On robust parameter estimation in finite-time without persistence of excitation

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***Abstract*—The problem of adaptive estimation of constant parameters in the linear regressor model is studied without the hypothesis that regressor is Persistently Excited (PE). First, the initial vector estimation problem is transformed to a series of the scalar ones using the method of Dynamic Regressor Extension and Mixing (DREM). Second, several adaptive estimation algorithms are proposed for the scalar scenario. In such a case, if the regressor may be nullified asymptotically or in a finite time, then the problem of estimation is also posed on a finite interval of time. Robustness of the proposed algorithms with respect to measurement noise and exogenous disturbances is analyzed. The efficiency of the designed estimators is demonstrated in numeric experiments for academic examples.**

I. INTRODUCTION

Estimation and identification of model parameters of linear and nonlinear systems is an important problem, whose solution forms a basis for posterior state estimation and control synthesis [1]. One of the most popular problem statements is presented by the static linear regression model [1], [2] (the basic problem of on-line estimation of constant parameters of the n-dimensional linear regression):

$$\begin{aligned} x(t) &= \omega^\top(t)\theta,\ t\in\mathbb{R}, \\ y(t) &= x(t)+w(t), \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}$ is the model output, $\theta\in\mathbb{R}^n$ is the vector of unknown constant parameters that is necessary to estimate, $\omega:\mathbb{R}\to\mathbb{R}^n$ is the regressor function (usually assumed to be bounded and known), $y(t)\in\mathbb{R}$ is the signal available for measurements with a measurement noise $w:\mathbb{R}\to\mathbb{R}$ (here $\mathbb{R}$ denotes the set of real numbers). The noise $w(t)$ may also represent, for example, the time-varying deviations of $\theta(t)$ (if it is not a constant), then $w(t)$ is proportional to the derivative $\dot\theta(t)$. A conventional additional requirement, which is usually imposed on the regressor function ω, consists in its persistent excitation [2], [3], *i.e.* it is frequently assumed that there exist $\ell>0$ and $\vartheta>0$ such that

$$\int_t^{t+\ell}\omega(s)\omega^\top(s)ds\geq\vartheta I_n$$

for any $t\in\mathbb{R}$, where I_n denotes the identity matrix of dimension $n\times n$. It is a well-known fact [4], that if ω is PE, then for the linear estimation algorithm

$$\dot{\hat\theta}(t)=\gamma\omega(t)\left(y(t)-\omega^\top(t)\hat\theta(t)\right),\ \gamma>0, \tag{2}$$

where $\hat\theta(t)\in\mathbb{R}^n$ is the estimate of θ, its estimation error $e(t)=\theta-\hat\theta(t)$ dynamics is globally exponentially stable at the origin provided that $w(t)\equiv 0$ for all $t\in\mathbb{R}$; and it is input-to-state stable (ISS) for any essentially bounded noise w [3], [5]. There are several recent results attempting to relax the requirement on persistence of excitation and imposing some nonuniform in initial time restrictions on the regressor ω, which lead to a (nonuniform) global asymptotic stability of (2) [6], [7], and without introducing additional constraints they also guarantee the integral ISS property of $e(t)$ with respect to $w(t)$ [8] (with additional restrictions, a kind of nonuniform in initial time ISS can be recovered). The linear estimation algorithm (2) is one of the most popular methods to solve (1), but there are many other approaches, see for example [2], [3] or a recent work [9], and references there. However, almost all of these results are based on the assumption that the regressor $\omega(t)$ is PE or has an analogous property. For example, in the concurrent learning approach [10], [11], it is assumed that a kind of PE is satisfied on a finite interval of time only.

In this work we will consider the problem (1) without an assumption on interval excitation of ω. Moreover, for our design we will implicitly assume that the norm $|\omega(t)|$ of the regressor $\omega(t)$ might be converging to zero (asymptotically or in a finite time), which implies that the norm of the model output $|x(t)|\leq|\omega(t)||\theta|$ is also converging to zero that leads to a necessity of estimation of θ during the time interval when $|x(t)|\geq|w(t)|$, *i.e.* when initially the measured output $y(t)$ disposes the information about $x(t)$ and it is not hidden completely by the noise $w(t)$. Therefore, we will consider the problem of finite-time estimation in (1) for $t\in[0,T]$, where T is fixed, without persistence of excitation. For this purpose, first, we will apply DREM [9] to (1) in order to decouple this vector θ estimation problem on a series of independent problems of estimation of scalar components θ_i for $i=1,\ldots,n$. Such a decomposition is important since for scalar estimation algorithms it is possible to accelerate the convergence by parameter tuning [5], [8]. And, second, several algorithms are proposed for the scalar estimation without the regressor excitation and with the convergence in a finite-time. Robustness abilities of these algorithms are investigated (a preliminary version of this work was presented in [12], where the case $w(t)\equiv 0$ for all $t\geq 0$ was studied only). Efficiency of the proposed algorithms is demonstrated in simulations for regressors with exponentially converging upper bound and a bounded noise.

The outline of this note is as follows. Some preliminary results are introduced in section II. The problem statement is given in Section III. The estimation algorithms are designed in Section IV, where also the convergence and robustness conditions are established. Simple illustrating examples are considered in Section V.

Notation

- $\mathbb{R}_+=\{x\in\mathbb{R}:x\geq 0\}$, where $\mathbb{R}$ is the set of real number.
- $|x|$ denotes the absolute value for $x\in\mathbb{R}$ or a vector norm for $x\in\mathbb{R}^n$, and the corresponding induced matrix norm for a matrix $A\in\mathbb{R}^{n\times n}$ is denoted by $\|A\|$.

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- For a Lebesgue measurable and essentially bounded function $x : \mathbb{R} \to \mathbb{R}^n$ denote $\|x\|_\infty = \sup_{t\in\mathbb{R}} |x(t)|$, and define by $\mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$ the set of all such functions with finite norms $\|\cdot\|_\infty$; if
$$\int_{-\infty}^{+\infty} |x(t)|^2 dt < +\infty$$
then this class of functions is denoted by $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^n)$.
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing, a function $\alpha \in \mathcal{K}$ belongs to the class $\mathcal{K}_\infty$ if it is increasing to infinity. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing and $\lim_{t\to+\infty} \beta(s, t) = 0$ for each fixed $s > 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{GKL}$ if $\beta \in \mathcal{KL}$ and for each $s \in \mathbb{R}_+$ there is $T_s \in \mathbb{R}_+$ such that $\beta(s, t) = 0$ for all $t \geq T_s$.
- The identity matrix of dimension $n \times n$ is denoted as I_n.
- A sequence of integers $1, 2, ..., n$ is denoted by $\overline{1, n}$.
- Define $\mathbf{e} = \exp(1)$.
- Denote $\lceil s\rfloor^\alpha = |s|^\alpha \text{sign}(s)$ for any $s \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$.

II. Preliminaries

Consider a time-dependent differential equation [4]:

$$dx(t)/dt = f(t, x(t), d(t)),\ t \geq t_0,\ t_0 \in \mathbb{R}, \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the vector of external inputs and $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$; $f : \mathbb{R}^{n+m+1} \to \mathbb{R}^n$ is a continuous function with respect to x, d and piecewise continuous with respect to t, $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}$. A solution of the system (3) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}$ and some $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$ is denoted as $X(t, t_0, x_0, d)$, and we assume that f ensures definiteness and uniqueness of solutions $X(t, t_0, x_0, d)$ in forward time at least on some finite time interval $[t_0, t_0+T)$, where $T > 0$ may be dependent on the initial condition x_0, the input d and the initial time t_0.

A. Stability definitions

Let Ω, Ξ be open neighborhoods of the origin in $\mathbb{R}^n$, $0 \in \Omega \subset \Xi$.

Definition 1. [4], [13] At the steady state $x = 0$ the system (3) with $d = 0$ is said to be

(a) uniformly stable if for any $\epsilon > 0$ there is $\delta(\epsilon)$ such that for any $x_0 \in \Omega$ and $t_0 \in \mathbb{R}$, if $|x_0| \leq \delta(\epsilon)$ then $|X(t, t_0, x_0, 0)| \leq \epsilon$ for all $t \geq t_0$;

(b) uniformly asymptotically stable if it is uniformly stable and for any $\kappa > 0$ and $\epsilon > 0$ there exists $T(\kappa, \epsilon) \geq 0$ such that for any $x_0 \in \Omega$ and $t_0 \in \mathbb{R}$, if $|x_0| \leq \kappa$ then $|X(t, t_0, x_0, 0)| \leq \epsilon$ for all $t \geq t_0 + T(\kappa, \epsilon)$;

(c) uniformly finite-time stable[1] if it is uniformly stable and finite-time converging from Ω, *i.e.* for any $x_0 \in \Omega$ and $t_0 \in \mathbb{R}$ there exists $0 \leq T^{t_0,x_0} < +\infty$ such that $X(t, t_0, x_0, 0) = 0$ for all $t \geq T^{t_0,x_0}$. The function $T_0(t_0, x_0) = \inf\{T^{t_0,x_0} \geq 0 : X(t, t_0, x_0, 0) = 0\ \forall t \geq T^{t_0,x_0}\}$ is called the settling time of the system (3).

If $\Omega = \mathbb{R}^n$, then the corresponding properties are called global uniform stability/asymptotic stability/finite-time stability of $x = 0$.

[1] Another version of uniform finite-time stability has also been proposed in [14].

An equivalent formulation of these stability properties can be given using the classes of functions $\mathcal{KL}$ and $\mathcal{GKL}$ [4]: *i.e.* the steady state $x = 0$ of the system (3) is uniformly stable iff there is a function $\sigma \in \mathcal{K}$ such that $|X(t, t_0, x_0, 0)| \leq \sigma(|x_0|)$ for all $t \geq t_0$, any $t_0 \in \mathbb{R}$ and all $x_0 \in \Omega$; and uniformly asymptotically (finite-time) stable iff there is a function $\beta \in \mathcal{KL}$ ($\beta \in \mathcal{GKL}$) such that $|X(t, t_0, x_0, 0)| \leq \beta(|x_0|, t - t_0)$ for all $t \geq t_0$, any $t_0 \in \mathbb{R}$ and all $x_0 \in \Omega$.

In this work we will be also interested in a special stability notion defined not for all $t_0 \in \mathbb{R}$ as in Definition 1, but for a compact interval of initial times t_0 and only on a fixed interval of time [15], [16], [17], [18]:

Definition 2. [19] At the steady state $x = 0$ the system (3) with $d = 0$ is said to be

(a) short-time stable with respect to (Ω, Ξ, T^0, T_f) if for any $x_0 \in \Omega$ and $t_0 \in [-T^0, T^0]$, $X(t, t_0, x_0, 0) \in \Xi$ for all $t \in [t_0, t_0+T_f]$;

(b) short-finite-time stable with respect to (Ω, Ξ, T^0, T_f) if it is short-time stable with respect to (Ω, Ξ, T^0, T_f) and finite-time converging from Ω with the convergence time $T^{t_0,x_0} \leq t_0 + T_f$ for all $x_0 \in \Omega$ and $t_0 \in [-T^0, T^0]$;

(c) globally short-finite-time stable for $T^0 \geq 0$ if for any bounded set $\Omega \subset \mathbb{R}^n$ containing the origin there exist a bounded set $\Xi \subset \mathbb{R}^n$, $\Omega \subset \Xi$ and $T_f > 0$ such that the system is short-finite-time stable with respect to (Ω, Ξ, T^0, T_f).

The notions of Definition 2, as in the case of Definition 1, can be equivalently formulated using the functions from the classes $\mathcal{K}$ and $\mathcal{GKL}$. If we substitute $T^0 = +\infty$ and $T_f = +\infty$, then the stability properties given in Definition 2 can be reduced to their uniform counterparts from Definition 1.

In [15], [16], [17], [18] the short-time stability is considered for a fixed initial time instant t_0 only.

Remark 1. In the literature, short-time stability [17] is frequently called stability over a finite interval of time [15], [16], [18], but following [19], we prefer here the former notion to avoid a confusion with finite-time stability from [20], [21], since both concepts of stability are used in the paper.

Lemma 1. *[19] Let the system in* (3) *with* $d = 0$ *possess a Lyapunov function* $V : \mathbb{R}\times\Omega \to \mathbb{R}_+$*, where* $\Omega \subset \mathbb{R}^n$ *is an open neighborhood of the origin, such that for all* $x \in \Omega$ *and* $t \in \mathbb{R}$

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \quad \alpha_1, \alpha_2 \in \mathcal{K}_\infty;$$
$$\dot{V}(t, x) \leq -\alpha V^\eta(t, x) + k(\varpi t)V^\eta(t, x) \quad \alpha > 0,\ \varpi \in \mathbb{R},\ \eta \in (0, 1)$$

for a continuous $k : \mathbb{R} \to \mathbb{R}$, $k(0) = 0$*. Then there exist* $\varpi \in \mathbb{R}$ *and* $T^0 > 0$ *such that the system* (3) *is short-finite-time stable with respect to* (Ω', Ξ, T^0, T_f) *for some* $\Xi \subset \mathbb{R}^n$ *with* $\Omega' \subseteq \Omega \subset \Xi$ *and* $T_f > 0$.

B. Robust stability definitions

Consider the following definition of robust stability for (3) with $d \neq 0$:

Definition 3. The system (3) is said to be

(a) short-finite-time ISS with respect to (Ω, T^0, T_f, D) if there exist $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in \Omega$, all $d \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^m)$ with $\|d\|_\infty < D$ and $t_0 \in [-T^0, T^0]$:

$$|X(t, t_0, x_0, d)| \leq \beta(|x_0|, t - t_0) + \gamma(\|d\|_\infty) \quad \forall t \in [t_0, t_0 + T_f]$$

and $\beta(|x_0|, T_f) = 0$;

(b) globally short-finite-time ISS for $T^0 \geq 0$ if there exist $\beta \in \mathcal{GKL}$ and $\gamma \in \mathcal{K}$ such that for any bounded set $\Omega \subset \mathbb{R}^n$ containing the origin there is $T_f > 0$ such that the system is short-finite-time ISS with respect to $(\Omega, T^0, T_f, +\infty)$.

As we can conclude from Definition 3, if the system (3) is short-finite-time ISS, then for $d = 0$ there exists $\Xi \subset \mathbb{R}^n$, $\Omega \subset \Xi$ such that the system is short-finite-time stable with respect to (Ω, Ξ, T^0, T_f). The difference of global short-finite-time ISS and a conventional (finite-time or fixed-time) ISS [22], [23] is that in the former case the stability property is considered on a finite interval of time $[t_0, t_0 + T_f]$ only for $t_0 \in [-T^0, T^0]$.

Theorem 1. *Let the constants $\varrho > 0$, $D > 0$, $T_0, T_f \in \mathbb{R}_+$ and $\ell > 0$ be given. Let the system in* (3) *possess a Lyapunov function $V : \mathbb{R} \times \Omega \to \mathbb{R}_+$, where $\Omega = \{x \in \mathbb{R}^n : |x| \leq \varrho\}$, such that for all $x \in \Omega$, all $|d| \leq D$ and all $t \in [-T_0, T_0 + T_f]$:*

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \quad \alpha_1, \alpha_2 \in \mathcal{K}_\infty;$$
$$\dot{V}(t,x) \leq -u(t)V^\eta(t,x) + \kappa(|d|) \quad \kappa \in \mathcal{K},\ \eta \in (0,1)$$

for a function $u : \mathbb{R} \to \mathbb{R}_+$ satisfying

$$\int_t^{t+\ell} u(s)ds \geq \upsilon > 0$$

for all $t \in [-T_0, T_0 + T_f]$. Then the system (3) *is short-finite-time ISS with respect to (Ω', T^0, T_f, D) for*

$$\beta(s,t) = \alpha_1^{-1}\left(3^{\frac{\eta}{1-\eta}} 4 \left(\alpha_2^{1-\eta}(s) - \frac{1-\eta}{2^{1-\eta}}\frac{\upsilon}{4\ell}(t-\ell)\right)^{\frac{1}{1-\eta}}\right),$$
$$\gamma(s) = \alpha_1^{-1}\left(3^{\frac{\eta}{1-\eta}} 4 \left(\left(4\ell\upsilon^{-1}\kappa(s)\right)^{\frac{1}{\eta}} + \ell\kappa(s)\right)\right)$$

and

$$\Omega' = \{x \in \Omega : \beta(|x|, 0) + \gamma(D) \leq \varrho\}$$

provided that

$$T_f \geq \ell[1 + \frac{2^{1-\eta}}{1-\eta}\frac{4}{\upsilon}\sup_{x_0 \in \Omega'} \alpha_2^{1-\eta}(|x_0|)].$$

The condition imposed on u is a version of PE for a finite interval of time, while the last inequality simply checks that the time of convergence to zero of the proposed function $\beta \in \mathcal{GKL}$ from Ω' is less than given T_f.

Proof. First of all note that for any $t \in (0, T_f]$ such that $t = \nu\ell + \mu$ with an integer $\nu \geq 1$ and $\mu \in [0, \ell)$ we have:

$$\int_{t_0}^{t_0+t} u(t)dt = \sum_{i=1}^{\nu}\int_{t_0+(i-1)\ell}^{t_0+i\ell} u(t)dt + \int_{t_0+\nu\ell}^{t_0+t} u(t)dt$$
$$\geq \sum_{i=1}^{\nu}\int_{t_0+(i-1)\ell}^{t_0+i\ell} u(t)dt \geq \nu\upsilon = \nu\upsilon\frac{t}{t}$$
$$= \upsilon\frac{\nu}{\nu\ell+\mu}t \geq \upsilon\frac{\nu}{\nu+1}\frac{t}{\ell} \geq \frac{\upsilon}{2\ell}t.$$

Next, define two sets of instants of time:

$$\mathcal{T}_1 = \{t \in [t_0, t_0+T_f) : \frac{4\ell}{\upsilon}\kappa(\|d\|_\infty) < V^\eta(t,x)\},$$
$$\mathcal{T}_2 = \{t \in [t_0, t_0+T_f) : \frac{4\ell}{\upsilon}\kappa(\|d\|_\infty) \geq V^\eta(t,x)\},$$

then obviously $[t_0, t_0 + T_f) = \mathcal{T}_1 \cup \mathcal{T}_2$. Denote $V(t) = V(t, X(t, t_0, x_0, d))$ and consider $t \in [t_1, t_2) \subset \mathcal{T}_1$, then

$$\dot{V}(t) \leq -\left(u(t) - \frac{\upsilon}{4\ell}\right)V^\eta(t),$$

and integrating this inequality we obtain:

$$V^{1-\eta}(t) \leq V^{1-\eta}(t_1) - (1-\eta)\int_{t_1}^{t}\left(u(s) - \frac{\upsilon}{4\ell}\right)ds$$

for all $t \in [t_1, t_2)$. Thus,

$$V^{1-\eta}(t) \leq V^{1-\eta}(t_1) - (1-\eta)\frac{\upsilon}{4\ell}(t - t_1)$$

for all $t \in [t_1 + \ell, t_2)$. From the inequality

$$\dot{V}(t) \leq \kappa(|d|)$$

for all $t \in [t_1, t_1 + \ell)$ we get:

$$V(t) \leq V(t_1) + \kappa(\|d\|_\infty)(t - t_1)$$

or

$$V^{1-\eta}(t) \leq 2^{1-\eta}[V^{1-\eta}(t_1) + \ell^{1-\eta}\kappa^{1-\eta}(\|d\|_\infty)].$$

Therefore,

$$\begin{aligned} V^{1-\eta}(t) \quad &\leq \quad 2^{1-\eta}[V^{1-\eta}(t_1) + \ell^{1-\eta}\kappa^{1-\eta}(\|d\|_\infty)] \\ &\quad -(1-\eta)\frac{\upsilon}{4\ell}(t - t_1 - \ell) \end{aligned}$$

for all $t \in [t_1, t_2) \subset \mathcal{T}_1$, then the estimate that is satisfied for all $t \in [t_0, t_0 + T_f) = \mathcal{T}_1 \cup \mathcal{T}_2$ is

$$V^{1-\eta}(t) \leq 2^{1-\eta}\max\{V^{1-\eta}(t_0) - \frac{1-\eta}{2^{1-\eta}}\frac{\upsilon}{4\ell}(t - t_0 - \ell),$$
$$\left(4\ell\upsilon^{-1}\kappa(\|d\|_\infty)\right)^{\frac{1-\eta}{\eta}}\} + 2^{1-\eta}\ell^{1-\eta}\kappa^{1-\eta}(\|d\|_\infty)$$

and the system is ISS as needed for the given β and γ. The constraint on T_f follows from the estimate on the finite time of convergence. □

Finally, let us formulate a useful lemma:

Lemma 2. *[24] Let $x, y \in \mathbb{R}$ and $p > 0$, then for any $\kappa_1 \in (0,1)$ there exists $\kappa_2 > 0$ such that*

$$x\lceil x + y\rfloor^p \geq \kappa_1|x|^{p+1} - \kappa_2|y|^{p+1}.$$

In particular, $\kappa_2 = \max\{1 + \kappa_1, \frac{\kappa_1}{(1-\kappa_1^{1/p})^p}\}$.

C. Dynamic regressor extension and mixing method

Consider the estimation problem of the vector $\theta \in \mathbb{R}^n$ in (1) under the following hypothesis:

Assumption 1. *Let $\omega \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$ and $w \in \mathcal{L}_\infty(\mathbb{R}, \mathbb{R})$.*

As it has been proposed in [9], in order to overcome the limitations imposed by the condition that ω is PE and also to improve the transient performance, the DREM procedure transforms (1) to n new one-dimensional regression models, which allows the decoupled estimates of θ_i, $i = \overline{1,n}$ to be computed under a condition on the regressor ω that differs from the persistent excitation.

For this purpose $n-1$ linear operators $H_j : \mathcal{L}_\infty(\mathbb{R},\mathbb{R}) \to \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ are introduced for $j = \overline{1, n-1}$ (for instance an operator H_j can be chosen as a stable linear time-invariant filter with the transfer function $W_j(s) = \frac{\alpha_j}{s+\beta_j}$, where $s \in \mathbb{C}$ is a complex variable and $\alpha_j \neq 0$, $\beta_j > 0$ are selected to filter the noise w in (1); or it can realize the delay operation with the transfer function

$W_j(s) = e^{-\tau_j s}$ for $\tau_j > 0$). Note that $y \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ under Assumption 1, then these operators are applied to the measured output $y(t)$ of (1), and using the superposition principles (the operators H_j are linear) we obtain:

$$\tilde{y}_j(t) = H_j(y(t)) = \tilde{\omega}_j^\top(t)\theta + \tilde{w}_j(t),\ j = \overline{1, n-1},$$

where $\tilde{y}_j(t) \in \mathbb{R}$ is the j^{th} operator output, $\tilde{\omega}_j : \mathbb{R} \to \mathbb{R}^n$ is the j^{th} filtered regressor function and $\tilde{w}_j(t) : \mathbb{R} \to \mathbb{R}$ is the new j^{th} noise signal, which is composed by the transformation of the noise $w(t)$ by H_j and other exponentially converging components related to the initial conditions of the filters. By construction $\tilde{\omega}_j \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$ and $\tilde{w}_j \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ for all $j = \overline{1, n-1}$. Define new vector variables

$$\tilde{Y}(t) = [y(t)\ \tilde{y}_1(t) \dots \tilde{y}_{n-1}(t)]^\top \in \mathbb{R}^n,$$
$$\tilde{W}(t) = [w(t)\ \tilde{w}_1(t) \dots \tilde{w}_{n-1}(t)]^\top \in \mathbb{R}^n$$

and a time-varying matrix

$$M(t) = [\omega(t)\ \tilde{\omega}_1(t) \dots \tilde{\omega}_{n-1}(t)]^\top \in \mathbb{R}^{n\times n},$$

then stacking the original equation (1) with the $n-1$ filtered regressor models we construct an extended regressor system:

$$\tilde{Y}(t) = M(t)\theta + \tilde{W}(t).$$

For any matrix $M(t) \in \mathbb{R}^{n\times n}$ the following equality is true [25]:

$$\text{adj}\,(M(t))\,M(t) = \det\,(M(t))\,I_n,$$

even if $M(t)$ is singular, where adj $(M(t))$ is the adjugate matrix of $M(t)$ and $\det\,(M(t))$ is its determinant. Recall that each element of the matrix adj $(M(t))$,

$$\text{adj}\,(M(t))_{k,s} = (-1)^{k+s}\mathbf{M}_{k,s}(t)$$

for all $k, s = \overline{1, n}$, where $\mathbf{M}_{k,s}(t)$ is the (k, s) minor of $M(t)$, *i.e.* it is the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting the k^{th} row and the s^{th} column of $M(t)$. Define

$$Y(t) = \text{adj}\,(M(t))\,\tilde{Y}(t),\ W(t) = \text{adj}\,(M(t))\,\tilde{W}(t),$$
$$\phi(t) = \det\,(M(t))\,,$$

then multiplying from the left the extended regressor system by the adjugate matrix adj $(M(t))$ we get n scalar regressor models of the form:

$$Y_i(t) = \phi(t)\theta_i + W_i(t) \tag{4}$$

for $i = \overline{1, n}$. Again, by construction $Y \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$, $W \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$ and $\phi \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$. For the scalar linear regression model (4) the estimation algorithm (2) takes the form:

$$\dot{\hat{\theta}}_i(t) = \gamma_i\phi(t)\left(Y_i(t) - \phi(t)\hat{\theta}_i(t)\right),\ \gamma_i > 0 \tag{5}$$

for all $i = \overline{1, n}$, where now, contrarily to (2), the estimation processes for all components of θ are decoupled, and the adaptation gain γ_i can be adjusted separately for each element of θ. However, all these estimation algorithms are dependent on the same regressor $\phi(t)$ (determinant of $M(t)$).

Define the parameter estimation error as $e(t) = \theta - \hat{\theta}(t)$, then its dynamics admits the differential equation:

$$\dot{e}_i(t) = -\gamma_i\phi(t)\left(\phi(t)e_i(t) + W_i(t)\right),\ i = \overline{1, n} \tag{6}$$

and the following result can be proven for the DREM method:

Proposition 1. *Consider the linear regression system* (1) *under Assumption 1. Assume that for the selected operators* $H_j : \mathcal{L}_\infty(\mathbb{R},\mathbb{R}) \to \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$, $j = \overline{1, n-1}$*:*

$$\int_{t_0}^{+\infty} \phi^2(t)dt = +\infty \tag{7}$$

for any $t_0 \in \mathbb{R}$*, then the estimation algorithm* (5) *has the following properties:*

(A) *If* $\|W\|_\infty = 0$*, then the system* (6) *is globally asymptotically stable at the origin iff* (7) *is valid.*

(B) *For all* $W \in \mathcal{L}_2(\mathbb{R},\mathbb{R}^n)$ *we have* $e \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$*, in addition, for all* $t_0 \in \mathbb{R}$ *and* $e_i(t_0) \in \mathbb{R}$*:*

$$\begin{aligned} |e_i(t)| \ \leq\ & \mathbf{e}^{-\gamma_i\int_{t_0}^t \phi^2(\tau)d\tau}|e_i(t_0)| \\ & + \sqrt{\frac{\gamma_i}{2}}\sqrt{1 - \mathbf{e}^{-2\gamma_i\int_{t_0}^t \phi^2(\tau)d\tau}}\sqrt{\int_{t_0}^t W_i^2(s)ds} \end{aligned}$$

for all $t \geq t_0$ *and* $i = \overline{1, n}$*.*

Proof. If $\|W\|_\infty = 0$, then the system (6) can be rewritten as follows:

$$\dot{e}_i(t) = -\gamma_i\phi^2(t)e_i(t),\ i = \overline{1, n},$$

and its global uniform stability can be established considering a Lyapunov function $V(e) = e^\top \text{diag}\{\gamma_i^{-1}\}_{i=1}^n e$. The equivalence of global convergence of $e(t)$ to zero and (7) has been established in [9]. Thus, the part (A) is proven.

If $\|W\|_\infty \neq 0$, then solutions of (6) can be calculated analytically for all $t \geq t_0$ and $e_i(t_0) \in \mathbb{R}$:

$$e_i(t) = \mathbf{e}^{-\gamma_i\int_{t_0}^t \phi^2(\tau)d\tau}e_i(t_0) - \gamma_i\int_{t_0}^t \mathbf{e}^{-\gamma_i\int_s^t \phi^2(\tau)d\tau}\phi(s)W_i(s)ds$$

for all $i = \overline{1, n}$. Further, for all $t \geq t_0$ and $e_i(t_0) \in \mathbb{R}$:

$$|e_i(t)| \leq \mathbf{e}^{-\gamma_i\int_{t_0}^t \phi^2(\tau)d\tau}|e_i(t_0)| + \gamma_i\int_{t_0}^t |\mathbf{e}^{-\gamma_i\int_s^t \phi^2(\tau)d\tau}\phi(s)||W_i(s)|ds$$

$$\leq \mathbf{e}^{-\gamma_i\int_{t_0}^t \phi^2(\tau)d\tau}|e_i(t_0)| + \gamma_i\sqrt{\int_{t_0}^t \mathbf{e}^{-2\gamma_i\int_s^t \phi^2(\tau)d\tau}\phi^2(s)ds\int_{t_0}^t W_i^2(s)ds},$$

where the Cauchy–Schwarz inequality was used on the last step. Note that

$$\frac{\partial}{\partial s}\mathbf{e}^{-2\gamma_i\int_s^t \phi^2(\tau)d\tau} = 2\gamma_i\mathbf{e}^{-2\gamma_i\int_s^t \phi^2(\tau)d\tau}\phi^2(s),$$

hence,

$$2\gamma_i\int_{t_0}^t \mathbf{e}^{-2\gamma_i\int_s^t \phi^2(\tau)d\tau}\phi^2(s)ds = 1 - \mathbf{e}^{-2\gamma_i\int_{t_0}^t \phi^2(\tau)d\tau},$$

then the desired estimate of the part (B) follows. □

Obviously, if the signal $\phi(t)$ is PE, then the error dynamics is ISS with respect to $W \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$ and an exponential convergence rate can be guaranteed [3], [5], [8].

An interested reader is directed to [9] for a comparison of the condition (7) imposed for $\phi(t)$ and the requirement that $\omega(t)$ is PE, and also for the discussion about a possibility to select the operators H_j, $j = \overline{1, n-1}$ in a way enforcing the condition (7) for $\phi(t)$ while initially $\omega(t)$ does not admit a persistent excitation. Also an inverse question can be posed: assume that $\omega(t)$ is PE, is there a guarantee or restrictions to be imposed for the operators H_j, $j = \overline{1, n-1}$ that the condition (7) is satisfied, which is partially addressed in [26], [27]. In other words, can additional filtering, which leads to a decoupled scalar regressor model, destroys good estimation abilities in (1)? But since in this work we do not need the conditions of persistence of excitation, then there is no obstruction for us to use DREM, as it is stated in the problem statement below.

III. Problem statement

Consider the static linear regression model (1) under Assumption 1, and assume that the DREM method has been applied in order to reduce the initial problem of vector estimation to n scalar regressor models in the form (4). Note that $Y \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$, $W \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R}^n)$ and $\phi \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ under Assumption 1 and due to properties of the DREM approach. Relaxing the condition (7), which is also imposed on an infinite interval of time, assume that $\omega(t)$ may not admit PE, and that $\phi(t)$ is a converging function of time (more precise properties will be formulated separately for the designed algorithms).

It is also assumed that the set of admissible values for θ is known:

Assumption 2. *A constant $\bar{\theta} > 0$ is given such that $\theta \in \Omega = [-\bar{\theta},\bar{\theta}]^n$.*

It is necessary to propose an algorithm generating an estimate $\hat{\theta}(t) \in \mathbb{R}^n$ of the vector of unknown parameters $\theta \in \mathbb{R}^n$, and for $\|W\|_\infty = 0$ providing the property of short-finite-time stability with respect to (Ω,Ω,T^0,T) (see Definition 2) of the estimation error $e(t) = \theta - \hat{\theta}(t)$ dynamics under assumptions 1 and 2 for some given T^0 and T. If $\|W\|_\infty \neq 0$ then short-finite-time ISS with respect to $(\Omega,T^0,T,\mathcal{W})$ (see Definition 3) has to be guaranteed for some $\mathcal{W} > 0$.

Since by applying DREM method the problem is decoupled on n independent ones, for brevity of notation, we will further omit the index i in (4) by assuming that $n = 1$:

$$Y(t) = \phi(t)\theta + W(t), \tag{8}$$

then $\theta \in \mathbb{R}$, $Y \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$, $W \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$ and $\phi \in \mathcal{L}_\infty(\mathbb{R},\mathbb{R})$.

IV. Design of estimation algorithms converging in short-finite-time

Three different solutions to the posed estimation problem are proposed in this section, whose difference consists in the requirements imposed on excitation of $\phi(t)$ and on the guaranteed robustness abilities with respect to $W(t)$.

A. Algorithm 1

Consider an adaptive estimation algorithm proposed in [28], [29], [30], [31]:

$$\dot{\hat{\theta}}(t) = \gamma\phi(t)\left\lceil Y(t) - \phi(t)\hat{\theta}(t)\right\rfloor^\alpha, \ \gamma > 0, \ \alpha \in [0,1), \tag{9}$$

which admits the following properties:

Proposition 2. *Let assumptions 1 and 2 be satisfied, and*

$$\int_t^{t+\ell} |\phi(s)|^{1+\alpha} ds \geq \upsilon \tag{10}$$

for all $t \in [-T_0, T_0 + T]$ and some $\ell \in (0,T)$, $\upsilon > 0$. Take

$$\gamma \geq \frac{2^{\frac{5-3\alpha}{2}}}{1-\alpha}\frac{\bar{\theta}^{1-\alpha}}{\left(\frac{T}{\ell}-1\right)\upsilon},$$

then the estimation error $e(t) = \theta - \hat{\theta}(t)$ dynamics for (9) *with $\hat{\theta}(t_0) = 0$ is short-finite-time ISS with respect to $(\Omega,T^0,T,+\infty)$.*

Proof. The error dynamics for the estimation algorithm (9) can be written as follows:

$$\dot{e}(t) = -\gamma\phi(t)\left\lceil\phi(t)e(t) + W(t)\right\rfloor^\alpha.$$

Consider a Lyapunov function candidate $V(e) = 0.5e^2$, whose derivative has an upper estimate for some $\kappa_1 \in (0,1)$ and $\kappa_2 > 0$ coming from Lemma 2:

$$\begin{aligned}\dot{V}(t) &= -\gamma e(t)\phi(t)\left\lceil\phi(t)e(t) + W(t)\right\rfloor^\alpha \\ &\leq -\gamma\kappa_1|e(t)\phi(t)|^{1+\alpha} + \gamma\kappa_2|W(t)|^{1+\alpha} \\ &= -2^{\frac{1+\alpha}{2}}\gamma\kappa_1|\phi(t)|^{1+\alpha}V^{\frac{1+\alpha}{2}}(t) + \gamma\kappa_2|W(t)|^{1+\alpha}\end{aligned}$$

for any $e(t) \in \mathbb{R}$ and $W(t) \in \mathbb{R}$. Then under the imposed restrictions for ϕ, the system is short-finite-time ISS with respect to $(\Omega,T^0,T,+\infty)$ due to Theorem 1. The imposed restriction on γ guarantees that there exist $\kappa_1 \in (0,1)$ such that all trajectories from Ω converge to the origin faster than T if $\|W\|_\infty = 0$. ☐

It is important to clarify the differences between the restrictions imposed on $\phi(t)$ in propositions 1 and 2 for the case $\|W\|_\infty = 0$. Note that (7) allows us to establish the fact of asymptotic convergence of the estimation error $e(t)$, but it does not permit to evaluate the rate of convergence. Of course, the condition (7) can be formulated on a finite interval of time:

$$\int_{t_0}^{t_0+T}\phi^2(t)dt = +\infty$$

that implicitly implies unboundedness of $\phi(t)$ contrarily to its admissible convergence in (10). The condition (7) can also be strengthened to a usual PE condition (on a finite interval of time), then a practical stability can be obtained only.

B. Algorithm 2

To simplify the notation, suppose in this subsection that $\phi(t) > 0$ for all $t \in [t_0, t_0+T]$. An appealing idea to design a finite- or fixed-time converging system using time-varying feedbacks is presented in [32]. This idea in our context has an interpretation in the form of the following estimation algorithm:

$$\dot{\hat{\theta}}(t) = \gamma(\phi(t)-\beta)^{-\eta}\left(Y(t) - \phi(t)\hat{\theta}(t)\right), \ \gamma > 0, \ \eta > 1 \tag{11}$$

for $\hat{\theta}(t_0) = 0$, which has well-defined solutions while $\phi(t)$ is separated with $\beta > 0$.

Proposition 3. *Let assumption 1 be satisfied and there exist $\rho > 0$ and $\kappa > 0$ such that for any $t_0 \in [-T^0, T^0]$*

$$|\phi(t)| \geq \kappa\mathbf{e}^{-\rho(t-t_0)} \quad \forall t \in [t_0, t_0+T]$$

with $\beta = \kappa\mathbf{e}^{-\rho T}$, then dynamics of the estimation error $e(t) = \theta - \hat{\theta}(t)$ in (11) *for $\|W\|_\infty = 0$ is short-time stable with respect to (Ω,Ω,T^0,T) and asymptotically converging. In addition, if $\phi(t_0 + T) = \beta$, then the estimation error $e(t)$ of* (11) *is short-finite-time stable with respect to (Ω,Ω,T^0,T).*

Proof. The error dynamics for the estimation algorithm (11) can be written as follows:

$$\dot{e}(t) = -\gamma(\phi(t)-\beta)^{-\eta}\left(\phi(t)e(t) + W(t)\right).$$

For $\|W\|_\infty = 0$ the global uniform stability can be established considering a Lyapunov function $V(e) = 0.5e^2$. To prove convergence consider the new time variable $\tau = f(t)$, where

$$f(t) = \int_{t_0}^t \frac{\phi(s)}{(\phi(s)-\beta)^\eta}ds.$$

Such a transformation of time is well-defined under the restrictions introduced on $\phi(t)$ for $t\in[t_0,t_0+T)$ (*i.e.* $\frac{\phi(t)}{(\phi(t)-\beta)^\eta}>0$ for all $t\in[t_0,t_0+T)$). Define $\epsilon(\tau)=e(f^{-1}(\tau))$ as the estimation error variable in the new time τ, and recall the following equality

$$\frac{df^{-1}(\tau)}{d\tau}=\left(\frac{\phi(f^{-1}(\tau))}{(\phi(f^{-1}(\tau))-\beta)^\eta}\right)^{-1},$$

then

$$\frac{d\epsilon(\tau)}{d\tau}=\frac{de(f^{-1}(\tau))}{d\tau}=\frac{de(s)}{ds}\bigg|_{s=f^{-1}(\tau)}\frac{df^{-1}(\tau)}{d\tau}$$
$$=-\gamma\frac{\phi(f^{-1}(\tau))e(f^{-1}(\tau))+W(f^{-1}(\tau))}{(\phi(f^{-1}(\tau))-\beta)^\eta}\left(\frac{\phi(f^{-1}(\tau))}{(\phi(f^{-1}(\tau))-\beta)^\eta}\right)^{-1}$$
$$=-\gamma\left(e(f^{-1}(\tau))+\frac{W(f^{-1}(\tau))}{\phi(f^{-1}(\tau))}\right)=-\gamma\left(\epsilon(\tau)+\frac{W(f^{-1}(\tau))}{\phi(f^{-1}(\tau))}\right).$$

Therefore, the variable $\epsilon(\tau)$ is asymptotically decreasing for $\|W\|_\infty=0$:

$$e(f^{-1}(\tau))=\epsilon(\tau)=\mathbf{e}^{-\gamma(\tau-f(t_0))}\epsilon(f(t_0))+\int_{f(t_0)}^{\tau}\mathbf{e}^{-\gamma(\tau-s)}\frac{W(f^{-1}(s))}{\phi(f^{-1}(s))}ds$$
$$=\mathbf{e}^{-\gamma\tau}e(t_0)+\int_0^\tau\mathbf{e}^{-\gamma(\tau-s)}\frac{W(f^{-1}(s))}{\phi(f^{-1}(s))}ds,$$

where the property $f(t_0)=0$ has been used in the last step, and returning back to the time $t=f^{-1}(\tau)$ with a change of variables $s=f(\sigma)$ in the integral we obtain:

$$e(t)=\mathbf{e}^{-\gamma f(t)}e(t_0)+\int_{t_0}^{t}\mathbf{e}^{-\gamma(f(t)-f(\sigma))}\frac{W(\sigma)}{(\phi(\sigma)-\beta)^\eta}d\sigma \quad (12)$$

for all $t\in[t_0,t_0+T)$. If $\phi(t_0+T)=\beta$ and $\|W\|_\infty=0$ then $\lim_{t\to t_0+T}f(t)=+\infty$ and, consequently, $e(t_0+T)=0$. $\square$

An advantage of this lemma is that its result is independent in Assumption 2. For implementation of (11) it is enough that the condition $\phi(t_0+T)=\beta$ is verified approximately, *i.e.* $\phi(t_0+T)=\beta+\varepsilon$ for some $\varepsilon>0$ being close to machine computation precision.

A disadvantage of (11) consists in its weak robustness with respect to the measurement noise $W(t)\neq 0$. As we can conclude from (12), if $\phi(t_0+T)=\beta$ and $W(t_0+T)\neq 0$, then $e(t_0+T)=+\infty$ (recall that if $\phi(t_0+T)>\beta$ then (11) has an asymptotic convergence rate).

C. Algorithm 3

And, finally, let us introduce a united version of the algorithms (9) and (11), which borrows the nonlinear paradigm of the former and uses the time-varying feedback of the latter, but in a nonlinear fashion:

$$\dot{\hat{\theta}}(t)=\gamma\,\mathrm{sign}(\phi(t))\left\lceil Y(t)-\phi(t)\hat{\theta}(t)\right\rfloor^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}},\ \gamma>0,\ \varsigma>1, \quad (13)$$

where $\phi_{\max}=\max_{t\in[-T^0,T^0+T]}|\phi(t)|$ for given $T^0\in\mathbb{R}_+$ and $\hat{\theta}(t_0)=0$.

Proposition 4. *Let assumptions 1 and 2 be satisfied, and $\vartheta\in\mathcal{L}_\infty(\mathbb{R},\mathbb{R}_+)$ where $\vartheta(t)=\frac{W(t)}{\phi(t)}$. Take*

$$\gamma\geq\frac{\varsigma\max\{\bar{\theta}^{\frac{\varsigma-1}{\varsigma}},\bar{\theta}\}\mathbf{e}^{(\varsigma\mathbf{e})^{-1}}T^{-1}}{2\min\{1,\phi_{\max}^{\varsigma^{-1}}\}(\varsigma-1)},$$

then the estimation error $e(t)=\theta-\hat{\theta}(t)$ dynamics of (13) *is short-finite-time ISS with respect to $(\Omega,T^0,T,+\infty)$ for the input ϑ.*

Proof. The error dynamics for the estimation algorithm (13) can be written as follows:

$$\dot{e}(t)=-\gamma\,\mathrm{sign}(\phi(t))\left\lceil\phi(t)e(t)+W(t)\right\rfloor^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}.$$

Consider a Lyapunov function $V(e)=0.5e^2$ and observe that

$$r^{a(t)}(t)\geq\begin{cases}r^{a_{\min}}(t) & r(t)\geq 1\\ r^{a_{\max}}(t) & r(t)<1\end{cases}\geq\min\{1,r_{\min}^{a_{\max}}\}$$

for any $r:\mathbb{R}\to\mathbb{R}_+$ and $a:\mathbb{R}\to\mathbb{R}_+$ such that $r_{\min}=\inf_{t\in\mathbb{R}}r(t)$, $a_{\min}=\inf_{t\in\mathbb{R}}a(t)$ and $a_{\max}=\sup_{t\in\mathbb{R}}a(t)$ for some $r_{\min},a_{\min},a_{\max}\in\mathbb{R}_+$, then the time derivative of V for the estimation error dynamics admits an upper estimate:

$$\dot{V}(t)=-\gamma\,\mathrm{sign}(\phi(t))e(t)\left\lceil\phi(t)e(t)+W(t)\right\rfloor^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}$$
$$\leq-\gamma\kappa_1|\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}|e(t)|^{1+\frac{|\phi(t)|}{\varsigma\phi_{\max}}}+\gamma\kappa_2|\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}|\vartheta(t)|^{1+\frac{|\phi(t)|}{\varsigma\phi_{\max}}}$$
$$\leq-|\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}\xi\left(V(t)\right)+\sigma(\|\vartheta\|_\infty),$$

where

$$\xi(s)=\sqrt{2}\gamma\kappa_1\begin{cases}s^{0.5} & s\geq 1\\ s^{\frac{1+\varsigma}{2\varsigma}} & s<1\end{cases},$$
$$\sigma(s)=\gamma\kappa_2\max\{1,\phi_{\max}^{\varsigma^{-1}}\}\begin{cases}s^{1+\varsigma^{-1}} & s\geq 1\\ s & s<1\end{cases}$$

and $\kappa_1\in(0,1)$, $\kappa_2>0$ are from Lemma 2 (since $\phi(t)$ is upper bounded in amplitude such a κ_2 exists). Note that $\frac{1+\varsigma}{2\varsigma}<1$ for $\varsigma>1$. The last inequality can be represented as follows:

$$\dot{V}(t)\leq-\phi_{\max}^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}|\phi_{\max}^{-1}\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}\xi\left(V(t)\right)+\sigma(\|\vartheta\|_\infty)$$
$$\leq-\min\{1,\phi_{\max}^{\varsigma^{-1}}\}f_\varsigma(\phi_{\max}^{-1}|\phi(t)|)\xi\left(V(t)\right)+\sigma(\|\vartheta\|_\infty),$$

where

$$f_\varsigma(x)=x^{x/\varsigma}$$

for $x\in[0,1]$. Since $f_\varsigma(0)=f_\varsigma(1)=1$ and f_ς is a continuous function, then it has a critical point on the interval $[0,1]$. Calculate

$$\frac{\partial f_\varsigma(x)}{\partial x}=\frac{f_\varsigma(x)}{\varsigma}(1+\ln(x)),$$
$$\frac{\partial f_\varsigma^2(x)}{\partial x^2}=\frac{f_\varsigma(x)}{\varsigma}\left(\frac{1}{\varsigma}(1+\ln(x))^2+\frac{1}{x}\right),$$

then it is straightforward to conclude that it should be a minimum at the point $x_{\min}=\mathbf{e}^{-1}$ since it is the only point where $\frac{\partial f_\varsigma(x)}{\partial x}=0$ and $\frac{\partial f_\varsigma^2(x)}{\partial x^2}\geq 0$ for all $x\in[0,1]$. Therefore, $f_\varsigma(x_{\min})=\mathbf{e}^{-\frac{1}{\varsigma\mathbf{e}}}$ and

$$\inf_{\varsigma>1}f_\varsigma(x_{\min})=\mathbf{e}^{-\mathbf{e}^{-1}},$$

then

$$\dot{V}(t)\leq-\min\{1,\phi_{\max}^{\varsigma^{-1}}\}\mathbf{e}^{-\mathbf{e}^{-1}}\xi\left(V(t)\right)+\sigma(\|\vartheta\|_\infty)$$

and using the result of Theorem 1, the system is ISS as needed.

In order to evaluate more accurately the time of convergence to the origin assume that $\|W\|_\infty = 0$ then

$$\begin{aligned} \dot{V}(t) &\leq -\gamma\kappa_1|\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}|e(t)|^{1+\frac{|\phi(t)|}{\varsigma\phi_{\max}}} \\ &= -\gamma\kappa_1|\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}\bar{\theta}^{1+\frac{|\phi(t)|}{\varsigma\phi_{\max}}}|\bar{\theta}^{-1}e(t)|^{1+\frac{|\phi(t)|}{\varsigma\phi_{\max}}} \\ &\leq -\gamma\kappa_1|\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}\min\{\bar{\theta},\bar{\theta}^{1+\varsigma^{-1}}\}|\bar{\theta}^{-1}e(t)|^{1+\varsigma^{-1}} \\ &= -2^{\frac{1+\varsigma}{2\varsigma}}\gamma\kappa_1\min\{1,\bar{\theta}^{-\varsigma^{-1}}\}\phi_{\max}^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}|\phi_{\max}^{-1}\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}V^{\frac{1+\varsigma}{2\varsigma}}(t) \\ &\leq -\tau\gamma V^{\frac{1+\varsigma}{2\varsigma}}(t), \end{aligned}$$

where $\tau = 2^{\frac{1+\varsigma}{2\varsigma}}\kappa_1\min\{1,\bar{\theta}^{-\varsigma^{-1}}\}\min\{1,\phi_{\max}^{\varsigma^{-1}}\}\mathrm{e}^{-(\varsigma\mathrm{e})^{-1}}$. Consequently,

$$V(t) \leq \left(V^{\frac{\varsigma-1}{2\varsigma}}(t_0) - \gamma\frac{\tau}{\varsigma}(\varsigma-1)(t-t_0)\right)^{\frac{2\varsigma}{\varsigma-1}}$$

for all $t \in [t_0, t_0+T)$, and the restriction on γ follows ensuring the desired convergence time T. □

Thus, the idea of the algorithm (13) consists in the utilization of a nonlinearity such that the function $|\phi(t)|^{\frac{|\phi(t)|}{\varsigma\phi_{\max}}}$ becomes separated with zero overcoming the absence of excitation in the system. The price for that is the robustness with respect to a noise W with a well-defined ratio $\frac{W(t)}{\phi(t)}$. In particular, if $\phi(t) \geq \beta$ for all $t \in [t_0, t_0+T)$ for an arbitrary small $\beta > 0$, then the algorithm stays continuous being independent in the excitation of (1) and robust with respect to the noise W.

Remark 2. One of the most important features of estimation algorithms, after estimation error convergence in the ideal case, is their robustness with respect to measurement noises. In our case, since the regressor $\phi(t)$ may converge to zero, the appearance of $W(t) \neq 0$ additionally limit the time of convergence, since it is reasonable to use the output $Y(t)$ for estimation with $t \in [t_0, t_0+T]$ only while

$$|Y(t)| > |W(t)| + \varepsilon$$

for some $\varepsilon > 0$. If $Y(t)| \leq |W(t)| + \varepsilon$ (or $|Y(t)|$ is almost equal to $|W(t)|$ for a sufficiently small ε), then the measured output mainly contains the measurement noise, and it is ambiguous to ask an algorithm to estimate θ due to a bad ratio between the signal and the noise. In this sense the requirement, that the signal $\frac{W(t)}{\phi(t)}$ is well-defined, is not much restrictive (roughly speaking it just assumes that the ratio between the useful signal and the noise lies in reasonable limits).

If the regressor $\phi(t)$ is just asymptotically converging without crossing zero, then all these algorithms, (9), (11) and (13), can be applied for any finite $T > 0$ and $t_0 \in \mathbb{R}$. Let us check their performances in examples with converging regressors $\phi(t)$.

V. Example

Select

$$\theta = 1,\ \bar{\theta} = 1,\ t_0 = 0,\ T = 5,$$

and for the first scenario choose

$$\beta = 0.01,\ \phi(t) = 2\beta + \cos^2(t)\,\mathrm{e}^{-t}.$$

Let $\alpha = 0.5$ and $\gamma = 5$ for the algorithm (9), $\gamma = 5$ and $\eta = 1.5$ for the algorithm (11), $\gamma = 1$, $\varsigma = 2$ and $\phi_{\max} = 1 + 2\beta$ for the algorithm (13). For the case $\|W\|_\infty = 0$ the results of simulation are shown in Fig. 1, for $W(t) = 0.01\sin(10t)$ the results are given in Fig. 2, where the blue line represents the linear algorithm (5), the magenta line stays for (9), the green line corresponds to (11) and the red line is for (13). As we can conclude from these results, the linear algorithm (5) suffers from the absence of excitation, all the rest converge in a finite time to the ideal value of θ in the absence of noise, however, in the presence of noise the algorithm (11) becomes unstable (since the perturbation $W(t)$ is divided by the time-varying gain approaching 0), while the algorithms (9) and (13) generate bounded trajectories, but (13) becomes more sensitive to the noise since it is approaching discontinuity.

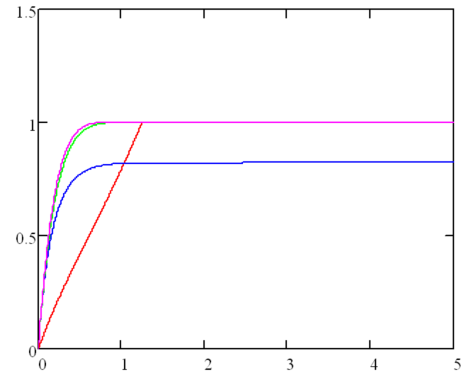

Figure 1. The results of simulation without noise in case 1

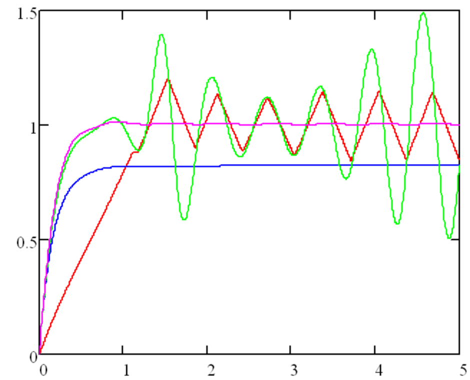

Figure 2. The results of simulation with noise in case 1

For the second scenario, let

$$\phi(t) = \cos(2t)\,\mathrm{e}^{-t},$$

then $\phi(t)$ crosses zero several times on the interval $[0, T)$. Thus, the algorithm (11) cannot be used in such a case, while the corresponding results for the algorithms (9), (13) and (5) (with the same values of parameters) are given in figures 3 and 4 for noise-free and noisy cases, respectively.

All these results confirm the theoretical findings of this note.

VI. Conclusions

The problem of adaptive estimation of a vector of constant parameters in the linear regressor model is studied without the hypothesis that regressor is PE. For this purpose, the initial estimation problem is transformed to a series of the scalar ones using the DREM approach. Three adaptive estimation algorithms are proposed for the scalar scenario converging in a short-finite-time, their robust abilities with respect to the measurement noise are analyzed using the notion

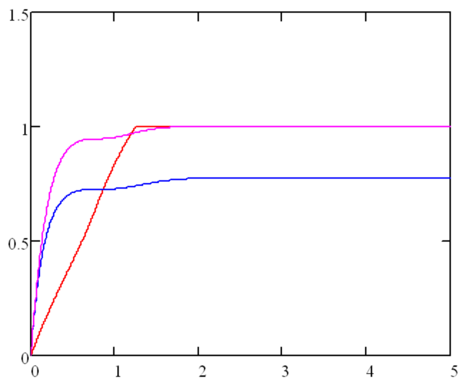

Figure 3. The results of simulation without noise in case 2

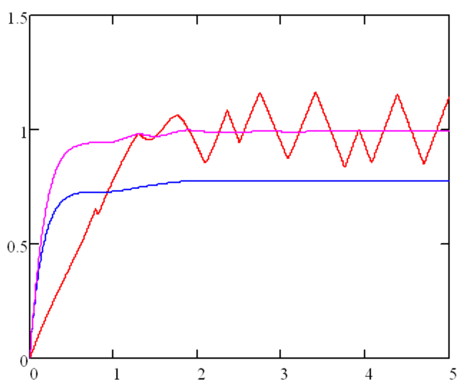

Figure 4. The results of simulation with noise in case 2

of short-finite-time ISS. A sufficient condition for the latter is also presented. Future directions of research have to include relaxation of the introduced hypotheses and tuning guidelines development for the proposed estimation algorithms.

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