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How long does it take for all users in a social network to choose their communities?

Jean-Claude Bermond ^{a*}, Augustin Chaintreau ^b, Guillaume Ducoffe ^{c†}, Dorian Mazauric ^{d‡}

(a) Université Côte d’Azur, CNRS, Inria, I3S, France

(b) Columbia University in the City of New York

(c) National Institute for Research and Development in Informatics, and Research Institute and Faculty of Mathematics and Computer Science of the University of Bucharest, București, România

(d) Université Côte d’Azur, Inria, France

Abstract

We consider a community formation problem in social networks, where the users are either friends or enemies. The users are partitioned into conflict-free groups (*i.e.*, independent sets in the conflict graph $G^- = (V, E)$ that represents the enmities between users). The dynamics goes on as long as there exists any set of at most k users, k being any fixed parameter, that can change their current groups in the partition simultaneously, in such a way that they all strictly increase their utilities (number of friends *i.e.*, the cardinality of their respective groups minus one). Previously, the best-known upper-bounds on the maximum time of convergence were $\mathcal{O}(|V|\alpha(G^-))$ for $k \leq 2$ and $\mathcal{O}(|V|^3)$ for $k = 3$, with $\alpha(G^-)$ being the independence number of G^- . Our first contribution in this paper consists in reinterpreting the initial problem as the study of a dominance ordering over the vectors of integer partitions. With this approach, we obtain for $k \leq 2$ the tight upper-bound $\mathcal{O}(|V|\min\{\alpha(G^-), \sqrt{|V|}\})$ and, when G^- is the empty graph, the exact value of order $\frac{(2|V|)^{3/2}}{3}$. The time of convergence, for any fixed $k \geq 4$, was conjectured to be polynomial [EGM12, KL13]. In this paper we disprove this. Specifically, we prove that for any $k \geq 4$, the maximum time of convergence is in $\Omega(|V|^{\Theta(\log |V|)})$.

Keywords: communities, social networks, integer partitions, coloring games, graphs, algorithms.

1 Introduction

Community formation is a fundamental problem in social network analysis. It has already been modeled in several ways, each trying to capture key aspects of the problem. The model studied in this paper has been proposed in [KL13] in order to reflect the impact of information sharing on the community formation process. Although it is a simplified model, we show that its understanding requires us to solve combinatorial problems that are surprisingly intricate. More precisely, we consider the following dynamics of formation of groups (communities) in social networks. Each group represents a set of users sharing about some information topic. We assume for simplicity that each user shares about a given topic in only one group. Therefore the groups will partition the set of users. We follow the approach of [KL13]. An important feature is the emphasis on incompatibility between some pairs of users that we will call enemies. Two enemies do not want to share information. Hence, they will necessarily belong to different groups. In the general model one considers different degrees of friendship or incompatibilities. Here we will restrict to the case where two users are

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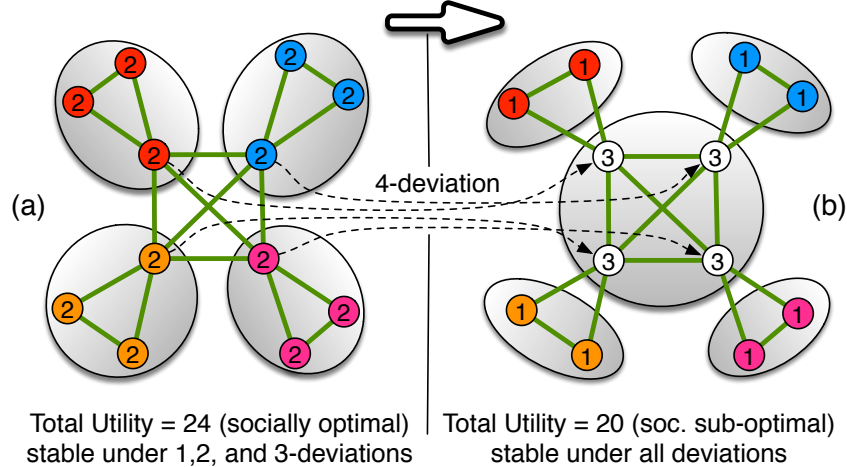


Figure 1: A friendship graph with 12 vertices (users). (a) 3-stable partition that is not 4-stable but it is optimal in terms of total utility. (b) k -stable partition for any $k \geq 1$ that is not optimal in terms of total utility.

13 either friends or enemies – as noted in [KL13], even a little beyond this case, the problem quickly becomes
 14 intractable. As an example, if we add a neutral (indifference) relation, there are instances for which there is
 15 no stability.

16 The social network is often modeled by the friendship graph G^+ where the vertices are the users and
 17 an edge represents a friendship relation. We will use this graph to present the first notions and examples.
 18 However, for the rest of the article and the proofs we will use the complementary graph, that we call the
 19 conflict graph and denote by G^- ; here the vertices represent users and the edges represent the incompatibility
 20 relation. We assign each user a *utility* which is the number of friends in the group to which she belongs.
 21 Equivalently, the utility is the size of the group minus one, as in a group there is no pair of enemies; in [KL13]
 22 this is modeled by putting the utility as $-\infty$ when there is an enemy in the group.

23 In the example of Figure 1, the graph depicted is the friendship graph: the edges represent the friendship
 24 relation, and if there is no edge then, it corresponds to a pair of enemies. Figure 1(a) depicts a partition of
 25 12 users that is composed of 4 non-empty groups each of size 3. The integers on the vertices represent the
 26 utilities of the users which are all equal to 2. This implies that the total utility is 24. Figure 1(b) depicts
 27 another partition consisting of 5 groups with one group of size 4 (where users have utility 3) and 4 groups
 28 of size 2 (where users have utility 1). Hence, the total utility is 20.

29 In this study we are interested in the dynamics of formation of groups. Another important feature
 30 of [KL13], taken into account in the dynamics, is the notion of bounded cooperation between users. More
 31 precisely, the dynamics is as follows: initially each user is alone in her own group. In the simplest case,
 32 a move consists for a specific user to leave the group to which she belongs to join another group but only
 33 if this action strictly increases her utility (acting in a selfish manner); in particular, it implies that a user
 34 does not join a group where she has an enemy. In the k -bounded mode of cooperation, a set of at most k
 35 users can leave their respective groups to join another group, again, only if each user strictly increases their
 36 utility. If the group they join is empty then, it corresponds to creating a new group. We call such a move a
 37 k -*deviation*. Note that this notion is slightly different from that of $(k + 1)$ -defection of [KL13]. We will say
 38 that a *partition is k -stable* if there does not exist a k -deviation for this partition.

39 The partition of Figure 1(a) is k -stable when $k \in \{1, 2, 3\}$. Indeed each user has at least one enemy in
 40 each non-empty other group. Hence, any user cannot join another group. Furthermore, when $k \leq 3$, if k
 41 users join an empty group then, their utility will be at most 2 and will not strictly increase. However, this
 42 partition is not 4-stable because there is a 4-deviation: the four central users can join an empty group and in
 43 doing so they increase their utilities from 2 to 3. The partition obtained after such a 4-deviation is depicted
 44 in Figure 1(b). This partition is k -stable for any $k \geq 1$. Note that the utility of the other users is now 1
 45 (instead of 2). Thus, we deduce that this partition is not optimal in terms of total utility (the total utility

46 has decreased from 24 to 20); but it is now stable under all deviations. This illustrates the fact that users
 47 act in a selfish manner as some increase their utility, but on the contrary the total utility decreases. For
 48 more information on the suboptimality of k -stable partitions, *i.e.*, bounds on the price of anarchy and the
 49 price of stability, the reader is referred to [KL13].

50 1.1 Related work.

51 This above dynamics has been also modeled in the literature with *coloring games*. A coloring game is played
 52 on the conflict graph. Players must choose a color in order to construct a proper coloring of the graph, and
 53 the individual goal of each agent is to maximize the number of agents with the same color as she has. On a
 54 more theoretical side, coloring games have been introduced in [PS08] as a game-theoretic setting for studying
 55 the chromatic number in graphs. Specifically, the authors in [PS08] have shown that for every coloring game,
 56 there exists a Nash equilibrium where the number of colors is exactly the chromatic number of the graph.
 57 Since then, these games have been used many times, attracting attention in the study of information sharing
 58 and propagation in graphs [CKPS10, EGM12, KL13]. Coloring games are an important subclass of the more
 59 general Hedonic games, of which several variations have been studied in the literature in order to model
 60 coalition formation under selfish preferences of the agents [CNS18, FMZ17, Haj06, HJ17, MS17, OBI⁺17].
 61 We stress that while every coloring game has a Nash equilibrium that can be computed in polynomial-
 62 time [PS08], deciding whether a given Hedonic game admits a Nash equilibrium is NP-complete [Bal04].
 63 In [DBHS06] the authors consider a special case of binary Hedonic games, close to our model, where every
 64 user considers another user either as a friend or as an enemy. However, unlike in our case, such preferences
 65 may be asymmetric (*i.e.*, u may consider v as a friend, whereas v considers u as an enemy). This implies
 66 that a stable partition may not always exist, whereas it always does in our model. The authors in [DBHS06]
 67 have thus focused on the existence of core partitions (a.k.a., k -stable partitions for any k) under different
 68 utility functions. For other works about the existence of core coalitions in such games, and related ones with
 69 neutral preferences, see [OBI⁺17] and the papers cited therein.

70 If the set of edges of the conflict graph is empty (edgeless conflict graph) then, there exists a unique
 71 k -stable partition, namely, that consisting of one group with all the users. In [KL13], it is proved that there
 72 always exists a k -stable partition for any conflict graph, but that it is NP-hard to compute one if k is part
 73 of the input (this result was also proved independently in [EGM12]). Indeed, if k is equal to the number of
 74 users then, a largest group in such a partition must be a maximum independent set of the conflict graph.
 75 In contrast, a k -stable partition can be computed in polynomial time for every fixed $k \leq 3$, by using simple
 76 *better-response dynamics* [PS08, EGM12, KL13]. In such an algorithm one does a k -deviation until there
 77 does not exist any one. That corresponds to the dynamics of formation of groups that we study in this work
 78 for larger values of k .

79 1.2 Additional related work and our results.

80 In this paper we are interested in analyzing in this simple model the convergence of the dynamics with
 81 k -deviations, in particular in the worst case. It has been proved implicitly in [KL13] that the dynamics
 82 always converges within at most $\mathcal{O}(2^n)$ steps. Let $L(k, G^-)$ be the size of a longest sequence of k -deviations
 83 on a conflict graph G^- . We first observe that the maximum value, denoted $L(k, n)$, of $L(k, G^-)$ over all
 84 the graphs with n vertices is attained on the edgeless conflict graph G^0 of order n . Prior to this work, no
 85 lower bound on $L(k, n)$ was known, and the analysis was limited to the use of a potential function that only
 86 applies when $k \leq 3$ [EGM12, KL13] giving upper bounds of $\mathcal{O}(n^2)$ in the case $k = 1, 2$ and $\mathcal{O}(n^3)$ in the
 87 case $k = 3$. In order to go further in our analysis, the key observation is that when the conflict graph is
 88 edgeless, the dynamics only depends on the size of the groups of the partitions generated. Following [Bry73],
 89 let an integer partition of $n \geq 1$, be a non-increasing sequence of integers $Q = (q_1, q_2, \dots, q_n)$ such that
 90 $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$ and $\sum_{i=1}^n q_i = n$. If we rank the groups by non-increasing order of their size then,
 91 there is a natural relation between partitions in groups and integer partitions (the sizes of the groups X_i
 92 corresponding to the integers q_i of the partition of n). By using this relation, we prove in Section 3 that
 93 the better response dynamics algorithm reaches a stable partition in p_n steps, where $p_n = \Theta((e^\pi \sqrt{2n/3})/n)$
 94 denotes the number of integer partitions. This upper bound is certainly not tight. Indeed, we considered
 95 a worst-case scenario where all possible partitions could be attained with a single sequence of k -deviations.

k	Prior to our work	Our results	
1	$\mathcal{O}(n^2)$ [KL13]	exact analysis, that implies $L(1, n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 8
2	$\mathcal{O}(n^2)$ [KL13]	exact analysis, that implies $L(2, n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 11
1-2	$\mathcal{O}(n\alpha(G^-))$ [PS08]	$L(k, G^-) = \Omega(n\alpha(G^-))$ for some G^- with $\alpha(G^-) = \mathcal{O}(\sqrt{n})$	Theorem 14
3	$\mathcal{O}(n^3)$ [EGM12, KL13]	$L(3, n) = \Omega(n^2)$	Theorem 16
≥ 4	$\mathcal{O}(2^n)$ [KL13]	$L(k, n) = \Omega(n^{\Theta(\ln(n))})$, $L(k, n) = \mathcal{O}(\exp(\pi\sqrt{2n/3})/n)$	Theorem 15

Table 1: Previous bounds and results we obtained on $L(k, n)$ and $L(k, G^-)$.

Nevertheless, this is already far less than 2^n , which was shown to be the best upper bound that one can obtain for $k \geq 4$ when using an additive potential function [KL13].

Table 1 summarizes our contributions described below.

- For $k = 1, 2$, we refine the relation between partitions into groups and integer partitions as follows.
 - In the case $k = 1$ (Section 4.1), we prove that there is a one to one mapping between sequences of 1-deviations in the edgeless conflict graph and chains in the dominance lattice of integer partitions. Then, we use the value of the longest chain in this dominance lattice obtained in [GK86] in order to determine exactly $L(1, n)$. More precisely, if $n = \frac{m(m+1)}{2} + r$, with $0 \leq r \leq m$, $L(1, n) = 2\binom{m+1}{3} + mr$. The latter implies in particular $L(1, n)$ is of order $\mathcal{O}(n^{3/2})$, thereby improving the previous bound $\mathcal{O}(n^2)$.
 - In Section 4.2, we prove that any 2-deviation can be “replaced” (in some precise way) either by one or two 1-deviations. Therefore, $L(2, n) = L(1, n)$.
 - For $k = 1, 2$ and a general conflict graph G^- , the value of $L(k, G^-)$ depends on the independence number $\alpha(G^-)$ (cardinality of a largest independent set) of the conflict graph. In [PS08] it was proved that the convergence of the dynamics is in $\mathcal{O}(n\alpha(G^-))$. In the case of the edgeless conflict graph, we have seen that $L(1, n) = \mathcal{O}(n^{3/2})$. Hence, the preceding upper bound was not tight. Thus, we inferred that the convergence of the dynamics was in $\mathcal{O}(n\sqrt{\alpha(G^-)})$. Yet in fact we prove in Section 4.3 that, for any $\alpha(G^-) = \mathcal{O}(\sqrt{n})$, there exists a conflict graph G^- with n vertices and independence number $\alpha(G^-)$ for which we reach a stable partition after a sequence of at least $\Omega(n\alpha(G^-))$ 1-deviations.
- Finally, our main contribution is obtained for $k \geq 3$. Prior to our work, it was known that $L(3, n) = \mathcal{O}(n^3)$, that follows from another application of the potential function method [KL13]. But nothing proved that $L(3, n) > L(2, n)$, and in fact it was conjectured in [EGM12] that both values are equal. In Section 5, we prove (Theorem 16) that $L(3, n) = \Omega(n^2)$ and thus we show for the first time that deviations can delay convergence and that the gap between $k = 2$ and $k = 3$ obtained from the potential function method is indeed justified. It was also conjectured in [KL13] that $L(k, n)$ was polynomial in n for k fixed. In Section 5.1 we disprove this conjecture and prove in Theorem 15 that $L(4, n) = \Omega(n^{\Theta(\ln(n))})$. This shows that 4-deviations are responsible for a sudden complexity increase, as no polynomial bounds exist for $L(4, n)$.

2 Notations

Conflict graph. We refer to [BM08] for standard graph terminology. For the remainder of the paper, we assume that we are given a conflict graph $G^- = (V, E)$ where V is the set of vertices (called users or players in the introduction) and edges represent the incompatibility relation (*i.e.*, an edge means that the two users are enemies). The number of vertices is denoted by $n = |V|$. The independence number of G^- , denoted $\alpha(G^-)$, is the maximum cardinality of an independent set in G^- . In particular, if $\alpha(G^-) = n$ then the conflict graph is edgeless, we denote it by $G^\emptyset = (V, E = \emptyset)$, and call it the empty graph.

132 **Partitions and utilities.** We consider any partition $P = X_1, \dots, X_i, \dots, X_n$ of the vertices into n
133 independent sets X_i called groups (colors in coloring games), with some of them being possibly empty. In
134 particular, two enemies are not in the same group. We rank the groups by non-increasing size, that is
135 $|X_i| \geq |X_{i+1}|$. For any $1 \leq i \leq n$ and for any vertex $v \in X_i$, the utility of v is the number of other vertices
136 in the same group as it, that is $|X_i| - 1$.

137 We use in our proofs two alternative representations of the partition P . The partition vector associated
138 with P is defined as $\vec{\Lambda}(P) = (\lambda_n(P), \dots, \lambda_1(P))$, where $\lambda_i(P)$ is the number of groups of size i . The integer
139 partition associated with P is defined as $Q = (q_1, q_2, \dots, q_n)$, where $q_i = |X_i|$ and satisfies $q_1 \geq q_2 \geq \dots \geq$
140 $q_n \geq 0$ and $\sum_{i=1}^n q_i = n$.

141 In the example of Figure 1(a) we have a partition P of the 12 vertices into 4 groups each of size 3.
142 Therefore, $\lambda_3(P) = 4$ and $\lambda_i(P) = 0$ for $i \neq 3$; i.e. $\vec{\Lambda}(P) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 4, 0, 0)$. The corresponding
143 integer partition is $Q(P) = (3, 3, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0)$. In the example of Figure 1(b) we have a partition
144 P' of the 12 vertices into one group of size 4 and 4 groups each of size 2. Hence, $\lambda_4(P') = 1$, $\lambda_2(P') = 4$ and
145 $\lambda_i(P') = 0$ for $i \notin \{2, 4\}$; in other words, $\vec{\Lambda}(P') = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 4, 0)$. The corresponding integer
146 partition is $Q(P') = (4, 2, 2, 2, 2, 0, 0, 0, 0, 0, 0, 0)$.

147 **k -deviations and k -stability.** We can think of a k -deviation as a move of at most k vertices which
148 leave the groups to which they belong in P , to join another group (or create a new group) with the necessary
149 condition that each vertex strictly increases its utility, thereby leading to a new partition P' . A k -stable
150 partition is simply a partition for which there exists no k -deviation. We write $L(k, G^-)$, resp. $L(k, n)$, for
151 the length of a longest sequence of k -deviations to reach a stable partition in G^- , resp. in any conflict graph
152 with n vertices. Observe that, as we want to maximize the longest sequence of k -deviations, we will always
153 start with the partition consisting of n groups of size 1, that is, $\vec{\Lambda}(P) = (\dots, 0, 0, 0, n)$.

154 We next define a natural vector representation for k -deviations. The difference vector $\vec{\varphi}$ associated with
155 a k -deviation φ from P to P' is equal to $\vec{\varphi} = \vec{\Lambda}(P') - \vec{\Lambda}(P)$. In concluding this section, we define the
156 k -deviations the most used in our proofs and explicit their difference vectors:

- 157 • $\alpha[p, q]$, the 1-deviation where a vertex leaves a group of size $q + 1$ for a group of size $p - 1$ (valid
158 when $p \geq q + 2$). The non-zero coordinates of the associated vector $\vec{\alpha}[p, q]$ are: when $p > q + 2$,
159 $\alpha_p = 1, \alpha_{p-1} = -1, \alpha_{q+1} = -1, \alpha_q = 1$; and for $p = q + 2$ (i.e., $p - 1 = q + 1$), $\alpha_p = 1, \alpha_{p-1} = \alpha_{q+1} =$
160 $-2, \alpha_q = 1$. For any $i \notin \{q, q + 1, p - 1, p\}$, $\alpha_i = 0$. (we omit for ease of reading the brackets $[p, q]$).
- 161 • $\gamma[p]$, the 3-deviation where one vertex in each of 3 groups of size $p - 1$ moves to a group of size $p - 3$
162 to form a new group of size p (valid if there are at least 3 groups of size $p - 1$ and one of size $p - 3$).
163 In that case the coordinates of $\vec{\gamma}[p]$ are: $\gamma_p = 1, \gamma_{p-1} = -3, \gamma_{p-2} = 3, \gamma_{p-3} = -1$, and $\gamma_i = 0$ for any
164 $i \notin \{p - 3, p - 2, p - 1, p\}$.
- 165 • $\delta[p]$, the 4-deviation where one vertex in each of 4 groups of size $p - 1$ moves to a group of size $p - 4$
166 to form a new group of size p (valid if there are at least 4 groups of size $p - 1$ and one of size $p - 4$).
167 In that case the coordinates of $\vec{\delta}[p]$ are: $\delta_p = 1, \delta_{p-1} = -4, \delta_{p-2} = 4, \delta_{p-4} = -1$, and $\delta_i = 0$ for any
168 $i \notin \{p - 4, p - 2, p - 1, p\}$. As an example, the move from the partition of Figure 1(a) to the partition
169 of Figure 1(b), is a 4-deviation with difference vector $\vec{\delta}[4]$.

170 3 Preliminary results

171 In [KL13], the authors prove that there always exists a k -stable partition. We give another proof of this
172 existence which will enable us to obtain a new upper bound. They also proved that it is NP-hard to
173 compute a k -stable partition if k is part of the input (this result was also proved independently in [EGM12]).
174 In contrast, a k -stable partition can be computed in polynomial time for every fixed $k \leq 3$, by using simple
175 *better-response dynamics* [PS08, EGM12, KL13]. The latter results question the role of the value of k in the
176 complexity of computing stable partitions.

177 Formally, a better-response dynamics proceeds as follows. We start from the trivial partition P_1 consisting
178 of n groups with one vertex in each of them. In particular, the partition vector $\vec{\Lambda}(P_1)$ is such that $\lambda_1(P_1) = n$
179 and, for all other $j \neq 1$, $\lambda_j(P_1) = 0$. Provided there exists a k -deviation with respect to the current partition

180 P_i , we pick any one of these k -deviations φ and we obtain a new partition P_{i+1} . If there is no k -deviation
 181 then, the partition P_i is k -stable. An algorithmic presentation is given in Algorithm 1.

Dynamics of the system (Algorithm 1)

Input: a positive integer $k \geq 1$, and a conflict graph G^- .

Output: a k -stable partition for G^- .

- 1: Let P_1 be the partition composed of n singletons groups.
 - 2: Set $i = 1$.
 - 3: **while** there exists a k -deviation for P_i **do**
 - 4: Set $i = i + 1$.
 - 5: Choose one k -deviation and compute the partition P_i after this k -deviation.
 - 6: Return the partition P_i .
-

182 We now prove in Proposition 1 that better-response dynamics can be used for computing a k -stable
 183 partition for every fixed $k \geq 1$ (but not necessarily in polynomial time). It shows that for every fixed $k \geq 1$,
 184 the problem of computing a k -stable partition is in the complexity class PLS (Polynomial Local Search),
 185 that is conjectured to lie strictly between P and NP [JPY88]. Recall that the problem becomes NP-hard
 186 when k is part of the input.

187 **Proposition 1.** *For any $k \geq 1$, for any conflict graph G^- , Algorithm 1 converges to a k -stable partition.*

188 *Proof.* Let P_i, P_{i+1} be two partitions for G^- such that P_{i+1} is obtained from P_i after some k -deviation φ .
 189 Let S be the set of vertices which move ($|S| \leq k$) and let j be the size of the group they join ($j = 0$ if they
 190 create a new group). Then, the new group obtained has size $p = j + |S|$. Note that all the vertices of S
 191 have increased their utilities and so, they belonged in P_i to groups of size $< p$. Therefore, the coordinates
 192 of the difference vector $\vec{\varphi}$ satisfy $\varphi_p = 1$ and $\varphi_j = 0$ for $j > p$. This implies that $\vec{\Lambda}(P_i) <_L \vec{\Lambda}(P_{i+1})$ where
 193 $<_L$ is the lexicographic ordering. Finally, as the number of possible partition vectors is finite, we obtain the
 194 convergence of Algorithm 1. \square

195 The proof of Algorithm 1 implies that $L(k, n)$ (the length of a longest sequence of k -deviations) is finite.
 196 We give here some elementary properties of $L(k, n)$.

197 **Property 2.** *$L(k, n)$ is always attained on the empty conflict graph G^0 of order n .*

198 *Proof.* It suffices to observe that any sequence of k -deviations on a conflict graph G^- is also a sequence in
 199 the empty conflict graph with the same vertices. \square

200 Note that the converse is not true as it can happen that some moves allowed in the empty conflict graph
 201 are not allowed in G^- as they bring two enemies in the same group.

202 **Property 3.** *If $k' > k$, then $L(k', n) \geq L(k, n)$.*

203 *Proof.* We recall that by definition, a k' -deviation consists of a move of at most k' vertices. In particular if
 204 $k' > k$ then, a k -deviation is also a k' -deviation. It implies that a sequence of k -deviations is also a sequence
 205 of k' -deviations. \square

206 In fact we will see after that $L(1, n) = L(2, n) < L(3, n) \ll L(4, n)$. One can note that k' -deviations are
 207 more "powerful" than k -deviations in the sense that, a single k' -deviation might produce the same modifica-
 208 tion as multiple k -deviations. This does not affect the length of a longest sequence needed to reach a stable
 209 partition, but can reduce the length of a shortest sequence. For instance if we consider the empty graph
 210 G^0 of order n and start with the partition consisting of n groups of size 1 then, we need at least $(n - 1)$
 211 1-deviations to reach the stable partition consisting of one group of size n , while we can reach the stable
 212 partition with only $\lceil (n - 1)/k \rceil$ k -deviations.
 213

214 Recall that we can associate with any partition $P = X_1, \dots, X_i, \dots, X_n$ of the vertices the integer
 215 partition $Q = (q_1, q_2, \dots, q_n)$ such that $q_1 \geq q_2 \geq \dots \geq q_n \geq 0$ and $\sum_{i=1}^n q_i = n$ by letting $q_i = |X_i|$.

216 The converse is not true in general; as an example it suffices to consider a partition with $q_1 > \alpha(G^-)$.
 217 However the converse is true when the conflict graph is empty; indeed it suffices to associate with an integer
 218 partition any partition of the vertices obtained by putting in the group X_i a set of q_i vertices .

219 Finally it is worth noting that partition vectors and integer vectors are equivalent representations of
 220 partitions as there is a one-to-one mapping between the partition vector and the integer partition associated
 221 with a partition.

222 We can now use the value p_n of the number of integer partitions (see [HW79]) in order to obtain the
 223 following proposition:

224 **Proposition 4.** $L(k, n) \leq p_n = \Theta((e^\pi \sqrt{\frac{2n}{3}})/n)$.

225 *Proof.* By the proof of Proposition 1, in any sequence of k -deviations the partition vectors of the partitions
 226 obtained are all different. Furthermore as noted above, with two different partition vectors are associated
 227 two different integer partitions. Therefore, $L(k, n)$ is bounded by the number of integer partitions $p(n)$ which
 228 is $\Theta((e^\pi \sqrt{\frac{2n}{3}})/n)$ (see [HW79]). \square

229 Note that this is already far less than 2^n , which was shown to be the best upper bound that one can
 230 obtain for $k \geq 4$ when using an additive potential function [KL13]. Note also that the number of integer
 231 partitions might be much larger than the maximum length of a sequence. Indeed, not all two consecutive
 232 (in the lexicographic ordering) partitions cannot appear in the same sequence. For instance for $n = 10$, if we
 233 have the partition consisting of two groups of size 5 then, we cannot reach with any k -deviation the partition
 234 with one group of size 6 and 4 groups of size 1. In Section 5.1 we give a lower bound $L(4, n) = \Omega(n^{\Theta(\ln(n))})$,
 235 but it still remains a large gap with the upper bound of Proposition 4.

236 4 Analysis for $k \leq 2$

237 In [KL13], the authors proved that for $k \leq 2$, Algorithm 1 converges to a stable partition in at most a
 238 quadratic time. They used the total utility (that is the sum of the utilities) as potential function. Indeed
 239 when performing a 1-deviation $\vec{\alpha}[p, q]$, a vertex moves from a group of size $q + 1$ to a group of size $p - 1$
 240 (with $p \geq q + 2$); the utility of this vertex increases by $p - q - 1$, the utility of the q other vertices of the
 241 group of size $q + 1$ decreases by 1, while the utility of the vertices of the group of size $p - 1$ increases by 1.
 242 Thus, the total utility (potential function) increases by $2p - 2q - 2 \geq 2$ as $p \geq q + 2$.

243 Furthermore, in any partition, the utility of a vertex is at most $n - 1$ and the total utility is at most
 244 $n(n - 1)/2$. Therefore the number of possible 1-deviations is bounded by $n(n - 1)/4$, that is $L(1, n) = \mathcal{O}(n^2)$.
 245 Similarly one can also prove that when performing a 2-deviation the total utility increases by at least 2 (see
 246 also Claim 10). Hence, $L(2, n) = \mathcal{O}(n^2)$.

247 In the next subsections we improve this result as we completely solve this case and give the exact (non-
 248 asymptotic) value of $L(k, n)$ when $k \leq 2$. The gist of the proof is to use a partial ordering that was introduced
 249 in [Bry73], and is sometimes called the dominance ordering.

250 4.1 Exact analysis for $k = 1$ and empty conflict graph

251 In [Bry73] the author has defined an ordering over the integer partitions, sometimes called the dominance
 252 ordering which creates a lattice of integer partitions. This ordering is a direct application of the theory of
 253 majorization to integer partitions [OM16].

254 **Definition 5.** (*dominance ordering*) Given two integer partitions of $n \geq 1$, $Q = (q_1, q_2, \dots, q_n)$ and $Q' =$
 255 $(q'_1, q'_2, \dots, q'_n)$, we say that Q' dominates Q if $\sum_{j=1}^i q'_j \geq \sum_{j=1}^i q_j$, for all $1 \leq i \leq n$.

256 The two next lemmas show that there is a one-to-one mapping between chains in the dominance lattice
 257 and sequences of 1-deviations in the empty conflict graph.

258 **Lemma 6.** Let P be a partition of the vertices and P' be the partition obtained after a 1-deviation φ . Then,
 259 the integer partition $Q' = Q(P')$ dominates $Q = Q(P)$.

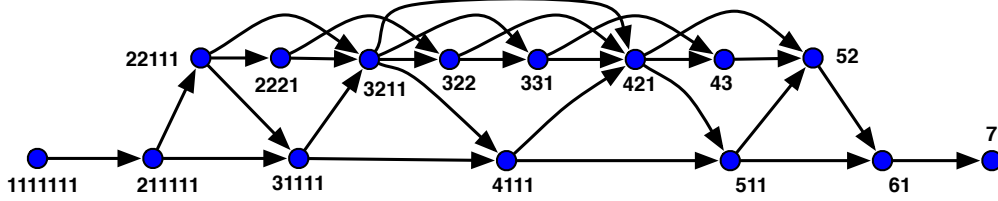


Figure 2: The lattice of integer partitions for $n = 7$, with the arcs associated with 1-deviations.

260 *Proof.* In the 1-deviation φ a vertex v moves from a group X_k to a group X_j with sizes $q_j = |X_j| \geq q_k = |X_k|$.
 261 W.l.o.g. we can assume that the groups (ranked in non-increasing order of size) are ranked in a such a way
 262 that X_j is the first group with size $|X_j|$ and X_k is the last group with size $|X_k|$. Thus, the integer partition
 263 $Q(P)$ associated with P satisfies $q_1 \geq q_2 \dots \geq q_{j-1} > q_j \geq q_{j+1} \geq \dots \geq q_k > q_{k+1} \geq \dots \geq q_n$. After the
 264 move the groups of P' are the same as those of P except we have replaced X_j with the group $X_j \cup \{v\}$ and
 265 X_k with $X_k - v$. Therefore the integer partition Q' associated with P' has the same elements as Q except
 266 $q'_j = q_j + 1$ and $q'_k = q_k - 1$. It implies that Q' dominates Q . Note that this lemma holds for any conflict
 267 graph. \square

268 In the example of Figure 2 (case $n = 7$), we have indicated the arcs between two integer partitions $Q(P)$
 269 and $Q(P')$, where P' is obtained from P after some 1-deviation. We did not write in the figure the integers
 270 equal to 0. As an example, consider the partition P with one group of size 3, one of size 2 and two of size 1.
 271 The integer partition associated with P is $Q(P) = (3, 2, 1, 1, 0, 0, 0)$. Let φ be the 1-deviation where a vertex
 272 in a group of size 1 moves to the other group of size 1. We obtain the partition P' with one group of size
 273 3 and two of size 2. The integer partition associated with P' is $Q(P') = (3, 2, 2, 0, 0, 0, 0)$ which dominates
 274 $Q(P)$. We can also consider the other possible 1-deviations where a vertex in a group of size 1 (resp. 1, 2)
 275 moves to the group of size 2 (resp. 3, 3) getting respectively the integer partitions $(3, 3, 1, 0, 0, 0, 0)$ (resp.
 276 $(4, 2, 1, 0, 0, 0, 0)$, $(4, 1, 1, 1, 0, 0, 0)$).

277 For our next result we need to refine Def. 5, as follows. Let Q, Q' be two integer partitions of $n \geq 1$. If Q
 278 dominates Q' and there is no other integer partition Q'' that simultaneously dominates Q' and is dominated
 279 by Q then, we say that Q covers Q' . For instance in Fig. 2, the integer partition $(3, 1, 1, 1, 1)$ dominates
 280 $(1, 1, 1, 1, 1, 1)$, but it does not cover this integer partition. Indeed, the integer partition $(2, 1, 1, 1, 1, 1)$
 281 dominates $(1, 1, 1, 1, 1, 1)$ and it is also dominated by $(3, 1, 1, 1, 1)$. However since no such integer partition
 282 exists “between” $(2, 1, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 1, 1)$, we have that $(2, 1, 1, 1, 1, 1)$ covers $(1, 1, 1, 1, 1, 1)$.
 283 Brylawski proved a nice characterization of the covering relation, that we will use in our next proof.

284 **Lemma 7.** Let G^0 be the empty conflict graph and let Q, Q' be two integer partitions of $n = |V|$ such that
 285 Q' dominates Q . For any partition P associated with Q , there exists another partition P' associated with Q'
 286 such that P' is obtained from P by doing a sequence of 1-deviations.

287 *Proof.* As proved in [Bry73], we have that if Q' dominates Q then there is a finite sequence of integer
 288 partitions $Q^0, \dots, Q^r, \dots, Q^s$, with $Q = Q^0$ and $Q' = Q^s$ such that for each $0 \leq r < s$, Q^{r+1} dominates Q^r
 289 and differs from it only in two elements j_r and k_r with $q_{j_r}^{r+1} = q_{j_r}^r + 1$ and $q_{k_r}^{r+1} = q_{k_r}^r - 1$. In fact as proved
 290 in [Bry73], this is equivalent to have Q^{r+1} that covers Q^r .

291 The proof is now by induction on r , starting from any partition $P^0 = P$ associated with Q . For $r > 0$,
 292 we consider the partition P^r associated with Q^r . Recall that Q^r and Q^{r+1} only differ in the two groups X_{j_r}
 293 and X_{k_r} . As $q_{j_r}^{r+1} = q_{j_r}^r + 1$ and $q_{k_r}^{r+1} = q_{k_r}^r - 1$, P^{r+1} can be obtained from P^r by moving a vertex from X_{k_r}
 294 to X_{j_r} . This move is valid as the conflict graph is empty. (Note that the lemma is not valid for a general
 295 conflict graph.) \square

296 As an example, consider the two integer partitions $Q = (2, 2, 2, 1, 0, 0, 0)$ and $Q' = (5, 1, 1, 0, 0, 0, 0)$ where
 297 Q' dominates Q . A possible sequence of integer partitions satisfying the conditions of the proof is $Q^0 = Q$,
 298 $Q^1 = (3, 2, 1, 1, 0, 0, 0)$, $Q^2 = (4, 1, 1, 1, 0, 0, 0)$, $Q^3 = Q' = (5, 1, 1, 0, 0, 0, 0)$. Partition P^1 is obtained from
 299 P^0 by moving a vertex in a group of size 2 to another group of size 2. Then, P^2 is obtained by moving a
 300 vertex of the group of size 2 to the group of size 3 and P' is obtained from P^2 by moving a vertex of one
 301 group of size 1 to that of size 4.

302 In summary we conclude that a sequence of 1-deviations with an empty conflict graph corresponds to
303 a chain of integer partitions, and vice versa. Therefore, by Property 2, the length of a longest sequence of
304 1-deviations with an empty conflict graph is the same as the length of a longest chain in the dominance
305 lattice of integer partitions. Since it has been proven in [GK86] that for $n = \frac{m(m+1)}{2} + r$, the longest chain
306 in the Dominance Lattice has length $2\binom{m+1}{3} + mr$, we obtain the exact value for $L(1, n)$.

307 **Theorem 8.** *Let m and r be the unique non-negative integers such that $n = \frac{m(m+1)}{2} + r$, and $0 \leq r \leq m$.
308 Then, $L(1, n) = 2\binom{m+1}{3} + mr$.*

309 In the example for $n = 7$, we have $m = 3$, $r = 1$ and so, $L(1, 7) = 11$. Such a longest sequence is
310 $(1, 1, 1, 1, 1, 1, 1)$, $(2, 1, 1, 1, 1, 1, 0)$, $(2, 2, 1, 1, 1, 0, 0)$, then the sequence in the upper part of Figure 2 until
311 $(5, 2, 0, 0, 0, 0, 0)$, then $(6, 1, 0, 0, 0, 0, 0)$ and $(7, 0, 0, 0, 0, 0, 0)$.

312 We note that the proof in [GK86] is not straightforward. One can think that the longest chain is ob-
313 tained by taking among the possible 1-deviations the one which leads to the smallest integer partition in
314 the lexicographic ordering, like for $n = 7$. Unfortunately this is not true. Indeed let $n = 9$. After 7 steps
315 we get the integer partition $(3, 3, 2, 1, 0, 0, 0, 0, 0)$. Then, by choosing the 1-deviation that gives the small-
316 est integer partition (in the lexicographic ordering), we get the integer partition $(3, 3, 3, 0, 0, 0, 0, 0, 0)$ and then
317 $(4, 3, 2, 0, 0, 0, 0, 0, 0)$. But there is a longer chain of length 3 from $(3, 3, 2, 1, 0, 0, 0, 0, 0)$ to $(4, 3, 2, 0, 0, 0, 0, 0, 0)$,
318 namely, $(4, 2, 2, 1, 0, 0, 0, 0, 0)$, $(4, 3, 1, 1, 0, 0, 0, 0, 0)$, $(4, 3, 2, 0, 0, 0, 0, 0, 0)$. However the proof in [GK86] im-
319 plies that the following simple construction works for any n .

320 **Proposition 9.** *A longest sequence of 1-deviations in the empty conflict graph is obtained by choosing, at
321 a given step, among all the possible 1-deviations, any one of which leads to the smallest increase of the
322 total utility.*

323 *Proof.* We need to introduce the terminology of [GK86]. Note that they consider their chains starting from
324 the end, and so, we need to reverse the steps in their construction in order to make them correspond to
325 1-deviations.

- 326 • A V -step is corresponding to a user leaving her group of size $p-1$ for another group of size $p-1$, thereby
327 increasing her utility from $p-2$ to $p-1$. In other words, the deviation vector of such 1-deviation is
328 $\vec{\alpha}[p, p-2]$ for some p .
- 329 • An H -step is corresponding to a user leaving her group X_i of size $q+1$ for another group X_j of size
330 $p-1 \geq q+1$, but only if there is no other group of size between $q+1$ and $p-1$; in particular, if groups
331 are ordered by non-increasing size, this means that $j = i-1$. Furthermore, note that an H -step can
332 also be a V -step.

333 The relationship between V -steps, H -steps and our construction is as follows. At every 1-deviation, the
334 total utility has to increase by at least two, and this is attained if and only if the deviation vector is $\vec{\alpha}[p, p-2]$
335 for some p ; equivalently, this move is corresponding to a V -step.

336 Furthermore, if no such a move is possible, then we claim that any 1-deviation $\vec{\alpha}[p, q]$ that minimizes the
337 increase of the total utility is corresponding to an H -step. Indeed, we have $q+1 < p-1$ and, more generally,
338 all the groups have pairwise different sizes (otherwise, a V -step would have been possible). Furthermore, if
339 there were a group of size s , with $q+1 < s < p-1$, then this would contradict our assumption that the
340 1-deviation $\vec{\alpha}[p, q]$ minimizes the increase of the total utility.

341 As proved, *e.g.*, by Brylawski [Bry73], starting from any integer partition with at least two summands
342 (*i.e.*, $q_n \neq 1$), it is always possible to perform one of the two types of move defined above. These two types
343 of move actually correspond to the two cases when an integer partition can cover another one (see also the
344 proof of Lemma 7 for a different formulation).

345 Therefore, our strategy leads to a sequence of 1-deviations where all the moves correspond to either a
346 V -step or (only if the first type of move is not possible) to an H -step. By a commutativity argument (Lemma
347 3 in [GK86]) it can be proved that as soon as no move $\vec{\alpha}[p, p-2]$ (corresponding to V -steps) is possible for
348 any p , every ulterior move of this type will correspond to both a V -step and an H -step simultaneously (*i.e.*,
349 the deviation vector will be $\vec{\alpha}[p, p-2]$ for some p , and there will be no other groups of size $p-1$ than the two
350 groups involved in the 1-deviation). Therefore, the sequences we obtain are corresponding to a particular

351 case of the so-called *HV*-chains in [GK86]. Finally, the main result in [GK86] is that every *HV*-chain is of
 352 maximum length. \square

353 In what follows, we will reuse part of the construction in [GK86] for proving Theorem 12.

354 4.2 Analysis for $k = 2$

355 Interestingly we will prove that any 2-deviation can be replaced either by one or two 1-deviations. This will
 356 allow us to prove in Theorem 11 that $L(2, n) = L(1, n)$.

357 **Claim 10.** *If the conflict graph G^- is empty, then any 2-deviation can be replaced either by one or two*
 358 *1-deviations*

359 Proof. Consider a 2-deviation which is not a 1-deviation. In that case, two vertices u_i and u_j leave their
 360 respective group X_i and X_j (which can be the same) to join a group X_k . Let $|X_i| \geq |X_j|$; in order for the
 361 utility of the vertices to increase, we should have $|X_k| \geq |X_i| - 1 (\geq |X_j| - 1)$.

- 362 • Case 1: $|X_k| \geq |X_j|$. In that case the 2-deviation can be replaced by a sequence of two 1-deviations
 363 where firstly a vertex u_j leaves X_j to join X_k and then a vertex u_i leaves X_i to join the group $X_k \cup \{u_j\}$
 364 whose size is now at least that of X_i .
- 365 • Case 2: $|X_k| = |X_i| - 1 = |X_j| - 1 = p - 2$ and $X_i = X_j$. In that case, the effect of the 2-deviation
 366 is to replace the group X_i of size $p - 1$ with a group of size $p - 3$ and to replace the group X_k of
 367 size $p - 2$ with a group of size p . The difference vector $\vec{\varphi}$ associated with the 2-deviation has as
 368 non-null coordinates $\varphi_p = 1, \varphi_{p-1} = -1, \varphi_{p-2} = -1, \varphi_{p-3} = 1$. We obtain the same effect by doing
 369 the 1-deviation $\vec{\alpha}[p, p - 3]$ where a vertex leaves X_k to join X_i .
- 370 • Case 3: $|X_k| = |X_i| - 1 = |X_j| - 1 = p - 2$ and $X_i \neq X_j$. In that case, the effect of the 2-deviation is
 371 to replace the 2 groups X_i and X_j of size $p - 1$ with two groups of size $p - 2$ and to replace the group
 372 X_k of size $p - 2$ with a group of size p . The difference vector $\vec{\varphi}$ associated with the 2-deviation has as
 373 non-null coordinates $\varphi_p = 1, \varphi_{p-1} = -2, \varphi_{p-2} = 1$. We obtain the same effect by doing the 1-deviation
 374 $\vec{\alpha}[p, p - 2]$ where a vertex leaves X_j to join X_i .

375 Note that the fact that G^- is empty is needed for the proof. Indeed, in case 2 it might happen that all the
 376 vertices of X_k have some enemy in X_i and so, the 1-deviation we describe is not valid. Similarly, in case 3,
 377 it might happen that all the vertices of X_i have some enemy in X_j and so, the 1-deviation we describe is
 378 not valid. \diamond

379 **Theorem 11.** $L(2, n) = L(1, n)$.

380 *Proof.* Clearly, $L(2, n) \geq L(1, n)$ as any 1-deviation is also a 2-deviation. By Property 2, the value of $L(2, n)$
 381 is obtained when the conflict graph G^- is empty. In that case, Claim 10 implies that $L(2, n) \leq L(1, n)$. \square

382 4.3 Analysis for $k \leq 2$ and a general conflict graph

383 By using the potential function introduced at the beginning of this section, Panagopoulou and Spirakis
 384 ([PS08]) proved that for every conflict graph G^- with independence number $\alpha(G^-)$, the convergence of the
 385 dynamics is in $\mathcal{O}(n\alpha(G^-))$. Indeed as we have seen each 1-deviation increases the total utility by at least 2.
 386 But the total utility of a stable partition is at most $n(\alpha(G^-) - 1)$ as the groups have maximum size $\alpha(G^-)$.
 387 If the conflict graph is empty, we have seen that $L(1, n) = \Theta(n^{3/2})$, that is, in that case $\mathcal{O}(n\sqrt{\alpha(G^-)})$. This
 388 led one of us ([Duc16b], page 131) to conjecture that in the case of 1-deviations the worst time of convergence
 389 of the dynamics is in $\mathcal{O}(n\sqrt{\alpha(G^-)})$. We disprove the conjecture by proving the following theorem:

390 **Theorem 12.** *For $n = \binom{m+1}{2}$, there exists a conflict graph G^- with $\alpha(G^-) = m = \Theta(\sqrt{n})$ and a sequence*
 391 *of $\binom{m+1}{3}$ valid 1-deviations, that is, a sequence of $\Omega(n^{\frac{3}{2}}) = \Omega(n\alpha(G^-))$ 1-deviations.*

392 *Proof.* The reader can follow the proof on the example given after for $m = 4$ and on Figure 3. We will use
393 part of the construction of Greene and Kleitman ([GK86]). Namely, they prove that, if $n = \binom{m+1}{2}$, there
394 is a sequence of $\binom{m+1}{3}$ 1-deviations transforming the partition P_1 consisting of n groups each of size 1 (the
395 coordinates of $\vec{\lambda}(P_1)$ satisfy $\lambda_1 = n$) into the partition P_m consisting of m groups, one of each possible size
396 s for $1 \leq s \leq m$ (the coordinates of $\vec{\lambda}(P_m)$ satisfy $\lambda_s = 1$ for $1 \leq s \leq m$). Furthermore they prove that the
397 moves used are V -steps (see the proof of Proposition 9) which are nothing else than $\vec{\alpha}[p, p-2]$ for some p
398 (one vertex leaves a group of size $p-1$ to join a group of the same size $p-1$). One can note that in such a
399 move the utility increases exactly by 2, and as the total utility of P_m is $\sum_{i=1}^m i(i-1) = (m+1)m(m-1)/3$,
400 the number of moves is exactly $(m+1)m(m-1)/6$.

401 The conflict graph of the counterexample consists of m complete graphs $K^j, 1 \leq j \leq m$, where K^j has
402 exactly j vertices. An independent set is therefore formed by taking at most one vertex in each K^j and
403 $\alpha(G^-) = m$. We will denote the elements of K^j by $\{x_i^j\}$ with $1 \leq i \leq j \leq m$. The group of P_m of size s will
404 be $X_s[m] = \bigcup x_{m+1-s}^j$ with $m+1-s \leq j \leq m$. Thus, these groups are independent sets.

405 The idea of the proof is to create the groups of P_m by moving in successive phases $p, 1 \leq p \leq m-1$,
406 the vertices x_i^{p+1} starting from x_1^{p+1} until x_p^{p+1} . More precisely, recall that $n = m(m+1)/2$. For each
407 $p, 1 \leq p \leq m$, let us denote by P_p the partition consisting of 1 group of each size s for $1 \leq s \leq p$, namely
408 $X_s[p] = \bigcup x_{p+1-s}^j$ with $p+1-s \leq j \leq p$, (these groups form a partition of the vertices x_i^j with $1 \leq i \leq j \leq p$)
409 plus $n - p(p+1)/2$ groups of size 1 (corresponding to the vertices x_i^j with $1 \leq i \leq j$ and $p+1 \leq j \leq m$). We
410 will show in Claim 13 that we can transform the partition P_p into P_{p+1} with $p(p+1)/2$ valid 1-deviations.

411 **Claim 13.** *For any $1 \leq p \leq m-1$, there exists a sequence of $p(p+1)/2$ valid 1-deviations which transform
412 the partition P_p into P_{p+1} .*

413 Proof. (See example given after the proof.) In a first subphase, we move the vertex x_1^{p+1} successively to
414 the group $X_1[p]$, then to $X_2[p]$ and so on until $X_p[p]$. After these p 1-deviations we have created the group
415 $X_{p+1}[p+1] = X_p[p] \cup \{x_1^{p+1}\} = \bigcup x_1^j$ for $1 \leq j \leq p+1$. In the general subphase $1 \leq h \leq p$, we move via
416 $(p+1-h)$ 1-deviations the vertex x_h^{p+1} successively to the group $X_1[p]$, until $X_{p+1-h}[p]$, therefore creating
417 the group $X_{p+2-h}[p+1] = \bigcup x_h^j$ for $h \leq j \leq p+1$. To conclude the proof we have to note that all these
418 1-deviations are valid, as we move a vertex from K^{p+1} successively to groups containing vertices from $\bigcup K^j$,
419 for $1 \leq j \leq p$ (and there is no edge between this vertex and the vertices of these groups). \diamond

420 Thus, by applying $m-1$ times the previous claim, we can transform the partition P_1 into P_m with
421 $\sum_{p=1}^{m-1} p(p+1)/2 = \binom{m+1}{3}$ 1-deviations, therefore proving the theorem. \square

422 **Example for $m = 4$.** (See Figure 3.)

423 Partition P_1 consists of the group $X_1[1] = \{x_1^1\}$. Following the proof of Claim 13 for $p = 1$, the partition
424 P_2 is obtained by moving x_1^2 to $X_1[1]$, thereby creating the group $X_2[2] = \{x_1^1, x_1^2\}$ (the group of size 1 of P_2
425 being $X_1[2] = \{x_2^2\}$). By the proof of Claim 13 for $p = 2$, the partition P_3 is obtained by doing the 3 following
426 1-deviations: first we move x_1^3 to $X_1[2]$ and then $X_2[2]$, thereby creating the group $X_3[3] = \{x_1^1, x_1^2, x_1^3\}$; then
427 we move x_2^3 to $X_1[2]$ for creating the group $X_2[3] = \{x_2^2, x_2^3\}$ (the group of size 1 of P_3 being $X_1[3] = \{x_3^3\}$).
428 Similarly, the partition P_4 is obtained by doing the following 6 1-deviations: first we move x_1^4 to $X_1[3]$,
429 then $X_2[3]$ and $X_3[3]$ for creating the group $X_4[4] = \{x_1^1, x_1^2, x_1^3, x_1^4\}$; then we move x_2^4 to $X_1[3]$ and then
430 $X_2[3]$ for creating the group $X_3[4] = \{x_2^2, x_2^3, x_2^4\}$, and finally we move x_3^4 to $X_1[3]$ for creating the group
431 $X_2[4] = \{x_3^3, x_3^4\}$ (the group of size 1 of P_4 being $X_1[4] = \{x_4^4\}$).

432 We can prove a theorem analogous to Theorem 12 when $\alpha(G^-) = \mathcal{O}(\sqrt{n})$.

434 **Theorem 14.** *For any $\alpha = \mathcal{O}(\sqrt{n})$, there exists a conflict graph G^- with n vertices and independence
435 number $\alpha(G^-) = \alpha$ and a sequence of at least $\Omega(n\alpha)$ 1-deviations to reach a stable partition.*

436 *Proof.* Let G_0^- be the graph of Theorem 12 for $m = \alpha$. G_0^- has $n_0 = \mathcal{O}(\alpha^2)$ vertices, independence number
437 α , and furthermore there exists a sequence of $\Theta(\alpha^3)$ valid 1-deviations for G_0^- . Let G^- be the graph obtained
438 by taking the complete join of $k = n/n_0$ copies of G_0^- (i.e., we add all possible edges between every two
439 copies of G_0^-). By construction, G^- has order $n = kn_0 = \mathcal{O}(k\alpha^2)$ and the same independence number α as
440 G_0^- . Furthermore, there exists a sequence of $k\Theta(\alpha^3) = \Omega(n\alpha)$ valid 1-deviations for G^- . \square

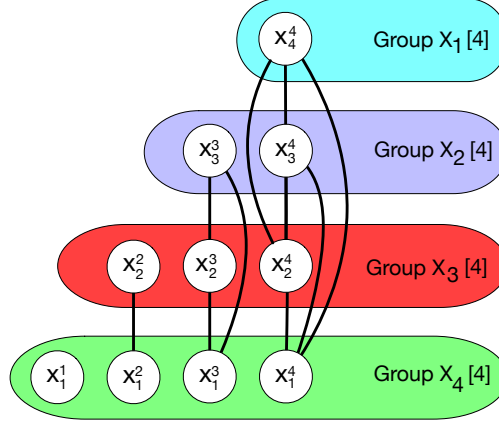


Figure 3: Illustration for Example 4.

441 Note that in any 2-deviation the total utility increases by at least 2. Hence, the number of 2 deviations is
 442 also at most $\mathcal{O}(n\alpha(G^-))$ when the conflict graph has independence number $\alpha(G^-)$. This bound is attained
 443 by using only 1-deviations as proved in Theorem 14, which is also valid for $k = 2$.

444 5 Lower bounds for $k > 2$

445 The classical dominance ordering does not suffice to describe all k -deviations as soon as $k \geq 3$. Indeed
 446 Lemma 6 is no more valid. Let P be a partition of the vertices and P' be the partition obtained after a
 447 k -deviation φ . Then, if $k \geq 3$ the integer partition $Q' = Q(P')$ does not necessarily dominate $Q = Q(P)$.
 448 Let us give an example. As noted before, there is only one k -stable partition P_{max} in the empty conflict
 449 graph G^\emptyset , namely, the one consisting of one group of size n , with integer partition $Q_{max} = (n, 0, \dots, 0)$ and
 450 partition vector $(1, 0, \dots, 0)$. Let $d(Q)$ be the length of a longest sequence in the dominance lattice from the
 451 integer partition Q to the integer partition Q_{max} . For $k = 4$, let P be the partition consisting of 4 groups of
 452 size 4 and one group of size 1. We have $Q(P) = Q = (4, 4, 4, 4, 1)$ (we omit the 0's). Apply the 4-deviation
 453 $\delta[5]$ where one vertex of each group of size 4 joins the group of size 1; it leads to the partition P' with
 454 integer partition $Q' = (5, 3, 3, 3, 3)$. Q is uniquely covered in the dominance lattice by the integer partition
 455 $(5, 4, 4, 3, 1)$ while Q' is at distance 3 from it via $(5, 4, 3, 3, 2)$ and $(5, 4, 4, 2, 2)$ and so, $d(Q') \geq d(Q) + 2$.
 456 Furthermore we note that the total utility of P is 48 while that of P' is 44, a smaller value.

457 Prior to our work, it was known that $L(3, n) = \mathcal{O}(n^3)$ [KL13]. But it was unknown whether $L(3, n) >$
 458 $L(2, n)$, and in fact it was conjectured in [EGM12] that both values are equal. It was also conjectured
 459 in [KL13] that $L(k, n)$ was polynomial in n for k fixed. We disprove this conjecture and prove in Theorem 15
 460 that 4-deviations are responsible for a sudden complexity increase. Indeed, we prove that no polynomial
 461 bounds exist for $L(4, n)$ (Theorem 15).

462 **Theorem 15.** $L(4, n) = \Omega(n^{\Theta(\ln(n))})$.

463 The tools developed in the proof of Theorem 15 can also be used in the case $k = 3$ in order to prove for
 464 the first time that deviations can delay convergence and that the gap between $k = 2$ and $k = 3$ obtained
 465 from the potential function method is indeed justified (Theorem 16).

466 **Theorem 16.** $L(3, n) = \Omega(n^2)$.

467 The main idea of the proofs of these theorems is to do repeated ordered sequences (called cascades) of
 468 deviations similar to the one given in the example above (all these sequences are obtained by using a similar
 469 procedure, but with a first group of different size).

470 We give the proof of Theorem 16 in Section 5.2. In the following, we give the proof of Theorem 15 for
 471 $k = 4$. We use ordered sequences (cascades) of 4-deviations $\delta[p]$ (defined in the introduction) and various
 472 additional tricks such as the repetition of the process by using cascades of cascades. For completeness, we

473 recall here the definition of $\delta[p]$.

474

475 **Definition of $\delta[p]$:** Consider a partition P containing at least 4 groups of size $p - 1$ and 1 group of size
 476 $p - 4$. In the 4-deviation $\delta[p]$ one vertex in each of the 4 groups of size $p - 1$ moves to the group of size $p - 4$
 477 to form a new group of size p .

478 The example given at the beginning of this section corresponds to the case $p = 5$ and the one given in
 479 Figure 1 to the case $p = 4$. Our motivation for using $\delta[p]$ as a basic building block for our construction is
 480 that it is the only type of 4-deviation which decreases the total utility.

481 Figure 4 gives a visual description of the cascades used in the proof. The proof uses a value t which will
 482 be chosen at the end of the proof. In the figure, $t = 16$. Here we first do an ordered sequence of t translated
 483 4-deviations, with associated vectors $\vec{\delta}[p], \vec{\delta}[p - 1], \dots, \vec{\delta}[p - t + 1]$ represented by black rectangles. The
 484 cascade obtained, called $\vec{\delta}^1[p, t]$, is represented by the red big rectangle. Then we do an ordered sequence
 485 of $(t - 2)$ such cascades, i.e. $\vec{\delta}^1[p, t], \vec{\delta}^1[p, t - 1], \dots, \vec{\delta}^1[p, t - 3]$, that are represented by red rectangles.
 486 In doing so we get the cascade $\vec{\delta}^2[p, t - 2]$ which contains $16 \times 14 = 224$ 4-deviations. We apply some 1-
 487 deviations represented in white to get a deviation called $\vec{\zeta}^2[p]$, represented by the big yellow rectangle, which
 488 will have the so-called Nice Property (Definition 24) (few non-zero coordinates and symmetry properties)
 489 enabling us to do recursive constructions.

490 We repeat the process by doing ordered sequences of this new cascade $\vec{\zeta}^2$, but with translations of 2
 491 instead of 1. More precisely we do successively the 7 cascades $\vec{\zeta}^2[p], \vec{\zeta}^2[p - 2], \vec{\zeta}^2[p - 4], \dots, \vec{\zeta}^2[p - 12]$
 492 represented by yellow rectangles. After some additional 1-deviations (represented in white) we get the cascade
 493 called $\vec{\zeta}^3[p]$, that is represented by the big blue rectangle; this cascade still has the Nice property. Then, we
 494 do a cascade of 5 of these cascades, but now with a translation of 3, namely $\vec{\zeta}^3[p], \vec{\zeta}^3[p - 3], \dots, \vec{\zeta}^3[p - 15]$
 495 (represented by blue rectangles). After some additional 1-deviations (represented in white) we get the
 496 cascade called $\vec{\zeta}^4[p]$ (represented by the big green rectangle) that still has the Nice property. Finally the
 497 figure represent the ordered sequence of 3 of these cascades (with a translation of 5), namely $\vec{\zeta}^4[p], \vec{\zeta}^4[p -$
 498 $5], \vec{\zeta}^4[p - 10]$, that are represented by green rectangles. After some 1-deviations (represented in white) we
 499 get the bigger cascade called $\vec{\zeta}^5[p]$ (represented by the turquoise rectangle). We continue the construction
 500 with a sequence of 3 of them (with a translation of 8), namely $\vec{\zeta}^5[p], \vec{\zeta}^5[p - 8], \vec{\zeta}^5[p - 16]$, plus some 1-
 501 deviations in order to get a cascade $\vec{\zeta}^6[p]$ (represented by the big grey rectangle). The reader has to realize
 502 that, in this example, $\vec{\zeta}^6[p]$ contains 3 cascades $\vec{\zeta}^5[p]$, each containing 3 cascades $\vec{\zeta}^4[p]$, each containing
 503 5 cascades $\vec{\zeta}^3[p]$, each containing 7 cascades $\vec{\zeta}^2[p]$. Altogether the cascade $\vec{\zeta}^6[p]$ of this example contains
 504 $3 \times 3 \times 5 \times 7 \times 224 = 70560$ 4-deviations $\delta[p]$ plus many 1-deviations.

505 5.1 Case $k = 4$. Proof of Theorem 15

506 Note that after the 4-deviation $\delta[p]$ we get a new partition P' with one more group of size p , 4 less groups of
 507 size $p - 1$, 4 more groups of size $p - 2$, and one less group of size $p - 4$. This is expressed by the coordinates
 508 of the associated difference vector (where we omit the bracket $[p]$ for ease of reading).

509 **Difference vector $\vec{\delta}[p]$:** The difference vector $\vec{\delta}[p]$ has the following coordinates: $\delta_p = 1, \delta_{p-1} =$
 510 $-4, \delta_{p-2} = 4, \delta_{p-4} = -1$, and $\delta_j = 0$ for all other values of j . See Table 2.

...	δ_p	δ_{p-1}	δ_{p-2}	δ_{p-3}	δ_{p-4}	...
...0	1	-4	4	0	-1	0...

Table 2: Difference vector of $\delta[p]$.

511 **The cascade $\vec{\delta}^1[p, t]$:** We first do a cascade consisting of an ordered sequence of t translated 4-deviations
 512 $\delta[p], \delta[p - 1], \dots, \delta[p - t + 1]$, for some parameter $t \geq 4$, that will be chosen later to give the maximum number
 513 of 4-deviations.

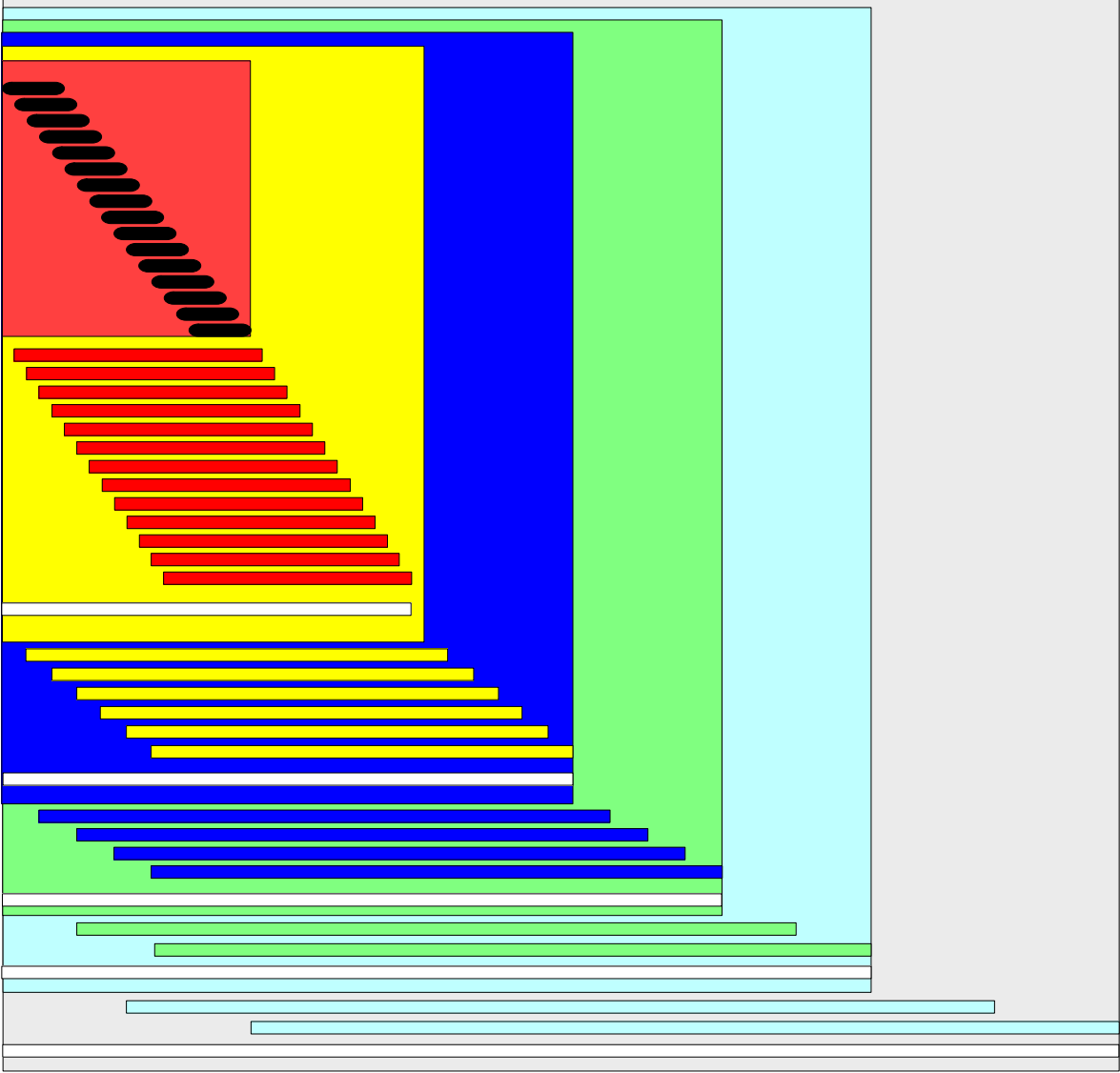


Figure 4: Cascades of cascades.

514 In what follows we represented a cascade Φ by the vector $\vec{\Phi}$ (as we are mainly interested in the sequence
515 obtained) but we implicitly consider Φ as an ordered set of deviations which corresponding vector $\vec{\Phi}$ is equal
516 to the sum of the corresponding vectors of the deviations.

517 The reader can follow the construction in Table 3 with $t = 7$. The coordinates of $\vec{\delta}^1[p, t]$ are given in
518 Claim 17 and Table 4. We note that there are a lot of cancellations and only 8 non-zero coordinates. Indeed
519 consider the groups of size $p - i$ for $4 \leq i \leq t - 1$; we have deleted such a group when doing the 4-deviation
520 $\vec{\delta}^1[p + 4 - i]$, then created 4 such groups with $\vec{\delta}^1[p + 2 - i]$, then deleted 4 such groups with $\vec{\delta}^1[p + 1 - i]$, and
521 finally created one with $\vec{\delta}^1[p - i]$. The reader can follow these cancellations in Table 3 for $i = 4, 5, 6$. The
522 variation of the number of groups of a given size $p - i$ (which corresponds to the coordinate δ_{p-i}^1) is obtained
523 by summing the coefficients appearing in the corresponding column and is 0 for $p - 4, p - 5, p - 6$.

524 **Claim 17.** For $4 \leq t \leq p - 3$, the coordinates of the cascade $\vec{\delta}^1[p, t] = \sum_{i=0}^{t-1} \vec{\delta}^1[p - i]$ satisfy: $\delta_p^1 = 1$,
525 $\delta_{p-1}^1 = -3$, $\delta_{p-2}^1 = 1$, $\delta_{p-3}^1 = 1$, $\delta_{p-t}^1 = -1$, $\delta_{p-t-1}^1 = 3$, $\delta_{p-t-2}^1 = -1$, $\delta_{p-t-3}^1 = -1$, and $\delta_j^1 = 0$ for all the
526 other values of j (see Table 4).

	...	p	$p-1$	$p-2$	$p-3$	$p-4$	$p-5$	$p-6$	$p-7$	$p-8$	$p-9$	$p-10$...
$\vec{\delta}[p]$...0	1	-4	4	0	-1	0	0	0	0	0	0	0...
$+ \vec{\delta}[p-1]$...0	0	1	-4	4	0	-1	0	0	0	0	0	0...
$+ \vec{\delta}[p-2]$...0	0	0	1	-4	4	0	-1	0	0	0	0	0...
$+ \vec{\delta}[p-3]$...0	0	0	0	1	-4	4	0	-1	0	0	0	0...
$+ \vec{\delta}[p-4]$...0	0	0	0	0	1	-4	4	0	-1	0	0	0...
$+ \vec{\delta}[p-5]$...0	0	0	0	0	0	1	-4	4	0	-1	0	0...
$+ \vec{\delta}[p-6]$...0	0	0	0	0	0	0	1	-4	4	0	-1	0...
$= \vec{\delta}^1[p, 7]$...0	1	-3	1	1	0	0	0	-1	3	-1	-1	0...

Table 3: Computation of $\delta^1[p, 7]$.

...	δ_p^1	δ_{p-1}^1	δ_{p-2}^1	δ_{p-3}^1	...	δ_{p-t}^1	δ_{p-t-1}^1	δ_{p-t-2}^1	δ_{p-t-3}^1	...
...0	1	-3	1	1	0...0	-1	3	-1	-1	0...

Table 4: Difference vector $\delta^1[p, t]$.

Proof. We have $\delta_j^1 = \sum_{i=0}^{t-1} \delta_j[p-i]$. For a given j , $\delta_j[p-i] = 0$ except for the following values of i such that $0 \leq i \leq t-1$: $i = p-j$ where $\delta_j[j] = 1$; $i = p-j-1$ where $\delta_j[j+1] = -4$; $i = p-j-2$ where $\delta_j[j+2] = 4$; $i = p-j-4$ where $\delta_j[j+4] = -1$ (in the table it corresponds to the non-zero values in a column, whose number is at most 4). Therefore, for $j > p$: $\delta_j^1 = 0$; $\delta_p^1 = 1$; $\delta_{p-1}^1 = -4 + 1 = -3$ (since $t \geq 2$); $\delta_{p-2}^1 = 4 - 4 + 1 = 1$ (since $t \geq 3$); $\delta_{p-3}^1 = 0 + 4 - 4 + 1 = 1$ (since $t \geq 4$); for $p-4 \geq j \geq p-t+1$, $\delta_{p-j}^1 = -1 + 0 + 4 - 4 + 1 = 0$; $\delta_{p-t}^1 = -1 + 0 + 4 - 4 = -1$ (since $t \geq 4$ and $t \leq p$); $\delta_{p-t-1}^1 = -1 + 0 + 4 = 3$ (since $t \geq 3$ and $t \leq p-1$); $\delta_{p-t-2}^1 = -1 + 0 = -1$ (since $t \geq 2$ and $t \leq p-2$); $\delta_{p-t-3}^1 = -1$ (since $t \geq 1$ and $t < p-3$) and, for $j \leq p-t-4$, $\delta_j^1 = 0$. \diamond

Validity of the cascades. We have to see when the cascades are valid, that is, to determine how many groups we need at the beginning. For the cascade $\vec{\delta}^1[p, t]$ we note that the coordinates of any subsequence of the cascade, *i.e.*, the coordinates of some $\vec{\delta}^1[p, r]$, are all at least -1 except δ_{p-1}^1 : which is -4 when $r = 1$ and then -3 . Therefore such a cascade is valid as soon as we have at least 4 groups of size $p-1$ and one group of each other size $p-i$ ($4 \leq i \leq t+3$). Let us now introduce the notion of an h -balanced sequence to deal with the general validity of cascades.

Definition 18. Let h be a positive integer and let $\vec{\Phi} = \sum_{j=1}^s \vec{\varphi}^j$ be a cascade consisting of s ordered k -deviations. We call this cascade h -balanced if, for any $1 \leq r \leq s$, the sum of the r first vectors, namely, $\sum_{j=1}^r \vec{\varphi}^j$, has all its coordinates greater than or equal to $-h$.

The interest in this notion lies in the following Property 19:

Property 19. Let p_{\max} be the largest index j that satisfies $\vec{\Phi}_j \neq 0$. If we start from a partition with at least h groups of each size j such that $1 \leq j < p_{\max}$, then an h -balanced sequence is valid.

Note that the p_{\max} coordinate is positive as it corresponds to a creation of a group and so we do not need the existence of groups of size p_{\max} . For example, the cascade $\vec{\delta}^1[p, t]$ described before is 4-balanced. Thus, if we have 4 groups of each size j , $1 \leq j < p$, then the sequence is valid. We have seen above that in fact we do need so many groups for a size $\leq p-2$; but the use of h -balanced sequences will simplify the computations and enable to get recursive properties.

Note that a sequence is itself composed of subsequences, and the following lemma will be useful to bound the value h of a sequence.

Lemma 20. Let $\vec{\Phi}^1$ be an h_1 -balanced sequence and $\vec{\Phi}^2$ be an h_2 -balanced sequence. Then, $\vec{\Phi}^1 + \vec{\Phi}^2$ is a $(\max\{h_1, h_2 - \min_i \Phi_i^1\})$ -balanced sequence.

557 Proof. As $\vec{\Phi}^1$ is h_1 -balanced, the coordinates of any subsequence of $\vec{\Phi}^1$ are greater than or equal to $-h_1$.
558 Consider a subsequence $\vec{\Phi}^1 + \vec{\Phi}^3$ where $\vec{\Phi}^3$ is a subsequence of $\vec{\Phi}^2$. The j th coordinate is $\Phi_j^1 + \Phi_j^3$; by
559 definition $\Phi_j^3 \geq -h_2$ and so, $\Phi_j^1 + \Phi_j^3 \geq \Phi_j^1 - h_2 \geq \min_i \Phi_i^1 - h_2$. \diamond

560 **The cascade $\vec{\delta}^2[p, t-2]$:** We do now the following sequence of $t-2$ cascades $\vec{\delta}^2[p, t-2] = \sum_{i=0}^{t-3} \vec{\delta}^1[p-$
561 $i, t]$. Altogether we have a sequence of $t(t-2)$ 4-deviations. There are a lot of cancellations, and in fact,
562 as shown in Claim 21, $\vec{\delta}^2[p, t-2]$ has only 10 non-zero coordinates. Table 5 describes an example of
563 computation of $\vec{\delta}^2[p, t-2]$ with $t = 7$.

	...	p	$p-1$			$p-5$			$p-9$			$p-13$	$p-14$...			
$\vec{\delta}^1[p, 7]$...0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...			
$+\vec{\delta}^1[p-1, 7]$...0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...		
$+\vec{\delta}^1[p-2, 7]$...0	0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...	
$+\vec{\delta}^1[p-3, 7]$...0	0	0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0	...
$+\vec{\delta}^1[p-4, 7]$...0	0	0	0	0	1	-3	1	1	0	0	0	-1	3	-1	-1	0...
$=\vec{\delta}^2[p, 5]$...0	1	-2	-1	0	0	-1	2	0	2	1	0	0	1	-2	-1	0...

Table 5: Computation of $\delta^2[p, 5]$.

564 **Claim 21.** For $5 \leq t \leq \frac{p}{2}$, the coordinates of the cascade $\vec{\delta}^2[p, t-2] = \sum_{i=0}^{t-3} \vec{\delta}^1[p-i, t]$ satisfy: $\delta_p^2 = 1$,
565 $\delta_{p-1}^2 = -2$, $\delta_{p-2}^2 = -1$, $\delta_{p-t+2}^2 = -1$, $\delta_{p-t+1}^2 = 2$, $\delta_{p-t}^2 = 2$, $\delta_{p-t-1}^2 = 1$, $\delta_{p-2t+2}^2 = 1$, $\delta_{p-2t+1}^2 = -2$,
566 $\delta_{p-2t}^2 = -1$, and $\delta_j^2 = 0$ for all the other values of j (see Table 6). Furthermore this cascade is 4-balanced.

...	δ_p^2	δ_{p-1}^2	δ_{p-2}^2	...	δ_{p-t+2}^2	δ_{p-t+1}^2	δ_{p-t}^2	δ_{p-t-1}^2	δ_{p-t-2}^2	...	δ_{p-2t+2}^2	δ_{p-2t+1}^2	δ_{p-2t}^2	...
0	1	-2	-1	0	-1	2	0	2	1	0	1	-2	-1	0

Table 6: Difference vector of $\delta^2[p, t-2]$.

567 Proof. We have $\delta_j^2 = \sum_{i=0}^{t-3} \delta_j^1[p-i, t]$. Furthermore, by the hypothesis we have $4 \leq t \leq p-i-4$ for every
568 $0 \leq i \leq t-3$, and so, we can apply Claim 17. By using the values of $\delta_j^1[p-i, t]$, we get that: for $j > p$,
569 $\delta_j^2 = 0$; $\delta_p^2 = 1$; $\delta_{p-1}^2 = -3 + 1 = -2$; $\delta_{p-2}^2 = 1 - 3 + 1 = -1$ (since $t \geq 5$); for $p-3 \geq j \geq p-t+3$,
570 $\delta_j^2 = 1 + 1 - 3 + 1 = 0$; $\delta_{p-t+2}^2 = 1 + 1 - 3 = -1$; $\delta_{p-t+1}^2 = 1 + 1 = 2$; $\delta_{p-t}^2 = -1 + 1 = 0$; $\delta_{p-t-1}^2 = 3 - 1 = 2$;
571 $\delta_{p-t-2}^2 = -1 + 3 - 1 = 1$; for $p-t-3 \geq j \geq p-2t+3$, $\delta_j^2 = -1 - 1 + 3 - 1 = 0$; $\delta_{p-2t+2}^2 = -1 - 1 + 3 = 1$;
572 $\delta_{p-2t+1}^2 = -1 - 1 = -2$, $\delta_{p-2t}^2 = -1$, and for $j < p-2t$, $\delta_j^2 = 0$.

573 By using Lemma 20 we get that $\vec{\delta}^2[p, t-2]$ is 7-balanced; but a careful analysis shows that this sequence
574 is in fact 4-balanced. Indeed, we will prove by induction that $\vec{\delta}^2[p, r] = \sum_{i=0}^{r-1} \vec{\delta}^1[p-i, t]$ is 4-balanced for
575 any $r \leq t-2$. That is true for $r = 1$, as $\vec{\delta}^1[p, t]$ is 4-balanced. Suppose that it is true for r . We have to
576 prove that any subsequence $\vec{\Phi}$ of $\vec{\delta}^2[p, r+1] = \vec{\delta}^2[p, r] + \vec{\delta}^1[p-r-1, t]$ has all its coordinates ≥ -4 .
577 That is true by induction hypothesis if $\vec{\Phi}$ is a subsequence of $\vec{\delta}^2[p, r]$. Otherwise $\vec{\Phi}$ consists of $\vec{\delta}^2[p, r]$
578 plus a subsequence of $\vec{\delta}^1[p-r-1, t]$. All the coordinates of $\vec{\delta}^2[p, r]$ are at least -3 , because the sum of
579 the coordinates in any consecutive subsequence of $\vec{\delta}^1[p, t]$ is at least -3 . Furthermore, as seen before the
580 coordinates of $\vec{\delta}^1[p-r-1, t]$ are greater than -1 , except, for $j = p-r-2$, where the coordinate is ≥ -4 .
581 Therefore, for all values of $j \neq p-r-2$, $\Phi_j \geq -3 - 1 = -4$. For $j = p-r-2$, we have $\delta_{p-r-2}^2[p, r] = 1$ (case
582 $r = 1$) or 2 (case $r > 1$) and so, $\Phi_{p-r-2} \geq 1 - 4 = -3$. In summary all the coordinates of any subsequence
583 $\vec{\Phi}$ of $\vec{\delta}^2[p, r+1]$ are at least -4 . Therefore by induction $\vec{\delta}^2[p, t-2]$ is 4-balanced. \diamond

584 At this stage we could continue and do a cascade of $\vec{\delta}^2[p, t-2]$, but there is no more the phenomenon
585 of cancellation. In fact we will use the following ‘‘symmetrization’’ trick. We will transform the cascade

586 $\vec{\delta}^2[p, t-2]$ into a sequence $\vec{\zeta}^2[p]$ by doing some sequence of 1-deviations whose coordinates are given in
587 Claim 22. The sequence obtained has only 8 non-zero coefficients (4 with values 1 and 4 with values -1)
588 arranged in a very symmetric nice way (that we will call Nice Property). Furthermore we will be able to
589 iterate a cascade process on it many times while preserving the Nice property.

590 For $p \geq q+2$, we recall that $\vec{\alpha}[p, q]$ is the 1-deviation where a vertex leaves a group of size $q+1$ for a
591 group of size $p-1$ (valid as $p \geq q+2$). Let $\vec{\alpha}^1[p, q, r] = \sum_{i=0}^{r-1} \vec{\alpha}[p-i, q+i]$ denote a cascade of r such
592 1-deviations (we need $p-r+1 \geq q+r+2$). The coordinates of $\vec{\alpha}^1[p, q, r]$ are given in the following Claim 22.

593 **Claim 22.** For $p-r \geq q+r+1$, $\vec{\alpha}^1[p, q, r] = \sum_{i=0}^{r-1} \vec{\alpha}[p-i, q+i]$ has only 4 non-zero coordinates, namely,
594 $\alpha_p^1 = 1$, $\alpha_{p-r}^1 = -1$, $\alpha_{q+r}^1 = -1$, and $\alpha_q^1 = 1$.

595 Table 7 shows an example with $t = 7$.

	...	p	$p-1$...				$p-7$	$p-8$...				$p-15$...		
$\vec{\delta}^2[p, 5]$	0	1	-2	-1	0	0	-1	2	0	2	1	0	0	1	-2	-1	0
$+\vec{\alpha}[p-1, p-15, 5]$	0	0	1	0	0	0	0	-1	0	0	0	-1	0	0	0	0	1
$+\vec{\alpha}[p-1, p-14, 2]$	0	0	1	0	-1	0	0	0	0	0	0	-1	0	-1	0	1	0
$+\vec{\alpha}[p-5, p-13, 1]$	0	0	0	0	0	0	1	-1	0	0	0	0	0	-1	1	0	0
$+\vec{\alpha}[p-7, p-10, 1]$	0	0	0	0	0	0	0	0	1	-1	-1	1	0	0	0	0	0
$= \vec{\zeta}^2[p]$	0	1	0	-1	-1	0	0	0	1	1	0	0	0	-1	-1	0	1

Table 7: Computation of $\zeta^2[p]$ with $t = 7$.

596 **Claim 23.** For $5 \leq t \leq \frac{p-1}{2}$, the coordinates of the sequence $\vec{\zeta}^2[p] = \vec{\delta}^2[p, t-2] + \vec{\alpha}^1[p-1, p-2t-1, t-2]$
597 $+ \vec{\alpha}^1[p-1, p-2t, 2] + \vec{\alpha}^1[p-t+2, p-2t+1, 1] + \vec{\alpha}^1[p-t, p-t-3, 1]$ satisfy: $\zeta_p^2 = 1$, $\zeta_{p-2}^2 = -1$,
598 $\zeta_{p-3}^2 = -1$, $\zeta_{p-t}^2 = 1$, $\zeta_{p-t-1}^2 = 1$, $\zeta_{p-2t+2}^2 = -1$, $\zeta_{p-2t+1}^2 = -1$, $\zeta_{p-2t-1}^2 = 1$ (see Table 8). Furthermore this
599 cascade is still 4-balanced.

...	ζ_p^2	ζ_{p-1}^2	ζ_{p-2}^2	ζ_{p-3}^2	...	ζ_{p-t}^2	ζ_{p-t-1}^2	...	ζ_{p-2t+2}^2	ζ_{p-2t+1}^2	ζ_{p-2t}^2	ζ_{p-2t-1}^2	...
...0	1	0	-1	-1	0...0	1	1	0...0	-1	-1	0	1	0...

Table 8: Difference vector $\zeta^2[p]$.

600 **Proof.** By Claim 22, we have the following coordinates:

- 601 • for $\vec{\alpha}^1[p-1, p-2t-1, t-2]$, $\alpha_{p-1}^1 = 1$, $\alpha_{p-t+1}^1 = -1$, $\alpha_{p-t-3}^1 = -1$, $\alpha_{p-2t-1}^1 = 1$;
- 602 • for $\vec{\alpha}^1[p-1, p-2t, 2]$, $\alpha_{p-1}^1 = 1$, $\alpha_{p-3}^1 = -1$, $\alpha_{p-2t+2}^1 = -1$, $\alpha_{p-2t}^1 = 1$;
- 603 • for $\vec{\alpha}^1[p-t+2, p-2t+1, 1]$, $\alpha_{p-t+2}^1 = 1$, $\alpha_{p-t+1}^1 = -1$, $\alpha_{p-2t+2}^1 = -1$, $\alpha_{p-2t+1}^1 = 1$;
- 604 • for $\vec{\alpha}^1[p-t, p-t-3, 1]$, $\alpha_{p-t}^1 = 1$, $\alpha_{p-t-1}^1 = -1$, $\alpha_{p-t-2}^1 = -1$, $\alpha_{p-t-3}^1 = 1$.

605 Therefore, by using these values and the values of the coordinates of δ_j^2 given in Claim 21, we get $\zeta_p^2 = 1$,
606 $\zeta_{p-1}^2 = -2 + 1 + 1 = 0$, $\zeta_{p-2}^2 = -1$, $\zeta_{p-3}^2 = 0 - 1 = -1$, $\zeta_{p-t+2}^2 = -1 + 1 = 0$, $\zeta_{p-t+1}^2 = 2 - 1 - 1 = 0$,
607 $\zeta_{p-t}^2 = 0 + 1 = 1$, $\zeta_{p-t-1}^2 = 2 - 1 = 1$, $\zeta_{p-t-2}^2 = 1 - 1 = 0$, $\zeta_{p-t-3}^2 = 0 - 1 + 1 = 0$, $\zeta_{p-2t+2}^2 = 1 - 1 - 1 = -1$,
608 $\zeta_{p-2t+1}^2 = -2 + 1 = -1$, $\zeta_{p-2t}^2 = -1 + 1 = 0$, $\zeta_{p-2t-1}^2 = 0 + 1 = 1$.

609 To prove that $\vec{\zeta}^2[p]$ is 4-balanced, apply Lemma 20 with $\vec{\Phi}^1 = \vec{\delta}^2[p, t-2]$ and $\vec{\Phi}^2 = \vec{\alpha}^1[p-1, p-2t-1, t-2] + \vec{\alpha}^1[p-1, p-2t, 2] + \vec{\alpha}^1[p-t+2, p-2t+1, 1] + \vec{\alpha}^1[p-t, p-t-3, 1]$. We have that $h_1 = 4$ and
610 furthermore all the coefficients of $\vec{\Phi}^1$ are greater than -2 . Furthermore, we claim that $\vec{\Phi}^2$ is 2-balanced.
611 Indeed, the four sequences of 1-deviations we use are clearly 1-balanced, and all their coordinates are at
612 least -1 . Therefore, the sum of any two such sequences is always 2-balanced. Now, we can observe that all
613 the coordinates in $\vec{\alpha}^1[p-1, p-2t-1, t-2] + \vec{\alpha}^1[p-1, p-2t, 2]$ are at least -1 . Therefore, we can apply
614 Lemma 20 to $\vec{\alpha}^1[p-1, p-2t-1, t-2] + \vec{\alpha}^1[p-1, p-2t, 2]$ with $\vec{\alpha}^1[p-t, p-t-3, 1]$. Again, we can observe
615

616 that all the coordinates in $\vec{\alpha}^1[p-1, p-2t-1, t-2] + \vec{\alpha}^1[p-1, p-2t, 1] + \vec{\alpha}^1[p-t, p-t-3, 1]$ are at least -1 .
617 Therefore, we can apply Lemma 20 to $\vec{\alpha}^1[p-1, p-2t-1, t-2] + \vec{\alpha}^1[p-1, p-2t, 1] + \vec{\alpha}^1[p-t, p-t-3, 1]$
618 with $\vec{\alpha}^1[p-t+2, p-2t+1, 1]$.

619 Hence, $\vec{\zeta}^2[p]$ is $\max(4, 2+2) = 4$ -balanced. \diamond

620 **Definition 24. Nice Property:** Let $k \geq 2$ be a positive integer. We say that the sequence $\vec{\zeta}^k[p]$ has
621 the Nice Property, if there exist three integers $a(k)$, $b(k)$, and $s(k)$ satisfying $1 < a(k) < b(k) < 2a(k)$ and
622 $b(k) < s(k) - 1 < p/2$ and all coordinates of $\vec{\zeta}^k$ are null except for:

- 623 • $\zeta_p^k = \zeta_{p+1-2s(k)}^k = 1$,
- 624 • $\zeta_{p-a(k)}^k = \zeta_{p-b(k)}^k = \zeta_{p+1-2s(k)+b(k)}^k = \zeta_{p+1-2s(k)+a(k)}^k = -1$, and
- 625 • $\zeta_{p+1-s(k)}^k = \zeta_{p-s(k)}^k = 1$.

626 We note the symmetry of the coordinates, as for any j , $\zeta_{p-j}^k = \zeta_{p+1-2s(k)+j}^k$. As an example, the sequence
627 $\vec{\zeta}^2[p]$ satisfies the Nice Property with $a(2) = 2$, $b(2) = 3$ and $s(2) = t + 1$ and is 4-balanced. Now we will
628 show how starting with a sequence $\vec{\zeta}^k[p]$ satisfying the Nice Property we can construct a sequence $\vec{\zeta}^{k+1}[p]$
629 having still the Nice Property.

630 **Claim 25. Main construction:** Let $\vec{\zeta}^k[p]$ be a sequence satisfying the Nice Property with parameters
631 $a(k), b(k), s(k)$ and with $p \geq 3s(k) - 2 - b(k)$. Then we can construct a sequence $\vec{\zeta}^{k+1}[p]$ satisfying the
632 following properties:

- 633 • $\vec{\zeta}^{k+1}[p]$ satisfies the Nice Property with parameters :
 - 634 • $a(k+1) = b(k)$,
 - 635 • $b(k+1) = b(k) + a(k)$,
 - 636 • $s(k+1) = s(k) + a(k)r(k)/2$, where $r(k)$ is the largest even integer such that $r(k)a(k) + b(k) <$
637 $s(k) - 1$;
- 638 • if $\vec{\zeta}^k[p]$ is $h(k)$ -balanced, then $\vec{\zeta}^{k+1}[p]$ is $(h(k) + 1)$ -balanced;
- 639 • $\vec{\zeta}^{k+1}[p]$ contains $r(k) + 1$ sequences $\vec{\zeta}^k[p]$.

640 Proof. (The reader can follow the proof on the example of Figure 4 that is detailed after the proof.) We will
641 first do a cascade of $\vec{\zeta}^k[p]$, but we will take values of the parameters differing by a multiple of $a(k)$ in order for
642 some of the coordinates to cancel (translation of $a(k)$). Specifically, let us define $\vec{\Psi}^r = \sum_{j=0}^r \vec{\zeta}^k[p - ja(k)]$.
643 We suppose for reasons explained below that r is even and non-zero such that $p - b(k) - ra(k) > p + 1 - s(k)$.
644 Using the values of Definition 24, we get the following values for the non-zero coordinates:

- 645 (1) $\psi_p^r = 1$; $\psi_{p-ja(k)}^r = -1 + 1 = 0$ for $0 < j \leq r$ (cancellation); $\psi_{p-(r+1)a(k)}^r = -1$;
- 646 (2) $\psi_{p+1-2s(k)+a(k)}^r = -1$; $\psi_{p+1-2s(k)-(j-1)a(k)}^r = 1 - 1 = 0$ for $0 < j \leq r$ (cancellation); $\psi_{p+1-2s(k)-ra(k)}^r =$
647 1 ;
- 648 (3) for $0 \leq j \leq r$, $\psi_{p-b(k)-ja(k)}^r = -1$;
- 649 (4) for $0 \leq j \leq r$, $\psi_{p+1-2s(k)+b(k)-ja(k)}^r = -1$;
- 650 (5) for $0 \leq j \leq r$, $\psi_{p+1-s(k)-ja(k)}^r = \psi_{p-s(k)-ja(k)}^r = 1$.

651 Since $a(k) < b(k) < 2a(k)$, all the indices of the coordinates are different provided we choose r even and
652 non-zero such that $p - b(k) - ra(k) > p + 1 - s(k)$ (that is equivalent to $ra(k) + b(k) < s(k) - 1$). Furthermore
653 we claim that all the indices are positive. Indeed on one hand the smallest index is $p + 1 - 2s(k) - ra(k)$.

654 However on the other hand we have $p + 1 - 2s(k) - ra(k) > p + 2 - 3s(k) + b(k) \geq 0$ as we assume
655 $p \geq 3s(k) - 2 - b(k)$.

656 Let us denote $a(k+1) = b(k)$; $b(k+1) = b(k) + a(k)$ and $s(k+1) = s(k) + a(k)r/2$.

657 Among the $4r + 8$ non-zero coordinates of $\vec{\Psi}^r$, 8 of them correspond to the non-zero coordinates of a
658 sequence having the Nice property with parameters $a(k+1), b(k+1)$ and $s(k+1)$, namely:

- 659 • $\psi_p^r = 1$ by (1) and $\psi_{p+1-2s(k+1)}^r = 1$ by (2) with $j = r$ (as $2s(k+1) = 2s(k) + ra(k)$);
- 660 • $\psi_{p-a(k+1)}^r = \psi_{p-b(k)}^r = -1$, $\psi_{p-b(k+1)}^r = \psi_{p-b(k)-a(k)}^r = -1$ by (3) with $j = 0, 1$;
- 661 • $\psi_{p+1-2s(k+1)+b(k+1)}^r = -1$, $\psi_{p+1-2s(k+1)+a(k+1)}^r = -1$ by (4) with $j = r - 1, r$;
- 662 • $\psi_{p+1-s(k+1)}^r = \psi_{p-s(k+1)}^r = 1$ by (5) with $j = r/2$.

663 We claim that the $4r$ remaining non-zero values can be written as follows: for the values -1 , in the form
664 $\psi_{p-x_m}^r$ and $\psi_{p+1-2s(k+1)+x_m}^r$; and for the values 1 , in the form $\psi_{p-y_m}^r$ and $\psi_{p+1-2s(k+1)+y_m}^r$ with $x_m < y_m$
665 ($0 \leq m \leq r-1$). The proof of that assertion is technical and can be followed on the example given after the
666 proof with $k = 2$.

667 Indeed the $4r$ non-zero coordinates can be partitioned as follows: there are r values equal to -1 , namely
668 $\psi_{p-b(k)-ja(k)}^r = -1$, for $2 \leq j \leq r$, and $\psi_{p-(r+1)a(k)}^r = -1$; then there are $2r$ values equal to 1 , namely
669 $\psi_{p+1-s(k)-ja(k)}^r = \psi_{p-s(k)-ja(k)}^r = 1$, for $0 \leq j \leq r, j \neq r/2$; finally there are r values equal to -1 , that are
670 $\psi_{p+1-2s(k)+a(k)}^r = -1$ and $\psi_{p+1-2s(k)+b(k)-ja(k)}^r = -1$, for $0 \leq j \leq r-2$, respectively. We note that these
671 values are disposed in a very symmetric way.

672 Let $x_m = b(k) + (m+2)a(k)$ for $0 \leq m \leq r-2$ and $x_{r-1} = (r+1)a(k)$. For $2 \leq j \leq r$, let $m = j-2$;
673 then we get $p - b(k) - ja(k) = p - x_m$. We also have $p - (r+1)a(k) = p - x_{r-1}$. On the other side,
674 for $0 \leq j \leq r-2$ let $m = r - j - 2$; then we get $p + 1 - 2s(k) + b(k) - ja(k) = p + 1 - 2s(k) +$
675 $b(k) - ra(k) + (m+2)a_k = p + 1 - 2s(k+1) + x_m$ (recall that $2s(k+1) = 2s(k) + ra(k)$). We also have
676 $p + 1 - 2s(k) + a(k) = p + 1 - 2s(k+1) + (r+1)a(k) = p + 1 - 2s(k+1) + x_{r-1}$.

677 The choice for the values y_m , $0 \leq m \leq r-1$ depends on the parity of m . Let $y_{2m'} = s(k) - 1 + m'a(k)$
678 and $y_{2m'+1} = s(k) + m'a(k)$ where $0 \leq m' \leq r/2 - 1$. For $0 \leq j \leq r/2 - 1$, let $m' = j$; then we get
679 $p + 1 - s(k) - ja(k) = p - (s(k) - 1 + m'a(k)) = p - y_{2m'}$ and $p - s(k) - ja(k) = p - (s(k) + m'a(k)) =$
680 $p - y_{2m'+1}$. In the same way for $r/2 + 1 \leq j \leq r$, let $m' = r - j$; then we get $p + 1 - s(k) - ja(k) =$
681 $p + 1 - s(k) - ra(k) + m'a(k) = p + 1 - 2s(k+1) + s(k) + m'a(k) = p + 1 - 2s(k+1) + y_{2m'+1}$ and
682 $p - s(k) - ja(k) = p - s(k) - ra(k) + m'a(k) = p + 1 - 2s(k+1) + s(k) - 1 + m'a(k) = p + 1 - 2s(k+1) + y_{2m'}$.

683
684 Furthermore, these r quadruples of values can be canceled by adding to $\vec{\Psi}^r$ the r sequences $\vec{\alpha}^1[p -$
685 $x_m, p + 1 - 2s(k+1) + x_m, y_m - x_m]$.

686 We claim that the resulting sequence with partition vector $\vec{\Psi}^r + \sum_{m=0}^{r-1} \vec{\alpha}^1[p - x_m, p + 1 - 2s(k+1) +$
687 $x_m, y_m - x_m]$ satisfies the Nice Property with parameters $a(k+1), b(k+1)$ and $s(k+1)$. Indeed the non-
688 zero coordinates are exactly the 8 listed before. Furthermore, $a(k+1) = b(k) < b(k) + a(k) = b(k+1)$,
689 $b(k+1) = b(k) + a(k) < b(k) + b(k) < 2a(k+1)$ and $b(k+1) = b(k) + a(k) < s(k) - 1 + a(k) \leq s(k+1) - 1$ as $r \geq 2$.
690 Finally, since we choose $p \geq 3s(k) - 2 - b(k)$, we get $p \geq (2s(k) + ra(k)) + (s(k) - 2 - b(k) - ra(k)) > 2s(k+1)$
691 as $ra(k) + b(k) < s(k) - 1$ and $p \geq 2s(k+1) - 1$.

692 We now prove that $\vec{\zeta}^{k+1}[p]$ is $(h(k) + 1)$ -balanced. We first prove by induction that $\vec{\Psi}^r$ is $(h(k) + 1)$ -
693 balanced. That is true for $r = 0$ as $\vec{\zeta}^k[p]$ is $h(k)$ -balanced. Then suppose it is true for some r ; we
694 apply Lemma 20 with $\vec{\Phi}^1 = \vec{\Psi}^r$ and $\vec{\Phi}^2 = \vec{\zeta}^k[p - (r+1)a(k)]$. We have that $h_1 = h(k) + 1$ by our
695 induction hypothesis and furthermore all the coefficients of $\vec{\Phi}^1$ are greater than -1 ; furthermore $\vec{\Phi}^2$ is
696 $h(k)$ -balanced and so, $\vec{\Psi}^{r+1}$ is $(\max(h(k) + 1, h(k) + 1) = h(k) + 1)$ -balanced. We add the 1-deviation
697 $\vec{\alpha}^1[p - x_m, p + 1 - 2s(k+1) + x_m, y_m - x_m]$ which is 1-balanced. Since $\max(h(k) + 1, 1 + 1) = h(k) + 1$, we
698 still get a $(h(k) + 1)$ -balanced sequence.

699 Finally, by construction, we get that $\vec{\zeta}^{k+1}[p]$ contains $r(k) + 1$ sequences $\vec{\zeta}^k[p]$. \diamond

700 **Example with $t = 16$.** (See Figure 4.) We start with $\vec{\zeta}^2[p]$, which satisfies the Nice property with
701 $a(2) = 2, b(2) = 3$ and $s(2) = 17$, and whose coordinates are $\zeta_p^2 = \zeta_{p-33}^2 = 1, \zeta_{p-2}^k = \zeta_{p-3}^k = \zeta_{p-30}^k = \zeta_{p-31}^k =$
702 -1 , and $\zeta_{p-16}^k = \zeta_{p-17}^k = 1$. $r(2)$ is the largest even integer such that $2r(2) + 3 < 16$, that is $r(2) = 6$.

703 Thus, we can do a cascade of 7 sequences $\vec{\zeta}^2[p]$, translated by 2, and in doing so we get $\vec{\Psi}^6 = \sum_{j=0}^6 \vec{\zeta}^2[p-$
704 $2j]$. We have $\psi_p^6 = \psi_{p-45}^6 = 1, \psi_{p-3}^6 = \psi_{p-5}^6 = \psi_{p-40}^6 = \psi_{p-42}^6 = -1$, and $\psi_{p-22}^6 = \psi_{p-23}^6 = 1$. We also
705 have 12 values -1 namely: $\psi_{p-7}^6 = \psi_{p-9}^6 = \psi_{p-11}^6 = \psi_{p-13}^6 = \psi_{p-15}^6 = \psi_{p-14}^6$ and $\psi_{p-31}^6 = \psi_{p-30}^6 = \psi_{p-32}^6 =$
706 $\psi_{p-34}^6 = \psi_{p-36}^6 = \psi_{p-38}^6$, which can be written on the form $p - x_m$ and $p - 45 + x_m$ with $x_0 = 7, x_1 =$
707 $9, x_2 = 11, x_3 = 13, x_4 = 15, x_5 = 14$. We have 12 values 1 namely: $\psi_{p-16}^6 = \psi_{p-17}^6 = \psi_{p-18}^6 = \psi_{p-19}^6 =$
708 $\psi_{p-20}^6 = \psi_{p-21}^6$ and $\psi_{p-24}^6 = \psi_{p-25}^6 = \psi_{p-26}^6 = \psi_{p-27}^6 = \psi_{p-28}^6 = \psi_{p-29}^6$, which can be written on the
709 form $p - y_m$ and $p - 45 + y_m$ with $y_0 = 16, y_1 = 17, y_2 = 18, y_3 = 19, y_4 = 20, y_5 = 21$. Finally, the blue
710 cascade $\vec{\zeta}^3[p] = \vec{\Psi}^6 + \sum_{m=0}^5 \vec{\alpha}^1[p - x_m, p + 1 - 2s(k + 1) + x_m, y_m - x_m]$ satisfies the Nice property with
711 $a(3) = 3, b(3) = 5$ and $s(3) = 23$.

712 Then $r(3)$, the largest even integer such that $3r(2) + 5 < 22$, satisfies $r(3) = 4$. Applying the claim we
713 get the green cascade $\vec{\zeta}^4[p]$, which satisfies the Nice property with $a(4) = 5, b(4) = 8$ and $s(4) = 29$. Then
714 $r(4)$, the largest even integer such that $5r(4) + 8 < 28$, satisfies $r(4) = 2$. We apply once again the claim and
715 in doing so we get the grey cascade $\vec{\zeta}^5[p]$, which satisfies the Nice property with $a(5) = 8, b(5) = 13$ and
716 $s(5) = 34$. We can apply the claim again as $r(5)$, the largest even integer such that $8r(5) + 13 < 33$, satisfies
717 $r(5) = 2$. Doing so, we get a cascade $\vec{\zeta}^6[p]$ which satisfies the Nice property with $a(6) = 13, b(6) = 21$ and
718 $s(6) = 42$.

719 *Proof. (End of the proof of Theorem 15)* At this stage we have built a sequence $\vec{\zeta}^2[p]$ which satisfies
720 the Nice Property with $a(2) = 2, b(2) = 3$ and $s(2) = t + 1$. Since $h(2) = 4$, it is 4-balanced. Furthermore,
721 it contains $t(t - 2)$ 4-deviations (see Claim 23). Then, for some well-chosen K (to be defined later) we can
722 apply $K - 2$ times the main construction (Claim 25) to construct a sequence $\vec{\zeta}^K[p]$ which satisfies the Nice
723 Property with parameters $a(K), b(K)$ and $s(K)$ and is $h(K)$ -balanced.

On one hand since we want to upper bound the number of vertices, we need to upper bound $a(K), b(K)$
and $s(K)$. We have $a(K) = b(K - 1), b(K) = b(K - 1) + a(K - 1) = b(K - 1) + b(K - 2)$ and so, we recognize
the Fibonacci recurrence relation. The j th Fibonacci number $F(j)$ satisfies:

$$F(j) = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^j - \left(\frac{1 - \sqrt{5}}{2} \right)^j \right).$$

724 Then, as $a(2) = 2 = F(3)$ and $b(2) = 3 = F(4)$, we get $a(K) = F(K + 1)$ and $b(K) = F(K + 2)$. In fact in
725 what follows we will only use that $a(K) \leq 2^{K-1}$ and $b(K) \leq 2^K$. We have $s(K + 1) = s(K) + a(K)r(K)/2$;
726 but $a(K)r(K) < s(K) - 1 - b(K) < s(K) - 1$ and so, $s(K + 1) < (3/2) \times s(K)$. Therefore, we prove by
727 induction that $s(K) < s(2)(3/2)^{K-2} = (t + 1)(3/2)^{K-2}$.

728 Recall that we should have $p \geq 2s(K) - 1$. Hence, we choose $p = 2s(K)$. Furthermore by induction
729 we have that $h(K) = K + 2$. Thus, by Proposition 19, it suffices to start with a partition containing
730 at least $K + 2$ groups of each size $i, 1 \leq i \leq p$. It is easy to obtain such a starting partition from the
731 initial partition — which consists of n groups of size 1 — by doing a sequence of 1-deviations of size
732 $(K + 2)p(p - 1)/2$; indeed we can create a group of any size i with $(i - 1)$ 1-deviations. Therefore, we will
733 take $n = (K + 2)p(p - 1)/2 = (K + 2)s(K)(2s(K) - 1)$. By using the inequality $s(K) < (t + 1)(3/2)^{K-2}$ we
734 get that:

$$n = \mathcal{O}(t^2 K (3/2)^{2K}). \quad (1)$$

735 On the other hand, we have to lower bound the number of deviations. By construction we have, for any
736 value $k, \vec{\zeta}^{k+1}[p]$ contains $r(k) + 1$ sequences $\vec{\zeta}^k[p]$. Furthermore, $\vec{\zeta}^2[p]$ contains $t(t - 2)$ 4-deviations. As
737 a result, $\vec{\zeta}^K[p]$ contains $t(t - 2) \prod_{k=2}^{K-1} (r(k) + 1)$ 4-deviations.

738 We also recall that, for any $k \leq K, r(k)$ is the largest even integer r such that $ra(k) + b(k) < s(k) - 1$.
739 In particular, $r(k) \geq \lfloor \frac{s(k) - 1 - b(k)}{a(k)} \rfloor - 1$. By using the facts that $b(k) + 1 \leq 2a(k), s(k) > s(2) - 1 = t$, and

740 $a(k) \leq a(K) < 2^{K-1}$, we get $r(k) \geq \frac{t}{2^{K-1}} - 3$. Then $\prod_{k=2}^{K-1} (r(k) + 1) \geq (\frac{t}{2^{K-1}} - 2)^{K-2}$ and the number D
741 of deviations satisfies:

$$D = \Omega \left(t^2 \left(\frac{t}{2^{K-1}} - 2 \right)^{K-2} \right). \quad (2)$$

742 We have now to choose K as a function of t . In order for the number of deviations (given by Equation (2))
743 to increase we need that 2^{K-1} is small compared to t , that is, $K \ll \log_2(t)$. However in view of Equation (1)
744 we want to choose the largest possible K . Therefore, a good choice is $K = \log_2(t)/2$.

745 On one hand, we get by Equation 1 that $n = \mathcal{O}(t^2 \log_2(t) (3/2)^{\log_2(t)})$. Let $f(t) = t^2 \log_2(t) (3/2)^{\log_2(t)}$;
746 then $\log_2(f(t)) = 2 \log_2(t) + \log_2(\log_2(t)) + \log_2(t) (\log_2(3) - \log_2(2))$. By using that $\log_2(3) - \log_2(2) < 0.585$
747 and the fact that for t large enough $\log_2(\log_2(t)) < 0.014 \log_2(t)$, we get for t large enough $\log_2(f(t)) <$
748 $2.6 \log_2(t)$ or equivalently $f(t) < t^{2.6}$. As a result, $n = \mathcal{O}(t^{2.6})$.

749 On the other hand we get by Equation 2: $D = \Omega((t^{1/2})^{\log_2(t)/2}) = \Omega(t^{\log_2(t)/4})$. Therefore, $D =$
750 $\Omega(n^{c \log_2(n)})$ with $c = \frac{1}{4 \times (2.6)^2} \simeq 1/27$, thereby proving Theorem 15. \square

751 5.2 Case $k = 3$. Proof of Theorem 16

752 The proof uses the same idea that for $k = 4$ but is simpler as we can only do a limited iteration of cascades,
753 which use a lot of 3-deviations $\gamma[p]$ (the only 3-deviation which does not increase the global utility). For
754 completeness, we recall here the definition of $\gamma[p]$.

755 **Definition of $\gamma[p]$:** Consider a partition P containing at least 3 groups of size $p - 1$ and a group of size
756 $p - 3$. In the 3-deviation $\gamma[p]$ one vertex in each of the 3 groups of size $p - 1$ moves to the group of size $p - 3$
757 to form a new group of size p .

758 Let P be the partition consisting of 3 groups of size 3 and one group of size 1. We have $Q(P) = Q =$
759 $(3, 3, 3, 1)$ (we omit the 0s). Apply the 3-deviation $\gamma[4]$ where one vertex of each group of size 3 joins the
760 group of size 1; it leads to the partition P' with integer partition $Q' = (4, 2, 2, 2)$. Note that Q and Q' have
761 the same total utility 18.

762 Thus, after this 3-deviation we get a new partition P' with one more group of size p , 3 less groups of size
763 $p - 1$, 3 more groups of size $p - 2$, and one less group of size $p - 3$. This is expressed by the coordinates of the
764 associated difference vector (where we omit the bracket $[p]$ for ease of reading). Note that such a deviation
765 is valid only if there are 3 groups of size $p - 1$ and one group of size $p - 3$.

766 **Difference vector $\vec{\gamma}[p]$:** The difference vector $\vec{\gamma}[p]$ has the following coordinates: $\gamma_p = 1, \gamma_{p-1} =$
767 $-3, \gamma_{p-2} = 3, \gamma_{p-3} = -1$, and $\gamma_j = 0$ for all other values of j . See Table 9.

...	γ_p	γ_{p-1}	γ_{p-2}	γ_{p-3}	...
...0	1	-3	3	1	0...

Table 9: Difference vector of $\gamma[p]$.

769 **The cascade $\vec{\gamma}^1[p, t]$:** Like for $k = 4$, we do a cascade consisting of t ordered 3-deviations $\gamma[p], \gamma[p -$
770 $1], \gamma[p - 2], \dots, \gamma[p - t + 1]$. We will denote this cascade by its difference vector $\vec{\gamma}^1[p, t] = \sum_{i=0}^{t-1} \vec{\gamma}[p - i]$.

771 The reader can follow the construction in Table 10 with $t = 5$. The coordinates of $\vec{\gamma}^1[p, t]$ are given in
772 Claim 26 and Table 11. We note that there are a lot of cancellations and only 6 non-zero coordinates.

	p	$p - 1$	$p - 2$	$p - 3$	$p - 4$	$p - 5$	$p - 6$	$p - 7$	$p - 8$	$p - 9$	$p - 10$	$p - 11$	$p - 12$	$p - 13$
$\vec{\gamma}[p]$	1	-3	3	-1										
$\vec{\gamma}[p - 1]$		1	-3	3	-1									
$\vec{\gamma}[p - 2]$			1	-3	3	-1								
$\vec{\gamma}[p - 3]$				1	-3	3	-1							
$\vec{\gamma}[p - 4]$					1	-3	3	-1						
$\vec{\gamma}^1[p, 5]$	1	-2	1	0	0	-1	2	-1	0	0	0	0	0	0

Table 10: Computation of $\gamma^1[p, 5]$.

773 **Claim 26.** For $3 \leq t \leq p-3$, the coordinates of the cascade $\vec{\gamma}^1[p, t] = \sum_{i=0}^{t-1} \vec{\gamma}[p-i]$ satisfy: $\gamma_p^1 = 1$,
774 $\gamma_{p-1}^1 = -2$, $\gamma_{p-2}^1 = 1$, $\gamma_{p-t}^1 = -1$, $\gamma_{p-t-1}^1 = 2$, $\gamma_{p-t-2}^1 = -1$, and $\gamma_j^1 = 0$ for all the other values of j .
775 Furthermore, this cascade is 3-balanced (see Table 11).

\dots	γ_p^1	γ_{p-1}^1	γ_{p-2}^1	\dots	γ_{p-t}^1	γ_{p-t-1}^1	γ_{p-t-2}^1	\dots
$\dots 0$	1	-2	1	0...0	-1	2	-1	0...

Table 11: Difference vector $\gamma^1[p, t]$.

776 Proof. We have $\gamma_j^1 = \sum_{i=0}^{t-1} \gamma_j[p-i]$. For a given j , $\gamma_j[p-i] = 0$ except for the following values of i such that
777 $0 \leq i \leq t-1$: $i = p-j$ where $\gamma_j[j] = 1$; $i = p-j-1$ where $\gamma_j[j+1] = -3$; $i = p-j-2$ where $\gamma_j[j+2] = 3$;
778 $i = p-j-3$ where $\gamma_j[j+3] = -1$ (in the table it corresponds to the consecutive non-zero values in a column
779 which are at most 4). Therefore, for $j > p$: $\gamma_j^1 = 0$; $\gamma_p^1 = 1$; $\gamma_{p-1}^1 = -3 + 1 = -2$; $\gamma_{p-2}^1 = 3 - 3 + 1 = 1$; for
780 $p-3 \geq j \geq p-t+1$, $\gamma_{p-j}^1 = -1 + 3 - 3 + 1 = 0$; $\gamma_{p-t}^1 = -1 + 3 - 3 = -1$; $\gamma_{p-t-1}^1 = -1 + 3 = 2$; $\gamma_{p-t-2}^1 = -1$
781 and, for $j < p-t-2$, $\gamma_j^1 = 0$. Finally we note that the coordinates of any subsequence of the cascade, i.e.
782 the coordinates of $\vec{\gamma}^1[p, r]$, are all at least -1 except γ_{p-1}^1 : which is -3 when $r = 1$ and then -2 , thereby
783 proving that the cascade is 3-balanced. \diamond

784 In our example we can do the cascade until $t = p-3$. However, it is better to choose t smaller (we
785 will see below that a good value for t should satisfy $p-4t+3 \geq 0$) and repeat the previous cascade but
786 translated (i.e., we do a cascade of cascades). More precisely, we now do the following sequence of $t-1$
787 cascades $\vec{\gamma}^2[p, t-1] = \sum_{i=0}^{t-2} \vec{\gamma}^1[p-i, t]$. Altogether we have a sequence of $t(t-1)$ 3-deviations.

788 In the example (see Table 12), we choose $t = 5$ and after having done $\vec{\gamma}^1[p, 5]$ we do $\vec{\gamma}^1[p-1, 5]$,
789 $\vec{\gamma}^1[p-2, 5]$ and $\vec{\gamma}^1[p-3, 5]$. We can see again a phenomenon of cancelation as the coordinate γ_{p-2}^2 of this
790 cascade of cascades equals 0. Similarly the coordinates γ_{p-3}^2 , γ_{p-7}^2 and γ_{p-8}^2 of this cascade of cascades equal
791 0. This cancelation stays for all the other deviations. In the general case there are a lot of cancelations and
792 in fact, as shown in the next Claim 27, $\vec{\gamma}^2[p, t-1]$ has only 6 non-zero coordinates.

793

	p	$p-1$	$p-2$	$p-3$	$p-4$	$p-5$	$p-6$	$p-7$	$p-8$	$p-9$	$p-10$	$p-11$	$p-12$	$p-13$
$\vec{\gamma}^1[p, 5]$	1	-2	1	0	0	-1	2	-1						
$\vec{\gamma}^1[p-1, 5]$		1	-2	1	0	0	-1	2	-1					
$\vec{\gamma}^1[p-2, 5]$			1	-2	1	0	0	-1	2	-1				
$\vec{\gamma}^1[p-3, 5]$				1	-2	1	0	0	-1	2	-1			
$\vec{\gamma}^2[p, 4]$	1	-1	0	0	-1	0	1	0	0	1	-1			
$\vec{\gamma}^2[p-1, 4]$		1	-1	0	0	-1	0	1	0	0	1	-1		
$\vec{\gamma}^2[p-2, 4]$			1	-1	0	0	-1	0	1	0	0	1	-1	
$\vec{\gamma}^3[p, 3]$	1	0	0	-1	-1	-1	0	1	1	1	0	0	-1	
$\vec{\alpha}^1[p-3, p-12, 4]$	0	0	0	1	0	0	0	-1	-1	0	0	0	1	
$\vec{\gamma}^3[p, 3]$	1	0	0	0	-1	-1	0	0	0	1	0	0	0	
$\vec{\gamma}^3[p-1, 3]$		1	0	0	0	-1	-1	0	0	0	1	0	0	
$\vec{\gamma}^3[p-2, 3]$			1	0	0	0	-1	-1	0	0	0	1	0	
$\vec{\gamma}^3[p-3, 3]$				1	0	0	0	-1	-1	0	0	0	1	0
$\vec{\gamma}^3[p-4, 3]$					1	0	0	0	-1	-1	0	0	0	1
$\vec{\gamma}^4[p, 5]$	1	1	1	1	0	-2	-2	-2	-2	0	1	1	1	1

Table 12: Example of cascade of cascades.

794 **Claim 27.** For $3 \leq t \leq \frac{p-1}{2}$, the coordinates of the cascade $\vec{\gamma}^2[p, t-1] = \sum_{i=0}^{t-2} \vec{\gamma}^1[p-i, t]$ satisfy: $\gamma_p^2 = 1$,
795 $\gamma_{p-1}^2 = -1$, $\gamma_{p-t+1}^2 = -1$, $\gamma_{p-t-1}^2 = 1$, $\gamma_{p-2t+1}^2 = 1$, $\gamma_{p-2t}^2 = -1$ and $\gamma_j^2 = 0$ for all the other values of j .
796 Furthermore this cascade is 3-balanced.

797 Proof. We have $\gamma_j^2 = \sum_{i=0}^{t-2} \gamma_j^1[p-i, t]$. By using the values of $\gamma_j^1[p-i, t]$ given in Claim 26, we get that: for
798 $j > p$, $\gamma_j^2 = 0$; $\gamma_p^2 = 1$; $\gamma_{p-1}^2 = 1 - 2 = -1$; for $p-2 \geq j \geq p-t+2$, $\gamma_j^2 = 1 - 2 + 1 = 0$; $\gamma_{p-t+1}^2 = 1 - 2 = -1$;
799 $\gamma_{p-t}^2 = -1 + 0 + 1 = 0$; $\gamma_{p-t-1}^2 = 2 - 1 = 1$; for $p-t-2 \geq j \geq p-2t+2$, $\gamma_j^2 = -1 + 2 - 1 = 0$;
800 $\gamma_{p-2t+1}^2 = -1 + 2 = 1$; $\gamma_{p-2t}^2 = -1$ and for $j < p-2t$, $\gamma_j^2 = 0$.

...	γ_p^2	γ_{p-1}^2	...	γ_{p-t+1}^2	γ_{p-t-1}^2	...	γ_{p-2t+1}^2	γ_{p-2t}^2	...	
0	1	-1	0	-1	0	1	0	1	-1	0

Table 13: Difference vector $\gamma^2[p, t - 2]$.

801 Here again we can see that after any number r of 3-deviations the coordinates of the sequence are
802 always at least -3 . Indeed a subsequence either consists of a cascade $\vec{\gamma}^2[p, r] = \sum_{i=0}^{r-2} \vec{\gamma}^1[p - i, t]$ for some
803 $r, 1 \leq r \leq t - 1$, which has coordinates all greater than -2]; or of a cascade $\vec{\gamma}^2[p, r] = \sum_{i=0}^{r-2} \vec{\gamma}^1[p - i, t]$ for
804 some $r, 1 \leq r \leq t - 1$, plus a subsequence $\sum_{j=0}^{r'} \vec{\gamma}[p - r - j]$. The coordinates of the last subsequence are
805 greater than -1 except the $p - r - 1$ coordinate equal to -3 if $r' = 0$ and -2 otherwise. But the $p - r - 1$
806 coordinate of $\vec{\gamma}^2[p, r]$ is 1 and all the coordinates of the considered subsequence are greater than -3 . \diamond

807 We can now repeat the cascade of translated cascades $\vec{\gamma}^2[p, t - 1]$. More precisely, we now do the fol-
808 lowing sequence of $t - 2$ cascades $\vec{\gamma}^3[p, t - 2] = \sum_{i=0}^{t-3} \vec{\gamma}^2[p - i, t - 1]$. Altogether we have a sequence of
809 $t(t - 1)(t - 2)$ 3-deviations. In the example (see Table 12), after having done $\vec{\gamma}^2[p, 4]$ we do $\vec{\gamma}^2[p - 1, 4]$
810 and $\vec{\gamma}^2[p - 2, 4]$. We can again see a phenomenon of cancelation as the coordinate γ_{p-1}^3 of this cascade of
811 cascades equals 0. Similarly the coordinates $\gamma_{p-2}^3, \gamma_{p-10}^3$ and γ_{p-11}^3 of this cascade of cascades equal 0. In
812 the general case there are a lot of cancelations. In fact, as shown in the next Claim 28, $\vec{\gamma}^3[p, t - 2]$ has only
813 8 non-zero coordinates.
814

815 **Claim 28.** For $3 \leq t \leq \frac{p+2}{3}$, the coordinates of the cascade $\vec{\gamma}^3[p, t - 2] = \sum_{i=0}^{t-3} \vec{\gamma}^2[p - i, t - 1]$ satisfy:
816 $\gamma_p^3 = 1, \gamma_{p-t+2}^3 = \gamma_{p-t+1}^3 = \gamma_{p-t}^3 = -1, \gamma_{p-2t+3}^3 = \gamma_{p-2t+2}^3 = \gamma_{p-2t+1}^3 = 1, \gamma_{p-3t+3}^3 = -1$, and $\gamma_j^3 = 0$ for all
817 the other values of j . Furthermore this cascade is 4-balanced.

...	γ_p^3	...	γ_{p-t+2}^3	γ_{p-t+1}^3	γ_{p-t}^3	...	γ_{p-2t+3}^3	γ_{p-2t+2}^3	γ_{p-2t+1}^3	...	γ_{p-3t+3}^3	...
0	1	0	-1	-1	-1	0	1	1	1	0	-1	0

Table 14: Difference vector $\gamma^3[p, t - 2]$.

818 **Proof.** We have $\gamma_j^3 = \sum_{i=0}^{t-3} \gamma_j^2[p - i, t - 1]$. By using the values of $\gamma_j^2[p - i, t - 1]$ given in Claim 27, we
819 get that: for $j > p, \gamma_j^2 = 0; \gamma_p^3 = 1$; for $p - 1 \geq j \geq p - t + 3, \gamma_j^3 = -1 + 1 = 0; \gamma_{p-t+2}^3 = 0 - 1 = -1$;
820 $\gamma_{p-t+1}^3 = -1 + 0 = -1; \gamma_{p-t}^3 = 0 - 1 = -1$; for $p - t - 1 \geq j \geq p - 2t + 4, \gamma_j^3 = 1 + 0 - 1 = 0; \gamma_{p-2t+3}^3 = 1 + 0 = 1$;
821 $\gamma_{p-2t+2}^3 = 0 + 1 = 1; \gamma_{p-2t+1}^3 = 1 + 0 = 1$; for $p - 2t \geq j \geq p - 3t + 4, \gamma_j^3 = -1 + 1 = 0; \gamma_{p-3t+3}^3 = 0 - 1 = -1$;
822 and for $j < p - 3t + 3, \gamma_j^3 = 0$.

823 Here again a careful but tedious analysis of all the subsequence of deviations indicates that all their
824 coordinates are at least -3 . However, by using Lemma 20 we can easily prove that this sequence is 4-
825 balanced. In fact, we will prove by induction that $\sum_{i=0}^r \vec{\gamma}^2[p - i, t - 1]$ is 4-balanced for any $r \leq t - 3$.
826 That is true for $r = 0$, as $\vec{\gamma}^2[p, t - 1]$ is 3-balanced. Suppose that $\sum_{i=0}^r \vec{\gamma}^2[p - i, t - 1]$ is 4-balanced for
827 some $r \leq t - 3$. We apply Lemma 20 with $\vec{\Phi}^1 = \sum_{i=0}^r \vec{\gamma}^2[p - i, t - 1]$ and $\vec{\Phi}^2 = \vec{\gamma}^2[p - r - 1, t - 1]$. By
828 our induction hypothesis, we have $h_1 = 4$. Furthermore, all the coefficients of $\vec{\Phi}^1$ are greater than -1 by
829 Claim 27 when $r = 0$ or Claim 28 when $r > 0$. Therefore, $\min_i \Phi_i^1 = -1$. Finally $\vec{\Phi}^2$ is 3-balanced and
830 $\sum_{i=0}^{r+1} \vec{\gamma}^2[p - i, t - 1]$ is also $\max(4, 3 + 1) = 4$ -balanced. By induction $\vec{\gamma}^3[p, t - 2]$ is 4-balanced. \diamond

831 Again we can repeat t times the cascades $\vec{\gamma}^3[p, t - 2]$ for creating a sequence of cascades that we call
832 $\vec{\gamma}^4[p, t] = \sum_{i=0}^{t-1} \vec{\gamma}^3[p - i, t - 2]$. This is enough to prove Theorem 16. Nevertheless, in order to reduce the
833 number of non-zero coordinates to four, we will complete our construction with some trick already used in
834 the previous subsection (Case $k = 4$); namely, we will complete $\vec{\gamma}^3[p, t - 2]$ with a sequence of 1-deviations.
835 Note that in doing so, we improve the ratio between the number of 3-deviations and the number of vertices.
836 Thus, after having done $\vec{\gamma}^3[p, t - 2]$, we apply the cascade $\vec{\alpha}^1[p - t + 2, p - 3t + 3, t - 1]$ (see Claim 22). Let
837 $\vec{\tau}^3[p, t - 2]$ be the sequence we obtain. As shown in Claim 29, it only has 4 non-zero coordinates.

838 **Claim 29.** For $3 \leq t \leq (p+2)/3$ the coordinates of $\vec{\tau}^3[p, t-2] = \vec{\gamma}^3[p, t-2] + \vec{\alpha}^1[p-t+2, p-3t+3, t-1]$
839 satisfy: $\tau_p^3 = 1, \tau_{p-t+1}^3 = \tau_{p-t}^3 = -1, \tau_{p-2t+1}^3 = 1$, and $\tau_j^3 = 0$ for all the other values of j . Furthermore it is
840 4-balanced.

841 Proof. Compared to $\vec{\gamma}^3[p, t-2]$ only 4 coordinates have been changed. We get $\tau_{p-t+2}^3 = \gamma_{p-t+2}^3 + \alpha_{p-t+2}^1 =$
842 $-1 + 1 = 0$, $\tau_{p-2t+3}^3 = \tau_{p-2t+2}^3 = 1 - 1 = 0$ and $\tau_{p-3t+3}^3 = -1 + 1 = 0$. All the other coordinates remain
843 the same. To prove that $\vec{\tau}^3[p, t-2]$ is 4-balanced, we apply Lemma 20 with $\vec{\Phi}^1 = \vec{\gamma}^3[p, t-2]$ and
844 $\vec{\Phi}^2 = \vec{\alpha}^1[p-t+2, p-3t+3, t-1]$. We have proved that $\vec{\Phi}^1$ is 4-balanced and by Claim 28 $\min_i \Phi_i^1 = -1$.
845 Furthermore, $\vec{\Phi}^2$ is 1-balanced. Hence, $\vec{\tau}^3[p, t-2]$ is $\max(4, 1+1) = 4$ -balanced. \diamond

846 Now we can do t cascades of $\vec{\tau}^3$ in order to obtain the cascade $\vec{\tau}^4[p, t] = \sum_{i=0}^{t-1} \vec{\tau}^3[p-i, t-2]$.

847 **Claim 30.** For $3 \leq t \leq (p+3)/4$ the coordinates of $\vec{\tau}^4[p, t] = \sum_{i=0}^{t-1} \vec{\tau}^3[p-i, t-2]$ satisfy: for $p \geq j \geq$
848 $p-t+2$, $\tau_j^4 = 1$; for $p-t \geq j \geq p-2t+2$, $\tau_j^4 = -2$; for $p-2t \geq j \geq p-3t+2$, $\tau_j^4 = 1$; and $\tau_j^4 = 0$ for all
849 the other values of j . Furthermore it is 6-balanced.

850 Proof. This easily follows from the values of the coordinates of $\vec{\tau}^3[p-i, t-2]$. Note that $\tau_{p-t+1}^4 = -1+1 = 0$
851 and $\tau_{p-2t+1}^4 = 1-1 = 0$.

852 We will prove by induction that $\sum_{i=0}^r \vec{\tau}^3[p-i, t-2]$ is 6-balanced for any $r \leq t-1$. That is true for
853 $r = 0$ as $\vec{\tau}^3[p, t-2]$ is 4-balanced. Suppose that $\sum_{i=0}^r \vec{\tau}^3[p-i, t-2]$ is 6-balanced for some $r \leq t-2$.
854 We apply Lemma 20 with $\vec{\Phi}^1 = \sum_{i=0}^r \vec{\tau}^3[p-i, t-2]$ and $\vec{\Phi}^2 = \vec{\tau}^3[p-r-1, t-2]$. By our induction
855 hypothesis, we have $h_1 = 6$. Furthermore, all the coefficients of $\vec{\Phi}^1$ are greater than -2 , that follows from
856 Claim 29 when $r = 0$ and from Claim 30 when $r > 0$. Hence, $\min_i \Phi_i^1 = -2$. Finally, $\vec{\Phi}^2$ is 4-balanced.
857 Hence, $\sum_{i=0}^{r+1} \vec{\tau}^3[p-i, t-2]$ is also $\max(6, 4+2) = 6$ -balanced and by induction $\vec{\tau}^4[p, t]$ is 6-balanced. \diamond

858 *Proof. (End of the proof of Theorem 16)* The cascade $\vec{\tau}^4[p, t]$ consists of t cascades $\vec{\tau}^3[p-i, t-2]$,
859 each of them consisting of a cascade $\vec{\gamma}^3[p-i, t-2]$ and $(t-1)$ 1-deviations. $\vec{\gamma}^3[p-i, t-2]$ itself consists
860 of $t-2$ cascades $\vec{\gamma}^2[p-i, t-1]$, each of them consisting of $t-1$ cascades $\vec{\gamma}^1[p-i, t-1]$ each of them
861 consisting of t 3-deviations $\vec{\gamma}[p-i]$. Thus, the cascade $\vec{\tau}^4[p, t]$ contains $t^2(t-1)(t-2) = \theta(t^4)$ 3-deviations
862 (plus $t(t-1)$ 1-deviations, but that is negligible).

863 By Claim 30, $\vec{\tau}^4[p, t]$ is 6-balanced. If we choose as starting partition one with 6 groups of each size
864 $i, 1 \leq i \leq p-1$, then the cascade is valid. It is easy to obtain such a starting partition from the initial
865 partition which consists of n groups of size 1. Indeed we can create a group of any size i ; for that we choose
866 a specific group of size 1 and successively move with $(i-1)$ 1-deviations one element of $i-1$ other groups of
867 size 1 to form a group of size i . We do it 6 times for each size $i, 1 \leq i \leq p-1$. Of course that is possible
868 only if $n \geq 6p(p-1)/2$.

869 Finally, note that in order for all the coordinates in the cascade to have a meaning we must choose p and
870 t such that $p-4t+3 \geq 0$. Let us choose $p = 4t-3$; then the number of vertices is $n = 6 \sum_{i=0}^{p-1} i = 6p(p-1)/2$
871 $= (48t-36)(t-1) = \Theta(t^2)$.

872 In summary we have built a cascade which contains $\Theta(t^4) = \Theta(n^2)$ 3-deviations. Therefore, $L(3, n) =$
873 $\Omega(n^2)$. \square

874 6 Conclusion

875 Our analysis shows that, in the case of a polarized society (*i.e.*, with only friends and enemies), even modest
876 cooperations between $k = 4$ users can surprisingly delay the convergence time of the dynamics of formation
877 of groups. This result was obtained by using an all new combinatorial approach for this problem that may
878 be of independent interest for further studies. In particular, we would find it interesting to know whether
879 our techniques in this paper could be applied to establish the asymptotic worst-case time of convergence for
880 this dynamics, for every $k \geq 3$.

881 From the complexity point of view, the main open question is to determine the complexity of computing
882 a k -stable partition for $k \geq 4$. In particular, is this problem PLS-complete? The reduction presented
883 in [Duc16a] – where it is proved that computing a 1-stable partition is already PTIME-hard – may be
884 helpful in order to answer this open question.

885 Finally, we intend to study several possible generalizations of the present model of group formation.
886 In [ABK⁺16], Angel et al. introduced the more general MAX k -COLORED CLUSTERING problem, where in-
887 formally speaking, there are now different categories of information considered (represented by edge-colors).
888 A game-theoretic study of this problem remains to be done. Kleinberg and Ligett also proposed in [KL13]
889 to include in the model different levels of friendship and conflict, as well as the possibility for neutral inter-
890 actions (represented by positive, negative and zero edge-weights, respectively). In a forthcoming paper, we
891 study this generalization. Our main results there relate the existence of a k -stable partition, for any fixed k ,
892 with the palette of social evaluations (= edge-weights) available.

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896 References

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