  $\nu := \nu_R + i\nu_I = \rho_\nu \exp(i\theta_\nu)$ , with  $\rho_\nu > 0$  and  $\theta_\nu \in (-\frac{\pi}{2}, \frac{\pi}{2})$ , one has  
 1204  $\lambda_{\mathbf{e}} = \rho_\nu^{1/2} \exp(i\theta_\nu/2)$ , so that  $\Re(\sqrt{\nu}) \geq 2^{-1/2} \rho_\nu^{1/2}$ . One may check that

$$1205 \quad \Re(\sqrt{\nu}) \geq 2^{-1/2} \left( \frac{(\int_{\mathbb{S}^1} \xi_R |\partial_\theta \psi|^2 d\theta)^2 + (\int_{\mathbb{S}^1} \xi_I |\partial_\theta \psi|^2 d\theta)^2}{(\int_{\mathbb{S}^1} \xi_R |\psi|^2 d\theta)^2 + (\int_{\mathbb{S}^1} \xi_I |\psi|^2 d\theta)^2} \right)^{1/4}.$$

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Hence, one gets the lower bound

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$$\tau_{\mathbf{e}}(\varepsilon_{\sigma}) \geq 2^{-1/2} c_P \left( \frac{\varepsilon_{min}^2 \omega^2 + \sigma_{min}^2}{\varepsilon_{max}^2 \omega^2 + \sigma_{max}^2} \right)^{1/4}.$$

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**B.2. Boundary domain.** We consider now that the domain  $\mathcal{O}$  on which a local problem is defined is a *boundary domain*, for which  $\partial\mathcal{O} \cap \partial\Omega \neq \emptyset$ . The main difference with the interior domain case is that the local problems now come with a (homogeneous) boundary condition: Dirichlet boundary condition for  $p_0$ , resp. Neumann boundary condition for  $q_0$ . On the other hand, the theory is quite similar to the one of §B.1. As in the interior domain case, one must consider edge problems, and corner problems. Below, we suppose explicitly that  $\mathcal{O} \cap \Sigma \neq \emptyset$ .

For the *corner problem*, one looks for homogeneous solutions that belong to  $S_{\mathbf{e}}^{\lambda}(\Gamma)$ , but now  $\partial\Gamma \neq \emptyset$ , and  $G = \Gamma \cap \mathbb{S}^2$  is a strict subset of  $\mathbb{S}^2$ . Then, one solves an eigenproblem like (B.1), with the relation  $\nu = \lambda_{\mathbf{c}}(\lambda_{\mathbf{c}} + 1)$ , now set in the function space  $H_0^1(G)$  (Dirichlet boundary condition), resp. in  $H_{zmv}^1(G)$  (Neumann boundary condition). The main observation for the corner problem set in a boundary domain is that, since there holds a Poincaré inequality in both function spaces, one may still apply the previous analysis (interior domain), to draw the conclusions. Namely, one finds that

$$\tau_{\mathbf{c}}(\mu) > 1/2, \quad \tau_{\mathbf{e}}(\varepsilon_{\sigma}) > 1/2,$$

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*independently of the values of the coefficients.*

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For the *edge problem*, one looks for homogeneous solutions that belong to  $S_{\mathbf{e}}^{\lambda}(\Gamma)$ , where  $\partial\Gamma \neq \emptyset$ , and  $G$  is a strict subset of  $\mathbb{S}^1$ . One solves an eigenproblem like (B.7), with the relation  $\nu = \lambda_{\mathbf{e}}^2$ , set in the function space  $H_0^1(G)$  (Dirichlet boundary condition), resp. in  $H_{zmv}^1(G)$  (Neumann boundary condition). Since there holds a Poincaré inequality in both function spaces, one may again apply the previous analysis (interior domain), to draw the conclusions. One finds the lower bounds:

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$$\tau_{\mathbf{e}}(\mu) \geq c_{P,Neu} \left( \frac{\mu_{min}}{\mu_{max}} \right)^{1/2}, \quad \tau_{\mathbf{e}}(\varepsilon_{\sigma}) \geq 2^{-1/2} c_{P,Dir} \left( \frac{\varepsilon_{min}^2 \omega^2 + \sigma_{min}^2}{\varepsilon_{max}^2 \omega^2 + \sigma_{max}^2} \right)^{1/4}.$$

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Finally, if there is no interface in  $\mathcal{O}$ , ie.  $\mathcal{O} \cap \Sigma = \emptyset$ , one simply considers that the coefficient  $\xi$  is constant on  $\mathcal{O}$ . In this case the value of  $\tau_{\mathbf{c}}, \tau_{\mathbf{e}} \in (\frac{1}{2}, 1]$  is determined by the geometry of the boundary. Precisely, if  $\mathcal{O}$  is defined as the intersection of  $\Omega$  with a ball, one finds that  $\tau_{\mathbf{c}} < 1$  or  $\tau_{\mathbf{e}} < 1$  if, and only if,  $\mathcal{O}$  is not convex.

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**B.3. Behavior of the best exponent.** From the previous studies, we conclude that one derives the actual value of the best regularity exponent by taking  $\tau_{(\varepsilon, \mu, \sigma)}^{opt} := \min(1, \min_{\mathbf{e}} \tau_{\mathbf{e}}, \min_{\mathbf{c}} \tau_{\mathbf{c}})$ . In particular, one may compute numerically the value of  $\tau_{(\varepsilon, \mu, \sigma)}^{opt}$ . As for the local approach (see §6.5), let us study what happens when  $\Theta$  is not reduced to a singleton. Again, let us consider that all its elements  $(\varepsilon, \mu, \sigma)$  are such that  $\varepsilon, \sigma, \mu$  are piecewise-constant on a *fixed partition*. We see that if condition

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$$(B.8) \quad \sup_{(\varepsilon, \mu, \sigma) \in \Theta} \frac{\mu_{max}}{\mu_{min}} + \sup_{(\varepsilon, \mu, \sigma) \in \Theta} \left( \frac{\varepsilon_{max}^2 \omega^2 + \sigma_{max}^2}{\varepsilon_{min}^2 \omega^2 + \sigma_{min}^2} \right)^{1/2} < \infty$$

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holds, then  $\inf_{(\varepsilon, \mu, \sigma) \in \Theta} \tau_{(\varepsilon, \mu, \sigma)}^{opt} > 0$ .