Analogy between concepts

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Abstract

Reasoning by analogy plays an important role in human thinking, in exploring parallels between situations. It enables us to explain by comparing, to draw plausible conclusions, or to create new devices or concepts by transposing old ones in new contexts. A basic form of analogy, called Analogical Proportion (AP), describes a particular relation between four objects of the same kind, e.g. "A calf is to a bull as a foal is to a stallion". It is only recently that researchers have started to study APs in a formal way and to use their properties in different tasks of artificial intelligence (AI). This paper follows this line of research. Specifically, we are interested in giving the definition and some properties of an AP in lattices, a widely used structure in AI. We give general results before focusing on Concept Lattices, with the goal to investigate how analogical reasoning could be introduced in the framework of Formal Concept Analysis (FCA). This leads us to define an AP between formal concepts and to give algorithms to compute them, but also to point to special subcontexts, called analogical complexes. They are themselves organized as a lattice, and we show that they are closely related to APs between concepts, while not needing the complete construction of the lattice. To finish, we relate them to another form of analogy, called Relational Proportion, which involves two universes of discourse, e.g. "Carlsen is to chess as Mozart is to music", which leads to the more compact way of saying "Carlsen is the Mozart of chess", which is not anymore a relation between four objects of the same kind, but can be interpreted as well in FCAs framework.

Keywords: Analogy, analogical reasoning, analogical proportion, analogy in lattices, formal concept analysis.

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1. Introduction

Analogical reasoning exploits parallels between situations. It enables us to state analogies for explanation purposes, to draw plausible conclusions, or to create new devices by transposing old ones in new contexts. As such, reasoning by analogy plays an important role in human thinking, as it is widely acknowledged. For this reason, it has been studied for a long time, in philosophy, e.g., [12, 23], in cognitive psychology, e.g. [20, 25, 26], and in artificial intelligence, e.g., [22, 24, 38], under various approaches [17, 52, 37].

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The classical view of analogy describes the parallel between two situations (or universes), which are described in terms of objects, properties of the objects, and relations linking the objects, by identifying one-to-one correspondences between objects in situation 1 and objects in situation 2, on the basis of the properties and relations that hold both for the objects in situation 1 and for the objects in situation 2. This is the basis of the structure-mapping theory in cognitive psychology [19, 15], see also [66] in artificial intelligence. Usual illustrations of this view are Rutherford's analogy between the atom structure and the solar system, or the similarity between electricity and hydraulics equations. A formal account of this view, using second order logic, has been provided in the HDTP model [21].

An analogical parallel between situations may be also expressed by comparing proportions or relations. In that respect, a key pattern which is associated with the idea of analogical reasoning, is what medieval scholastics called "analogy of proportionality", namely statements of the form "A is to B as C is to D", as in the examples "a calf is to a bull as a foal is to a stallion", or "gills are to fishes as lungs to mammals".

In the first example, the four items involved are animals, which are thus pairwise comparable. In this first case, the "is to" suggests that A and B are compared on the one hand, that C and D are compared on the other hand, while the 'as' expresses that the results of these two comparisons are identical. In such a statement where the four items A, B, C, D belong to *the same* category, we shall speak of *analogical proportion*. Analogical proportions can be viewed as a symbolic counterpart of numerical proportions, in particular arithmetic proportions (i.e., $a - b = c - d$) and geometric proportions (i.e., $\frac{a}{b} = \frac{c}{d}$, or "the product of the extremes is equal to the product of the means"). Indeed, besides symmetry, this view agrees with a crucial postulate of analogical proportions, namely that one can exchange B and C in the proportion (as well as A and D). This similarity of behavior between numerical proportions and analogical proportion was already noticed in Aristotle's time.

In the second example above, the four items belong to two *different* categories, here A and C are organs and B and D classes of animals. In that latter case, the 'is to' refers to some relationship(s) existing between two items belonging to the two categories respectively, A and B on the one hand, C and D on the other hand, and the 'as' expresses the identity of this/these relationship(s). In this second type of analogical statement where two categories are present, we shall speak of *relational proportion*.

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Analogical proportions can be encountered in a number of works on analogy. In that respect, the COPYCAT project [24] is especially worth mentioning for the development of an analogy-making program that was able to complete triples of character strings with a fourth string in order to build plausible analogical proportions. The emphasis was not on the formal modeling of a complete analogical proportion, but rather on their building from three items by choosing the fourth one among competing alternatives by means of an optimization procedure. A proposal for a formal modeling in terms of a mapping between algebras that generate the different items can be found in [10] and an algorithm is also given that computes a fourth pattern such that an identical relation holds between the items of the two pairs making the analogical proportion. A category-based view has been also advocated earlier in formal anthropology [36]. However, there exists another trend of research aiming at a simple modeling of analogical proportions.

If we except the largely ignored pioneering work of S. Klein [27] and some other less advanced works in formal anthropology (see [52] for references), it is only at the very end of the XX^{th} century and at the beginning of the XXI^{st} century that researchers working in computational linguistics have started to propose simple models of analogical proportions and to use them [16, 34, 31, 58, 29]. This has led to different types of factorization-based algebraic formalizations and their generalization to various structures ranging from semi-groups to lattices, including words over finite alphabets and finite trees [41, 5, 2]. A Boolean logic view of analogical proportions precisely expressing that the difference between A and B (resp. B and A) is the same as the difference between C and D (resp. D and C) [43], has been further investigated in [51], and extended to multiple-valued logics for handling numerical features [13]. There also exist probabilistic views of analogical proportions [62, 3]. Besides, analogical proportions have not been only developed at the formal level. They have been shown as being of particular interest for classification tasks [40, 6, 7], in computational linguistics [30, 33], or for solving IQ tests [9].

All the above reported works deal with analogical proportions. Aristotle [1] discussed metaphors as particular forms of analogies. Let us quote an excerpt that includes often cited examples:

> "Thus the cup is to Dionysus as the shield to Ares. The cup may, therefore, be called "the shield of Dionysus", and the shield "the cup of Ares". Or, again, as old age is to life, so is evening to day. Evening may therefore be called "the old age of the day", and old age, "the evening of life", or, in the phrase of Empedocles, "life's setting sun".

The sentence "the cup is to Dionysus as the shield to Ares" is an example of analogies between the two pairs (cup, Dionysus) and (shield, Ares), where each pair is made of two elements belonging to two different categories, namely objects and (Greek) gods respectively. This example may suggest a linkage between two particular areas of research in artificial intelligence, namely, on the one hand, Formal Concept Analysis (FCA) [18] where formal concepts are extracted from a formal context which is nothing but a binary relation linking two distinct universes, and on the other hand, analogical reasoning, in particular based on analogical proportions. It is worth noting that these two areas are respectively related to two basic mental activities: categorization and analogy making.

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Following this line, we have worked on the hypothesis that it could be fruitful to study how an analogical reasoning can be implemented in the framework of FCA. Since the formal concepts extracted from a formal context are structured as a lattice, the first point was to investigate how an analogical relation can be defined in lattices. This point had been briefly discussed in [58], but only a sketchy definition by factorization had been given. We have completed this definition, and shown that it is equivalent to a sort of equality between the "products" of the "extremes" and that of the "means", but only when the lattice is distributive.

This property is verified in Boolean lattices, but is generally not true in lattices produced by FCA. Hence, we have investigated the properties of analogical proportions in non distributive lattices.

The lattices produced by FCA have special properties, due to the fact that a formal concept is a pair of maximal finite subsets (one of objects and one of attributes). Therefore, we have analyzed how these properties influence the definition of analogical proportions in such lattices.

We also have kept in mind that analogical proportions are used for reasoning, and we have tried to relate their common definitions as figures of speech to the framework of FCA.

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In Section 2, we give the necessary background for the presented work. We briefly recall how analogy is used with examples stated in natural language and define several types of analogy, including Analogical Proportion (AP) and Relational Proportion (RP). We give basic notions on the formalization of an analogical proportion, namely the axioms which are commonly accepted for an AP, their Boolean counterpart as well as a function-based view of analogical proportion, and an algebraic definition based on a factorization property, which characterizes APs in commutative semigroups. We also recall the definition of the lattice of Formal Concepts extracted from a Formal Context (which describes how objects are related to attributes).

In Section 3, we give a complete definition of what is an AP in a lattice and give some original results. Firstly, we go back to a factorization-based formalization of analogical proportions proposed in [59, 57]. On this basis, these authors proposed a definition of analogical proportions in different settings such as sets, sets of sequences, set of trees, and distributive lattices. As shown in this paper, their definition suggested for the lattice setting was incomplete. We correct and complete this definition for general lattices. We show that it encompasses the Boolean lattice case that corresponds to the propositional logic encoding of analogical proportions. Then, we study the general case, introduce so-called canonical proportions, and show how analogical proportions can be decomposed into canonical ones. We also illustrate the approach in the case of a distributive lattice induced by fuzzy sets. We also introduce the notion of Weak Analogical Proportion (WAP) and we show that it is equivalent to that of factorization-based proportion only in distributive lattices.

In Section 4, we study the issue of the solving of analogical equations, i.e. the discovery of the fourth term of an analogical proportion, when the three others are given; this is a key issue for application to algorithms for analogical reasoning.

In Section 5, we study how APs can occur among Formal Concepts, computed from a Formal Context. The particular structure of the concepts induces special properties with respect to the structure of the lattice. Since concept lattices are generally non distributive, we study how WAPs can be discovered between formal concepts and give examples in simple Formal Contexts.

In Section 6, we relate the notion of analogical proportion between concepts to that of analogical complex (a notion introduced in [42]). An analogical complex is a subcontext of a Formal Context with a special (4×4) pattern. We give algorithms to identify complexes at the core of a WAP, and reciprocally construct several WAPs from a complex. Besides the formal interest, it is interesting to notice that WAPs between concepts can be constructed without the knowledge of the whole lattice of concepts, but directly from the context, with a low complexity.

Finally, in Section 7, we investigate how the notion of Relational Proportion can be formalized in terms of Formal Concept Analysis, thanks to the notion of Analogical Complex. This section, like the previous, aims at giving basis to an analogical approach of reasoning in a Formal Concept setting.

The idea of connecting analogical proportions and formal concept analysis has its roots in [44]. Moreover, a preliminary version of some of the results in this paper can be found in [39], or in [42] for Section 6.

2. Analogical proportion: basics

In this background section, we first introduce a distinction between analogical proportions that involve four objects of the same kind, and relational proportions that exhibit pairs of items referring to two different categories. We use examples stated in natural language. However, note that we are not interested here in the computational linguistics task of discovering such analogies by taking advantage of domain and function similarities between words [64, 61, 63], or by using a parallelogram-based modeling of analogical proportions in numerical settings, where words are represented by vectors of great dimension (e.g.,

[45]), following the pioneering ideas of Rumelhart and Abrahamson [56]. So, in this paper, we are neither discussing how formal contexts are obtained and how such contexts could be built by mining natural languages corpora, nor in evaluating on the basis of such corpora if a series of four words can be considered as making an analogical proportion together or not. Our aim is rather to provide a formal model in order to analyse analogical proportions and to connect them with FCA. The English sentences used as illustrations are examples that have not been automatically extracted from any corpus. We now discuss the formal modeling of analogical proportions in a Boolean perspective, before extending it to a more general lattice structure suitable for FCA.

2.1. Analogical proportion and relational proportion

An analogy is a figure of speech that compares two things that are mostly different from each other but have some traits in common. It can take different forms, which can be informally listed as follows.

Analogical Proportion (AP). An Analogical Proportion is a figure of speech stating a relation between four objects of the same kind. For example: "A calf is to a bull as a foal is to a stallion".

The general syntactic form of an AP is: $\boxed{A \text{ is to } B \text{ as } C \text{ is to } D}$ which is often denoted $(A : B :: C : D)$.

In this example, the AP could be the answer to the question "What is a calf?", assuming that bull, foal and stallion are part of the common knowledge.

It means that the dissimilarities between calf and bull are the same as between foal and stallion. Implicitly, it defines a calf as a young bovine.

The links and dissimilarities between A and B and between C and D are part of the common background knowledge. They can be deduced from the four objects.

In an AP, the four objects involved are of the same kind, i.e. they can be described in terms of the same set of applicable features. In the example above, the four items are animals. This allows strong properties to hold: inversion around the *as* connector and permutation of the extremes or of the means leave the AP true. For example "A stallion is to a bull as a foal is to a calf" can be deduced from "A calf is to a bull as a foal is to a stallion".

Relational Proportion (RP) The general syntactic form of this figure of speech is $\boxed{A \text{ is to } a \text{ as } B \text{ is to } b}$ (in complete form) or $\boxed{A \text{ is the } B \text{ of } a}$ (in reduced form). A complete Relational Proportion[1] uses four objects, of the same kind by pairs.

[1]We use this term to enforce that we distinguish this proportion from the analogical proportion. Usually, both are referred as "analogy of proportionality", e.g. see Hesse [23].

For example: “Carlsen is the Mozart of chess” or “Carlsen is to chess as Mozart is to music” introduces Carlsen as a precocious virtuoso of chess, a quality that Mozart is well known to have concerning music.

Generally speaking, a RP introduces A as an object having the same relationship with property a that object B has to property b. When b (music) is obvious from B (Mozart), it can be eliminated to give the reduced form.

The connector *as* is not exactly symmetric, since object B, attribute b and their relationship are part of the common knowledge. Also, permutations of the extremes or the means are meaningless (“music is to chess as Mozart is to Carlsen”) or awkward (“Carlsen is to Mozart as chess is to music”).

In the last section, we will introduce the notation $A \updownarrow a \upuparrows\!\!\downdownarrows B \updownarrow b$ for the Relational Proportion in the framework of Formal Context Analysis.

A Relational Proportion and an Analogical Proportion have the same use in natural language: that of introducing object A by comparing it to object B. But the situations where both can indifferently be used are not frequent. One would not spontaneously say: “The calf is the foal of the bovine”, nor “Carlsen is to an average chess player as Mozart is to an average musician”, although these phrases have a good semantics. We will examine in the framework of Formal Concept Analysis how an AP can be syntactically transformed into a RA, and vice versa, while keeping the semantics. But the choice of one phrasing or the other is a matter of language pragmatics.

Simile. A simile contains a comparison word, such as “like” or “as”. It aims at transferring a property obvious for an object to another object for which it is unexpected. The syntax form is either $\boxed{A \text{ is like } B\text{: both are } a}$ or $\boxed{A \text{ is as } a \text{ as } B}$.

Two examples: “I am as hungry as a wolf” and “Withdrawal of U.S. troops will become like salted peanuts to the American public; the more U.S. troops come home, the more will be demanded”.

As in the first example, a comparison generally works in the following manner: B is well-known to have the quality a (however a is mentioned). Then it states that A has also the quality a. In the second example, a is an unexpected quality that gives a humorous flavor to the phrase.

It could be seen as a special case of a Relational Proportion: A is to a as B is to a. Still, A differs from B, which makes it a bit troublesome. Since A and B share the quality a the reduced form of the RP “A is the B of a” becomes meaningless.

Metaphor. A metaphor is also a figure of speech that describes an object in a way that isn't literally true. It is more allusive than a simile. The syntax is $\boxed{A \text{ is } B}$ or $\boxed{A \text{ is like } B}$.

> For example "My teacher is a dragon" means that my teacher has the well known quality of a dragon: to frighten. This metaphor is a short-cut for the simile: "My teacher is as frightening as a dragon".
>
> Besides, a standard RP may have an elliptic form with the same syntax, when the background knowledge is strong. For example: "Carlsen is like Mozart" makes sense among chess and music amateurs.

Our understanding here of simile and metaphor does not claim to contribute to a study of the linguistic nature of these notions. On this latter aspect, see e.g., [28]. Actually, in the following, we will especially be interested in analogical proportions. We will investigate how it can be defined between formal concepts, which are composed of a set of objects and a set of attributes (an object possesses or not an attribute). We also define a relational proportion between two (sets of) objects and two (sets of) attributes in the setting of formal concept analysis.

2.2. Formal modeling of analogical proportions

We have seen that, when "A stallion is to a bull as a foal is to a calf" is true, then "A calf is to a bull as a foal is to a stallion" is also true. The permutation properties of an AP naturally lead to a formal definition of an AP. Another formal definition, stronger or equivalent according to the formal domains, has also been recently given in [39].

2.2.1. Axiomatic definition

Analogical proportions are usually characterized by three axioms. The first two axioms acknowledge the symmetrical role played by the pairs (x, y) and (z, t) in the proportion "x is to y as z is to t", and enforce the idea that y and z can be interchanged if the proportion is valid, just as in the equality of two numerical ratios where means can be exchanged. This view dates back to Aristotle [12]. A third (optional) axiom, called determinism, insists on the uniqueness of the solution $t = y$ for completing the analogical proportion in t: $(x : y :: x : t)$. These axioms are studied in [32].

Definition 1 (Analogical proportion). *An analogical proportion (AP) on a set X is a quaternary relation on X, i.e. a subset of X^4. An element of this subset, written $(x : y :: z : t)$, which reads "x is to y as z is to t", must obey the following axioms:*

1. Reflexivity of "as": $(x : y :: x : y)$
2. Symmetry of "as": $(x : y :: z : t) \Leftrightarrow (z : t :: x : y)$
3. Exchange of means: $(x : y :: z : t) \Leftrightarrow (x : z :: y : t)$

Then, thanks to symmetry, it can be easily seen that $(x : y :: z : t) \Leftrightarrow (t : y :: z : x)$ should also hold (exchange of the extremes). According to the first two axioms, four other formulations are equivalent to the canonical form $(x : y :: z : t)$. Finally, the eight equivalent forms of an analogical proportion are:

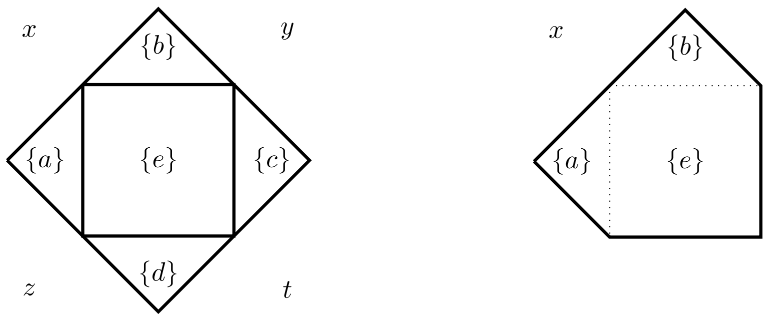


Figure 1: Venn diagram of four sets x, y, z and t in AP (left). The central square is the intersection of the four sets, and the upper triangle is $x \cap y = \{b\}$. On the right, only the set x is represented. The representations of the three other sets are obtained by rotation.

$$\begin{array}{llll} (x:y::z:t) & (z:t::x:y) & (y:x::t:z) & (t:z::y:x) \\ (z:x::t:y) & (t:y::z:x) & (x:z::y:t) & (y:t::x:z) \end{array}$$

Example. Let us consider the set $X = 2^\Sigma$ where Σ is a finite set, and four elements x, y, z, t of X. When saying of x, y, z and t that "x is to y as z is to t", we express that x differs from y in the same way that z differs from t.

If we take for example $\Sigma = \{a,b,c,d,e\}$, $x = \{a,b,e\}$ and $y = \{b,c,e\}$, we see that to transform x into y, we have to remove element a and add element c. Now, if $z = \{a,d,e\}$, we can construct t with the same operations, to obtain $t = \{c,d,e\}$. Figure 1 gives a representation of these sets in AP.

In more formal terms, with this definition, x, y, z and t have to satisfy the following properties (with $x \setminus y = x \cap \neg y$[2]):

$$x \setminus y = z \setminus t \qquad \text{and} \qquad y \setminus x = t \setminus z\,.$$

This relation linking x, y, z, t is clearly symmetrical, and satisfies the exchange of the means. Hence it is a correct definition of the analogical proportion in the Boolean setting [43], as further discussed in the following. The general representation as a Venn diagram is that of Figure 1, where each of the five regions can have any number of elements (including zero).

Boolean point of view. When the set Σ is a set of binary properties, a subset x of Σ can be used to characterize some object described by the corresponding true properties. Conversely, for an object x, each property takes the Boolean value 1 or 0 on x. If we take four objects x, y, z and t, we can also describe them by a set of vectors of size 4, one vector per property. For example, the vector (1 0 1 1) means that the associated property is false on object y and true on the three other objects (its first entry is associated to x, its second one to y, its third one to z and its last one to t). This leads to the following definition.

[2]We denote the complement in 2^Σ by the symbol $\neg$

Definition 2. *The four objects are in AP if and only if all the 4-component vectors are equal to one of the 6 following vectors, among the* $2^4 = 16$ *possibilities:*

$$\begin{array}{ccc} (0\,0\,0\,0) & (0\,1\,0\,1) & (0\,0\,1\,1) \\ (1\,1\,1\,1) & (1\,0\,1\,0) & (1\,1\,0\,0) \end{array}$$

Note that Definition 1 is not sufficient for entailing the above definition. However the 6 above 4-tuple patterns is the *minimal* Boolean model obeying the three requirements of Definition 1 for analogical proportions [53]. Indeed one may add (0 1 1 0) and (1 0 0 1) in particular, without violating the requirements of Definition 1.

It has been proved in [43] that, if the four objects are in AP, any vector $(\alpha\ \beta\ \gamma\ \delta)$ describing the four binary values of some attribute on the four objects verifies the two equivalent logical expressions:

$$\begin{array}{ccc} (\alpha \to \beta \equiv \gamma \to \delta) & \wedge & (\beta \to \alpha \equiv \delta \to \gamma) \\ (\alpha \wedge \delta \equiv \beta \wedge \gamma) & \wedge & (\alpha \vee \delta \equiv \beta \vee \gamma) \end{array}$$

The first corresponds to the fact that, as explained above, four subsets in AP are characterized by the property $(x \setminus y = z \setminus t) \wedge (y \setminus x = t \setminus z)$. The second has a less intuitive interpretation but is easy to check on Figure 1, each zone representing a subset of binary properties.

Note that in the Boolean setting, there are cases where the analogical equation in t, $x : y :: z : t$, may have no solution (e.g., if $x = 0$, and $y = z = 1$), and when there is a solution, it is unique.

A function-based view. As often mentioned (see, e.g., [50]), $u : f(u) :: v : f(v)$ looks like a good prototype of analogical proportion. Indeed a statement of the form "u is to $f(u)$ as v is to $f(v)$" sounds as making sense, namely one applies the same function f for obtaining $f(u)$ and $f(v)$ from u and v respectively. Then applying the central permutation axiom, one obtains "u is to v as $f(u)$ is to $f(v)$", which suggests that f should be injective (one-to-one) for making sure that $f(u) \neq f(v)$ as soon as $u \neq v$.

Thus, if we consider four items x, y, z, t, and we are wondering if $x : y :: z : t$ can be stated, one may think in terms of the change from x to y (and z to t), hypothesizing that y is obtained by the application of some unknown function f, i.e., $y = f(x)$. Such an intuition is implicitly underlying approaches such as the ones developed in COPYCAT [24], or in perception problems [10], for completing x, y, z with a plausible t. So t should be obtained as $f(z)$, when $y = f(x)$. This means there is no harm to assume that f is onto. Thus f is bijective and can be inverted. Still, it is also natural, especially when trying to complete x, y, z, to look at the change from x to z and to hypothesize that z is obtained from x by the application of some unknown function g, i.e., $z = g(x)$. This leads to $x : f(x) :: g(x) : f(g(x))$, which indeed sounds right. However, due to central permutation postulates we have $x : g(x) :: f(x) : f(g(x))$, and thus we should also have $a : g(x) :: f(x) : g(f(x))$. This means that $f(g(x)) = g(f(x))$, i.e., f and g commute. Moreover g, as f, is bijective and can be inverted. In such a case, completing x, y, z has a unique solution.

Let us consider the particular case where x, y, z, t are subsets of a set Σ as above. Then if $x : y :: z : t$ holds true, we can state $x \wedge \neg y = z \cap \neg t = \alpha$ and $y \cap \neg x = t \cap \neg z = \beta$, where $\alpha \cap \beta = \varnothing$. Similarly, since $x : z :: y : t$ holds also true, let $x \cap \neg z = y \cap \neg t = \varphi$ and $z \cap \neg x = t \cap \neg y = \psi$, where $\varphi \cap \psi = \varnothing$. Thus, we can introduce the two Boolean functions $f(u) = (u \cap \neg\alpha) \cup \beta$ and $g(u) = (u \cap \neg\varphi) \cup \psi$, and check that we indeed have $x : z :: y : t = x : f(x) :: g(x) : f(g(x)) = x : f(x) :: g(x) : g(f(x))$, which shows the agreement of the Boolean view with the function-based view.

2.2.2. Definition by factorization

Stroppa and Yvon [60, 57] have given a particular definition of the analogical proportion, based on the notion of *factorization*, when the set of objects is a commutative semigroup $(X, \oplus)$.

Definition 3 (Factorial analogical proportion). *A 4-tuple (x, y, z, t) in a commutative semigroup $(X, \oplus)$ is a Factorial Analogical Proportion (FAP) $(x : y :: z : t)$ when:*

1. *either $(y, z) \in \{(x, t), (t, x)\}$,*
2. *or there exists $(x_1, x_2, t_1, t_2) \in X^4$ such that $x = x_1 \oplus x_2$, $y = x_1 \oplus t_2$, $z = t_1 \oplus x_2$ and $t = t_1 \oplus t_2$.*

This definition satisfies the basic axioms of the analogical proportion (Definition 1). Hence, four elements in FAP are also in AP.

For example, in $(X, \oplus) = (\mathbb{N}^+, \times)$, with $x_1 = 2$, $x_2 = 3$, $t_1 = 5$ and $t_2 = 7$, one has $((2 \times 3) : (2 \times 7) :: (5 \times 3) : (5 \times 7))$, i.e. $(6 : 14 :: 15 : 35)$, a multiplicative analogical proportion. Note that this particular proportion corresponds equivalently to the equality: $6 \times 35 = 14 \times 15$.

Stroppa and Yvon [60, 57] have extended this definition to non commutative semigroups, in order to study APs between sequences. This subject has in particular been investigated for a practical use in [4].

The factorization-based definition coincides in the Boolean case with the 6 patterns given previously, as we shall see in the next section.

3. Analogical proportions in lattices

In this section, we are interested in studying how the definition of an analogical proportion by factorization applies to lattices. In particular we show that whether the lattice is distributive or not is an important property with respect to the property of the analogical proportion. To begin, we recall some useful basic definitions about lattices and formal concept analysis. Then we define and study analogical proportions in this framework.

3.1. Lattices

Lattices are mathematical structures commonly encountered in the semantics of representation and programming languages, in formal concept analysis, machine learning, data mining, and in other areas of computer sciences.

Definition 4. $(L, \vee, \wedge)$ *is a* lattice *when [11]:*

1. *the set L has at least two elements,*
2. *$\wedge$ and $\vee$ are two binary internal operations, both idempotent, commutative, associative, and satisfying the absorption laws: $x \vee (x \wedge y) = x \wedge (x \vee y) = x$ for all x and y in L.*

Usually, $\vee$ is called the *join* operation, and $\wedge$ the *meet* operation. Equivalently $(L, \vee, \wedge)$ can be defined as the partial ordered set $(L, \leq)$ according to the order relation $\leq$:

$$x \leq y \;\Leftrightarrow\; x \vee y = y \;\Leftrightarrow\; x \wedge y = x\,.$$

Therefore, every pair of elements (x, y) has a supremum and an infimum since $x \wedge y \leq x \leq x \vee y$ and $x \wedge y \leq y \leq x \vee y$. Due to the partial order relation, the notion of antichain is well defined: a subset A of L is an *antichain* when it is impossible to find in A two elements x and y such that $x \leq y$.

A lattice is *distributive* when $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, or equivalently $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y and z in L.

A *bounded* lattice has a greatest (or maximum) and least (or minimum) element, denoted $\top$ and $\bot$. A bounded lattice is *complemented* if for each element x, there exists has some (though not necessary unique) complementary element[3] y such that $x \wedge y = \bot$ and $x \vee y = \top$. A distributive, bounded and complemented lattice is called a *Boolean* lattice, wherein the complement is unique. A *sublattice generated by a subset* S of a lattice L is the intersection of all sublattices of L containing S.

In order to help the reading, the partial order will be added in the notation of a lattice L in the following manner: $(L, \vee, \wedge, \leq)$.

Duality theorem. If a theorem T is true in a lattice, then the dual of T is also true. This dual is obtained by replacing all occurrences of $\wedge$ (resp. $\vee$, $\leq$) by $\vee$ (resp. $\wedge$, $\geq$).

Examples.

1. $(2^{\Sigma}, \cup, \cap, \subseteq)$, where Σ is a set, is a Boolean lattice.
2. $(\mathbb{N}^{+}, \mathrm{lcm}, \mathrm{gcd}, |)$ where $(x \mid y)$ iff x divides y is a distributive lattice, with the minimum element 1 but no maximum element.

[3]Without confusing with the set theory notations, a complementary element associated to x is denoted $\neg x$.

3. The set $\mathcal{S}$ of closed intervals on $\mathbb{R}$, including $\emptyset$ and $\mathbb{R}$, is a non-distributive lattice when $\wedge$ is the set intersection and $[a, b] \vee [c, d] = [\min(a, c), \max(b, d)]$, where min and max are defined according to the order in $\mathbb{R}$.

Let us recall the following well-known property (see e.g. [11]) which is useful to study the analogical proportions in the case of Boolean lattice.

Proposition 1. *A finite Boolean lattice is isomorphic to* $(2^{\Sigma}, \cap, \cup, \subseteq)$*, where* Σ *is a finite set.*

3.2. Basics on formal concept analysis

Formal concept analysis starts with a binary relation R defined between a set $\mathcal{O}$ of objects and a set $\mathcal{A}$ of attributes. The tuple $(\mathcal{O}, \mathcal{A}, R)$ is called a *formal context*. The notation $(o, a) \in R$ or oRa means that object o has attribute a. We denote $o^{\uparrow} = \{a \in \mathcal{A} | (o, a) \in R\}$ the set of attributes of object o and $a^{\downarrow} = \{o \in \mathcal{O} | (o, a) \in R\}$ the set of objects having attribute a.

Similarly, for any subset $\mathbf{o}$ of objects, $\mathbf{o}^{\uparrow}$ is defined as $\{a \in \mathcal{A} | a^{\downarrow} \supseteq \mathbf{o}\}$. Then a *formal concept* is defined as a pair $(\mathbf{o}, \mathbf{a})$, such that $\mathbf{a}^{\downarrow} = \mathbf{o}$ and $\mathbf{o}^{\uparrow} = \mathbf{a}$. One calls $\mathbf{o}$ the *extension* of the concept and $\mathbf{a}$ its *intension*.

The set of all formal concepts is equipped with a partial order (denoted $\leq$) defined as: $(\mathbf{o}_1, \mathbf{a}_1) \leq (\mathbf{o}_2, \mathbf{a}_2)$ iff $\mathbf{o}_1 \subseteq \mathbf{o}_2$ (or, equivalently, $\mathbf{a}_2 \subseteq \mathbf{a}_1$). Then it is structured as a lattice, called the *concept lattice* of R.

Example. Let us consider the relation R, shown in Figure 2, between eight attributes $a_1, \ldots, a_8$ and nine objects $o_1, \ldots, o_9$. There is a "$\times$" in the cell corresponding to an object o and to an attribute a iff o has attribute a, in other words the "$\times$"s describe the relation R (or context). A "blank" cell corresponds to the fact that $(o, a) \notin R$, i.e., it is known that o has not attribute a.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
o_1							$\times$	$\times$
o_2							$\times$	$\times$
o_3					$\times$	$\times$	$\times$	$\times$
o_4					$\times$	$\times$	$\times$	$\times$
o_5					$\times$	$\times$	$\times$	$\times$
o_6					$\times$	$\times$		
o_7	$\times$	$\times$	$\times$	$\times$				
o_8			$\times$	$\times$				
o_9			$\times$	$\times$				

Figure 2: A context R with 7 formal concepts (including the 2 trivial cases where $\mathbf{o} = \emptyset$ or $\mathbf{a} = \emptyset$).

From a geometric point of view, a formal concept is a maximal rectangle of $\times$ in the formal context (assuming that columns and lines are adequately permuted). Indeed, it can be also shown that formal concepts are maximal pairs $(\mathbf{o}, \mathbf{a})$ (in the sense of inclusion) such that $\mathbf{o} \times \mathbf{a} \subseteq R$.

There are 5 non trivial formal concepts in R (and the corresponding rectangles of $\times$ are visible in the context). Two concepts (with $\mathbf{o} = \varnothing$ or $\mathbf{a} = \varnothing$) are added to complete the lattice. For instance, consider $\mathbf{o} = \{o_1, o_2, o_3, o_4, o_5\}$, we have $\mathbf{o}^{\uparrow} = \{a_7, a_8\}$. Likewise, if $\mathbf{a} = \{a_7, a_8\}$, then $\mathbf{a}^{\downarrow} = \{o_1, o_2, o_3, o_4, o_5\}$.

The concept lattice of R is displayed in Figure 3.

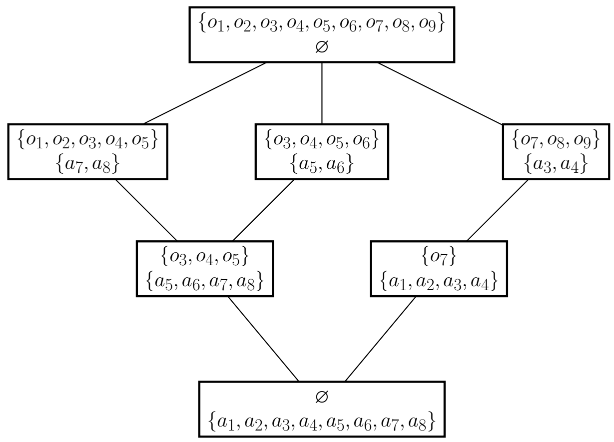


Figure 3: The formal concept lattice of the context R, described in Figure 2.

3.3. Analogical proportions in lattices: definition and properties

Given a lattice $(L, \vee, \wedge, \leq)$, we can extract two structured sets $(L, \vee)$ and $(L, \wedge)$, which are commutative semigroups. Therefore we can derive from Definition 3 the following definition of an analogical proportion in a lattice.

Definition 5 (Factorial analogical proportion in a lattice)**.**
A 4-tuple (x, y, z, t) in a lattice $(L, \vee, \wedge, \leq)$ is a FAP $(x : y :: z : t)$ when:

1. *there exists $(x_1, x_2, t_1, t_2) \in L^4$ such that $x = x_1 \vee x_2$, $y = x_1 \vee t_2$, $z = t_1 \vee x_2$ and $t = t_1 \vee t_2$,*
2. *and there exists $(x'_1, x'_2, t'_1, t'_2) \in L^4$ such that $x = x'_1 \wedge x'_2$, $y = x'_1 \wedge t'_2$, $z = t'_1 \wedge x'_2$ and $t = t'_1 \wedge t'_2$.*

Note that when $x_2 = t_2$ then $y = x$ and $z = t$ and that when $x_1 = t_1$ then $y = t$ and $z = x$. Hence we can have $(y, z) = (x, t)$ or $(y, z) = (t, x)$.

Since the rest of this article deals with AP in lattices, and the main AP we will use is the FAP, we will in the following use the notation $(x : y :: z : t)$ for a FAP and omit when there is no ambiguity the adjective "factorial".

Examples.

1. In $(\mathbb{N}^+, \text{lcm}, \text{gcd}, |)$, we have the FAP $(20 : 4 :: 60 : 12)$, with $x_1 = 20$, $x_2 = t_1 = 60$, $t_2 = 12$, $x'_1 = t'_2 = 4$, $x'_2 = 20$ and $t'_1 = 12$.

2. In the lattice $\mathcal{S}$ of closed intervals on $\mathbb{R}$, we have the FAP $([0,3] : \{3\} :: [0,4] : [3,4])$ with $x_1 = \{3\}$, $x_2 = \{0\}$, $t_1 = [3,4]$, $t_2 = \varnothing$, $x'_1 = [0,3]$, $x'_2 = [0,4]$, $t'_1 = [0,4]$ and $t'_2 = [3,4]$.

We give now a result leading to an equivalent definition of a FAP in a lattice.

Proposition 2. *A 4-tuple (x,y,z,t) of $(L,\vee,\wedge,\leq)^4$ is a FAP $(x : y :: z : t)$ iff:*

$$\begin{array}{rclrcl} x &=& (x\wedge y)\vee(x\wedge z) & x &=& (x\vee y)\wedge(x\vee z)\\ y &=& (x\wedge y)\vee(y\wedge t) & y &=& (x\vee y)\wedge(y\vee t)\\ z &=& (z\wedge t)\vee(x\wedge z) & z &=& (z\vee t)\wedge(x\vee z)\\ t &=& (z\wedge t)\vee(y\wedge t) & t &=& (z\vee t)\wedge(y\vee t) \end{array}$$

Proof. Let us suppose that the FAP $(x : y :: z : t)$ holds and show that $x = (x\wedge y)\vee(x\wedge z)$. Since $x = x_1 \vee x_2$ and $y = x_1 \vee t_2$, we have $x_1 \leq x$ and $x_1 \leq y$. Then $x_1 \leq x\wedge y$. Similarly, factor x_2 satisfies $x_2 \leq x\wedge z$. Hence, $x \leq (x\wedge y)\vee(x\wedge z)$. Besides, x being greater than $(x\wedge y)$ and $(x\wedge z)$, $(x\wedge y)\vee(x\wedge z) \leq x$. The antisymmetry of $\leq$ implies that $x = (x\wedge y)\vee(x\wedge z)$. We show the other equalities in the same manner.
Reciprocally, taking $x_1 = x\wedge y$, $x_2 = x\wedge z$, $t_1 = z\wedge t$, $t_2 = y\wedge t$, $x'_1 = x\vee y$, $x'_2 = x\vee z$, $t'_1 = z\vee t$ and $t'_2 = y\vee t$ shows directly that there exist factors satisfying Definition 5. □

Comment. In [57, 60], an incomplete definition of an analogical proportion in a lattice has been given. Actually, only four equalities of Definition 5 were given, and only four equalities of Proposition 2 were demonstrated (in a different manner from here). This definition was flawed, since for example in the lattice $(\{0,1\},\vee,\wedge,\leq)$ it would have given $(0 : 1 :: 1 : 1)$ as an AP.

3.4. Sublattice generated by a FAP

We show here that if there exists a FAP in a lattice, then the lattice has a strong property.

Proposition 3. *The sublattice generated by four elements in FAP is a Boolean lattice.*

Proof. There are two cases: either the four elements form an antichain, or not.

In the first case, consider four elements x, y, z and t of a lattice L such that $(x : y :: z : t)$ holds. We define the subset P composed of x, y, z, t and the associated elements of their decompositions given by Proposition 2:

$$P = \{x,y,z,t,x\vee y,x\vee z,y\vee t,z\vee t,x\wedge y,x\wedge z,y\wedge t,z\wedge t\}\,. \quad (1)$$

Due to Proposition 2 and the fact that x, y, z and t form an antichain, the 12 elements of P are necessarily distinct. Moreover, from Proposition 2, P is a poset according to the partial order relation $\leq$ displayed in Figure 4. But it does not define a sublattice. Therefore, we can build the smallest lattice compatible with $\leq$ using the Dedekind MacNeille completion (see details in Appendix A). The result is given in Figure 5: it is a Boolean lattice. There exists an embedding of this lattice in L.

In the second case, when x, y, z and t are not in antichain but form a FAP, a further result, given by Proposition 31, indicates that the only possibilities are that the cardinal of P equals 1 or 2. In these cases, the associated sublattice is obviously Boolean. □

As a consequence, the sublattice generated by a FAP is distributive.

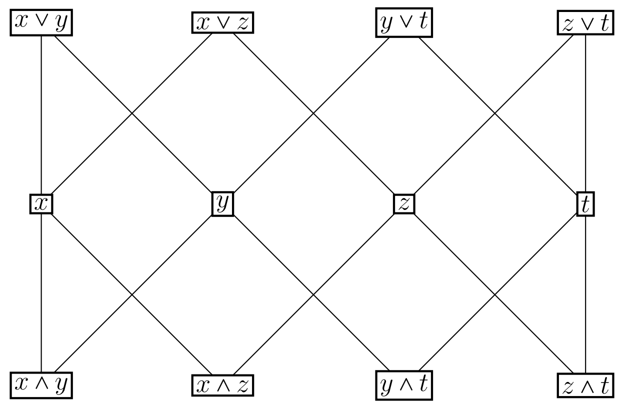


Figure 4: Hasse diagram of the partial order induced by a FAP.

3.5. Weak analogical proportion

We present in this section an alternative definition of an analogical proportion in a lattice. In some cases (e.g. a Boolean lattice) the two definitions can be proved equivalent. But in general, the new definition is only implied by the first one. That is why call it "Weak Analogical Proportion". We begin with an original proposition.

Proposition 4. *Let (x, y, z, t) be an element of $(L, \vee, \wedge, \leq)^4$,*

1. *If there is the FAP $(x : y :: z : t)$ then we have the two equalities:*

$$x \vee t = y \vee z \quad \text{and} \quad x \wedge t = y \wedge z. \tag{2}$$

2. *If the lattice is distributive, the converse is true.*

Proof. Equalities (2) can be easily checked using the factorizations given by Proposition 2. Conversely, by absorption law and distributivity, we have $x = x \wedge (x \vee t) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. The other equations of Proposition 2 can be obtained in a similar way. □

Comment. In the particular case of Boolean lattices, Proposition 4 has been shown in [43].

Definition 6 (Weak analogical proportion)**.** *A 4-tuple (x, y, z, t) of the lattice $(L, \vee, \wedge, \leq)^4$ is a Weak Analogical Proportion (WAP) when $x \wedge t = y \wedge z$ and $x \vee t = y \vee z$. It is denoted $(x, t)\ WAP\ (y, z)$ or $x : y\ WAP\ z : t$.*

Interestingly enough, J. Piaget (see appendix of [46] and [47]) has defined the WAP (under the name of "Logical Proportion") in Boolean lattices, but without making any relation with the idea of analogy.

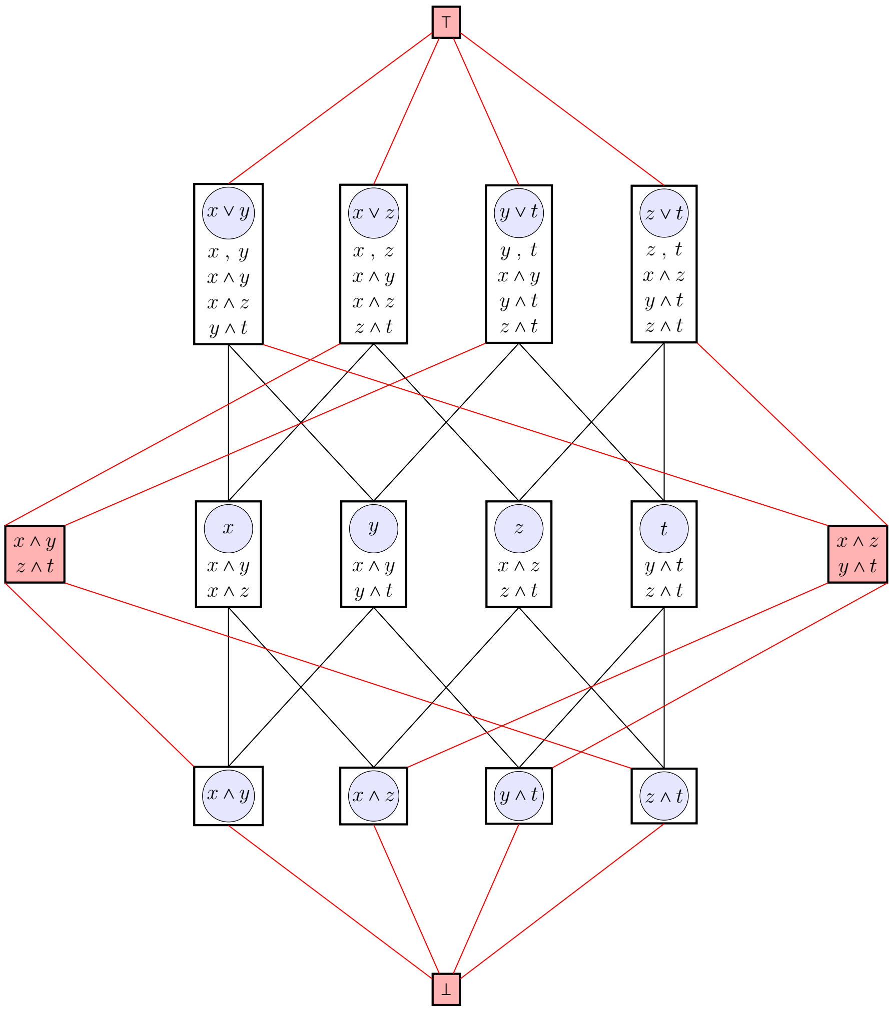


Figure 5: Hasse diagram of the completion of the partial order induced by the FAP $(x : y :: z : t)$: it is a Boolean lattice. The elements of this Boolean lattice correspond to subsets of P (defined in Equation (1)), that are detailed in rectangular boxes. When it is possible, the supremum of each subset is indicated by a circle. There exists an embedding of this Boolean lattice in the initial lattice containing P: in particular, the above subsets of P which admit a supremum are associated to their supremum element.

Comments.

1. In the notation $(x,t)\ WAP\ (y,z)$, the relation WAP is an equivalence between non ordered pairs of elements in a lattice.

2. A WAP can be read as "the product of the extremes is equal to the product of the means", where "product" is replaced either by "join" or by "meet".

3. In a non distributive lattice, a WAP is generally not a FAP. For example, let us consider the elements $x = [2,3]$, $y = [2,6]$, $z = [8,9]$ and t=$[6,9]$ of the lattice $\mathcal{S}$ of closed intervals on $\mathbb{R}$. We have $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$. However, the conditions of Proposition 2 are not checked since $x \neq (x \vee y) \wedge (x \vee z)$.

Proposition 5. *Let x_1, x_2, y, z and t be five elements of a lattice, with $x_1 \leq x_2$, such that*

$$x_1 : y\, WAP\, z : t \quad \text{and} \quad x_2 : y\, WAP\, z : t\,.$$

Then, for every x' such that $x_1 \leq x' \leq x_2$ we have

$$x' : y\, WAP\, z : t$$

Proof. Since $x_1 \leq x' \leq x_2$, we have $x_1 \wedge t \leq x' \wedge t \leq x_2 \wedge t$ and $x_1 \vee t \leq x' \vee t \leq x_2 \vee t$. Moreover, from Definition 6, we get $x_1 \wedge t = x_2 \wedge t = y \wedge z$ and $x_1 \vee t = x_2 \vee t = y \vee z$, and straightforwardly $x' \wedge t = y \wedge z$ and $x' \vee t = y \vee z$. □

3.6. Determinism

The axioms of Definition 1 are straightforwardly verified by Definition 5. We are interested now in a property reinforcing the first axiom in Definition 1 (reflexivity), which has been called *determinism* in [32].

Proposition 6 (Determinism)**.** *Let x and y be two elements of a lattice, the FAP equation in z: $(x : x :: y : z)$ has the unique solution $z = y$. This is also true for the FAP equation $(x : y :: x : z)$.*

Proof. Let z be such that $(x : x :: y : z)$, we have

$$y = (y \wedge z) \vee (x \wedge y) \tag{3}$$
$$y = (y \vee z) \wedge (x \vee y) \tag{4}$$
$$z = (y \wedge z) \vee (x \wedge z) \tag{5}$$
$$z = (y \vee z) \wedge (x \vee z) \tag{6}$$

from Proposition 2. Consequently, using (4) and absorption law,

$$\begin{aligned} x \wedge y &= x \wedge (y \vee z) \wedge (x \vee y) \\ &= (x \wedge (x \vee y)) \wedge (y \vee z) \\ &= x \wedge (y \vee z) \end{aligned}$$

and similarly, using (6)

$$\begin{aligned} x \wedge z &= x \wedge (y \vee z) \wedge (x \vee z) \\ &= x \wedge (y \vee z)\,. \end{aligned}$$

Therefore, $x \wedge y = x \wedge z$. From (3) and (5), we obtain that $y = z$. □

Comment. Note that the determinism for FAPs in the previous proposition holds even if the lattice is non distributive. But the determinism is not true for WAPs. For instance, in the lattice of closed intervals on $\mathbb{R}$, we have $[2,3] : [2,3]\, WAP\, [7,9] : [8,9]$.

3.7. Transitivity

We are interested in this section in developing a notion of transitivity between 4-tuples in AP. First, two kinds of transitivity are defined before being studied in a case of the factorial and weak analogical proportions.

Definition 7 (C- and S-transitivity of an AP)**.** *Let L be a set on which is defined an AP relation. If this relation satisfies the following first implication*

$$\forall (x,y,z,t,u,v) \in L^6 \quad ((x : y :: z : t) \text{ and } (x : u :: v : t)) \Rightarrow (u : y :: z : v)$$

then it is said to be C-transitive. If the relation satisfies this second implication

$$\forall (x,y,z,t,u,v) \in L^6 \quad ((x : y :: z : t) \text{ and } (x : y :: u : v)) \Rightarrow (u : v :: z : t)$$

then it is said to be S-transitive[4].

We require that the two APs on the left part of the implications have two elements in common. Actually, according to the axioms verified by an AP, these two transitivities are the only ones with this property.

In a lattice, both are strong properties and would be probably useful in the prospective of designing algorithms to explore it with respect to its analogical properties. An elementary result is the following.

Proposition 7. *The WAP relation is C-transitive.*

Proof. Let us consider six elements x, y, z, t, u and v of a lattice such that $(x : y\, WAP\, z : t)$ and $(x : u\, WAP\, v : t)$. These WAPs imply by definition $x \wedge t = y \wedge z = u \wedge v$ and $x \vee t = y \vee z = u \vee v$. Hence, $(x : u\, WAP\, v : t)$ is true. □

Interestingly, the FAP relation is not always C-transitive, as we shall see in the next section. On the other hand, the WAP relation is not always S-transitive, as we show now.

Counterexample (failure of S-transitivity for a FAP and for a WAP). Let us consider the lattice $\mathcal{S}$ of closed intervals on $\mathbb{R}$ (see Section 3.1). We have the FAP $([0,3] : [3,3] :: [0,0] : \varnothing)$ by considering Definition 5 and $x_1 = [3,3]$, $x_2 = [0,0]$, $t_1 = \varnothing$, $t_2 = \varnothing$, $x'_1 = x'_2 = [0,3]$, $t'_1 = [0,0]$ and $t'_2 = [3,3]$. Similarly, the FAP $([0,0] : \varnothing :: [0,4] : [4,4])$ holds by setting $x_1 = \varnothing$, $x_2 = [0,0]$, $t_1 = [4,4]$, $t_2 = \varnothing$, $x'_1 = [0,0]$, $x'_2 = [0,4]$, $t'_1 = [0,4]$ and $t'_2 = [4,4]$. However, the FAP $([0,3] : [3,3] :: [0,4] : [4,4])$ is not true because it is impossible to satisfy the second condition of Definition 5. Indeed, if there exist four elements x'_1,

[4] C ans S refer to Center and Side

x'_2, t'_1 and t'_2 of $\mathcal{S}$ such that $[0,3] = x'_1 \wedge x'_2$, $[0,0] = x'_1 \wedge t'_2$, $[0,4] = t'_1 \wedge x'_2$ and $[4,4] = t'_1 \wedge t'_2$, the closed interval t'_2 contains 0 and 4 and then $[0,4] \subset t'_2$. Moreover, $[0,4] \subset t'_1$. Consequently, $t'_1 \wedge t'_2 \neq [4,4]$. Besides, the weak analogical proportions $([0,3] : [3,3]\ WAP\ [0,0] : \varnothing)$ and $([0,0] : \varnothing\ WAP\ [0,4] : [4,4])$ result from Proposition 4, but $([0,3] : [3,3]\ WAP\ [0,4] : [4,4])$ is false since $[0,3] \wedge [4,4] = \varnothing$ and $[3,3] \wedge [0,4] = [3,3]$.

3.8. Canonical proportions

We present in this section a basic form of AP in a lattice for which S-transitivity is true.

Proposition 8 (Canonical proportions)**.** *Let x and y be two arbitrary elements of a lattice. Then the proportion*

$$x : x \vee y :: x \wedge y : y$$

is true and is called Canonical Analogical Proportion (CAP).

Proof. The first equality of Proposition 2, namely $x = (x \wedge (x \vee y)) \vee (x \wedge (x \wedge y))$, is true since the right member is equal to $(x) \vee (x \wedge y) = x$. The verification of the seven other equalities of Proposition 2 is similar, using the absorption laws. □

Example. In $(\mathbb{N}^+, lcm, gcd, |)$, an obvious example of a canonical proportion is: $2 : 10 :: 6 : 30$.

As for every analogical proportion, there are 8 equivalent analogical proportions (due to the axioms of symmetry and exchange of means) of a CAP, that correspond to 4 different syntactic forms (see Figure 6):

1. two APs are such that the first term is $x \wedge y$, namely $(x \wedge y) : x :: y : (x \vee y)$ and $(x \wedge y) : y :: x : (x \vee y)$. Since $\wedge$ and $\vee$ operations are commutative, they have the same syntactic form, called CAP of the first type, or CAP1.
2. Two APs are such that $x \wedge y$ is the second term, with the syntactic form $x : (x \wedge y) :: (x \vee y) : y$. We call them CAPs of the second type, or CAP2s.
3. There are also two CAP3s with the syntax $x : (x \vee y) :: (x \wedge y) : y$,
4. and two CAP4s with the syntactic form $(x \vee y) : x :: y : (x \wedge y)$.

The reason we have to distinguish between these different syntactic forms of the same proportion is given in the following property.

Proposition 9 (Transitivity of canonical proportions)**.** *The family of canonical proportion of the i-th type, for $i = 1, 4$, is closed under S-transitivity.*

In other words, for example, if $(x : y :: z : t)$ and $(z : t :: u : v)$ are two canonical proportions of the i-th type in a lattice, then $(x : y :: u : v)$ is a canonical proportion of the i-th type.

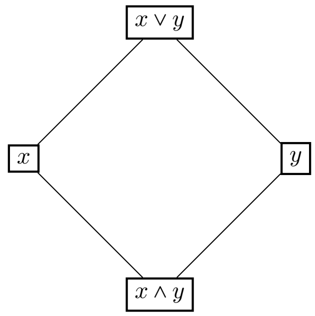


Figure 6: A canonical proportion (CAP) in a lattice. The FAP expression $(x : x{\vee}y :: x{\wedge}y : y)$ is a CAP3 syntactic form, while $(x \wedge y : y :: x \vee z : x \vee z)$ is a CAP1.

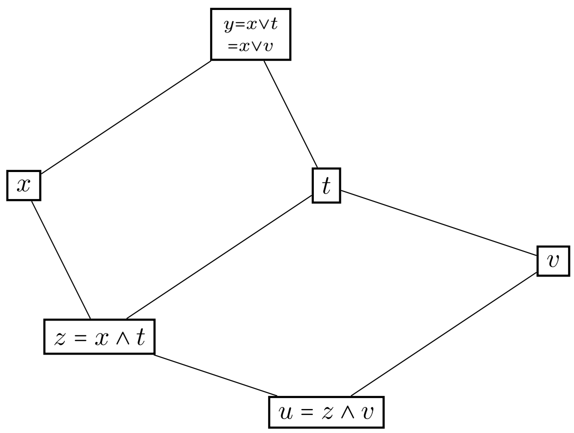


Figure 7: S-Transitivity of canonical proportions of the same type: if $x : y :: z : t$, $z : t :: u : v$ are two CAPi (canonical proportions of type i) then $x : y :: u : v$ also a CAPi.

Proof. Let us take the two canonical proportions of the first type. We know that $y = x \vee t$, $z = x \wedge t$, $t = z \vee v$ and $u = z \wedge v$. Hence, $v \leq t$ and $u = x \wedge t \wedge v = x \wedge v$. Similarly, $z \leq x$ and $y = x \vee z \vee v = x \vee v$. Then, $x : y :: u : v$ is a canonical proportion. The proof is analogous for the three other types. □

If we take two canonical proportions of different types, the S-transitivity is either syntactically impossible, either giving a result which is not in general a CAP of any type. For example, let us consider again the lattice $\mathcal{S}$ of closed intervals on $\mathbb{R}$. The proportion $([0,3] : [3,3] :: [0,0] : \varnothing)$ is a CAP4 and $([0,0] : \varnothing :: [0,4] : [4,4])$ is a CAP2. The result of S-transitivity would be $([0,3] : [3,3] :: [0,4] : [4,4])$ which is false and is not even a WAP, since $[0,3] \wedge [4,4] = \varnothing$ and $[3,3] \wedge [0,4] = [3,3]$.

The situation is different when considering the C-transitivity. If we have two CAP1s $(x \wedge y : x :: y : x \vee y)$ and $(x \wedge y : u :: v : x \vee y)$, it implies only straightforwardly that ($x : u\ WAP\ v : y$). The same holds for CAP2, CAP3 and CAP4. For example, consider the context composed of four objects and four attributes and its associated lattice displayed in Figure 8. We have the two CAP1s ($\bot : x :: y : \top$) and ($\bot : u :: v : \top$) but the AP ($u : x :: y : v$) is false (and is not a CAP1). However, since $x \wedge y = u \wedge v = \bot$ and $x \vee y = u \vee v = \top$, ($u : x\ WAP\ y : v$) holds. Note that this WAP is not a FAP, giving an additional example of the non-S-transitivity of the FAP in general.

	a_1	a_2	a_3	a_4
o_1	×			
o_2		×		
o_3			×	
o_4				×

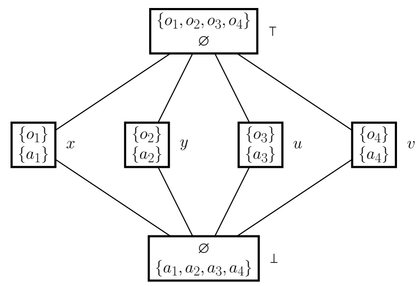


Figure 8: In this non distributive concept lattice, the FAPs $x : \top :: \bot : y$ and $\bot : y :: z : \top$ are true but $x : \top :: z : \top$ is false. Moreover, the FAP equation in u that writes ($x : \top :: \bot : u$) admits three solutions $u = y$, z and t.

In addition to having the S-transitivity property, canonical proportions can be viewed as a "building block" of the general proportion, providing a decomposition form of the factorial analogical proportion. This point is detailed in Appendix B.

Note that it is almost impossible to give a linguistic meaning to a canonical analogy. To explain that a mule is both a horse and a donkey, one would not phrase that "a mule ($x \wedge y$) is to donkeys (x) as horses (y) are to equines ($x \vee y$)". In such cases, an analogical definition is useless. There are only considered here for technical reasons.

3.9. Analogical proportions in Boolean lattices

In this section, we focus on a lattice of great use in Artificial Intelligence: the Boolean lattice of the propositional logic. Another lattice of its extensions to multiple valued logic is detailed in Appendix C.

We have given the basics of AP in Boolean lattices at the end of Section 2.2.1. We recall now a proposition demonstrated by Y. Lepage in [32], which gives a necessary and sufficient condition to the existence of a solution t of the FAP equation ($x : y :: z : t$). Moreover, this unique solution is explicitly derived.

Proposition 10. *Let x, y, z and t be four elements of $(2^\Sigma, \cup, \cap, \subseteq)$, the FAP equation in t: $(x : y :: z : t)$ has a solution if and only if $y \cap z \subseteq x \subseteq y \cup z$. In this case, the solution is unique and has the value*

$$\begin{aligned} t &= ((y \cup z) \setminus x) \cup (y \cap z) \\ &= (z \setminus x) \cup (y \setminus x) \cup (y \cap z)\,. \end{aligned}$$

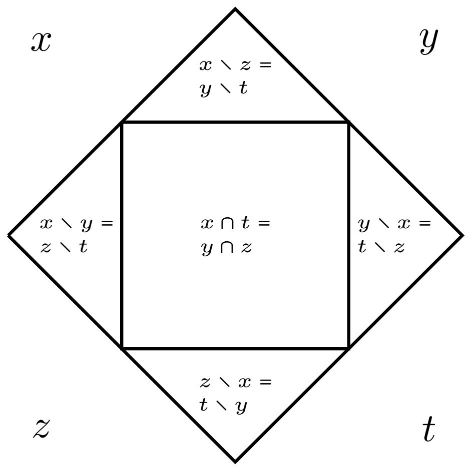


Figure 9: Geometric representation of an analogical proportion in a Boolean lattice.

Proof. A proof, different from the one derived in [32], is given in Appendix D. □

When considering the lattice $(2^\Sigma, \cup, \cap, \subseteq)$, the quantities involved in Definition 2 can be described more precisely (see [32, 41]), as explained below.

Proposition 11. *A 4-tuple (x, y, z, t) in the Boolean lattice $(2^\Sigma, \cup, \cap, \subseteq)$ is in the FAP $(x : y :: z : t)$ iff there exists a partition of Σ composed of six subsets (a, b, c, d, e, f) such that $x = a \cup c \cup e$, $y = b \cup c \cup e$, $z = a \cup d \cup e$ and $t = b \cup d \cup e$.*

The link with Definition 5 is made by taking: $x_1 = c \cup e$, $x_2 = a \cup e$, $t_1 = d \cup e$ and $t_2 = b \cup e$, and $x'_1 = \neg d \cap \neg f$, $x'_2 = \neg b \cap \neg f$, $t'_1 = \neg c \cap \neg f$ and $t'_2 = \neg a \cap \neg f$.

It is also easy to check that this definition is equivalent to Definition 5. We can express in different manners the six subsets in function of the elements of the proportion, for example as follows:

$$\begin{array}{ll} a = x \setminus y = x \cap \neg y & d = z \setminus x \\ b = y \setminus x & e = x \cap t = y \cap z \\ c = x \setminus z & f = \neg(x \cup y \cup z \cup t) \end{array}$$

A geometric representation of an analogical proportion in a Boolean lattice of finite subsets is given by Figure 9: for example, x can be written as the union made of the three surrounding subsets $x = (x \setminus y) \cup (x \setminus z) \cup (x \cap t)$.

Thus, one can say that x, y, z and t are respectively factorized under the form of pairs of disjoint subsets, namely $(a, c \cup e)$ for x, $(b, c \cup e)$ for y, $(a, d \cup e)$ for z, and $(b, d \cup e)$ for t, which perfectly parallels the equality of two numerical ratios of the form $\frac{\alpha \times \gamma}{\beta \times \gamma} = \frac{\alpha \times \delta}{\beta \times \delta}$.

Moreover, the above decomposition using the partition of the referential into six subsets exactly corresponds to the truth table of the analogical proportion in a Boolean propositional setting [43, 49] defined equivalently by

$$\begin{array}{rcccc} (x : y :: z : t) & = & (x \to y \equiv z \to t) & \wedge & (y \to x \equiv t \to z) \\ \text{or } (x : y :: z : t) & = & (x \wedge t \equiv y \wedge z) & \wedge & (x \vee t \equiv y \vee z) \end{array}$$

Indeed, in the Boolean lattice associated to the two truth values $0, 1$, the FAP $(x : y :: z : t)$ is true (i.e., is equal to '1') for the six patterns $(x, y, z, t) = (1, 0, 1, 0)$, $(x, y, z, t) = (0, 1, 0, 1)$, $(x, y, z, t) = (1, 1, 0, 0)$, $(x, y, z, t) = (0, 0, 1, 1)$, $(x, y, z, t) = (1, 1, 1, 1)$ and $(x, y, z, t) = (0, 0, 0, 0)$, and false for the ten other possible patterns which are $(x, y, z, t) = (1, 0, 0, 1)$, $(x, y, z, t) = (0, 1, 1, 0)$ and the eight patterns having an odd number of '1' and '0' (e.g., $(x, y, z, t) = (0, 0, 1, 0)$ or $(x, y, z, t) = (0, 1, 1, 1)$). The six above patterns which make $(x : y :: z : t)$ true clearly correspond to the subsets a, b, c, d, e, f. Moreover, the solution of an analogical proportion equation $(x : y :: z : t)$ exists if $x \equiv y \vee x \equiv z$ is true, and the expression given in Proposition 10 has a simpler form, yet equivalent, in terms of logical equivalence connective, since $t : x \equiv y \equiv z$. This latter result was first hinted in [27] by S. Klein.

We come now to an additional property of Boolean lattices: the transitivity of the FAP relation.

Proposition 12. *In a finite Boolean lattice, the factorial analogical proportion is S-transitive and C-transitive.*

Proof. The C-transitivity results from Propositions 4 and 7: WAPs and FAPs are equivalent in a distributive lattice and the WAP relation is C-transitive. As for the S-transitivity, we use Proposition 1 and consider any six subsets of a finite set, named $x, \cdots, v$, such that the two following FAPs hold: $(x : y :: z : t)$ and $(x : y :: u : v)$. We want to show that $(z : t :: u : v)$. Due to Proposition 10, it is sufficient to verify that $(t \cap u) \subseteq z \subseteq (t \cup u)$ and $v = (u \setminus z) \cup (t \setminus z) \cup (t \cap u)$.

For that, we use the decomposition of elements in the FAPs $(x : y :: z : t)$ and $(x : y :: u : v)$ given by Proposition 10 and displayed in Figure 9:

$$\begin{array}{rcl} u & = & (x \setminus y) \cup (x \cap v) \cup (u \setminus x) \\ t & = & (y \setminus x) \cup (x \cap t) \cup (z \setminus x) \\ z & = & (x \setminus y) \cup (x \cap t) \cup (z \setminus x) \,. \end{array}$$

It is then easy to check that $z \subseteq (t \cup u)$. Moreover, by distributivity,

$$\begin{array}{rcl} u \cap t & = & ((x \setminus y) \cap t) \cup (x \cap v \cap t) \cup ((u \setminus x) \cap t) \\ & \subseteq & (x \setminus y) \cup (x \cap t) \cup (t \setminus y) \ = \ z \end{array}$$

since $((u \setminus x) \cap t) = ((v \setminus y) \cap t) \subseteq (t \setminus y) = (z \setminus x)$. Hence, we have $(t \cap u) \subseteq z \subseteq (t \cup u)$.

Due to $(x : y :: u : v)$, v can be written as $v = (y \setminus x) \cup (x \cap v) \cup (u \setminus x)$. Since $(y \setminus x) = (t \setminus z)$, we only have to prove that $(u \setminus x) \cup (x \cap v) = (u \setminus z) \cup (t \cap u)$ in order to derive $v = (u \setminus z) \cup (t \setminus z) \cup (t \cap u)$.

Since $x \cap v = y \cap u$ and also $x \setminus y = z \setminus t$, we have on one hand $(u \setminus x) \cup (y \cap u) = (u \cap \neg x) \cup (y \cap u) = u \cap (\neg x \cup y)$ and on the other hand $(u \setminus z) \cup (t \cap u) = u \cap (\neg z \cup t) = u \cap (\neg x \cup y)$. At last, the S-transitivity is proved. □

Both transitivities can also be checked from the truth table of an analogical proportion in the Boolean lattice $(0, 1, \cup, \cap)$, see [51].

4. Solving an analogical equation in general lattices

In this section, we consider the following question: given three elements of an AP, can we find the fourth one? This is an important issue in analogical reasoning, which we have already considered in the particular case of Boolean lattices in Proposition 10.

4.1. General results

Let us consider three elements x, y, z of a lattice $(L, \vee, \wedge, \leq)$ and the FAP equation in t: $(x : y :: z : t)$. Due to Proposition 2, two conditions linking x, y and z are necessary:

$$x = (x \wedge y) \vee (x \wedge z) \text{ and } x = (x \vee y) \wedge (x \vee z) .$$

However, they are not sufficient to guarantee that a solution exists, as explained below, even in a distributive case. We briefly introduce here the main results concerning the existence and uniqueness of the solution and more details can be found in Appendix D.

Proposition 13 (Existence of a solution)**.** *Let x, y and z be elements of a lattice $(L, \vee, \wedge, \leq)$, such that $x = (x \wedge y) \vee (x \wedge z)$ and $x = (x \vee y) \wedge (x \vee z)$. The existence of a solution t to the FAP equation $(x : y :: z : t)$ is not guaranteed, even in the distributive lattice case.*

In the general case, if a solution exists, it is not necessarily unique as shown in Figure 8. However, it is different in a distributive case as stated in the following proposition.

Proposition 14 (Uniqueness of the solution)**.** *Let x, y and z be three elements of a distributive lattice $(L, \vee, \wedge, \leq)$. If there is a solution t to the FAP equation $(x : y :: z : t)$, it is unique.*

Proof. According to Proposition 4, WAP and FAP relations are equivalent in a distributive lattice. If there are two solutions t and t' to the equation $(x : y \, WAP \, z : t)$, then $x \wedge t = x \wedge t'$ and $x \vee t = x \vee t'$. In a distributive lattice, these equalities imply $t = t'$ (see for example [14]). □

In the case of comparable elements, the FAP equation is easily solved as described in the next proposition.

Proposition 15. *Let x, y, z and t be four elements of a lattice $(L, \vee, \wedge, \leq)$ such that they form the FAP $(x : y :: z : t)$.*

- *If $\{x, y, z, t\}$ is not an antichain, then the FAP corresponds to either a trivial AP $x : x :: x : x$, or a simple AP $x : x :: y : y$, or a CAP or associated APs using the properties of symmetry and exchange of means.*
- *If $\{x, y, z\}$ is an antichain, $\{x, y, z, t\}$ is also an antichain.*

Proof. See Appendix D. □

The second item of Proposition 15 provides a way to reduce the search space to solve the FAP equation in t: $(x : y :: z : t)$.

4.2. Reducing the search space

Let x, y and z be three elements of a lattice such that $x = (x \wedge y) \vee (x \wedge z)$ and $x = (x \vee y) \wedge (x \vee z)$. We are looking for a solution t of the FAP equation $(x : y :: z : t)$. Due to the previous subsection, we only consider the case where x, y and z are not comparable. Necessarily, $y \wedge z \leq t \leq y \vee z$ and the four elements x, y, z and t form an antichain (due to Proposition 15). Even if these conditions are necessary to lead a FAP, they are not sufficient as shown in the formal concept analysis example of Figure 10. However, they provide a reduction of the search space of the solution(s).

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
o_1	×			×	×			
o_2		×		×				
o_3			×		×			
o_4				×				
o_5					×			
o_6	×	×		×	×	×		
o_7	×		×	×	×		×	
o_8								×

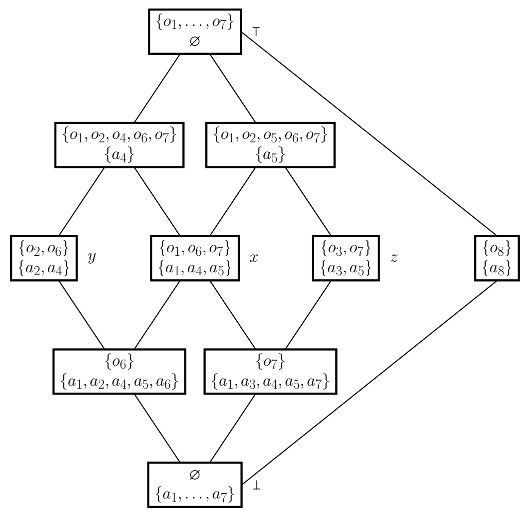


Figure 10: Even if $\{x, y, z, t\}$ is an antichain, $x = (x \wedge y) \vee (x \wedge z)$, $x = (x \vee y) \wedge (x \vee z)$ and $y \wedge z \leq t \leq y \vee z$, the FAP $(x : y :: z : t)$ does not hold. In this lattice, there is no FAP with x, y and z.

Firstly, we assume that we know a procedure able to decide whether two elements u and v of the lattice are such that $u \leq v$. The implementation of this procedure depends on the type of the lattice, but in the case of a lattice given by the list of the pairs defining this relation, it can be performed through a direct access.

Secondly, it is also easy to devise a procedure $\texttt{CondSons}(u, v)$ which, given a lattice and two elements u and v of this lattice, with $u \leq v$, produces the list of all elements w that are such that $w \leq v$ and form an antichain with u. This is displayed in Figure 11.

Similarly, we can devise a procedure $\texttt{CondFathers}(u, v)$ which, given a lattice and two elements u and v of this lattice, with $u \geq v$, produces the list of all elements w that are such that $w \geq v$ and form an antichain with u.

Now, the search for t is easy. One can compute the lists $X = \texttt{CondSons}(x, y \vee z)$, $Y = \texttt{CondSons}(y, y \vee z)$ and $Z = \texttt{CondSons}(z, y \vee z)$. Only the elements t such that $x \vee t = y \vee z$ are kept in each of these lists. If a list is empty,

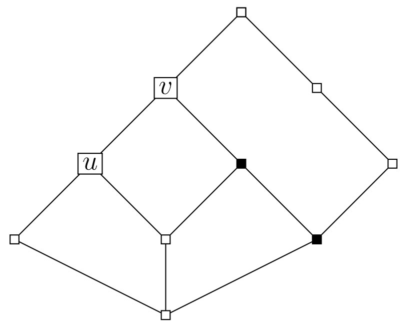


Figure 11: Given a lattice and two elements u and v of this lattice with $u \leq v$, algorithm `CondSons`(u, v) produces the list of all elements w (in black) that are such that $w \leq v$ and form an antichain with u.

the equation has no solution. Similarly, the lists X' = `CondFathers`$(x, y \wedge z)$, Y' = `CondFathers`$(y, y \wedge z)$ and Z' = `CondFathers`$(z, y \wedge z)$ are computed and only the elements t such that $x \wedge t = y \wedge z$ are kept in each of these lists. The potential solution t is in the intersection of these six lists of elements. However, all elements t in this intersection are not solutions of the AP $(x : y :: z : t)$ as it can be observed in the example displayed in Figure 10. Consequently, when looking for all solutions, one has to check on every element whether or not it is a solution.

5. Analogical proportions and formal concept analysis

We are interested in this section to study how the previous results can be applied in the framework of formal concept analysis, more precisely how analogical proportions can be described between formal concepts.

The lattices of concepts constructed from a formal context are quite particular, as seen in Section 3.2. We know that a concept is a pair of subsets, one of attributes and the other of objects. In the first part of this section, we will revisit some elementary properties of a concept lattice and propose a characterization of a Weak Analogical Proportion between four formal concepts. We will afterwards propose another analogical proportions, stronger than the WAP and easier to use.

The main motivation of this section is to give new tools to explore concept lattices, and contribute to their study. We think that it is interesting to develop a theory and some algorithms to connect the notions of analogy and that of formal concepts, since both are of interest in artificial intelligence, and the latter are extensively used in data mining. Moreover, analogical proportions and formal concepts refer to basic issues in cognitive sciences, in relation with analogical reasoning and conceptual categorization respectively. Indeed, this idea leads us to propose in Section 7 an interpretation of the figure of speech called *relational proportion* (see Section 1) in terms of a particular subcontext of a formal context.

5.1. Analogical proportions in concept lattices

In this paragraph, several analogical proportions in a concept lattice are characterized and their links are considered. This study is based on basic properties of concepts, introduced in Appendix E as preliminary propositions.

Proposition 16. *Let $x = (\mathbf{o}_x, \mathbf{a}_x)$, $y = (\mathbf{o}_y, \mathbf{a}_y)$, $z = (\mathbf{o}_z, \mathbf{a}_z)$ and $t = (\mathbf{o}_t, \mathbf{a}_t)$ be elements of a concept lattice, one has:*

1. $x \vee t = y \vee z$ *iff* $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$,
2. $x \wedge t = y \wedge z$ *iff* $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$.

Proof. Thanks to Preliminary 2 (see preliminaries given in Appendix E), equality $x \vee t = y \vee z$ clearly implies $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$. Conversely, due to Preliminary 2, $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ implies $\mathbf{a}_{x \vee t} = \mathbf{a}_{y \vee z}$. Using Preliminary 3, it results that $x \vee t = y \vee z$.

The proof of the second equivalence can be done in a similar manner. □

Proposition 17 (Characterization of a WAP in a concept lattice)**.** *Four elements $x = (\mathbf{o}_x, \mathbf{a}_x)$, $y = (\mathbf{o}_y, \mathbf{a}_y)$, $z = (\mathbf{o}_z, \mathbf{a}_z)$ and $t = (\mathbf{o}_t, \mathbf{a}_t)$ of a concept lattice are in WAP iff they are such that*

$$\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z \quad \textit{and} \quad \mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z \,.$$

Proof. This proposition follows directly from Definition 6 and Proposition 16. □

As a consequence, the attributes and the objects present in a WAP are organized in the Venn diagram displayed in Figure 12.

Proposition 18. *Let $x = (\mathbf{o}_x, \mathbf{a}_x)$, $y = (\mathbf{o}_y, \mathbf{a}_y)$, $z = (\mathbf{o}_z, \mathbf{a}_z)$ and $t = (\mathbf{o}_t, \mathbf{a}_t)$ be elements of a concept lattice, if one of the following APs between subsets holds*

$$\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t \qquad \textit{or} \qquad \mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$$

then x, y, z and t form the weak analogical proportion $(x : y \, WAP \, z : t)$.

Comment. In the previous proposition, equation $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$, respectively $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$, refers to the analogical proportions in the lattice $(2^{\mathcal{A}}, \cup, \cap, \subseteq)$ where $\mathcal{A}$ is the set of attributes, respectively $(2^{\mathcal{O}}, \cup, \cap, \subseteq)$ where $\mathcal{O}$ is the set of objects.

Proof. Let x, y, z and t be concepts such that $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$, or equivalently $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ and $\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z$. Thanks to Proposition 16, $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ is equivalent to $x \vee t = y \vee z$. As for equation $\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z$, it implies $(\mathbf{a}_x \cup \mathbf{a}_t)^{\downarrow} = (\mathbf{a}_y \cup \mathbf{a}_z)^{\downarrow}$. Besides, using Preliminary 1, we have

$$\begin{aligned} (\mathbf{a}_x \cup \mathbf{a}_t)^{\downarrow} &= (\mathbf{a}_x)^{\downarrow} \cap (\mathbf{a}_t)^{\downarrow} \\ &= \mathbf{o}_x \cap \mathbf{o}_t \,. \end{aligned}$$

Similarly $(\mathbf{a}_y \cup \mathbf{a}_z)^{\downarrow} = \mathbf{o}_y \cap \mathbf{o}_z$. Thus, $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$, and Proposition 17 permits to conclude.

In the case of four concepts x, y, z and t such that $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$, the proof is similar since we also have $(\mathbf{o}_x \cup \mathbf{o}_t)^{\uparrow} = \mathbf{a}_x \cap \mathbf{a}_t$ and $(\mathbf{o}_y \cup \mathbf{o}_y)^{\uparrow} = \mathbf{a}_y \cap \mathbf{a}_z$. □

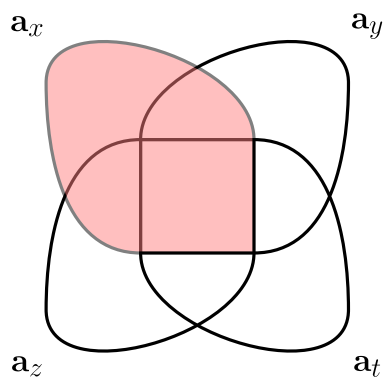


Figure 12: The organization of the attributes in a WAP, consequence of the property $\mathbf{a}_x \cap \mathbf{a}_y = \mathbf{a}_y \cap \mathbf{a}_z$. For instance, $\mathbf{a}_x$ is partitioned in the four subsets $(\mathbf{a}_x \cup \mathbf{a}_y \cup \mathbf{a}_z \cup \mathbf{a}_t) \smallsetminus (\mathbf{a}_y \cup \mathbf{a}_z \cup \mathbf{a}_t)$ (attributes proper to subset $\mathbf{a}_x$), $\mathbf{a}_\cap = \mathbf{a}_x \cap \mathbf{a}_y \cap \mathbf{a}_z \cap \mathbf{a}_t$ (attributes common to the four subsets), $(\mathbf{a}_x \cap \mathbf{a}_z) \smallsetminus \mathbf{a}_\cap$ and $(\mathbf{a}_x \cap \mathbf{a}_y) \smallsetminus \mathbf{a}_\cap$.

Comment. The converse is false. Let us consider the following formal context

	a_1	a_2	a_3	a_4	a_5
o_1			×	×	
o_2	×		×		
o_3		×		×	
o_4	×	×			×

its concept lattice is displayed on Figure 13. The concepts $x = (\{o_1\}, \{a_3, a_4\})$, $y = (\{o_2\}, \{a_1, a_3\})$, $z = (\{o_3\}, \{a_2, a_4\})$ and $t = (\{o_4\}, \{a_1, a_2, a_5\})$ are in (complete) WAP, due to Proposition 17. However, the Boolean APs $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$ and $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$ are both false. The WAP between concepts is less restrictive than the AP between sets of attributes: in a WAP, objects are allowed to possess attributes which are not shared by any other object concerned in the WAP. We study now the case when the sets of attributes or objects or both are in Boolean AP.

Definition 8 (AP on attributes or on objects)**.** *When four concepts $x = (\mathbf{o}_x, \mathbf{a}_x)$, $y = (\mathbf{o}_y, \mathbf{a}_y)$, $z = (\mathbf{o}_z, \mathbf{a}_z)$ and $t = (\mathbf{o}_t, \mathbf{a}_t)$ are such that*

$$\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t \quad or \quad \mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t\,,$$

they are said to be in analogical proportion on attributes or on objects, which writes: $x : y \; AP_{\mathbf{a}} \; z : t$ or $x : y \; AP_{\mathbf{o}} \; z : t$.

We give now a proposition which leads us to a corollary in which is defined yet another analogical proportion between formal concepts, the strongest of all.

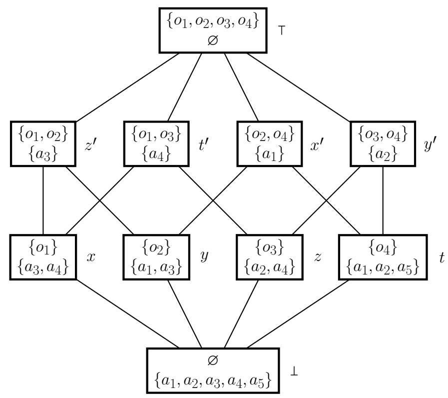


Figure 13: In this lattice, x, y, z and t are in WAP but $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$ and $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$ are both false. Besides, the concepts x', y', z' and t' are in WAP and $\mathbf{o}_{x'} : \mathbf{o}_{y'} :: \mathbf{o}_{z'} : \mathbf{o}_{t'}$ is true, but $\mathbf{a}_{x'} : \mathbf{a}_{y'} :: \mathbf{a}_{z'} : \mathbf{a}_{t'}$ and the FAP $x' : y' :: z' : t'$ are both false.

Proposition 19. *Let $x = (\mathbf{o}_x, \mathbf{a}_x)$, $y = (\mathbf{o}_y, \mathbf{a}_y)$, $z = (\mathbf{o}_z, \mathbf{a}_z)$ and $t = (\mathbf{o}_t, \mathbf{a}_t)$ be elements of a concept lattice. If*

$$\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z \qquad \textit{and} \qquad \mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z ,$$

then the concepts x, y, z and t form the FAP $x : y :: z : t$.

Proof. Let x, y,z and t be four concepts such that $\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z$ and $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z$. From $\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z$ and Preliminary 2, we have

$$\begin{aligned} \mathbf{a}_x &= \mathbf{a}_x \cap (\mathbf{a}_x \cup \mathbf{a}_t) \\ &= \mathbf{a}_x \cap (\mathbf{a}_y \cup \mathbf{a}_z) \\ &= (\mathbf{a}_x \cap \mathbf{a}_y) \cup (\mathbf{a}_x \cap \mathbf{a}_z) \\ &= \mathbf{a}_{x \vee y} \cup \mathbf{a}_{x \vee z} . \end{aligned}$$

It results that, using Preliminaries 1 and 2,

$$\begin{aligned} \mathbf{o}_x &= (\mathbf{a}_x)^{\downarrow} \\ &= (\mathbf{a}_{x \vee y} \cup \mathbf{a}_{x \vee z})^{\downarrow} \\ &= \mathbf{o}_{x \vee y} \cap \mathbf{o}_{x \vee z} \\ &= \mathbf{o}_{(x \vee y) \wedge (x \vee z)} . \end{aligned}$$

Consequently, Preliminary 3 permits to obtain $x = (x \vee y) \wedge (x \vee z)$. Similarly, from $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z$, it follows that $\mathbf{o}_x = (\mathbf{o}_x \cap \mathbf{o}_y) \cup (\mathbf{o}_x \cap \mathbf{o}_z)$. Combined with Preliminaries 1 and 2, this factorization of $\mathbf{o}_x$ implies that

$$\begin{aligned} \mathbf{a}_x &= (\mathbf{o}_x)^{\uparrow} \\ &= ((\mathbf{o}_x \cap \mathbf{o}_y) \cup (\mathbf{o}_x \cap \mathbf{o}_z))^{\uparrow} \\ &= (\mathbf{o}_{x \wedge y} \cup \mathbf{o}_{x \wedge z})^{\uparrow} \\ &= \mathbf{a}_{x \wedge y} \cap \mathbf{a}_{x \wedge z} \\ &= \mathbf{a}_{(x \wedge y) \vee (x \wedge z)} . \end{aligned}$$

Thus, $x = (x \wedge y) \vee (x \wedge z)$ thanks to Preliminary 3. The other equalities of Proposition 2 are checked in a similar way. □

Comment. Considering the proof of Proposition 19, we can note that the assertion $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z$ implies the four factorisations on the left part of Proposition 2. Similarly, $\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z$ implies the four factorisations on the right part of Proposition 2. If the combination of both assertions provides a FAP and then a WAP, only one of them is not enough to obtain a WAP. For instance, in Figure 13, $\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_\perp$ but $x : y \; WAP \perp : t$ does not hold.

Corollary 1 (Strong analogical proportion between concepts)**.** *Let* $x = (\mathbf{o}_x, \mathbf{a}_x)$, $y = (\mathbf{o}_y, \mathbf{a}_y)$, $z = (\mathbf{o}_z, \mathbf{a}_z)$ *and* $t = (\mathbf{o}_t, \mathbf{a}_t)$ *be elements of a concept lattice. The following two pairs of equalities are equivalent:*

$$\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z \quad \text{and} \quad \mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z \,,$$

$$\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t \quad \text{and} \quad \mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t \,.$$

This characterizes a particular case of a FAP between concepts that we call a Strong Analogical Proportion *(SAP). It is denoted* $x : y \; SAP \; z : t$*.*

In other words, four concepts in analogical proportion on attributes and on objects are said to be in strong analogical proportion.

Proof. Let x, y, z and t, four concepts such that $\mathbf{a}_x \cup \mathbf{a}_t = \mathbf{a}_y \cup \mathbf{a}_z$ and $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z$, Proposition 19 implies the FAP $x : y :: z : t$. Hence, using Propositions 4 and 17, we have $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ and $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$. Consequently, $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$ and $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$.

The converse is trivial. □

Comments. From Proposition 19 and its corollary, the analogical proportions $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a}_t$ and $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t$ imply the FAP $x : y :: z : t$. However, the reciprocal is false. Let us consider once more the concept lattice displayed in Figure 13. The concepts $y = (\{o_2\}, \{a_1, a_3\})$, $\top = (\{o_1, o_2, o_3, o_4\}, \varnothing)$, $\perp = (\varnothing, \{a_1, a_2, a_3, a_4, a_5\})$ and $z = (\{o_3\}, \{a_2, a_4\})$ form the FAP $y : \top :: \perp : z$ (which is a CAP3). However, $\mathbf{o}_y \cup \mathbf{o}_z \neq \mathbf{o}_\top \cup \mathbf{o}_\perp$ and $\mathbf{a}_y \cup \mathbf{a}_z \neq \mathbf{a}_\top \cup \mathbf{a}_\perp$. Furthermore, the concepts $x' = (\{o_2, o_4\}, \{a_1\})$, $y' = (\{o_3, o_4\}, \{a_2\})$, $z' = (\{o_1, o_2\}, \{a_3\})$ and $t' = (\{o_1, o_3\}, \{a_4\})$ satisfy $\mathbf{o}_{x'} : \mathbf{o}_{y'} :: \mathbf{o}_{z'} : \mathbf{o}_{t'}$. That implies $x' : y' \; WAP \; z' : t'$ from Proposition 18. However, this single AP between sets of objects is not sufficient to obtain the FAP between the associated concepts. Indeed, $\mathbf{a}_{x'} : \mathbf{a}_{y'} :: \mathbf{a}_{z'} : \mathbf{a}_{t'}$ and $x' : y' :: z' : t'$ are not true.

These observations stem from the fact that the FAP and WAP between concepts are directly related to the lattice whereas the SAP, $\text{AP}_\mathbf{a}$ and $\text{AP}_\mathbf{o}$ depend on the formal context.

As a consequence, *even in a Boolean lattice, FAP and SAP are not equivalent properties*, as it will be illustrated by the following example:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
o_1		×	×	×			×	×	×
o_2	×		×	×	×		×		×
o_3	×	×		×		×		×	×
o_4	×	×	×		×	×			×
o_5	×	×	×	×	×	×	×	×	×

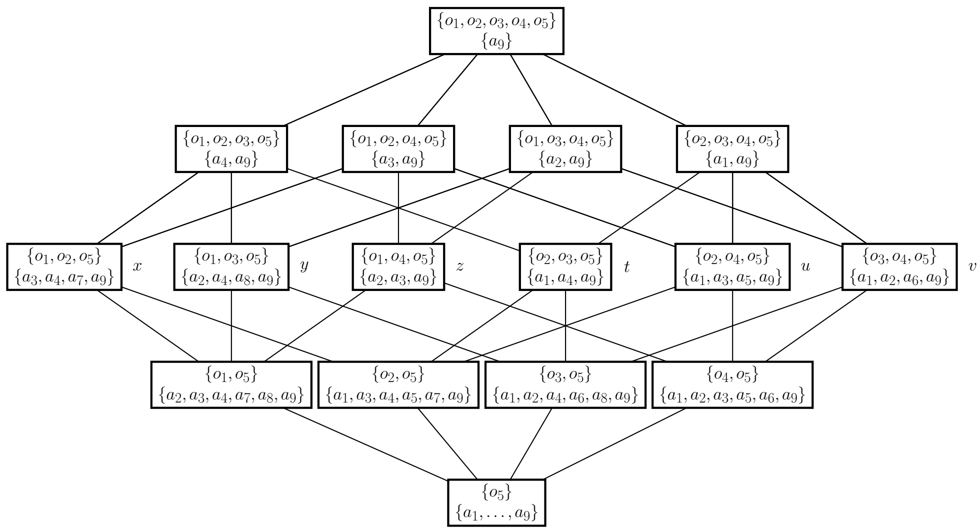


Figure 14: Example of Boolean concept lattice where the FAP $x : y :: u : v$ is not a SAP. The AP $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_u : \mathbf{o}_v$ is true but $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_u : \mathbf{a}_v$ does not hold.

The associated Boolean concept lattice is displayed in Figure 14. The concepts $x = (\{o_1, o_2, o_5\}, \{a_3, a_4, a_7, a_9\})$, $y = (\{o_1, o_3, o_5\}, \{a_2, a_4, a_8, a_9\})$, $u = (\{o_2, o_4, o_5\}, \{a_1, a_3, a_5, a_9\})$ and $v = (\{o_3, o_4, o_5\}, \{a_1, a_2, a_6, a_9\})$ form a FAP but are not in SAP. Indeed, the AP $\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_u : \mathbf{a}_v$ does not hold:

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
$\mathbf{a}_x$			×	×			×		×
$\mathbf{a}_y$		×		×				×	×
$\mathbf{a}_u$	×		×		×				×
$\mathbf{a}_v$	×	×				×			×

However, the formal context can be reduced to the form given in Figure 15 without changing the lattice.

The modified concept composition associated to the reduced formal context is described in Figure 15 and the six central concepts exhibit the three proportions $x : y\ SAP\ u : v$, $x : z\ SAP\ t : v$ and $y : z\ SAP\ t : u$. As an example of such a formal context, we can take the four animals Bat, Eagle, Shrew and Sparrow, and the four attributes: to walk, to fly, to kill other animals, to have a weight less than one kilo (assuming that an eagle and a sparrow can walk, but not a bat). An analogical proportion would be: "bats and shrews are to bats and sparrows as shrews and eagles are to sparrows and eagles". It is difficult to make sense out of such a sentence, especially because the analogical proportion involves pairs of not equivalent objects. To say it informally, the pair of objects are too intertwined, i.e. share too many attributes. Notice that there is no WAP between the four concepts involving only one object (or one attribute), since we have: $\{a_2, a_3, a_4\} \cap \{a_1, a_2, a_3\} \neq \{a_1, a_3, a_4\} \cap \{a_1, a_2, a_4\}$.

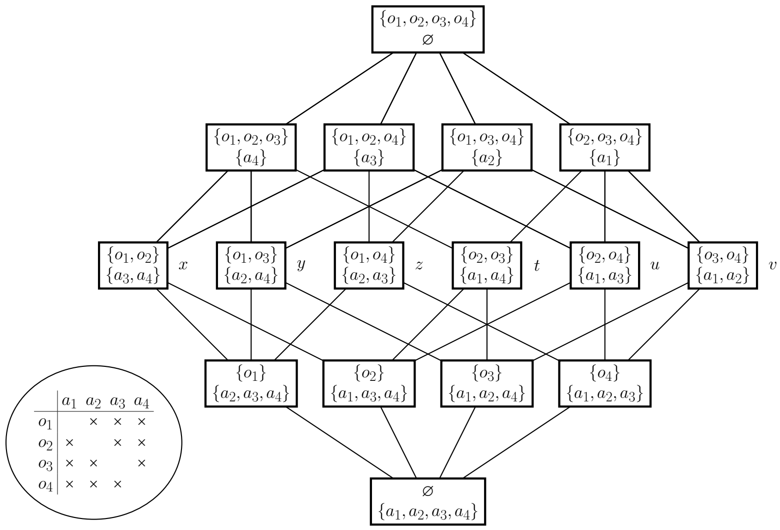


Figure 15: Example of Boolean concept lattice where $x : y \, SAP \, u : v$, $x : z \, SAP \, t : v$ and $y : z \, SAP \, t : u$.

In order to help the reader, Figure 16 sums up the previous relations and remarks between the different analogical proportions. So as to avoid overloading this figure, an ordinary single headed arrow corresponds to an implication with no reciprocal relation. However, the use of single headed arrows is not sufficient to mention all the false implications between the considered propositions and some crossed arrows (in orange color) explicitly indicate these additional false implications. The other undisplayed false and true implications can be easily derived from the sketched ones. For instance,

- $(x : y :: z : t)$ does not imply $(\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t)$: let us suppose that $(x : y :: z : t)$ is a sufficient condition to $(\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}_t)$. By transitivity of the implication relation, $(x : y :: z : t)$ also implies $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_t$, which is in contradiction with the crossed arrow displayed between these assertions. Similarly, it is not necessary to draw a crossed arrow from $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_t$ to $(x : y :: z : t)$.

- Since $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_t$ implies the four factorisations detailed in box (a) in Figure 16 but does not implies a WAP relation, it is useless to sketch a crossed arrow from box (a) to $(x : y \, WAP \, z : t)$.

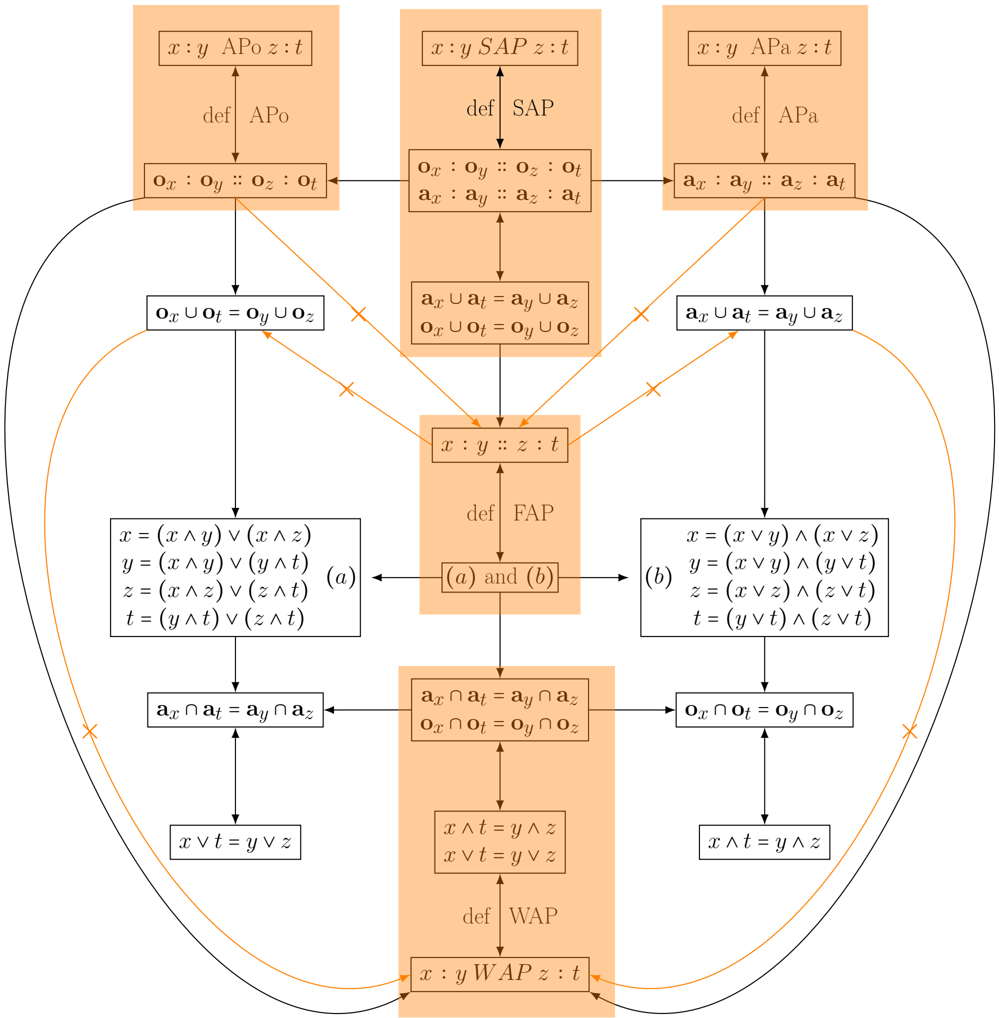


Figure 16: Graph of the properties of a concept lattice, in relation with the five types of analogical proportions between concepts: Factorial (FAP), Strong (SAP), AP on attributes ($\mathrm{AP}_{\mathbf{a}}$), AP on objects ($\mathrm{AP}_{\mathbf{o}}$) and Weak (WAP). An arrow labeled with "def" displays a definition. An ordinary single headed arrow indicates an implication and a false reciprocal, whereas a double headed arrow corresponds to an equivalence relation. A crossed arrow (in orange color) indicates, if non obvious, that there is no implication. Since WAP and FAP are equivalent in Boolean lattices, the nature of the AP between subsets is not made precise.

5.2. Comparison of the different analogical proportions between concepts

The intrinsic properties of the lattices of formal concepts have led us to define some particular analogical proportions. The general Factorial AP, as we have seen in Section 3.4, is closely related to the distributivity property, which is in general not true in concept lattices. Hence, it is interesting to make a distinction between the FAP and a weaker proportion that we have called WAP or "the product of the extremes is equal to the product of the means". In a non distributive lattice, four concepts in FAP are also in WAP, but the reverse is not generally true.

As we have seen, the WAP is a good candidate to define an analogical proportion between concepts, since it is easy to describe in terms of lattice properties. However, as we shall see in Section 5.4, since the solution is generally non unique, solving an analogical equation in the WAP framework leads either to a set of solutions, or to a "best" solution if some criterion is optimized during the solving algorithm (for example, the number of objects of the solution).

The $\mathrm{AP}_{\mathbf{a}}$ (or $\mathrm{AP}_{\mathbf{o}}$, we will keep to the former in the rest of this section) is stronger than the WAP: four concepts in $\mathrm{AP}_{\mathbf{a}}$ are also in WAP, whereas the reverse is generally false. But it has several important advantages which follow directly from of its definition (it is a FAP between subsets).

1. Like FAP between concepts, it obeys to the axiom of determinism. Moreover, as detailed in the next section, the solution to an $\mathrm{AP}_{\mathbf{a}}$ (or $\mathrm{AP}_{\mathbf{o}}$) equation, if it exists, is unique (contrary to FAP equation) and is straightforwardly computed: one has only look in the formal context.

2. It is S-transitive, in the sense of Section 3.8, which means that $(x : y \,\mathrm{AP}_{\mathbf{a}}\, z : t)$ and $(z : t \,\mathrm{AP}_{\mathbf{a}}\, u : v)$ imply $(x : y \,\mathrm{AP}_{\mathbf{a}}\, u : v)$[5].

5.3. Resolution of a SAP, $AP_{\mathbf{a}}$ or $AP_{\mathbf{o}}$ equation

In this section, we return to the resolution of an analogical equation, in the more constrained case of strong analogical proportion. Let us consider a context $(\mathcal{O}, \mathcal{A}, R)$ and three concepts x, y and z of the associated concept lattice. Due to the definition of SAP, a way to find concept t such that $(x : y \,SAP\, z : t)$ is to solve the corresponding $\mathrm{AP}_{\mathbf{a}}$ equation and check if its solution t is in SAP with x, y and z.

From Definition 8 of $\mathrm{AP}_{\mathbf{a}}$, solving equation in t: $(x : y \,\mathrm{AP}_{\mathbf{a}}\, z : t)$ is equivalent to looking for subset $\mathbf{a} \subseteq \mathcal{A}$ such that $\mathbf{a}^{\downarrow\uparrow} = \mathbf{a}$ and $(\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a})$. Since this last equation refers to analogical proportion in the Boolean lattice $(2^{\mathcal{A}}, \cup, \cap, \subseteq)$, it follows from Proposition 10 that there exists $\mathbf{a} \subseteq \mathcal{A}$ such that $(\mathbf{a}_x : \mathbf{a}_y :: \mathbf{a}_z : \mathbf{a})$ iff $\mathbf{a}_y \cap \mathbf{a}_z \subseteq \mathbf{a}_x \subseteq \mathbf{a}_y \cup \mathbf{a}_z$, or equivalently $\mathbf{a}_{y \vee z} \subseteq \mathbf{a}_x \subseteq \mathbf{a}_y \cup \mathbf{a}_z$. Let us point out that, using Preliminary 2, condition $\mathbf{a}_y \cap \mathbf{a}_z \subseteq \mathbf{a}_x$ is equivalent to $x \leq y \vee z$, whereas condition $\mathbf{a}_x \subseteq \mathbf{a}_y \cup \mathbf{a}_z$ implies $y \wedge z \leq x$. Under these conditions, Proposition 10 permits to derive the unique solution $\mathbf{a} \subseteq \mathcal{A}$:

$$\mathbf{a} = ((\mathbf{a}_y \cup \mathbf{a}_z) \setminus \mathbf{a}_x) \cup (\mathbf{a}_{y \vee z}) \,.$$

If $\mathbf{a}$ exactly corresponds to an attribute set of a concept t, the latter is the unique solution of $x : y \,\mathrm{AP}_{\mathbf{a}}\, z : t$, otherwise this $\mathrm{AP}_{\mathbf{a}}$ has no solution.

[5]Since a SAP is both an $\mathrm{AP}_{\mathbf{a}}$ and an $\mathrm{AP}_{\mathbf{o}}$, it is of course S-transitive as well.

It is clear that the resolution of an $\text{AP}_{\mathbf{o}}$ equation can be done in a similar manner. Let us note equation $\mathbf{o}_x : \mathbf{o}_y :: \mathbf{o}_z : \mathbf{o}$ admits a solution $\mathbf{o} \subseteq \mathcal{O}$ iff $\mathbf{o}_{y\wedge z} \subseteq \mathbf{o}_x \subseteq \mathbf{o}_y \cup \mathbf{o}_z$. However, to solve a SAP equation, it is not necessary to solve the associated $\text{AP}_{\mathbf{a}}$ and $\text{AP}_{\mathbf{o}}$ equations. It is sufficient to solve one of these equations, for instance the $\text{AP}_{\mathbf{a}}$ equation. If the $\text{AP}_{\mathbf{a}}$ equation admits a solution t, it directly follows from Corollary 1 that $(x : y\ SAP\ z : t)$ is true iff $\mathbf{o}_x \cup \mathbf{o}_t = \mathbf{o}_y \cup \mathbf{o}_z$.

Examples.

1. Let us consider the context and its lattice described in Figure 15, concepts $y = (\{o_1, o_3\}, \{a_2, a_4\})$, $u = (\{o_2, o_4\}, \{a_1, a_3\})$ and $y \vee t = (\{o_1, o_2, o_3\}, \{a_4\})$ are such that $\mathbf{a}_y \cap \mathbf{a}_u \subseteq \mathbf{a}_{y\vee t} \subseteq \mathbf{a}_y \cup \mathbf{a}_u$. Equation

$$\mathbf{a}_{y\vee t} : \mathbf{a}_y :: \mathbf{a}_u : \mathbf{a}$$

admits as solution $\mathbf{a} \subseteq \{a_1, \dots, a_4\}$

$$\begin{aligned}\mathbf{a} &= ((\mathbf{a}_y \cup \mathbf{a}_u) \setminus \mathbf{a}_{y\vee t}) \cup (\mathbf{a}_{y\vee u}) \\ &= \{a_1, a_2, a_3\} \\ &= \mathbf{a}_{z\wedge u}\end{aligned}$$

since $z \wedge u = (\{o_4\}, \{a_1, a_2, a_3\})$. Consequently, $((y \vee t) : y\ \text{AP}_{\mathbf{a}}\ u : (z \wedge u))$ is true. Besides,

$$\begin{aligned}\mathbf{o}_{y\vee t} \cup \mathbf{o}_{z\wedge u} &= \{o_1, o_2, o_3\} \cup \{o_4\} \\ &= \{o_1, o_3\} \cup \{o_2, o_4\} \\ &= \mathbf{o}_y \cup \mathbf{o}_u\,.\end{aligned}$$

Due to Corollary 1, it follows that $((y \vee t) : y\ \text{AP}_{\mathbf{o}}\ u : (z \wedge u))$, and, at last, $((y \vee t) : y\ SAP\ u : (z \wedge u))$.

2. As an additional illustration, we shall consider the context and the lattice displayed in Figure 14 and concepts $y = (\{o_1, o_3, o_5\}, \{a_2, a_4, a_8, a_9\})$, $u = (\{o_2, o_4, o_5\}, \{a_1, a_3, a_5, a_9\})$ and $y \vee t = (\{o_1, o_2, o_3, o_5\}, \{a_4, a_9\})$. Since $\mathbf{a}_y \cap \mathbf{a}_u \subseteq \mathbf{a}_{y\vee t} \subseteq \mathbf{a}_y \cup \mathbf{a}_u$, equation $\mathbf{a}_{y\vee t} : \mathbf{a}_y :: \mathbf{a}_u : \mathbf{a}$ admits as solution $\mathbf{a} \subseteq \{a_1, \dots, a_9\}$

$$\begin{aligned}\mathbf{a} &= ((\mathbf{a}_y \cup \mathbf{a}_u) \setminus \mathbf{a}_{y\vee t}) \cup (\mathbf{a}_{y\vee u}) \\ &= \{a_1, a_2, a_3, a_5, a_8, a_9\}\,.\end{aligned}$$

Since $\mathbf{a}$ does not match with any intension of concept, there is no $\text{AP}_{\mathbf{a}}$ (and no SAP) using the tuple $(y \vee t, y, u)$.

Similarly, we have $\mathbf{o}_{y\wedge u} \subseteq \mathbf{o}_{y\vee t} \subseteq \mathbf{o}_y \cup \mathbf{o}_u$ and equation $\mathbf{o}_{y\vee t} : \mathbf{o}_y :: \mathbf{o}_u : \mathbf{o}$ admits as solution $\mathbf{o} \subseteq \{o_1, \dots, o_5\}$

$$\begin{aligned}\mathbf{o} &= ((\mathbf{o}_y \cup \mathbf{o}_u) \setminus \mathbf{o}_{y\vee t}) \cup (\mathbf{o}_{y\vee u}) \\ &= \{o_4, o_5\} \\ &= \mathbf{o}_{z\wedge u}\,.\end{aligned}$$

Therefore, the analogical proportion $((y \vee t) : y\ \text{AP}_{\mathbf{o}}\ u : (z \wedge u))$ holds.

5.4. Resolution of a WAP equation in formal concept analysis settings

We have seen in Section 4 the general case for solving an analogical equation in a lattice. We come back to that fundamental problem in the framework of a formal concept lattice in the particular case of weak analogical proportions. In this paragraph, we give an algorithm to solve the equation in t: $x : y \, WAP \, z : t$. It accepts the three first concepts and the formal context as an input and produces all solutions t. Recall that the uniqueness of a solution is not guaranteed: for example, in the concept lattice discussed in Figure 8, the equation $(x : \top :: \bot : ?)$ has three solutions. Moreover, even if $y \wedge z \leq x \leq y \vee z$ as required by the definition of WAP, equation $x : y \, WAP \, z : t$ may have no solution t as illustrated by the concept lattice of Figure D.26.

5.4.1. Basic procedure

Let us consider three concepts x, y and z such that $y \wedge z \leq x \leq y \vee z$ and the solving of the associated WAP equation. Since a FAP is also a WAP, the approach described in Section 4.2 could be used to reduce the search space. However, this 'pruning' process can exclude some WAP solutions since it lies on the search of FAP in antichain. Moreover, the framework of concept lattices permits a different approach: using the properties of the sets of attributes and objects associated to the concepts, the proposed algorithm mainly lies on basic set-theoretic operations, exploring some subsets of attributes instead of looking for all fathers or sons of given concepts, and does not require the prior construction of the whole lattice.

According to Definition 6, any solution t has to satisfy the inequalities $y \wedge z \leq t \leq y \vee z$, which are equivalent to

$$\mathbf{a}_y \cap \mathbf{a}_z \subseteq \mathbf{a}_t \subseteq \mathbf{a}_{y \wedge z} \,. \quad (7)$$

Moreover, from Proposition 17, $\mathbf{a}_t$ has also to satisfy the equality

$$\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z \,. \quad (8)$$

Equations (7) and (8) lead us to define the following disjoint subsets of $\mathbf{a}_x$:

$$\begin{aligned} U &= \mathbf{a}_x \cap \mathbf{a}_y \cap \mathbf{a}_z \\ &= \mathbf{a}_y \cap \mathbf{a}_z \\ V &= (\mathbf{a}_x \cap \mathbf{a}_y) \setminus \mathbf{a}_z \\ &= (\mathbf{a}_x \cap \mathbf{a}_y) \setminus U \\ W &= (\mathbf{a}_x \cap \mathbf{a}_z) \setminus \mathbf{a}_y \\ &= (\mathbf{a}_x \cap \mathbf{a}_z) \setminus U \end{aligned}$$

We notice that $\mathbf{a}_x$ is the disjoint union of U, V, W and $\mathbf{a}_x \setminus (U \cup V \cup W)$, likewise $\mathbf{a}_y$ is the disjoint union of U, W and $\mathbf{a}_y \setminus (U \cup V)$, and $\mathbf{a}_z$ is the disjoint union of U, W and $\mathbf{a}_z \setminus (U \cup W)$.

We consider now three other disjoints sets Y, Z and T such that

$$\begin{cases} Y & \subseteq & \mathbf{a}_{y \wedge z} \setminus (\mathbf{a}_x \cup \mathbf{a}_y \cup \mathbf{a}_z) \\ Z & \subseteq & \mathbf{a}_y \setminus (U \cup V) \ = \ \mathbf{a}_y \setminus \mathbf{a}_x \\ T & \subseteq & \mathbf{a}_z \setminus (U \cup W) \ = \ \mathbf{a}_z \setminus \mathbf{a}_x \end{cases} \tag{9}$$

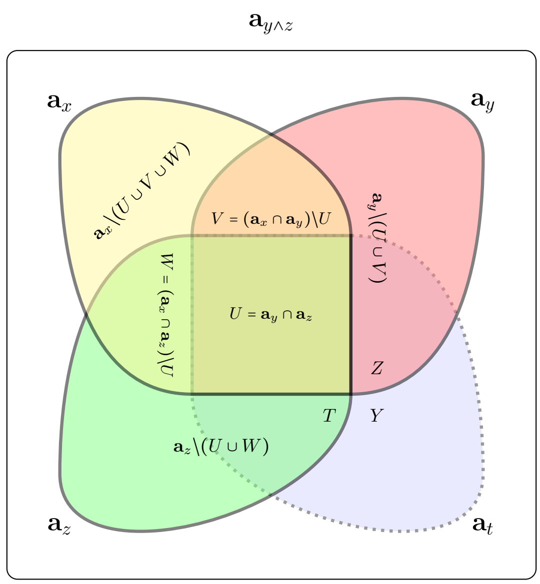


Figure 17: Search of set $\mathbf{a}_t$ for the resolution of the analogical equation between concepts $x : y \, WAP \, z : t$. Sets $\mathbf{a}_x$ and $\mathbf{a}_t$ have to contain $\mathbf{a}_y \cap \mathbf{a}_z$ and to be included $\mathbf{a}_{x \wedge y}$. Solutions $\mathbf{a}_t$ can be written as the cup of four disjoint sets: $U = \mathbf{a}_x \cap \mathbf{a}_y \cap \mathbf{a}_z$, $Z \subset \mathbf{a}_y \setminus \mathbf{a}_x$, $T \subset \mathbf{a}_z \setminus \mathbf{a}_x$ and Y, where Y is a subset of $\mathbf{a}_{y \wedge z}$ and has no element in common with $\mathbf{a}_x \cup \mathbf{a}_y \cup \mathbf{a}_z$.

Figure 17 explains all subsets introduced above. We now build the set of attributes $\mathbf{a} = U \cup Y \cup Z \cup T$, which is such that $\mathbf{a}_x \cap \mathbf{a} = \mathbf{a}_y \cap \mathbf{a}_z$. However, $\mathbf{a}$ is not necessarily a set of attributes corresponding to a concept. We still have to build the concept $t = (\mathbf{a}^{\downarrow}, \mathbf{a}^{\downarrow\uparrow})$ and to check whether or not $\mathbf{a}_t$ satisfies $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ and $\mathbf{o}_t$ satisfies $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$. If both conditions are fulfilled, t is a solution.

Hence, this algorithm is enumerative: all possible sets Y, Z and T that satisfy the subset relationship in equation (9) are built and checked. Its only efficiency comes from the pruning realized in taking $\mathbf{a}$ such as $\mathbf{a} = U \cup Y \cup Z \cup T$, which avoids to enumerating all subsets of the set of attributes. It is likely that a branch and bound amelioration is possible, but we have not investigated this possibility.

5.4.2. Example

To illustrate the previous procedure, we have extracted a subcontext from the Zoo data base (see [35]). It is described in Table 1, where the meaning of its attributes and objects is indicated, and the associated lattice, named *SmallZoo context*, is displayed in Figure 18.

Let us make two remarks about it. Firstly, the Zoo data base has been designed to challenge classification algorithms, and consequently animals with unusual features, such as penguin or platypus, are over-represented. Secondly, we use it uniquely to illustrate our model and algorithms: the sentences in natural language that we present have to be interpreted in taking into account that all their semantics comes from the set of 8 attributes.

SmallZoo		hair a_0	feathers a_1	eggs a_2	milk a_3	airborne a_4	aquatic a_5	predator a_6	toothed a_7
o_0	aardvark	×			×			×	×
o_1	chicken		×	×		×			
o_2	crow		×	×		×		×	
o_3	dolphin				×		×	×	×
o_4	duck		×	×		×	×		
o_5	fruitbat	×			×	×			×
o_6	kiwi		×	×				×	
o_7	mink	×			×		×	×	×
o_8	penguin		×	×			×	×	
o_9	platypus	×		×	×		×	×	

Table 1: The SmallZoo context

If we choose the following concepts

$$\begin{aligned} x &= C(29) \\ &= (\{o_5\}, \{a_0, a_3, a_4, a_7\}) \\ y &= C(25) \\ &= (\{o_0, o_7\}, \{a_0, a_3, a_6, a_7\}) \\ z &= C(19) \\ &= (\{o_1, o_2, o_4\}, \{a_1, a_2, a_4\}) \end{aligned}$$

they satisfy $\bot = y \wedge z \leq x \leq y \vee z = \top$ and equations (9) give

$$\begin{aligned} U &= \varnothing \\ Y &\subseteq \{a_5\} \\ Z &\subseteq \{a_6\} \\ T &\subseteq \{a_1, a_2\} \end{aligned}$$

As a consequence, we are looking for the concepts t such that

$$\left\{\begin{array}{rcl} \mathbf{a}_t & = & \mathbf{a}^{\downarrow\uparrow} \\ \mathbf{a} & \subseteq & \{a_1, a_2, a_5, a_6\} \\ \mathbf{o}_x \cap \mathbf{o}_t & = & \mathbf{o}_y \cap \mathbf{o}_z \\ \mathbf{a}_x \cap \mathbf{a}_t & = & \mathbf{a}_y \cap \mathbf{a}_z \end{array}\right. \tag{10}$$

Equations (10) are equivalent to

$$\left\{\begin{array}{lcl} \mathbf{a}_t & = & \mathbf{a}^{\downarrow\uparrow} \\ \mathbf{a} & \subseteq & \mathbf{a}_t \subseteq \{a_1, a_2, a_5, a_6\} \\ o_5 & \notin & \mathbf{o}_t \end{array}\right.$$

The solution set of (10) is given by

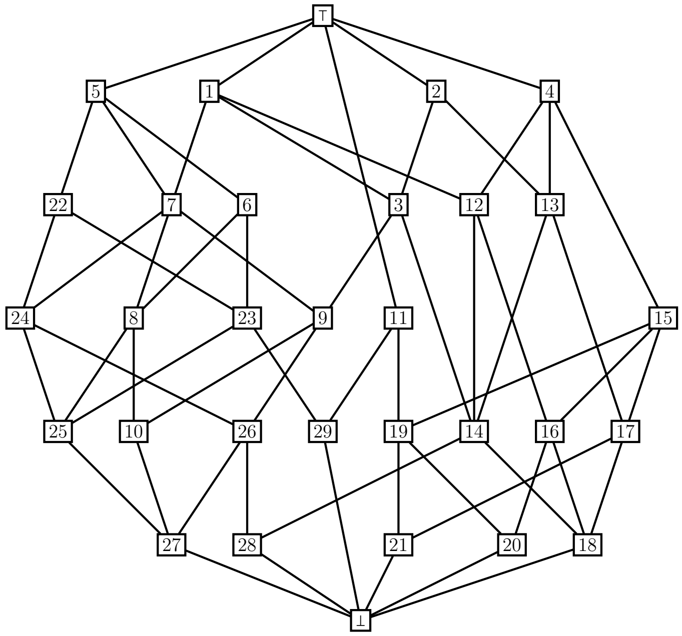
$$\{C(1), C(2), C(3), C(4), C(12), C(13), C(14), C(15), C(16), C(17), C(18)\}.$$

Among the obtained WAPs, there is no FAP since $x \neq (x \wedge y) \vee (x \wedge z)$, but the following AP between attributes holds:

$$C(29) : C(25) \ \mathrm{AP}_{\mathbf{a}} \ C(19) : C(16).$$

Besides, we can observe that, contrary to the FAP case (see Proposition 30 in Appendix D), even if $\{x, y, z\}$ is an antichain of three elements, some of solutions t of the WAP equation are comparable to one of these elements: it is the case for $C(1)$, $C(4)$ and $C(15)$.

This last remark will be exploited in the next section, where a special kind of WAP will be introduced, with in particular the property to be an antichain.



$C(1)$	$\big(\{o_0, o_2, o_3, o_6, o_7, o_8, o_9\},$ $\{a_6\}\big)$	$C(15)$	$\big(\{o_1, o_2, o_4, o_6, o_8\}, \{a_1, a_2\}\big)$
$C(2)$	$\big(\{o_3, o_4, o_7, o_8, o_9\}, \{a_5\}\big)$	$C(16)$	$\big(\{o_2, o_6, o_8\}, \{a_1, a_2, a_6\}\big)$
$C(3)$	$\big(\{o_3, o_7, o_8, o_9\}, \{a_5, a_6\}\big)$	$C(17)$	$\big(\{o_4, o_8\}, \{a_1, a_2, a_5\}\big)$
$C(4)$	$\big(\{o_1, o_2, o_4, o_6, o_8, o_9\}, \{a_2\}\big)$	$C(18)$	$\big(\{o_8\}, \{a_1, a_2, a_5, a_6\}\big)$
$C(5)$	$\big(\{o_0, o_3, o_5, o_7, o_9\}, \{a_3\}\big)$	$C(19)$	$\big(\{o_1, o_2, o_4\}, \{a_1, a_2, a_4\}\big)$
$C(6)$	$\big(\{o_0, o_3, o_5, o_7\}, \{a_3, a_7\}\big)$	$C(20)$	$\big(\{o_2\}, \{a_1, a_2, a_4, a_6\}\big)$
$C(7)$	$\big(\{o_0, o_3, o_7, o_9\}, \{a_3, a_6\}\big)$	$C(21)$	$\big(\{o_4\}, \{a_1, a_2, a_4, a_5\}\big)$
$C(8)$	$\big(\{o_0, o_3, o_7\}, \{a_3, a_6, a_7\}\big)$	$C(22)$	$\big(\{o_0, o_5, o_7, o_9\}, \{a_0, a_3\}\big)$
$C(9)$	$\big(\{o_3, o_7, o_9\}, \{a_3, a_5, a_6\}\big)$	$C(23)$	$\big(\{o_0, o_5, o_7\}, \{a_0, a_3, a_7\}\big)$
$C(10)$	$\big(\{o_3, o_7\}, \{a_3, a_5, a_6, a_7\}\big)$	$C(24)$	$\big(\{o_0, o_7; o_9\}, \{a_0, a_3, a_6\}\big)$
$C(11)$	$\big(\{o_1, o_2, o_4, o_5\}, \{a_4\}\big)$	$C(25)$	$\big(\{o_0, o_7\}, \{a_0, a_3, a_6, a_7\}\big)$
$C(12)$	$\big(\{o_2, o_6, o_8, o_9\}, \{a_2, a_6\}\big)$	$C(26)$	$\big(\{o_7, o_9\}, \{a_0, a_3, a_5, a_6\}\big)$
$C(13)$	$\big(\{o_4, o_8, o_9\}, \{a_2, a_5\}\big)$	$C(27)$	$\big(\{o_7\}, \{a_0, a_3, a_5, a_6, a_7\}\big)$
$C(14)$	$\big(\{o_8, o_9\}, \{a_2, a_5, a_6\}\big)$	$C(28)$	$\big(\{o_9\}, \{a_0, a_2, a_3, a_5, a_6\}\big)$
		$C(29)$	$\big(\{o_5\}, \{a_0, a_3, a_4, a_7\}\big)$

Figure 18: Formal concept lattice of formal context SmallZoo.

5.5. The complete WAP

The phrase "a stallion is to a ram as a lamb is to a calf" is an analogical proportion that makes sense: it can be used for example to define the word "ram" to a language learner. It is also meaningful in each of the eight possible forms. In this AP, the four animals involved are implicitly defined on the attributes "adult" vs "young", "equine" vs "ovine" and "male" vs "female" when adult.

Let us take the same set of attributes, plus "bovine" and consider the phrase: "a stallion is to a foal as a cow is to a calf". It is not as satisfactory as the previous one, since the word "stallion" is associated to an adult horse, but also to the male gender, whereas "cow" is associated to an adult bovine, and to the female gender. The form "a stallion is to a cow as a foal is to a calf" calls for a gender difference between a foal and a calf, which is not in the semantics of these two words.

Both examples can be gathered in the following formal context.

		equine a_1	male adult a_2	young a_3	bovine a_4	female adult a_5	ovine a_6
o_1	stallion	×	×				
o_2	foal	×		×			
o_3	crow				×	×	
o_4	calf			×	×		
o_5	ram		×				×
o_6	lamb			×			×

Its associated concept lattice is displayed in Figure 19.

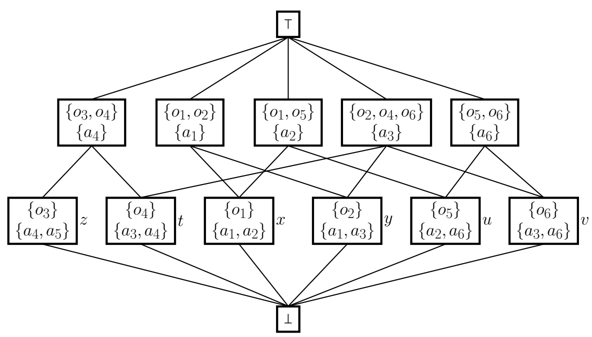


Figure 19: The weak analogical proportion $(x : y \text{ WAP } z : t)$ is not complete, but $(x : y \text{ WAP } u : v)$ is complete through attributes.

The phrase "a stallion is to a ram as a lamb is to a calf" corresponds to the weak analogical proportion $(x : y\, WAP\, u : v)$, that is to say

$$(\{o_1\}, \{a_1, a_2\}) : (\{o_2\}, \{a_1, a_3\})\, WAP\, (\{o_5\}, \{a_2, a_6\}) : (\{o_6\}, \{a_3, a_6\})\,.$$

As for the phrase "a stallion is to a foal as a cow is to a calf", it corresponds to $(x : y\, WAP\, z : t)$, i.e.

$$(\{o_1\}, \{a_1, a_2\}) : (\{o_2\}, \{a_1, a_3\})\, WAP\, (\{o_3\}, \{a_4, a_5\}) : (\{o_4\}, \{a_3, a_4\}),$$

but it has a different repartition of attributes.

This section formalizes this remark and defines what we call a *complete* WAP in a concept lattice. The first example will be a complete WAP, whereas the second will not, as far as the attribute "gender" is in the context. As explained previously, a WAP extracted from the implicit context of a linguistic phrase will be complete.

Definition 9 (Complete WAP)**.** *Let us consider a WAP $(x : y\, WAP\, z : t)$. We denote $\mathbf{a}_\cap$ and $\mathbf{o}_\cap$ the intersection of the four sets of attributes and that of the four sets of objects. We say that this WAP is* complete *when*

1. *either the four following subsets are all nonempty:*

   $(\mathbf{a}_x \cap \mathbf{a}_y) \setminus \mathbf{a}_\cap \quad (\mathbf{a}_x \cap \mathbf{a}_z) \setminus \mathbf{a}_\cap \quad (\mathbf{a}_y \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap \quad (\mathbf{a}_z \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap$

   (called complete WAP through attributes),

2. *or the four following subsets are all nonempty:*

   $(\mathbf{o}_x \cap \mathbf{o}_y) \setminus \mathbf{o}_\cap \quad (\mathbf{o}_x \cap \mathbf{o}_z) \setminus \mathbf{o}_\cap \quad (\mathbf{o}_y \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap \quad (\mathbf{o}_z \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap$

   (called complete WAP through objects).

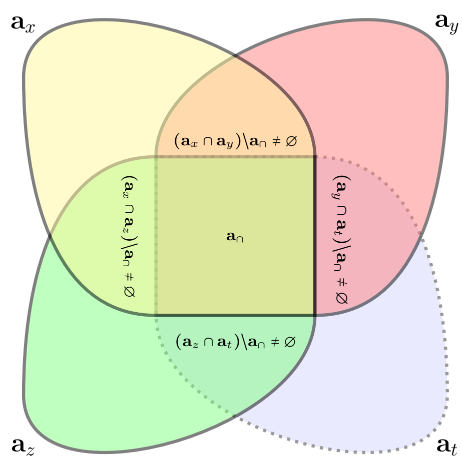


Figure 20: Representation of the attribute sets in case of a complete WAP $(x : y\, WAP\, z : t)$ through attributes.

Let us consider the case of the complete WAP $(x : y\, WAP\, z : t)$ through attributes. In order to obtain a substantial semantic interpretation, it appears that it is important that each pair of concepts (x, y), (z, t), (x, z) and (y, t) shares commun attributes which differentiate them from the other concepts that compose the WAP. In particular, the completeness permits to exclude WAPs where one concept is a "subconcept" of another as we shall see in Proposition 20.

Figure 20 depicts graphically the relation between attribute sets in the case of a complete WAP through attributes.

Example. Among the WAPs extracted from *SmallZoo context* and listed in Section 5.4.2,

$$C(29) : C(25)\, WAP\, C(19) : C(18)$$

is a complete WAP through attributes: $C(29)$ and $C(25)$ share attributes a_0, a_3 and a_7 (*hair, milk* and *toothed*), which are not present in $C(19)$ and $C(18)$. Similarly, $C(29)$ and $C(19)$ share a_4 (*airborne*), $C(25)$ and $C(18)$ share a_6 (*predator*), and $C(19)$ and $C(18)$ share a_1 and a_2 (*feathers* and *eggs*). These connections can also be summarized by the following subcontext.

			feathers a_1	eggs a_2	predator a_6	airborne a_4	hair a_0	milk a_3	toothed a_7	aquatic a_5
$\mathbf{o}_{C(29)}$	o_5	fruitbat				×	×	×	×	
$\mathbf{o}_{C(25)}$	o_0	aardvark			×		×	×	×	
	o_7	mink			×		×	×	×	
$\mathbf{o}_{C(19)}$	o_1	chicken	×	×		×				
	o_2	crow	×	×	×	×				
	o_4	duck	×	×		×				
$\mathbf{o}_{C(18)}$	o_8	penguin	×	×	×					×

Let us notice that, even though o_2 and a_6 are related in this subcontext, a_6 is not an element of $\mathbf{a}_{C(19)}$, and a_5 is only related to $C(18)$.

Consequently, taking into account these previous attributes, the analogical proportion "fruitbats are to aardvarks and minks as chickens, crows and ducks are to penguins" makes sense[6].

A complete WAP has the following properties.

Proposition 20. *Let x, y, z and t be four concepts such that $(x : y\, WAP\, z : t)$.*

1. *If the WAP is complete, x, y, z and t form an antichain.*
2. *If the WAP is complete through attributes, $(x \vee y)$, $(x \vee z)$, $(y \vee t)$ and $(z \vee t)$ form an antichain.*

[6] Note that the contrary AP "fruitbats are to aardvarks and minks as penguins are to chickens, crows and ducks" makes also sense if we think of airborne mammals and of non-airborne birds as exceptions in their category. In our case, the airborne/non-airborne attributes impose the way that the AP is formulated.

Similarly, if the WAP is complete through objects, $(x \wedge y)$, $(x \wedge z)$, $(y \wedge t)$ and $(z \wedge t)$ form an antichain.

3. *If the WAP is complete through attributes and objects, it is a FAP.*

Proof.

1. Let us suppose that $(x : y \, WAP \, z : t)$ and $x \leq y$. From Preliminary 2, we have $\mathbf{a}_x \cap \mathbf{a}_y = \mathbf{a}_{x \vee y}$, and then $\mathbf{a}_x \cap \mathbf{a}_y = \mathbf{a}_y$. Consequently, using Proposition 17,

$$\begin{aligned} \mathbf{a}_x \cap \mathbf{a}_z &= (\mathbf{a}_x \cap \mathbf{a}_y) \cap \mathbf{a}_z = \mathbf{a}_x \cap (\mathbf{a}_y \cap \mathbf{a}_z) \\ &= \mathbf{a}_x \cap (\mathbf{a}_x \cap \mathbf{a}_t) = \mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_\cap \end{aligned}$$

then $\mathbf{a}_x \cap \mathbf{a}_z \setminus \mathbf{a}_\cap = \varnothing$. and $(x : y \, WAP \, z : t)$ is not a complete WAP.

2. We take a complete WAP through attributes. We have $\mathbf{a}_{x \vee y} = \mathbf{a}_x \cap \mathbf{a}_y$, and three analog equalities. The organization of these attribute sets is displayed in Figure 12. Due to the completeness of the WAP, there is no inclusion relation between $\mathbf{a}_{x \vee y}$, $\mathbf{a}_{x \vee z}$, $\mathbf{a}_{z \vee t}$ and $\mathbf{a}_{y \vee t}$. Therefore, $x \vee y$, $x \vee z$, $y \vee t$ and $z \vee t$ are in antichain.

   Note that $x \vee y$, $x \vee z$, $y \vee t$ and $z \vee t$ are not in general a WAP. We come back to this point in Section 5.6.

3. Let us consider a complete WAP $(x : y \, WAP \, z : t)$ through attributes and objects. From the previous properties, concepts x, y, z and t form an antichain. Besides, $(x \vee y)$, $(x \vee z)$, $(y \vee t)$ and $(z \vee t)$ are also in antichain, as well as concepts $(x \wedge y)$, $(x \wedge z)$, $(y \wedge t)$ and $(z \wedge t)$. As a consequence, these 12 concepts are distinct and, in a similar manner as the proof of Proposition 3, it can be proved that they generate a Boolean sublattice. Therefore, because of the distributivity of this sublattice and Proposition 4, the WAP $(x : y \, WAP \, z : t)$ is a FAP. □

5.6. The analogical lattice

We have seen in Proposition 20 that when x, y, z and t are four concepts in complete WAP through attributes, $x \vee y$, $x \vee z$, $y \vee t$ and $z \vee t$ are an antichain. However, they are not in general a WAP.

We study in this section the particular case where both are a WAP. Firstly, we give the complete definition of the so-called *bi-complete* WAP. Secondly, we show that it corresponds to a special lattice, which is at the core of analogical proportions in concept lattices and of their semantic interpretation. It is also strongly related to the construction of a relational proportion from a formal context proposed in Section 7.

Definition 10. *The WAP $(x : y \, WAP \, z : t)$ is said to be bi-complete*

- *if it is complete through attributes and $((x \vee y) : (x \vee z) \, WAP \, (y \vee t) : (z \vee t))$ is complete through objects.*
- *or if it is complete through objects and $((x \wedge y) : (x \wedge z) \, WAP \, (y \wedge t) : (z \wedge t))$ is complete through attributes.*

Proposition 21. *Let x, y, z and t be four concepts,*

1. *if* $(x : y\ WAP\ z : t)$ *is complete through attributes and is a bi-complete WAP, then* $((x \vee y) : (x \vee z)\ WAP\ (y \vee t) : (z \vee t))$ *is complete through objects and is a bi-complete WAP.*

2. *if* $(x : y\ WAP\ z : t)$ *is complete through objects and is a bi-complete WAP, then* $((x \wedge y) : (x \wedge z)\ WAP\ (y \wedge t) : (z \wedge t))$ *is complete through attributes and is a bi-complete WAP.*

Proof. See Appendix H. □

The simplest context (in terms of number of attributes and objects) that produces such a bi-complete WAPs is given in Figure 21. We call it the *analogical context* and denote it $\mathcal{C}_{Analog}$.

	a_1	a_2	a_3	a_4
o_1	×	×		
o_2	×		×	
o_3		×		×
o_4			×	×

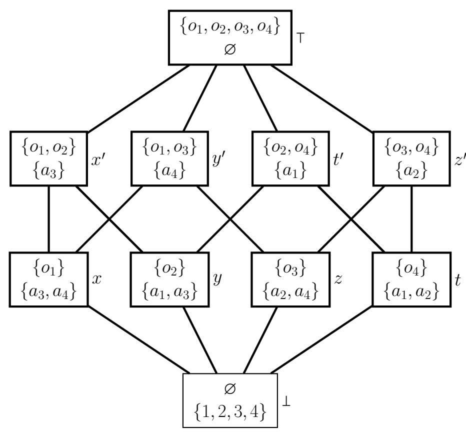


Figure 21: The analogical concept lattice $\mathcal{C}_{Analog}$ and the associated formal context. Concepts x, y, z and t are in complete WAP (through attributes), as well as concepts x', y', z' and t' (through objects).

The context $\mathcal{C}_{Analog}$ appeared first as a generator of WAPs between concepts in [44]. Note that this lattice is not distributive: if we consider concepts x, y and z described in Figure 21, one has $x \wedge (y \vee z) = x$ while $(x \wedge y) \vee (x \wedge z) = \bot$.

Many simple examples of such a formal context can be produced. Actually, it describes exactly the situation producing a linguistic analogical proportion, as for example between the four animals "calf", "bull", "foal" and "stallion".

Note that in such a (4×4) context, the attributes and the objects are opposed by pairs: for example, no animal is both an equine and a bovine. This opposition may result from the fact that the attributes come from semantic trees. This point is more detailed at the end of Section 7.

Note also that the algebraic symmetry between attributes and objects has not necessarily a linguistic equivalent. One cannot phrase a proportion between attributes such as "a bovine is to a young as an adult male is to an equine". It only means that the attributes, "young" and "adult male" have the same empty intersection as "bovine" and "equine".

We propose to give the name of *analogical lattice* to the corresponding lattice displayed in Figure 21. The comments before Definition 9 illustrate this structure is a good model for the analogical proportion in natural language, as far as formal concepts are concerned.

Let us remark that the context $\mathcal{C}_{Analog}$ can also be interpreted through the notion of semiproduct of two contexts, as explained in Appendix F.

The important role of the analogical lattice can also be enhanced by considering two other lattices. The first one is the Boolean lattice. We have seen in Section 5.1 (see Figure 15 and the associated comments) that a linguistic interpretation of one of the three WAPs in such a lattice is problematic. In a way, in a Boolean lattice with a reduced 4×4 context, the objects are too intertwined: the relationship between any pair of objects is two strong (these WAPs are also SAPs). Note also that there is no bi-complete WAP in a Boolean lattice.

On the other hand, consider the context (that we call "diagonal") and the lattice displayed in Figure 8. The four central concepts are indeed in a WAP, which is not complete. In a way, it is as incomplete as a WAP can be, since none of the four subsets that should be non empty are actually empty. Such a context can be obtained for example in taking randomly four objects and in filling up the context with attributes that are possessed by only the concerned object. Linguistically speaking, there is no analogy at all in such a situation. In particular, both the WAPs $x : y :: z : t$ and $y : x :: z : t$ hold, since all objects are independent with respect to the attributes.

The analogical context is the good balance between these two situations. The relationship between the objects is convenient with respect to a linguistic interpretation. It is the simplest context to contain a complete WAP, and this minimality implies that it contains actually a bi-complete WAP.

Note that the following context produces also a bi-complete WAP, as do all the intermediate context between this one and the analogical context.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
o_1	×	×			×			
o_2	×		×			×		
o_3		×		×			×	
o_4			×	×				×
o_5	×							
o_6		×						
o_7			×					
o_8				×				

Section 7 will give more linguistic soundness to the claim that the context $\mathcal{C}_{Analog}$ is the core of analogical proportions and relational proportions in the framework of formal concept analysis.

In Appendix G, we discuss how to construct a minimal analogical context associated to a given analogical proportion expressed in linguistic terms. As we will see in the following section, such a minimal context is based on a particular relation between objects and attributes that we call "analogical complex".

6. Analogical complexes and proportions between concepts

In the previous sections, it has turned out that the context of the analogical lattice is an interesting pattern, providing a natural semantic interpretation of the analogical proportion, and on which can be based the definition of the relational proportion. In this section, we consider a more general definition of this pattern, named *analogical complex*, introduced in [42] and we relate the WAP in lattices of formal concepts to the notion of analogical complex

A major interest of this connection is that analogical complexes are easy to discover and to enumerate from the sole context: there is no need in this section to construct the lattice of concepts.

We firstly recall the definitions given in the quoted article, and then examine the two-way relationships between WAPs and analogical complexes.

6.1. Analogical schema and analogical complex

An *analogical complex* is a subcontext of a formal context with a particular pattern described by

			×	×
		×		×
	×		×	
	×	×		

associated to the binary matrix

$$AS = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

called *analogical schema*. We write $AS(i,j)$ if the value at row i and column j of matrix AS is 1 (e.g. $AS(1,3)$ and $AS(1,4)$).

Definition 11 (Analogical complex)**.** *Given a formal context* $(\mathcal{O}, \mathcal{A}, R)$*, a set of objects* $\mathbf{o} \subseteq \mathcal{O}$*,* $\mathbf{o} = \mathbf{o}_1 \cup \mathbf{o}_2 \cup \mathbf{o}_3 \cup \mathbf{o}_4$*, a set of attributes* $\mathbf{a} \subseteq \mathcal{A}$*,* $\mathbf{a} = \mathbf{a}_1 \cup \mathbf{a}_2 \cup \mathbf{a}_3 \cup \mathbf{a}_4$*, and a binary relation R, the subcontext* $(\mathbf{o}, \mathbf{a})$ *forms an analogical complex* $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ *iff*

1. *the binary relation is compatible with the analogical schema AS:*
   $\forall i \in [1,4],\ \forall o \in \mathbf{o}_i,\ \forall j \in [1,4],\ \forall a \in \mathbf{a}_j,\ ((o,a) \in R) \Leftrightarrow AS(i,j)$.

2. *The context is maximal with respect to the first property (*$\oplus$ *denotes the exclusive or and* $\smallsetminus$ *the set-theoretic difference):*
   $\forall o \in \mathcal{O} \smallsetminus \mathbf{o}, \forall i \in [1,4],\ \exists j \in [1,4], \exists a \in \mathbf{a}_j, ((o,a) \in R) \oplus AS(i,j)$.
   $\forall a \in \mathcal{A} \smallsetminus \mathbf{a}, \forall j \in [1,4],\ \exists i \in [1,4], \exists o \in \mathbf{o}_i, ((o,a) \in R) \oplus AS(i,j)$.

Comments.

1. The cartesian products $\mathbf{o}_1 \times \ldots \times \mathbf{o}_4$ and $\mathbf{a}_1 \times \ldots \times \mathbf{a}_4$ are respectively denoted $\mathbf{o}_{1,4}$ and $\mathbf{a}_{1,4}$ in order to simplify the notations.

2. Definition 11 is still correct even if some sets $\mathbf{o}_i$ or $\mathbf{a}_j$ are empty. However, it does not correspond to interesting cases. For example, from the following formal context

	a_1	a_2	a_3
o_1			×
o_2		×	
o_3	×		×

the analogical complex below can be derived

		$\mathbf{a}_1$ $\{a_1\}$	$\mathbf{a}_2$ $\{a_2\}$	$\mathbf{a}_3$ $\{a_3\}$	$\mathbf{a}_4$ $\varnothing$
$\mathbf{o}_1$	$\{o_1\}$			×	×
$\mathbf{o}_2$	$\{o_2\}$		×		×
$\mathbf{o}_3$	$\{o_3\}$	×		×	
$\mathbf{o}_4$	$\varnothing$	×	×		

3. Any attribute subset $\{a\}$ and any object subset $\{o_1, o_2\}$ such that $(o_1, a) \in R$ and $(o_2, a) \notin R$ cannot form an analogical complex.

We come now to the definition of a particular complex, that we have called *complete* in [42], which will be related later to that of a bi-complete WAP.

Definition 12. *An analogical complex $\mathcal{C} = (\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ is* complete *if none of its eight sets are empty.*

By construction, if $\mathcal{C} = (\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ is a complete analogical complex and if $\mathcal{A} = \bigcup_{i=1,4} \mathbf{a}_i$, the following formula hold:

$\forall (o_1, o_2, o_3, o_4) \in \mathbf{o}_{1,4}, \forall (a_1, a_2, a_3, a_4) \in \mathbf{a}_{1,4}$

$$\begin{cases} (\{o_1\}^{\uparrow} \cap \{o_4\}^{\uparrow}) \cap \mathcal{A} &= (\{o_2\}^{\uparrow} \cap \{o_3\}^{\uparrow}) \cap \mathcal{A} &= \varnothing \\ (\{o_1\}^{\uparrow} \cup \{o_4\}^{\uparrow}) \cap \mathcal{A} &= (\{o_2\}^{\uparrow} \cup \{o_3\}) \cap \mathcal{A} &= \mathcal{A} \end{cases}$$

Example. The context SmallZoo, introduced in Section 5.4.2, contains the following subcontext (after reordering lines and columns[7]):

[7]Let us note that in an analogical table, the order within a column, as well as within a row, has no meaning.

		$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$
		a_4	a_5	a_0	a_6
$\mathbf{o}_1$	o_0			×	×
$\mathbf{o}_2$	o_3		×		×
	o_8		×		×
$\mathbf{o}_3$	o_5	×		×	
$\mathbf{o}_4$	o_4	×	×		

It is easy to check that no other attribute nor object can be added to these, still keeping the same matrix. Hence, this subcontext corresponds to an analogical complex with $\mathbf{o}_1 = \{o_0\}$, $\mathbf{o}_2 = \{o_3, o_8\}$, $\mathbf{o}_3 = \{o_5\}$, $\mathbf{o}_4 = \{o_4\}$ and $\mathbf{a}_1 = \{a_4\}$, $\mathbf{a}_2 = \{a_5\}$ $\mathbf{a}_3 = \{a_0\}$ and $\mathbf{a}_4 = \{a_6\}$. It is also complete, and can be displayed in different designs, like the table above, or those depicted below:

		$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$
		$\{a_4\}$	$\{a_5\}$	$\{a_0\}$	$\{a_6\}$
$\mathbf{o}_1$	$\{o_0\}$			×	×
$\mathbf{o}_2$	$\{o_3, o_8\}$		×		×
$\mathbf{o}_3$	$\{o_5\}$	×		×	
$\mathbf{o}_4$	$\{o_4\}$	×	×		

$\mathbf{o}_{1,4}$	$\mathbf{a}_{1,4}$
$\{o_0\}$	$\{a_4\}$
$\{o_3, o_8\}$	$\{a_5\}$
$\{o_5\}$	$\{a_0\}$
$\{o_4\}$	$\{a_6\}$

0	4
3, 8	5
5	0
4	6

The table on the left looks like a subcontext, the middle one is a more compact notation that we will use in the following with the name of *analogical table*. The table on the right is a simplified form of an analogical table that we will use in Figure 22.

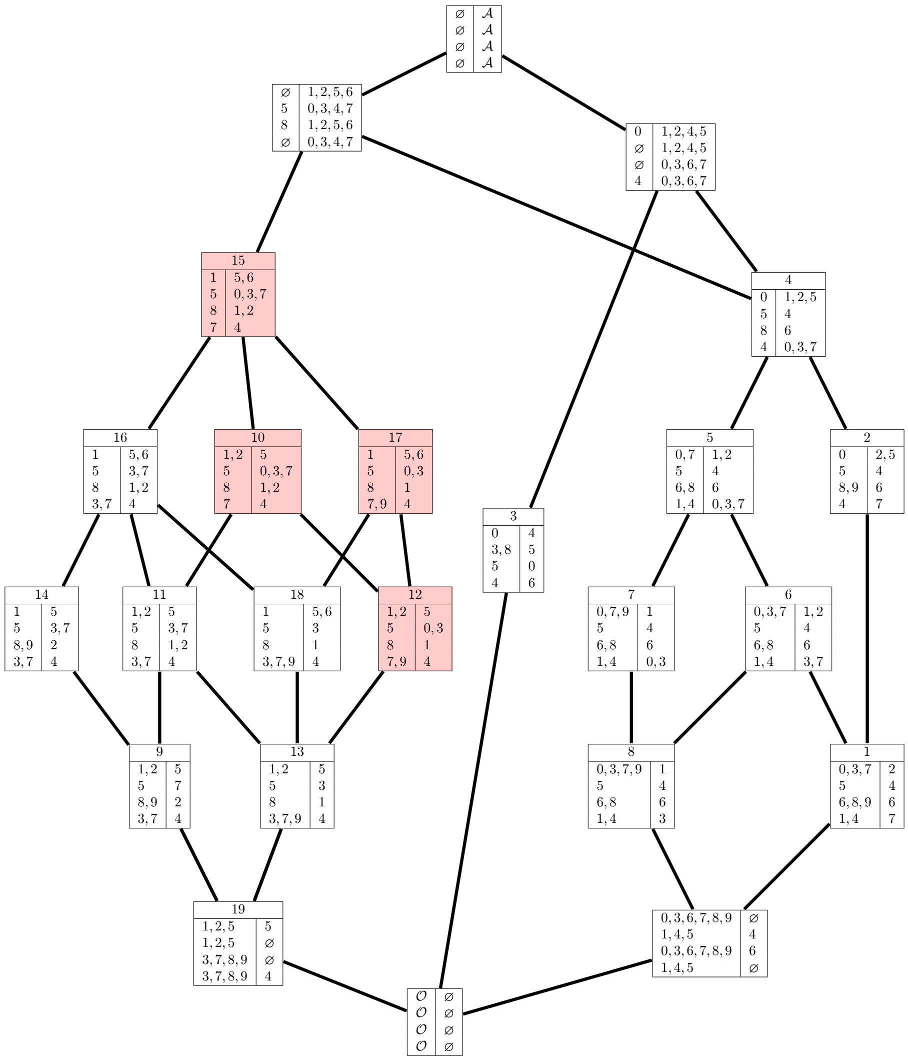


Figure 22: Hasse diagram of the analogical complex lattice for formal context SmallZoo. The complete complexes have been given a number to index them. In red are displayed the complete complexes that do not contain the objects o_0, nor o_3 nor o_6 (see the example of the algorithm `WAPtoCPLX`, Section 6.2).

Comment. Miclet and Nicolas [42] have shown that the set of the analogical complexes of any formal context is itself structured as a lattice. For example, the lattice of the analogical complexes from SmallZoo is given in Figure 22. It contains 24 elements, among them 18 are complete analogical complexes.

In the next section, we show that there are two complexes at the core of four concepts in WAP, and we also show, as a reciprocal, how to construct WAPs from an analogical complex. We also make a strong connection between the notion of a bi-complete WAP and that of a complete complex.

6.2. From a WAP to two lattices of analogical complexes

Figure 12 has displayed the organization of the attributes and objects of four concepts x, y, z and t in WAP. Let us define $\widetilde{\mathbf{o}_x} = (\mathbf{o}_x \cup \mathbf{o}_y \cup \mathbf{o}_z \cup \mathbf{o}_t) \setminus (\mathbf{o}_y \cup \mathbf{o}_z \cup \mathbf{o}_t)$, the set of objects proper to x, i.e. that appear in $\mathbf{o}_x$, but not in $\mathbf{o}_y$, $\mathbf{o}_z$ and $\mathbf{o}_t$. We define similarly $\widetilde{\mathbf{o}_y}$, $\widetilde{\mathbf{o}_z}$ and $\widetilde{\mathbf{o}_t}$, as well as $\widetilde{\mathbf{a}_x}$ to $\widetilde{\mathbf{a}_t}$.

With these definitions, we define the two following tables:

i	$\mathbf{o}_i$	$\mathbf{a}_i$
1	$\widetilde{\mathbf{o}_x}$	$(\mathbf{a}_z \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap$
2	$\widetilde{\mathbf{o}_y}$	$(\mathbf{a}_y \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap$
3	$\widetilde{\mathbf{o}_z}$	$(\mathbf{a}_x \cap \mathbf{a}_z) \setminus \mathbf{a}_\cap$
4	$\widetilde{\mathbf{o}_t}$	$(\mathbf{a}_x \cap \mathbf{a}_y) \setminus \mathbf{a}_\cap$

i	$\mathbf{o}'_i$	$\mathbf{a}'_i$
1	$(\mathbf{o}_z \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_x}$
2	$(\mathbf{o}_y \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_y}$
3	$(\mathbf{o}_x \cap \mathbf{o}_z) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_z}$
4	$(\mathbf{o}_x \cap \mathbf{o}_y) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_t}$

Table 2: Two tables extracted from a WAP.

Let us consider the first table. We can notice that if $(x : y\ WAP\ z : t)$ is complete through attributes, then for every $i \in \{1, \dots, 4\}$ set $\mathbf{a}_i$ is nonempty. Besides, by construction, every object of $\mathbf{o}_1$ is in relation with every attribute of $\mathbf{a}_3 \cup \mathbf{a}_4$. It is also the case between $\mathbf{o}_2$ and $\mathbf{a}_2 \cup \mathbf{a}_4$, $\mathbf{o}_3$ and $\mathbf{a}_1 \cup \mathbf{a}_3$, $\mathbf{o}_4$ and $\mathbf{a}_1 \cup \mathbf{a}_2$. For all the other combinations, for instance $\mathbf{o}_1$ and $\mathbf{a}_1$, for any $o \in \mathbf{o}_1$, there exists $a \in \mathbf{a}_1$ such that o and a are not in relation. This comes from the definition of a concept. These relations can be represented as follows:

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$
$\mathbf{o}_1$			×	×
$\mathbf{o}_2$		×		×
$\mathbf{o}_3$	×		×	
$\mathbf{o}_4$	×	×		

However, these properties do not guarantee that the subcontext $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ associated to this table is an analogical schema. Indeed, as explained above, a “blank” does not necessarily means that no object is in relation with no attribute concerned. It can exist an object $o \in \mathbf{o}_i$ in relation with an attribute $a \in \mathbf{a}_j$, where $(i,j) \in \{(1,1),(1,2),(2,1),(2,3),\ (3,2),(3,4),(4,3),(4,4)\}$. A similar reasoning can done from the second table, and if $(x : y\,WAP\,z : t)$ is complete through objects, then for every $i \in \{1, \dots, 4\}$ set $\mathbf{o}'_i$ is nonempty.

Actually, these tables do not yet represent exactly analogical complexes. They have to be postprocessed, as we shall illustrate it in the following example.

Example. Let us consider the four following concepts from the formal concept lattice associated to the context SmallZoo and displayed in Figure 18:

$C(19)$	$C(29)$	$C(17)$	$C(26)$
$\{o_1, o_2, o_4\}$	$\{o_5\}$	$\{o_4, o_8\}$	$\{o_7, o_9\}$
$\{a_1, a_2, a_4\}$	$\{a_0, a_3, a_4, a_7\}$	$\{a_1, a_2, a_5\}$	$\{a_0, a_3, a_5, a_6\}$

Concept $C(19)$, for example, can be interpreted as "chicken, crow and duck have feathers, lay eggs and can fly", while $C(26)$ is "mink and platypus have hair, produce milk and are aquatic predators".

Thanks to Proposition 17, it is easy to check that they are in WAP:

$$C(19) : C(29)\, WAP\, C(17) : C(26)\,.$$

Let us notice that it is a WAP complete through attributes but not a bi-complete WAP, and objects o_0, o_3 and o_6 do not appear in this WAP, though all attributes are present. From this WAP, we construct two tables in accordance with Table 2:

i	$\mathbf{o}_i$	$\mathbf{a}_i$
1	$\{o_1, o_2\}$	$\{a_5\}$
2	$\{o_5\}$	$\{a_0, a_3\}$
3	$\{o_8\}$	$\{a_1, a_2\}$
4	$\{o_7, o_9\}$	$\{a_4\}$

and

i	$\mathbf{o}'_i$	$\mathbf{a}'_i$
1	$\varnothing$	$\varnothing$
2	$\varnothing$	$\{a_7\}$
3	$\{o_4\}$	$\varnothing$
4	$\varnothing$	$\{a_6\}$

Table 3: The two tables extracted from $C(19) : C(29)\, WAP\, C(17) : C(26)$.

The second table means only that o_4 is not in relation with a_7 nor with a_6; we will come back later to it. The first table looks like a complete complex. Actually, it does not represent quite an analogical complex, since the subcontext constructed on o_7, o_9, a_1 and a_2 is as in Table 4, while it should be composed only of "blanks".

			$\mathbf{a}_3$	
			feathers	eggs
			a_1	a_2
$\mathbf{o}_4$	o_7	mink		
	o_9	platypus		×

Table 4: A conflict in a table: there is a "×" in a rectangle that should be entirely "blank" to derive an analogical schema from $\mathbf{o}_{1,4}$ and $\mathbf{a}_{1,4}$.

To postprocess the tables, one has firstly to remove either the attributes or the objects that lead the table to miss the analogical complex. In our example, this strategy applied to object o_9 (platypus) and attribute a_2 (eggs) produces two complete complexes (numbers 10 and 12 in Figure 22):

i	$\mathbf{o}_i$	$\mathbf{a}_i$
1	$\{o_1, o_2\}$ chicken, crow	$\{a_5\}$ aquatic
2	$\{o_5\}$ fruitbat	$\{a_0, a_3, a_7\}$ hair, milk, teeth
3	$\{o_8\}$ penguin	$\{a_1\}$ feathers
4	$\{o_7, o_9\}$ mink, platypus	$\{a_4\}$ airborne

i	$\mathbf{o}_i$	$\mathbf{a}_i$
1	$\{o_1, o_2\}$ chicken, crow	$\{a_5\}$ aquatic
2	$\{o_5\}$ fruitbat	$\{a_0, a_3\}$ hair, milk
3	$\{o_8\}$ penguin	$\{a_1, a_2\}$ feathers, eggs
4	$\{o_7\}$ mink	$\{a_4\}$ airborne

However, this first postprocessing may not be sufficient, since the obtained tables represent analogical schemas, but not necessarily analogical complexes. In other words, the maximality has to be checked.

For example, if there had been an attribute α in SmallZoo in the following relation with the objects (we recall also attribute a_4):

	o_0	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	o_9
a_4		×	×		×	×				
α		×	×			×	×			

then it would have been necessary to add α to a_4 for maximizing the previous complexes, although α would not have appeared in the WAP [8].

Then the second postprocessing has maximized the tables into complexes, adding attributes and objects chosen among those which do not appear in the four concepts in WAP.

Finally, this double postprocessing produces a lattice of complexes, each of one deriving from the original table by removing and/or adding of a set of attributes and a set of objects. The top (resp. the bottom) of this lattice is obtained by removing only objects (resp. attributes) and adding only attributes (resp. objects) in the two steps of the postprocessing (in our example, this sublattice is composed of only two complexes, noted $C(10)$ and $C(12)$ in Figure 22).

This sublattice is included in another one: the sublattice of all complexes that can be produced by using only the subsets of objects and attributes present in the WAP (in our example, it is composed of the four complexes $C(10)$, $C(12)$ $C(15)$ and $C(17)$).

We come back now to the second table produced at the beginning of the procedure. On our example, it represents only the following subcontext:

	a_6	a_7
o_4		

Postprocessing this table into a lattice of complexes would produce the sublattice in Figure 22 including complexes number 1 to 4, for the only reason that they share the "negative" subcontext above. Obviously, the link of this sublattice

[8]Attribute α would have had an intriguing semantics: shared only by "chicken", "crow", "fruitbat" and "kiwi" in SmallZoo.

of complexes with the WAP is much weaker than that of the other sublattice obtained: it is not based on relations in the context (then implicitly on concepts), but on the absence of relations. Note that the top and bottom of this second lattice are both non complete complexes.

To sum up, the algorithm that constructs complexes from a WAP, that we call `WAPtoCPLX` is as follows:

Algorithm `WAPtoCPLX`

`input:` A WAP between four concepts.

`step 1:` Construct the two tables displayed in Table 2.

`step 2:` As a first preprocessing, remove from the two tables all attributes that are in conflict with one or several objects, as in the example of Table 4.

`step 3:` As a second postprocessing, add objects to the two tables to transform them into two complexes.

`step 4:` Run again `step 2` and `step 3`, replacing "attribute" by "object" and vice versa.

`output:` Four complexes. Two of them are the top and the bottom of one sublattice of complexes, the other two are the top and the bottom of another sublattice of complexes.

Comment. We can note that the order between steps 2 and 3 is important. However, step 4 could be done before steps 2 and 3, since attributes and objects have a symmetrical role.

The following property shows that a bi-complete WAP (see Definition 10) produces complete complexes.

Proposition 22. *Let a bi-complete WAP be processed by the algorithm* `WAPtoCPLX`*. One of the output lattices is entirely composed of complete analogical complexes.*

Proof.

1. Since the WAP is bi-complete, one of the tables has no empty subset. See the proof at the end of Appendix H.
2. Step 2: when a conflict occurs, it is only because of some × to be removed, which is done by removing the corresponding attributes. Can this removal lead to an "blank" rectangle in the complex? No, because we show now that there is at least one object and one attribute that are not in relation in the "blank" rectangles produced by the previous table.

   Indeed, suppose that is false. Then, one object or one attribute would be in a number $n > 2$ of concepts. $n = 4$ is impossible by definition of a table. In the case $n = 3$, there would be no WAP. Hence step 2 will keep all subsets of the table non empty.
3. Step 2 transforms a table with no empty subset into a complex, without creating any new "blank" rectangle. Hence the result is a complete complex C_M.

4. Step 3 produces from the same table another complete complex C_m.
5. The complexes greater than C_m and lesser than C_M compose a sublattice of complete complexes.

□

At the end of Appendix H, a detailed description of a bi-complete WAP and of the associated tables are given, which illustrates this proof. Note that the assumption to produce a complete complex lies on the bi-completeness of the WAP as input. For instance, in the Boolean lattice displayed in Figure 15, we have $x : y \, WAP \, u : v$ where

$$x = (\{o_1, o_2\}, \{a_3, a_4\}), \qquad y = (\{o_1, o_3\}, \{a_2, a_4\}),$$
$$u = (\{o_2, o_4\}, \{a_1, a_3\}), \qquad v = (\{o_3, o_4\}, \{a_1, a_2\}).$$

This WAP (it is also a SAP) is not bi-complete and does not produce any complete complex.

6.3. From an analogical complex to some WAPs

We come now to the reciprocal issue: produce WAPs from an analogical complex $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ extracted from a formal context. The complex has not necessarily to be complete.

Reading the first line in the table of the complex, we form the pair $(\mathbf{o}_1, \mathbf{a}_3 \cup \mathbf{a}_4)$ and we transform it into the concept $h_1 = \big((\mathbf{a}_3 \cup \mathbf{a}_4)^{\downarrow}, ((\mathbf{a}_3 \cup \mathbf{a}_4)^{\downarrow})^{\uparrow}\big)$. We can construct 8 concepts in the same way by reading lines and columns:

$$\begin{array}{lcl} h_1 & = & \big((\mathbf{a}_3 \cup \mathbf{a}_4)^{\downarrow}, ((\mathbf{a}_3 \cup \mathbf{a}_4)^{\downarrow})^{\uparrow}\big) \\ h_2 & = & \big((\mathbf{a}_2 \cup \mathbf{a}_4)^{\downarrow}, ((\mathbf{a}_2 \cup \mathbf{a}_4)^{\downarrow})^{\uparrow}\big) \\ h_3 & = & \big((\mathbf{a}_1 \cup \mathbf{a}_3)^{\downarrow}, ((\mathbf{a}_1 \cup \mathbf{a}_3)^{\downarrow})^{\uparrow}\big) \\ h_4 & = & \big((\mathbf{a}_1 \cup \mathbf{a}_2)^{\downarrow}, ((\mathbf{a}_1 \cup \mathbf{a}_2)^{\downarrow})^{\uparrow}\big) \end{array} \qquad \begin{array}{l} g_1 = \big((\mathbf{a}_1)^{\downarrow}, ((\mathbf{a}_1)^{\downarrow})^{\uparrow}\big) \\ g_2 = \big((\mathbf{a}_2)^{\downarrow}, ((\mathbf{a}_2)^{\downarrow})^{\uparrow}\big) \\ g_3 = \big((\mathbf{a}_3)^{\downarrow}, ((\mathbf{a}_3)^{\downarrow})^{\uparrow}\big) \\ g_4 = \big((\mathbf{a}_4)^{\downarrow}, ((\mathbf{a}_4)^{\downarrow})^{\uparrow}\big) \end{array}$$

But we can also take the same pair $(\mathbf{o}_1, \mathbf{a}_3 \cup \mathbf{a}_4)$ and transform it into the concept $h'_1 = \big((\mathbf{o}_1)^{\uparrow}, ((\mathbf{o}_1)^{\uparrow})^{\downarrow}\big)$. In that way, we can construct the 8 concepts:

$$\begin{array}{lcl} h'_1 & = & \big(((\mathbf{o}_1)^{\uparrow})^{\downarrow}, (\mathbf{o}_1)^{\uparrow}\big) \\ h'_2 & = & \big(((\mathbf{o}_2)^{\uparrow})^{\downarrow}, (\mathbf{o}_2)^{\uparrow}\big) \\ h'_3 & = & \big(((\mathbf{o}_3)^{\uparrow})^{\downarrow}, (\mathbf{o}_3)^{\uparrow}\big) \\ h'_4 & = & \big(((\mathbf{o}_4)^{\uparrow})^{\downarrow}, (\mathbf{o}_4)^{\uparrow}\big) \end{array} \qquad \begin{array}{l} g'_1 = \big(((\mathbf{o}_3 \cup \mathbf{o}_4)^{\uparrow})^{\downarrow}, (\mathbf{o}_3 \cup \mathbf{o}_4)^{\uparrow})\big) \\ g'_2 = \big(((\mathbf{o}_2 \cup \mathbf{o}_4)^{\uparrow})^{\downarrow}, (\mathbf{o}_2 \cup \mathbf{o}_4)^{\uparrow})\big) \\ g'_3 = \big(((\mathbf{o}_1 \cup \mathbf{o}_3)^{\uparrow})^{\downarrow}, (\mathbf{o}_1 \cup \mathbf{o}_3)^{\uparrow})\big) \\ g'_4 = \big(((\mathbf{o}_1 \cup \mathbf{o}_2)^{\uparrow})^{\downarrow}, (\mathbf{o}_1 \cup \mathbf{o}_2)^{\uparrow})\big) \end{array}$$

This makes at most 16 different concepts, since all g_i's, h_i's, g'_i's and h'_i's may be different. The four g_i and the other 4-tuples are not necessarily structured as a WAP.

There is no real algorithm here, only the computation of these 16 concepts and the checking whether they combine or not into a WAP. However, we have the following properties.

Proposition 23. *Let $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ be an analogical complex in a formal context.*

1. *For $i \in [1,4]$, $\mathbf{o}_i$ is included in the set of objects of h_i and in that of h'_i. Similarly, $\mathbf{a}_i$ is included in the set of attributes of g_i and in that of g'_i.*
2. *For $i \in [1,4]$, $h'_i \leq h_i$ and $g'_i \leq g_i$*
3. *Let $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ be a complete analogical complex. If (h_1, h_2, h_3, h_4) is a WAP, then it is a complete WAP. The same holds for the four h'_i, the four g_i and the four g'_i.*

Proof.

1. Let us consider $\mathbf{o}_1$. By definition of an analogical complex, $\mathbf{o}_1 \subseteq (\mathbf{a}_3 \cup \mathbf{a_4})^{\downarrow}$. Moreover, by definition of extension and intension of a concept, $\mathbf{o}_1 \subseteq ((\mathbf{o}_1)^{\uparrow})^{\downarrow}$. To sum up, $\mathbf{o}_1$ is included in the set of objects of h_1 and h'_1. All verifications for $\mathbf{o}_i$ and $\mathbf{a}_i$, $i \in [1,4]$, are similar.
2. Since $\mathbf{o}_1 \subseteq (\mathbf{a}_3 \cup \mathbf{a_4})^{\downarrow}$, $\left((\mathbf{a}_3 \cup \mathbf{a_4})^{\downarrow}\right)^{\uparrow} \subseteq (\mathbf{o}_1)^{\uparrow}$. Therefore, $h'_1 \leq h_1$. The proof of the other inequalities is similar.
3. The initial complex is complete by hypothesis, i.e. none of its subsets are empty. To be clearer, we argue on the example given just below. The subset of objects $\mathbf{a}_1 = \{a_2, a_5\}$ is non empty and will be by construction in the intersection of the objects of h_3 and the objects of h_4, since the intension and extension operation only add attributes and objects. In the same way, $\mathbf{a}_2 = \{a_4\}$ will be in the intersection of the objects of h_3 and the objects of h_2 and h_4, etc. Hence, if the four h_is are in WAP, it is complete in attributes. The same holds for the h'_is, g_is and g'_is.

□

Example. We take the following complete analogical complex from the context SmallZoo (it has number 2 in Figure 22):

			$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$
			eggs aquatic	airborne	predator	toothed
			$\{a_2, a_5\}$	$\{a_4\}$	$\{a_6\}$	$\{a_7\}$
$\mathbf{o}_1$	$\{o_0\}$	aadvark			×	×
$\mathbf{o}_2$	$\{o_5\}$	fruitbat		×		×
$\mathbf{o}_3$	$\{o_8, o_9\}$	penguin platypus	×		×	
$\mathbf{o}_4$	$\{o_4\}$	duck	×	×		

Starting from the pair $\{o_0\}, \{a_6, a_7\}$, we obtain $\{a_6, a_7\}^{\downarrow} = \{o_0, o_3, o_7\}$ and $\{o_0, o_3, o_7\}^{\uparrow} = \{a_6, a_7\}$, then $h_1 = C(8)$.

We also have $\{o_0\}^{\uparrow} = \{a_0, a_3, a_6, a_7\}$ and $\{a_0, a_3, a_6, a_7\}^{\downarrow} = \{o_0, o_7\}$ then $h'_1 = C(25)$.

We eventually obtain ten different concepts[9], over a maximum of 16:

[9] For readability, the braces (set symbols) are omitted.

$h_1 = C(8)$	$h_2 = h_2' = C(29)$	$h_3 = h_3' = C(14)$	$h_4 = h_4' = C(21)$
$\{o_0, o_3, o_7\}$	$\{o_5\}$	$\{o_8, o_9\}$	$\{o_4\}$
$\{a_6, a_7\}$	$\{a_0, a_3, a_4, a_7\}$	$\{a_2, a_5, a_6\}$	$\{a_1, a_2, a_4, a_5\}$

$g_1 = g_1' = C(13)$	$g_2 = g_2' = C(11)$	$g_3 = g_3' = C(1)$
$\{o_4, o_8, o_9\}$	$\{o_1, o_2, o_4, o_5\}$	$\{o_0, o_2, o_3, o_6, o_7, o_8, o_9\}$
$\{a_2, a_5\}$	$\{a_4\}$	$\{a_6\}$

$g_4 = C(6)$	$g_4' = C(23)$	$h_1' = C(25)$
$\{o_0, o_3, o_5, o_7\}$	$\{o_0, o_5, o_7\}$	$\{o_0, o_7\}$
$\{a_3, a_7\}$	$\{a_0, a_3, a_7\}$	$\{a_0, a_3, a_6, a_7\}$

The sublattice generated by these concepts, which are derived from the chosen analogical complex, is displayed in Figure 23 (node g_1', for example, is not indicated, since it is the same as node g_1).

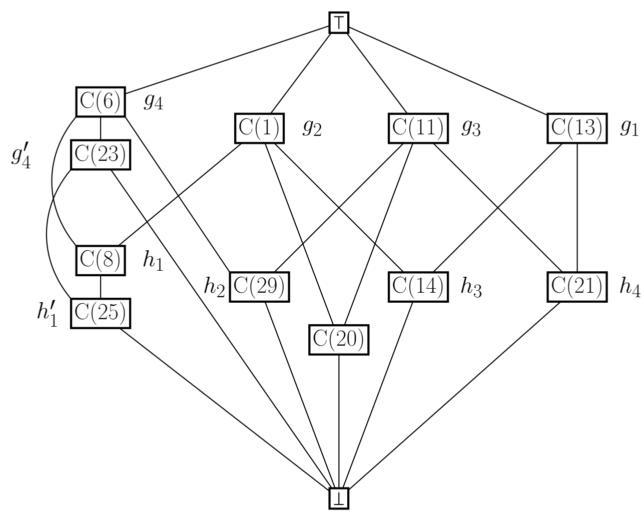


Figure 23: The sublattice of concepts induced by the analogical complex number 2 in Figure 22.

The four g_i's do not form a WAP, since $g_2 \wedge g_3 = C(20)$ and $g_1 \wedge g_4 = \perp$. This is due to the fact that o_2 (which does not appear in the complex 2) is an object common to g_1 and g_4, but is not in g_2 nor g_3. The same holds for the g_i''s.

At the contrary, the four h_i's and the four h_i''s are structured as two complete WAPs. They have the following meaning:

	Objects	Attributes
h_1'	aadvark, mink	hair, milk, predator, toothed
h_1	aadvark, dolphin, mink	predator, toothed
h_2	fruitbat	hair, milk, airborne, toothed
h_3	penguin, platypus	eggs, aquatic, predator
h_4	duck	feathers, eggs, airborne, aquatic

Following Proposition 5, we know that any concept h in the sublattice of concepts such that $h_1 \leq h \leq h_1'$ is such that (h, h_2, h_3, h_4) are four concepts in WAP.

As other examples, starting from the complete complex number 1 produces four 4-tuples that are all WAPs. Starting from complex number 12 produces no WAP at all.

We have seen the algorithms for moving from a complete WAP to an analogical complex and vice versa. With these tools we are now in position to reach the last step of our program, namely handling relational proportions in the setting of formal concept analysis.

7. Formal concepts and relational proportion

In this section, we are investigating whether we can instantiate in a concept lattice the notion of *Relational Proportion*, as introduced in Section 2.1. In other words, we wonder whether an expression "A is the B of a" could be syntactically extracted from a formal context, with a semantic connection to the ordinary meaning of a relational proportion.

We shall see that the notion of analogical complex plays a key role also in this section.

7.1. From an analogical complex to a relational proportion

Let us start with the example "the fins are the wings of the fish". How can this analogy be expressed in the framework of formal concept analysis? Firstly, "fins" and "wings" are comparable, since they are organs useful to move in a fluid. They however are different since "fins" is related to a "fish" and "wings" to a "bird" (this last word does not appear in the phrase "the fins are the wings of the fish", but is implicit). Hence, it is possible to consider "wings" and "fins" as objects (resp. attributes) and "bird" and "fish" as attributes (resp. objects).

A fish and a bird can not only be distinguished by the way they move. They do not breathe in the same manner, either. Oxygen in air is transferred in the blood by the means of lungs, while in water it is transferred by gills. Also, the skin of a fish is protected by scales, while that of a bird is by feathers.

A small formal context can be now extracted from this knowledge, as follows:

		a_1	a_2	a_3	a_4	a_5
o_1:	fins	×		×		
o_2:	wings		×	×		
o_3:	gills	×			×	
o_4:	lungs		×		×	
o_5:	scales	×				×
o_6:	feathers		×			×

where

where	a_1:	is a part of a fish
	a_2:	is a part of a bird
	a_3:	is a mobility part
	a_4:	is a breathing part
	a_5:	is a covering part

We have now three similar relational proportions: "the fins are the wings of the fish", "the scales are the feathers of the fish" and "the gills are the lungs of the fish". The corresponding lattice is displayed in Figure 24.

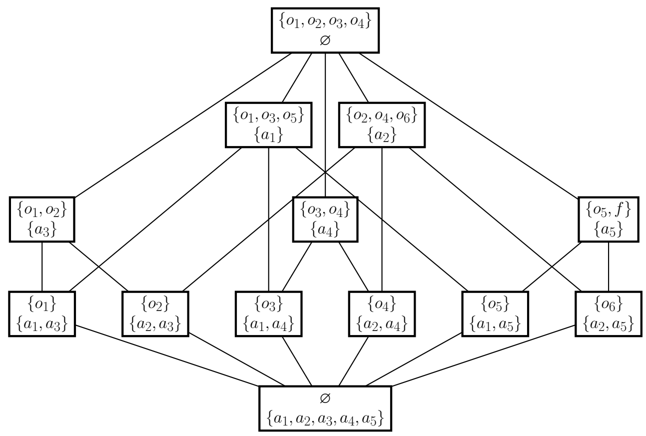


Figure 24: The lattice of the context describing some analogies between a bird and a fish.

Therefore, we are looking for a syntactic structure that is common to the three relational proportions "o_1 is the o_2 of a_1", "o_3 is the o_4 of a_1" and "o_5 is the o_6 of a_1". Actually, this structure appears when we extract two analogical complexes from the previous context, as follows:

	$\mathbf{a}_1$ $\{a_4\}$	$\mathbf{a}_2$ $\{a_2\}$	$\mathbf{a}_3$ $\{a_1\}$	$\mathbf{a}_4$ $\{a_3\}$
$\mathbf{o}_1$ $\{o_1\}$			×	×
$\mathbf{o}_2$ $\{o_2\}$		×		×
$\mathbf{o}_3$ $\{o_3\}$	×		×	
$\mathbf{o}_4$ $\{o_4\}$	×	×		

	$\mathbf{a}_1$ $\{a_5\}$	$\mathbf{a}_2$ $\{a_2\}$	$\mathbf{a}_3$ $\{a_1\}$	$\mathbf{a}_4$ $\{a_3\}$
$\mathbf{o}_1$ $\{o_1\}$			×	×
$\mathbf{o}_2$ $\{o_2\}$		×		×
$\mathbf{o}_3$ $\{o_5\}$	×		×	
$\mathbf{o}_4$ $\{o_6\}$	×	×		

The formal definition of a relational proportion can now be derived quite straightforwardly from that of an analogical complex.

Definition 13 (Formal relational proportion). *Let* $(\mathbf{o}_{1,4}, \mathbf{a}_{1,4})$ *be a complete analogical complex in a formal context.*

The sets of objects $\mathbf{o}_1$ *and* $\mathbf{o}_2$ *and the sets of attributes* $\mathbf{a}_2$ *and* $\mathbf{a}_3$ *are said to be in the* formal Relational Proportion*, or RP,* $\big(\mathbf{o}_1$ *is to* $\mathbf{a}_3$ *as* $\mathbf{o}_2$ *is to* $\mathbf{a}_2\big)$*, and we write:* $\big(\mathbf{o}_1 \updownarrow \mathbf{a}_3 \upuparrows\!\!\downdownarrows \mathbf{o}_2 \updownarrow \mathbf{a}_2\big)$.

This definition leads to some remarks:

1. The reduced form of the relational proportion (see Section 2.1) "A is the B of a" would be $\big(\mathbf{o}_1$ is the $\mathbf{o}_2$ of $\mathbf{a}_3\big)$. We keep the complete form in the following.

2. We can derive an analogical complex (hence some RPs) from a WAP between concepts, as in Section 6.2. If we ask the WAP to be complete, we moreover ensure that the analogical complex is complete (see Proposition 22).

3. According to the definition, the operator $\updownarrows$ is clearly commutative, while permuting the extreme and the means in a relational proportion is not allowed. It may indeed lead to awkward phrasings, as expected, see Section 2.1. Here are the eight phrases when this RP is transformed by the permutations valid in an AP (see Definition 1). Only the first two are fully satisfactory.

   Fins are to fish as wings are to bird.
   Wings are to bird as fins are to fish.

   Fins are to wings as fish is to bird.
   Bird is to fish as wings are to fins
   Fish is to bird as fins are to wings.
   Wings are to fins as bird is to fish.
   Bird is to wings as fish is to fins.
   Fish is to fins as bird is to wings.

4. From the same complex, we can also extract the 4 following formal relational proportions $(\mathbf{o}_1 \updownarrow \mathbf{a}_4 \updownarrows \mathbf{o}_3 \updownarrow \mathbf{a}_1)$, $(\mathbf{o}_2 \updownarrow \mathbf{a}_4 \updownarrows \mathbf{o}_4 \updownarrow \mathbf{a}_1)$ and $(\mathbf{o}_3 \updownarrow \mathbf{a}_3 \updownarrows \mathbf{o}_4 \updownarrow \mathbf{a}_2)$. Since the operator $\updownarrows$ is commutative, it gives a total of 8.

Examples. Let us take complex numbered 10 from the SmallZoo context lattice (see Figure 22):

		aquatic $\{a_5\}$	hair, milk, toothed $\{a_0, a_3, a_7\}$	feathers, eggs $\{a_1, a_2\}$	airborne $\{a_4\}$
$\{o_1,$ $o_2\}$	chicken crow			×	×
$\{o_5\}$	fruitbat		×		×
$\{o_8\}$	penguin	×		×	
$\{o_7\}$	mink	×	×		

We have seen, in Section 6.2, that this complex is one of the output complexes provided by `WAPtoCPLX` from the input $C(19) : C(29)\ WAP\ C(17) : C(26)$. Complex 10 implies all attributes but a_6 (predator). For example, we extract from this complex the following RPs.

- "A fruitbat is to airborne animals as a mink is to aquatic animals", or in reduced form "the fruitbat is the mink of airborne animals":

  $$\{\text{fruitbat}\} \updownarrow \{\text{airborne}\} \updownarrows \{\text{mink}\} \updownarrow \{\text{aquatic}\}\,.$$

  This RP means that a fruitbat and a mink are haired and toothed mammals, but that a mink is aquatic, contrary to fruitbats.

- “A penguin is to oviparous animals with feathers as a mink is to haired and toothed mammals”:

$$\{\text{penguin}\} \updownarrow \{\text{feathers,eggs}\} \updownarrow\!\updownarrow \{\text{mink}\} \updownarrow \{\text{hair,milk,toothed}\}\,.$$

  A penguin and a mink are aquatic animals, both are not airborne animals, but a mink is a haired and toothed mammal, contrary to penguins which have feathers and produce eggs.

To give an additional illustration from the SmallZoo database, let us now consider complex numbered 16

		aquatic, predator $\{a_5, a_6\}$	milk, toothed $\{a_3, a_7\}$	feathers, eggs $\{a_1, a_2\}$	airborne $\{a_4\}$
$\{o_1\}$	chicken			×	×
$\{o_5\}$	fruitbat		×		×
$\{o_8\}$	penguin	×		×	
$\{o_3, o_7\}$	dolphin mink	×	×		

from which, for instance, the following RPs can be derived:

- “A chicken is to airborne animals as a penguin is to aquatic predators”. Both chicken and penguin are oviparous animals with feathers, but a penguin is aquatic, contrary to a chichen.
- “Dolphins and minks are to toothed mammals as penguins are oviparous animals with feathers”. Dolphins, minks and penguins are aquatic predators but dolphins and minks are toothed mammals, contrary to penguins.

Of course, the interest of such phrases has to be taken in context: the SmallZoo data base is supposed to be the only knowledge.

7.2. From a formal relational proportion to concepts in analogical proportion

In the previous section, we have shown how to construct relational proportions from analogical complexes, and consequently from WAPs. In this section, we study the converse: assuming we have a relational proportion “A is the B of a”, or “A is to a as B is to b” is it possible to deduce formal concepts in WAP and an analogical complex from this knowledge?

As an example, we have found in a web magazine[10] the following proportion “Massimiliano Alajmo is the Mozart of Italian cooking”. The background knowledge allowing to understand this relational proportion is the following:

[10] http://www.slate.fr/story/43841/massimiliano-alajmo

- Music and Italian cooking are disciplines practiced by humans.
- Such disciplines can be practiced with different levels of ability.
- Mozart is a musician.
- Mozart is a genius[11].

Since the quality "to be a genius" is not possessed by everybody, there must exist many "ordinary gifted" musicians. Then, the background knowledge can be expressed by the following formal context

	a_1	a_2	a_3
o_1	×	×	
o_2	×		×

where o_1 stands for Mozart, o_2 for one of "ordinary gifted" musicians, a_1 is the attribute "practices music", a_2 "is a genius" and a_3 "has an ordinary ability". Note that a_2 and a_3 refer here to two mutually exclusive properties.

Now, when the new data "Alajmo is the Mozart of Italian cooking" is introduced, the knowledge extends as follows:

- Alajmo practices Italian cooking.
- Alajmo has something in common with Mozart that is not Italian cooking.

The relational proportion "Alajmo is the Mozart of Italian cooking" is a reduced form of "Alajmo is to Italian cooking as Mozart is to music", which can be formally written by

$$o_3 \updownarrow a_4 \upuparrows\!\!\downdownarrows o_1 \updownarrow a_1$$

where object o_3 stands for Alajmo and attribute a_4 for "does Italian cooking".

Since Mozart has only the other attribute "is a genius", Alajmo must have it. And this deduction is the main information brought by the proportion. Moreover, since cooking is a discipline practiced by humans, there must exist some people ordinary gifted for Italian cooking: let us call him (or her) o_4.

This is in accordance with what was said in the last section: we must introduce the notion of non-genius in our universe. If we do not, we implicitly suppose that everybody is a genius for some activity, and Picasso would become an ordinary painter and Mozart an ordinary musician, and that would destroy the meaning of the analogy. The knowledge is now as follows

[11] Note that the background knowledge has to be an implicit pragmatical consensus. Mozart is probably the archetype of the genius in music in all the world. But when a French newspaper writes "Juppé is the Poulidor of politics", as we recently saw, it is hardly understandable outside of the community of French people, and surely not by all of it :Juppé has been defeated several times as a runner for French presidency and Poulidor is famous for never have won the Tour de France.

	a	b	c	$\tilde{c}$
A	×		×	
B		×	×	

(a) The context $\mathcal{C}_{A \neq B}$: "A is not B", or "A has a and not B".

	a	b	c	$\tilde{c}$
A	×		×	
B		×	×	
$\tilde{A}$		×		×
$\tilde{B}$	×			×

(b) The apposition of $\mathcal{C}_{A \neq B}$ and of its complementary $\mathcal{C}_{\tilde{A} \neq \tilde{B}}$.

	a	c	$\tilde{c}$	b
A	×	×		
$\tilde{B}$	×		×	
B		×		×
$\tilde{A}$			×	×

(c) The previous context is actually $\mathcal{C}_{Analog}$.

Figure 25: Deriving the analogical context $\mathcal{C}_{Analog}$ from the proportion "A is the B of a".

		a_1	a_2	a_3	a_4
o_1:	Mozart	×	×		
o_2:	Ordinary musician	×		×	
o_3:	Alajmo		×		×
o_4:	Ordinary cook			×	×

where

a_1:	practises music
a_2:	is a genius
a_3:	has an ordinary hability
a_4:	does Italian cooking

Hence, we have reconstructed the analog context, and the corresponding analogical proportions, from a relational proportion. An analogical proportion between concepts derived from the previous context is for example:

$$(\{o_1\}, \{a_1, a_2\}) : (\{o_2\}, \{a_1, a_3\}) \; WAP \; (\{o_3\}, \{a_2, a_4\}) : (\{o_4\}, \{a_3, a_4\})$$

which translates into "Mozart is to some ordinary musician as Alajmo is to some ordinary cook".

This construction is semantically possible since Mozart is strongly connoted as a characteristic genius among musicians. The result of this analogy is that we naturally deduce that the Italian cook named Alajmo is a genius in his discipline. Notice that we implicitly assume that only one discipline is associated to an human, whatever his ability.

In more formal terms, from the relational proportion "A is to a as B is to b", or the reduced form "A is the B of a", we can derive a (4×4) formal context as indicated in Figure 25.

Firstly, we construct the context that we can call $\mathcal{C}_{A \neq B}$. It is composed of the two objects A and B, described by four attributes. Attribute a is possessed by A and not by B. Attribute b is possessed by B and not by A. Attribute c is possessed both by A and B and $\tilde{c}$ is some attribute not possessed by A nor B.

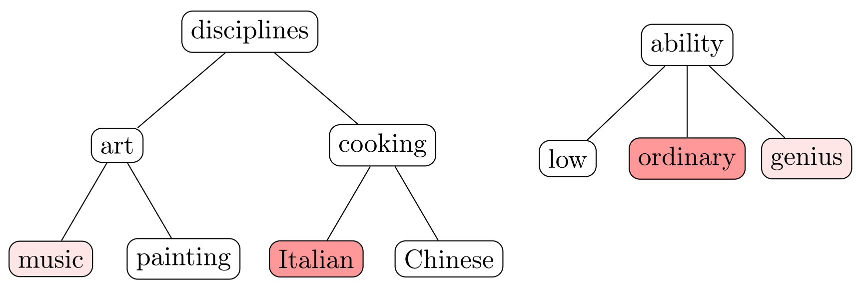
Secondly we complete $\mathcal{C}_{A \neq B}$ with two objects $\tilde{A}$ and $\tilde{B}$ that are the complements of A and B with respect to the four attributes. In other words, we augment the context by adding, for each line, another line that is its negation. After a permutation of lines and columns, the result is the analogical context (which is here a complex) $\mathcal{C}_{Analog}$.

In the corresponding concept lattice, from the relational proportion "A is the B of a", we construct the WAP between concepts

$$(\{A\}, \{a, c\}) : (\{B\}, \{b, c\}) \; WAP \; (\{\tilde{B}\}, \{a, \tilde{c}\}) : (\{\tilde{A}\}, \{b, \tilde{c}\}) \,.$$

Let us notice that this formalization respects the symmetry of "as" inherent in the RP "A is to a as B is to b" and permits the permutations between (A, a) and (B, b) corresponding to the reduced form "B is the b of A".

Comment. We have said that object $\tilde{A}$ is the complement of A and B with respect to the four attributes. Another way of seeing this point is to imagine that the attributes are organized in semantic trees. Taking back the previous example, we can assume that there is a semantic tree describing the different disciplines and another one describing the different level of ability. In a very simplistic manner, they could be as follows:



Mozart (object o_1), who is a musician with genius, is opposed to $\tilde{A}$ (object o_4) in both attributes, since o_4 is an ordinary cook in Italian cooking. The term "opposition" means here that the attributes of A and $\tilde{B}$ are different nodes in both trees. The attribute "music" is opposed to "Italian cooking", according to the first semantic tree, since they are different nodes. Note they are not logical contraries, but rather incompatible: no object, in this small semantic universe, can verify both of them.

8. Conclusion

Building analogies and categorizing items are important cognitive operations of the mind. On the one hand FCA provides an approach for formally defining the idea of a concept in a formal context describing the relation between objects and properties (which involves two universes). On the other hand, formal models for defining the analogical proportions have been proposed more recently, in particular in Boolean lattices, when the four items refer to the same universe of discourse. For bridging the two areas it has been necessary to define and study analogical proportions in non distributive lattices, since such lattices are currently encountered in FCA. This is the first contribution of this paper. Then it has been possible to define analogical proportions between formal concepts, and through the identification of the role of particular sub-contexts (called "analogical complex" here) to find all the relational proportions involving two universes of discourse. This is an important step towards the development of AI capabilities for the proper handling of concepts, analogies and metaphors in explanation tasks.

Our study of proportions between concepts uses the classical setting of FCA and explores a simple and fixed relation between concepts in a single lattice. It would be interesting to connect it with the more general framework of Relational Concept Analysis (see e.g., [55]), and with a recent proposal based on antichains [65]. Besides, a number of recent woks in computational linguistics (e.g., [45]) have been using a parallelogram-based modeling of analogical proportions (originated in [56]), in numerical settings, where words are represented by vectors of great dimension. Bridging this computational view of analogical proportions with the work presented here is certainly a challenging task for the future. The study of the links between relational proportions, analogical proportions, and semantic trees is a topic worth investigating in further research.

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Appendix A. Completion of the order relation derived from a FAP in a lattice

Let $(S, \leq)$ be a partially ordered finite set and $A \subset S$. The smallest lattice $\mathcal{L}$ compatible with $(S, \leq)$ is called the *Dedekind-MacNeille completion* of $(S, \leq)$. It can be computed in the following manner.

Define $A^u = \{x \in S \mid \forall t \in A, x \geq t\}$ and $A^l = \{x \in S \mid \forall t \in A, x \leq t\}$. Then, $\mathcal{L}$ is composed of all the subsets of A of S such that $(A^u)^l = A$.

Another manner to build $\mathcal{L}$ is to use the Formal Concept Analysis framework. Firstly, from $(P, \leq)$, construct the context $(P, P, \leq)$. In this context, the set of objects and the set of attributes are both P; an object o is in relation with the attribute a when $o \leq a$. Secondly, it is proved (e.g. [54] or [18]) that the concept lattice of this context is the completion of the partial order $(P, \leq)$.

We show here the context extracted from the order relation induced by a factorial analogical proportion in a lattice (see Section 3.4).

	$x \vee y$	$x \vee z$	$y \vee t$	$z \wedge t$	x	y	z	t	$x \wedge y$	$x \wedge z$	$y \wedge t$	$z \wedge t$
$x \vee y$	×											
$x \vee z$		×										
$y \vee t$			×									
$z \vee t$				×								
x	×	×			×							
y	×		×			×						
z		×		×			×					
t			×	×				×				
$x \wedge y$	×	×	×		×	×			×			
$x \wedge z$	×	×		×	×		×			×		
$y \wedge t$	×		×	×		×		×			×	
$z \wedge t$		×	×	×			×	×				×

The corresponding Boolean lattice is given in Figure 5.

Appendix B. Composition and decomposition of analogical proportions

In this section, we wonder whether an AP can be factorized into simpler APs, using a simple composition law.

Definition 14. *Let (x, y, z, t) and (x', y', z', t') be two 4-tuples of elements of a lattice $(L, \vee, \wedge, \leq)$, the $\vee$-composition and the $\wedge$-composition of these 4-tuples are defined as follows:*

$$\begin{array}{rcl} (x, y, z, t) \vee (x', y', z', t') & = & (x \vee x', y \vee y', z \vee z', t \vee t') \\ (x, y, z, t) \wedge (x', y', z', t') & = & (x \wedge x', y \wedge y', z \wedge z', t \wedge t')\,. \end{array}$$

Note that these operations are commutative and associative.

Definition 15 (Trivial and Simple analogical proportions)**.** *Let x and y be two elements of a lattice, we call the FAP*

- $x : x :: x : x$ *a trivial analogical analogy (TAP),*
- $x : x :: y : y$ *a simple analogical analogy of the first type (SAP1),*
- $x : y :: x : y$ *a simple analogical analogy of the second type (SAP2).*

We have the following results.

Proposition 24. *1. In a general lattice, the composition of a WAP and a TAP is a WAP.*

2. In a distributive lattice, the composition of two CAPi is a CAPi, for any $i \in \{1, \ldots, 4\}$.

Proof.

1. It is a direct consequence of Definition 6 and associativity of both operations $\vee$ and $\wedge$.
2. It results from the distributivity.

□

Proposition 25. *Let $(x : y :: z : t)$ be a FAP in a general lattice, it can be written as the following compositions of*

1. a CAP2, a CAP3 and two TAPs:

$$\big((x : x \wedge t :: x \vee t : t) \vee (y : y :: y : y)\big) \wedge \big((x : x \vee t :: x \wedge t : t) \vee (z : z :: z : z)\big)$$
$$\big((x : x \vee t :: x \wedge t : t) \vee (z : z :: z : z)\big) \wedge \big((x : x \wedge t :: x \vee t : t) \vee (y : y :: y : y)\big)$$
$$\big((x : x \vee t :: x \wedge t : t) \wedge (y : y :: y : y)\big) \vee \big((x : x \wedge t :: x \vee t : t) \wedge (z : z :: z : z)\big)$$
$$\big((x : x \wedge t :: x \vee t : t) \wedge (z : z :: z : z)\big) \vee \big((x : x \vee t :: x \wedge t : t) \wedge (y : y :: y : y)\big)$$

2. a CAP1, a CAP4 and two TAPs (similar to the precedent)

3. a SAP1 and a SAP2:

$$\big((x \vee y) : (x \vee y) :: (z \vee t) : (z \vee t)\big) \wedge \big((x \vee z) : (y \vee t) :: (x \vee z) : (y \vee t)\big)$$
$$\big((x \wedge y) : (x \wedge y) :: (z \wedge t) : (z \wedge t)\big) \vee \big((x \wedge z) : (y \wedge t) :: (x \wedge z) : (y \wedge t)\big)$$

Proof. These decompositions of a FAP rest on Definition 5. From Proposition 4 we get the four inequalities $x \wedge t \leq z \leq x \vee t$, $x \wedge t \leq y \leq x \vee t$, $y \wedge z \leq x \leq y \vee z$ and $y \wedge z \leq t \leq y \vee z$. For proving the first decomposition, one has to take in Definition 5 the following values: $x_1 = x \wedge y$, $x_2 = x \wedge z$, $t_1 = t \wedge z$, $t_2 = t \wedge y$, $x'_1 = x \vee y$, $x'_2 = x \vee z$, $t'_1 = t \vee z$ and $t'_2 = t \vee y$.

The other decompositions come from similar affectations to these eight basic factors.

□

Appendix C. The case of graded properties

The set of fuzzy sets $\mathcal{F}$, with membership functions defined from a referential U to a totally ordered scale $\mathcal{S}$, equipped with min and max for defining the intersection and the union respectively in a pointwise manner, namely $(\mathcal{F}, \min, \max)$, is a distributive lattice.

The description of an item, in terms of n numerical attributes properly renormalized in $\mathcal{S} = [0, 1]$, can be viewed as a fuzzy set (of more less satisfied properties) on a finite referential with n elements. The need for handling numerical attributes has motivated the extension of analogical proportions beyond Boolean lattices.

Indeed, analogical proportions have been also extended when properties are graded on a chain which is finite, or such as the unit interval $[0, 1]$ [48]. For

x	ω	1	ω	0
y	0	1	ω	1
z	ω	1	0	0
t	0	1	0	1

Table C.5: An example of a WAP in a distributive lattice for patterns graded on a 3-levels scale

instance, a property may be half-true. Then, in the case of a finite chain $\mathcal{S}$ with three elements $0 < \omega < 1$, two views make sense, for which the patterns having truth value '1' are respectively

- the 15 patterns, that includes the 6 of the binary case (1 0 1 0), (0 1 0 1), (1 1 0 0), (0 0 1 1), (1 1 1 1), (0 0 0 0), together with their 9 counterparts (ω 0 ω 0), (0 ω 0 ω), (1 ω 1 ω), (ω 1 ω 1), (ω ω 0 0), (0 0 ω ω), (1 1 ω ω), (ω ω 1 1), (ω ω ω ω)
- the 15 above patterns together with the 4 additional ones (1 ω ω 0), (0 ω ω 1), (ω 0 1 ω), (ω 1 0 ω).

In the second view, we acknowledge the fact that when there is a change from x to y there is *the same* change from z to t, and otherwise there is no change between x and y, and between z and t, but also the fact that the proportion still holds when the change from x to y has the same direction and intensity as the change from z to t (considering that ω is exactly in the "middle" between 0 and 1). It is easy to see that the lattice-based definition proposed here agrees with the first view only (where the 4 additional patterns do not make analogical proportions), while the second view, even it makes sense, does not fit with a lattice structure).

For instance, let us take $U = \{k, l, m, n\}$ and the four following fuzzy sets x, y, z, t defined by their respective membership grades (in $\{0, \omega, 1\}$) on the four elements k, l, m, n of U, namely $x = (\omega, 1, \omega, 0)$, $y = (0, 1, \omega, 1)$, $z = (\omega, 1, 0, 0)$, $t = (0, 1, 0, 1)$. Then it is to check that $\min(x, t) = \min(y, z) = (0, 1, 0, 0)$ and $\max(x, t) = \max(y, z) = (\omega, 1, \omega, 1)$. Moreover, if we consider x, y, z, t vertically in a component-wise manner, we recognize the patterns (ω 0 ω 0), (1 1 1 1), (ω ω 0 0), and (0 1 0 1), which are indeed valid analogical proportions, according to the first view (see Table C.5).

In the case of the unit interval $[0, 1]$, this leads to the following graded view of the analogical proportion:

$$(x : y :: z : t) = \min(1 - |\min(x, t) - \min(y, z)|, 1 - |\max(x, t) - \max(y, z)|).$$

It is easy to see that the above definition is a direct counterpart of the second form of the propositional logic expression of the analogical proportion given above. Moreover, it is equal to 1 only for the 15 patterns mentioned above in the first view.

Appendix D. Resolution of a FAP equation: general results

Let us consider three elements x, y, z of a lattice $(L, \vee, \wedge, \leq)$ and the FAP equation in t: $(x : y :: z : t)$. In this section, we present intermediate results and examples which permit to prove Propositions 13 and 15 given in Section 4.1 about the existence of a solution $t \in L$.

As already mentioned in Section 4, elements x, y and z have to satisfy the two following conditions stemming from Proposition 2:

$$x = (x \wedge y) \vee (x \wedge z) \text{ and } x = (x \vee y) \wedge (x \vee z)\,. \tag{D.1}$$

However, these conditions are not sufficient to guarantee that at least one solution exists, as it is shown by the lattice displayed in Figure D.26. Even in the more restrictive case of a distributive lattice, the FAP equation can have no solution: for example, $x = 12$, $y = 5$ and $z = 24$ are three elements of $(\mathbb{N}^+, \mathrm{lcm}, \mathrm{gcd}, |)$ such that $\mathrm{lcm}(\mathrm{gcd}(x, y), \mathrm{gcd}(x, z)) = x$ and $\mathrm{gcd}(\mathrm{lcm}(x, y), \mathrm{lcm}(x, z)) = x$. However, equation $12 : 5 :: 24 : t$ does not have any solution $t \in \mathbb{N}^+$, since $\mathrm{gcd}(12, t) = 1$ would imply t odd, while $\mathrm{lcm}(12, t) = 120$ would imply t even. Moreover, we can observe that equalities (D.1) imply $y \wedge z \leq x \leq y \vee z$, and the reciprocal holds in the distributive case.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
o_1	×			×	×		
o_2		×		×			
o_3			×		×		
o_4				×			
o_5					×		
o_6	×	×		×	×	×	
o_7	×		×	×	×		×

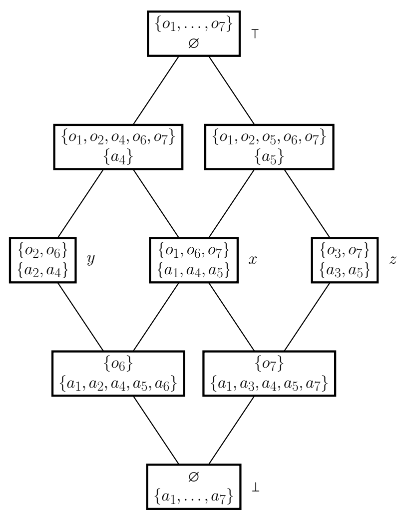

Figure D.26: Example of a concept lattice where the FAP equation $x : y :: z : t$ has no solution t, even if $x = (x \wedge y) \vee (x \wedge z)$ and $x = (x \vee y) \wedge (x \vee z)$. Let us observe that the equation $x : y \, WAP \; z : t$ has also no solution.

These two counterexamples of the existence of a solution t to the FAP equation $(x : y :: z : t)$ prove Proposition 13, that is: even under conditions (D.1), even in a distributive lattice, the FAP equation $(x : y :: z : t)$ can have no solution t.

According to Proposition 14, when the lattice is distributive, it turns out that there is at most one solution to an analogical equation and we give here a property that characterizes the potential solution.

Proposition 26. *Let x, y and z be three elements of a distributive lattice $(L, \vee, \wedge, \leq)$ such that $y \wedge z \leq x \leq y \vee z$. If there exists $\mathring{x} \in L$ such that:*

$$(\mathring{x} \vee x \geq y \vee z) \quad \text{and} \quad (\mathring{x} \wedge x \leq y \wedge z)$$

then

$$t = ((y \vee z) \wedge \mathring{x}) \vee (y \wedge z) = ((y \wedge z) \vee \mathring{x}) \wedge (y \vee z)$$

is the unique solution to the equation $(x : y :: z : t)$.

Moreover, the two equalities $(\mathring{x} \vee x = y \vee z)$ and $(\mathring{x} \wedge x = y \wedge z)$ are equivalent to the equality $(t = \mathring{x})$.

Proof. For convenience, let us denote $m = y \wedge z$ and $M = y \vee z$. Firstly, using distributivity, we show that $x \wedge t = m$ with the equalities:

$$\begin{aligned} x \wedge t &= x \wedge [(M \wedge \mathring{x}) \vee m] \\ &= x \wedge [(M \vee m) \wedge (\mathring{x} \vee m)] \\ &= x \wedge M \wedge (\mathring{x} \vee m) \\ &= (x \wedge M \wedge \mathring{x}) \vee (x \wedge m \wedge M) \\ &= (x \wedge \mathring{x}) \vee m \\ &= m\,. \end{aligned}$$

Secondly, the equality $x \vee t = M$ is demonstrated in the same manner, using $M \leq (\mathring{x} \vee x)$ instead of $(\mathring{x} \wedge x) \leq m$. Then, from Propositions 4 and 14, $t = (M \wedge \mathring{x}) \vee m$ is the unique solution to the equation $(x : y :: z : t)$.

Thirdly, from the equalities $t = ((y \vee z) \wedge \mathring{x}) \vee (y \wedge z) = ((y \wedge z) \vee \mathring{x}) \wedge (y \vee z)$, it is not difficult to show that $\mathring{x} = t$ implies $(\mathring{x} \vee x = y \vee z)$ and $(\mathring{x} \wedge x = y \wedge z)$, and the reciprocal is straightforward. □

Figure D.27 gives an illustration of Proposition 26.

Comment. In the more specific case of the Boolean lattice, the complement element $\neg x$ associated to x satisfies $(\neg x \vee x \geq y \vee z)$ and $(\neg x \wedge x \leq y \wedge z)$. Consequently, Proposition 10 (and also Proposition 13) can be proved using Proposition 26 with $\mathring{x} = \neg x$.

At present, we introduce several propositions from which results Proposition 15, providing an easy resolution of the FAP equation $(x : y :: z : t)$ when x, y, and z do not form an antichain.

Proposition 27. *Let x, y and z be three elements of a lattice. If $x \leq y \wedge z$, the FAP equation in t: $(x : y :: z : t)$ admits a solution if and only if $x = y \wedge z$, and the unique solution is $t = y \vee z$.*

Similarly, if $x \geq y \vee z$, the FAP equation admits a solution if and only if $x = y \vee z$, and the unique solution is $t = y \wedge z$.

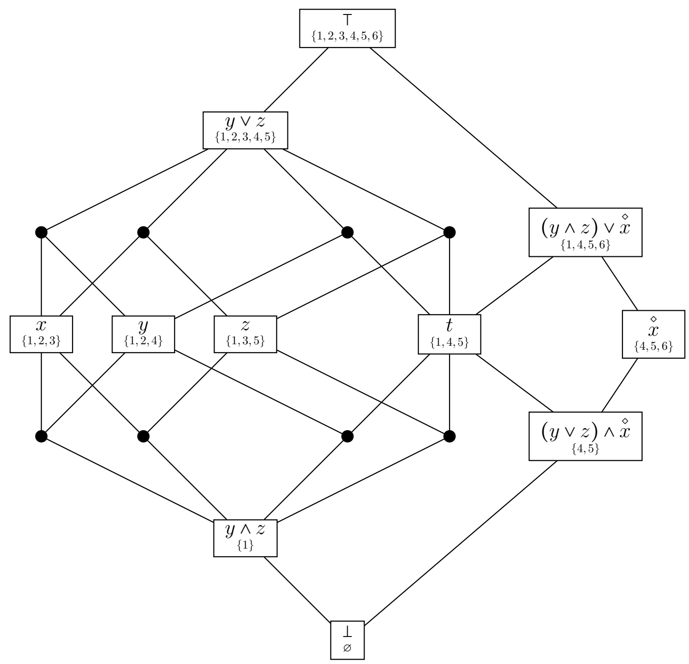

Figure D.27: Resolution of an analogical equation $(x : y :: z : t)$ in a distributive lattice according to Proposition 26. We display an extract of the Boolean lattice of the subsets of $\{1,2,3,4,5,6\}$. Taking $x = \{1,2,3\}$, $y = \{1,2,4\}$ and $z = \{1,3,5\}$ we have $y \wedge z \leq x \leq y \vee z$ and there exists $\overset{\diamond}{x}$ such that $(\overset{\diamond}{x} \vee x \geq y \vee z)$ and $(\overset{\diamond}{x} \wedge x \leq y \wedge z)$. Here we choose $\overset{\diamond}{x} = \{4,5,6\}$, but $\{1,4,5,6\}$ and $\{1,4,5\}$ are also right. Therefore, $t = ((y \vee z) \wedge \overset{\diamond}{x}) \vee (y \wedge z) = ((y \wedge z) \vee \overset{\diamond}{x}) \wedge (y \vee z) = \{1,4,5\}$ is the unique solution to the equation.

Proof. Let us consider the case where $x \leq y \wedge z$. From Proposition 4, the FAP $(x : y :: z : t)$ needs that $y \wedge z \leq x \leq y \vee z$ and $y \wedge z \leq t \leq y \vee z$. Thus, $x = y \wedge z$ and $x \leq t \leq y \vee z$. At last, using equality $x \vee t = y \vee z$, we obtain $t = y \vee z$. If $x \geq y \vee z$, the reasoning is similar. □

When three elements are comparable, the solutions of the FAP equation are severely constrained.

Proposition 28. *Let x, y, z and t be four elements of a lattice $(L, \vee, \wedge, \leq)$ such that they form the FAP $(x : y :: z : t)$. If the three first elements are comparable then $(x : y :: z : t)$ is more precisely*

1. *the CAP1 $(y \wedge z : y :: z : y \vee z)$ if $x \leq y \wedge z$, and the CAP4 $(y \vee z : y :: z : y \wedge z)$ if $x \geq y \vee z$.*

2. *a CAP3 $(x : x \vee t :: x \wedge t : t)$ for any element t such that*

$$\begin{cases} x \vee t &= y \\ x \wedge t &= z \end{cases}$$

if $z \leq x \leq y$

3. *a CAP2* $(x : x \wedge t :: x \vee t : t)$ *for any element* t *such that*

$$\begin{cases} x \wedge t &= y \\ x \vee t &= z \end{cases}$$

if $y \leq x \leq z$.

Proof.

1. See Proposition 27.
2. If $z \leq x \leq y$, $y = x \vee t$ and $z = x \wedge t$ using Proposition 4.
3. If $y \leq x \leq z$, the reasoning is similar to the previous case. □

Given three elements x, y and z of a lattice, we can remark that in the first item of Proposition 28, the solution t is explicitly derived as function of the others elements of the FAP. The next proposition is a direct consequence of Proposition 28.

Proposition 29. *Let* x, y, z *and* t *be four elements of a lattice* $(L, \vee, \wedge, \leq)$, *such that* $\{x, y, z, t\}$ *is a chain and the FAP* $(x : y :: z : t)$ *holds, then* $(x, t) = (y, z)$ *or* $(x, t) = (z, y)$.

When investigating the case where x, y and z form an antichain, we have the following property.

Proposition 30. *Let* x, y *and* z *be three elements of a lattice* $(L, \vee, \wedge, \leq)$ *such that* x, y *and* z *form an antichain. If there exists a solution* t *to the FAP equation* $(x : y :: z : t)$, *then* x, y, z *and* t *form also an antichain.*

Proof. The used reasoning is *reductio ad absurdum*. Let t be a solution to $(x : y :: z : t)$,

- if t and x are comparable, x is necessarily comparable to y and z since $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$ (from Proposition 4). This is inconsistent with the fact that x, y and z form an antichain.
- if t is comparable to y, we first assume that $t \leq y$. Due to the equality $x \wedge t = y \wedge z$, we have $t \geq y \wedge z$.
  - if $t = y \wedge z$, it results that $t = x \wedge t$. Therefore, t and x are comparable, that has been previously shown as absurd.
  - if $y \wedge z < t \leq y$, it follows that $z \wedge t = y \wedge z$ and $x \wedge t \leq x \wedge y$. Using Propositions 2 and 4, we then obtain

$$\begin{aligned} z &= (z \wedge t) \vee (x \wedge z) \\ &= (y \wedge z) \vee (x \wedge z) \\ &= (x \wedge t) \vee (x \wedge z) \\ &\leq (x \wedge y) \vee (x \wedge z) \\ &\leq x \end{aligned}$$

    Therefore, x and z are comparable, that is absurd.

    If we assume that $t \geq y$, the inequalities $y \leq t \leq y \vee z$ also drive the reasoning to the same absurdity.

- Due to the symmetry of equations in Propositions 2 and 4, the reasoning to compare t and z is similar to the previous one. □

Comment. As illustrated in Section 5.4.2, the previous proposition does not hold in a case of a WAP equation: even if $\{x, y, z\}$ is an antichain and there exists a concept t such that $x : y \; WAP \; z : t$, $\{x, y, z, t\}$ is not necessarily an antichain.

From the two last propositions and their proofs, it directly follows the next proposition, completing the demonstration of Proposition 15.

Proposition 31. *Let x, y, z and t be four elements of a lattice $(L, \vee, \wedge, \leq)$ such that they form the FAP $x : y :: z : t$. If $\{x, y, z, t\}$ is not an antichain, then the FAP corresponds to either a trivial AP $x : x :: x : x$, or a simple AP $x : x :: y : y$, or a CAP or associated APs using the properties of symmetry and exchange of means.*

Appendix E. Some properties of formal concept lattice

We give here a few simple and well known properties of a concept lattice, which will be useful for our study.

Preliminary 1. *Given two concepts $x = (\mathbf{o}_x, \mathbf{a}_x)$ and $y = (\mathbf{o}_y, \mathbf{a}_y)$ of a formal concept lattice, one has:*

$$\begin{aligned} (\mathbf{o}_x \cup \mathbf{o}_y)^{\uparrow} &= \mathbf{a}_x \cap \mathbf{a}_y \\ (\mathbf{a}_x \cup \mathbf{a}_y)^{\downarrow} &= \mathbf{o}_x \cap \mathbf{o}_y \end{aligned}$$

Proof. (Adapted from [8])

$$\begin{aligned} m \in (\mathbf{o}_x \cup \mathbf{o}_y)^{\uparrow} &\Leftrightarrow gRm \text{ for all } g \in \mathbf{o}_x \cup \mathbf{o}_y \\ &\Leftrightarrow gRm \text{ for all } g \in \mathbf{o}_x \text{ and } gRm \text{ for all } g \in \mathbf{o}_y \\ &\Leftrightarrow m \in (\mathbf{o}_x)^{\uparrow} \text{ and } m \in (\mathbf{o}_y)^{\uparrow} \\ &\Leftrightarrow m \in (\mathbf{o}_x)^{\uparrow} \cap (\mathbf{o}_y)^{\uparrow} \\ &\Leftrightarrow m \in \mathbf{a}_x \cap \mathbf{a}_y \end{aligned}$$

The second equality has a similar proof. □

Preliminary 2. *Given two concepts $x = (\mathbf{o}_x, \mathbf{a}_x)$ and $y = (\mathbf{o}_y, \mathbf{a}_y)$ of a formal concept lattice, one has:*

$$\mathbf{o}_x \cup \mathbf{o}_y \subseteq \mathbf{o}_{x \vee y}, \quad \mathbf{o}_x \cap \mathbf{o}_y = \mathbf{o}_{x \wedge y},$$
$$\mathbf{a}_x \cup \mathbf{a}_y \subseteq \mathbf{a}_{x \wedge y}, \quad \mathbf{a}_x \cap \mathbf{a}_y = \mathbf{a}_{x \vee y}.$$

Proof. A formal proof of this basic property is for example in [8]. The rationale is as follows: let $x = (\mathbf{o}_x, \mathbf{a}_x)$ and $y = (\mathbf{o}_y, \mathbf{a}_y)$ be two concepts of a lattice, it follows from $x \wedge y \leq x$ and $x \wedge y \leq y$ that $\mathbf{o}_{x \wedge y} \subseteq \mathbf{o}_x \cap \mathbf{o}_y$. Moreover, due to Preliminary 1, $\mathbf{o}_x \cap \mathbf{o}_y$ corresponds to the set of objects of a concept u. Therefore, $x \wedge y \leq u \leq x$ and $x \wedge y \leq u \leq y$. It results that $u = x \wedge y$ and then $\mathbf{o}_x \cap \mathbf{o}_y = \mathbf{o}_{x \wedge y}$.

□

Preliminary 3. *Let* **o** *(resp.* **a***) be a subset of* $\mathcal{O}$ *(resp.* $\mathcal{A}$*), there exists at most one concept* x *such that* $\mathbf{o}_x = \mathbf{o}$ *(resp.* $\mathbf{a}_x = \mathbf{a}$*).*

Proof. This is a direct consequence of the definition of a formal concept. □

These properties are actually simple consequences of the Main Theorem of Formal Concepts [8, 18].

Appendix F. The analogical lattice and the semiproduct of elementary contexts

In this section, we link the analogical lattice to the notion of semiproduct of contexts, which definition is as follows (see [18], section 1.4):

Definition 16. *Let* $\mathbb{K}_1 = (\mathcal{O}_1, \mathcal{A}_1, R_1)$ *and* $\mathbb{K}_2 = (\mathcal{O}_2, \mathcal{A}_2, R_2)$ *be two contexts, with* $\mathcal{A}_1 \cap \mathcal{A}_2 = \varnothing$*. Their semiproduct is defined as*

$$\mathbb{K}_1 ⧖ \mathbb{K}_2 \;=\; (\mathcal{O}_1 \times \mathcal{O}_2, \mathcal{A}_1 \cup \mathcal{A}_2, \nabla)$$

with

$$(o_1, o_2) \nabla m \quad \Leftrightarrow \quad \exists j \in \{1, 2\} : \; o_j \; R_j \; m$$

.

The analogical context is the semiproduct of two (2×2) "diagonal" contexts, as follows :

	a_1	a_4
o_1	×	
o_4		×

⧖

	a_2	a_3
o_2	×	
o_3		×

=

	a_1	a_2	a_3	a_4
(o_1, o_2)	×	×		
(o_1, o_3)	×		×	
(o_2, o_4)		×		×
(o_3, o_4)			×	×

In terms of lattices, an analogical lattice is the semiproduct of two elementary diamond shaped lattices. To say it informally, the bi-complete WAP in an analogical lattice is the semiproduct of two canonical proportions, as displayed in Figure F.28. Note the special role played by the bottom nodes, which do not take part to the cartesian product.

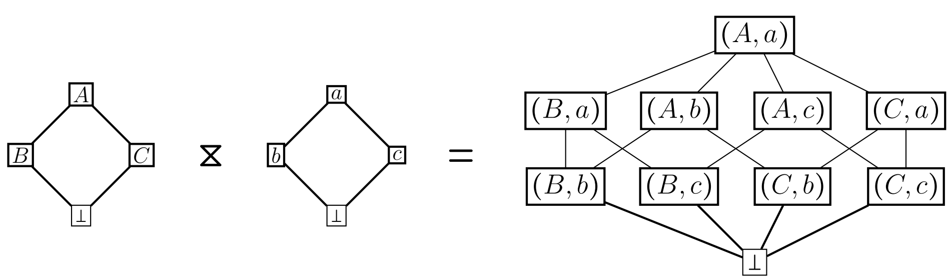

Figure F.28: A bi-complete WAP as the semiproduct of two canonical proportions. For example, the node B corresponds to the concept $(\{o_1\}, \{a_1\})$, the node b to $(\{o_2\}, \{a_2\})$, the node (B, c) to $(\{(o_1, o_3)\}, \{a_1, a_3\})$ and the node (B, a) to $(\{(o_1, o_2), (o_1, o_3)\}, \{a_1\})$.

Appendix G. From an AP to an analogical lattice

Suppose we know an analogical proportion, expressed in linguistic terms, such as "A sand kite board is to a sandyacht as a kite surf is to a sailing boat". Can we identify a set of attributes, construct a context and build a lattice in which there will exist the corresponding AP?

The minimal semantic domain is that of piloted vehicles moved by the wind. Using a sort of Occam razor argument, we do not have to extend it unless introducing useless complexity. Let us call the four objects kb, sy, ks and sb.

We know that the attributes will have to express a proportion. Hence, one of them must represent what kb and sy have in common, and that ws and sb do not possess. Symmetrically, another attribute must represent what ws and sb have in common, and that kb and sy do not possess. The simplest is to create two attributes, one meaning "the vehicle moves on the sand" (call it S), the other "the vehicle moves on the water" (call it W).

Note that there is no object (as far as we know) that shares S and W inside the semantic domain. But there exists in this domain the wind snowboard, which has not S nor W. More generally speaking, the attributes S and W must have an empty intersection, but do not cover necessarily the whole domain.

In the same way, we create the attributes "the pilot holds a kite" (call it K) and "there is a fixed mast" (call it M) to express that kb and ks have something in common that sy and sb do not possess, and the symmetrical property. Note that in the semantic domain, one can find vehicles with orientable mast, as a sailing board.

This leads directly to the following analogical context, which is an analogical complex:

		Sand S	Kite K	Mast M	Water W
sand kite board	kb	×	×		
sand yacht	sy	×		×	
kite surf	ks		×		×
sailing boat	sb			×	×

One of the two complete WAPs in the lattice is:

$$\boxed{\begin{matrix}\{kb\}\\ \{S,K\}\end{matrix}} : \boxed{\begin{matrix}\{sy\}\\ \{S,M\}\end{matrix}} \ \text{WAP}\ \boxed{\begin{matrix}\{ks\}\\ \{K,W\}\end{matrix}} : \boxed{\begin{matrix}\{sb\}\\ \{M,W\}\end{matrix}}$$

Appendix H. Bi-complete WAP.

Due to Definition 10 of the bi-complete WAP, Proposition 21 indicates that assertion (a) implies (b), where

(a) : $(x : y\ WAP\ z : t)$ is complete through attributes and is a bi-complete WAP.

$(b):$ $((x \vee y) : (x \vee z)\, WAP\, (y \vee t) : (z \vee t))$ is complete through objects and is a bi-complete WAP.

The proof of this implication is detailed below.

Proof. Let us assume assertion (a), that is to say that $(x : y\, WAP\, z : t)$ is complete through attributes and $((x \vee y) : (x \vee z)\, WAP\, (y \vee t) : (z \vee t))$ is complete through objects. In order to derive (b), it is sufficient to prove that $(x' : y'\, WAP\, z' : t')$ is complete through attributes where

$$\begin{aligned} x' &= (x \vee y) \wedge (x \vee z) \\ y' &= (x \vee y) \wedge (y \vee t) \\ z' &= (x \vee z) \wedge (z \vee t) \\ t' &= (y \vee t) \wedge (z \vee t) \end{aligned}$$

Since $(x : y\, WAP\, z : t)$ is complete through attributes, the associated attribute sets can be written as as unions of pairwise *disjoints* subsets of attributes as follows :

$$\begin{aligned} \mathbf{a}_x &= A_1 \cup A_2 \cup A_5 \cup A_9 \\ \mathbf{a}_y &= A_1 \cup A_3 \cup A_6 \cup A_9 \\ \mathbf{a}_z &= A_2 \cup A_4 \cup A_7 \cup A_9 \\ \mathbf{a}_t &= A_3 \cup A_4 \cup A_8 \cup A_9 \end{aligned}$$

with the property A_1, A_2, A_3 and A_4 non empty. Figure 20 illustrates this property.

Moreover, from Proposition 17, the object sets are such that $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$. Hence, there exist nine pairwise disjoint subsets of objects $O_1, \ldots, O_9$ such that

$$\begin{aligned} \mathbf{o}_x &= O_1 \cup O_2 \cup O_5 \cup O_9 \\ \mathbf{o}_y &= O_1 \cup O_3 \cup O_6 \cup O_9 \\ \mathbf{o}_z &= O_2 \cup O_4 \cup O_7 \cup O_9 \\ \mathbf{o}_t &= O_3 \cup O_4 \cup O_8 \cup O_9 \end{aligned}$$

Due to FCA properties, there also exist O_{10}, O_{11}, O_{12} and O_{13} such that $\mathbf{o}_{x\vee y}$, $\mathbf{o}_{x\vee z}$, $\mathbf{o}_{y\vee t}$ and $\mathbf{o}_{z\vee t}$ can be partitioned as

$$\begin{aligned} \mathbf{o}_{x\vee y} &= O_1 \cup O_2 \cup O_3 \cup O_5 \cup O_6 \cup O_9 \cup O_{10} \\ \mathbf{o}_{x\vee z} &= O_1 \cup O_2 \cup O_4 \cup O_5 \cup O_7 \cup O_9 \cup O_{11} \\ \mathbf{o}_{y\vee t} &= O_1 \cup O_3 \cup O_4 \cup O_6 \cup O_8 \cup O_9 \cup O_{12} \\ \mathbf{o}_{z\vee t} &= O_2 \cup O_3 \cup O_4 \cup O_7 \cup O_8 \cup O_9 \cup O_{13} \end{aligned}$$

As a consequence,

$$\begin{aligned} \mathbf{o}_{x\vee y} \cap \mathbf{o}_{z\vee t} &= (O_2 \cup O_3 \cup O_9) \cap (O_{10} \cap O_{13}) \\ \mathbf{o}_{x\vee z} \cap \mathbf{o}_{y\vee t} &= (O_1 \cup O_4 \cup O_9) \cap (O_{11} \cap O_{12}) \end{aligned}$$

Thanks to $((x \vee y) : (x \vee z)\, WAP\, (y \vee t) : (z \vee t))$, we have

$$\mathbf{o}_{x\vee y} \cap \mathbf{o}_{z\vee t} = \mathbf{o}_{x\vee z} \cap \mathbf{o}_{y\vee t}\,.$$

Due to the definition of $O_{10}, \ldots, O_{13}$, subsets $O_{10} \cap O_{13}$ and $O_{11} \cap O_{12}$ have no intersection with $O_2 \cup O_3 \cup O_9$ and $O_1 \cup O_4 \cup O_9$. Therefore, it follows that $O_1 = O_2 = O_3 = O_4 = \varnothing$ and $O_{10} \cap O_{13} = O_{11} \cap O_{12}$.

Hence, the eight concepts are as follows:

$$\begin{array}{rclcrcl}
\mathbf{o}_x & = & O_5 \cup O_9 & & \mathbf{o}_{x\vee y} & = & O_5 \cup O_6 \cup O_9 \cup O_{10} \\
\mathbf{o}_y & = & O_6 \cup O_9 & & \mathbf{o}_{x\vee z} & = & O_5 \cup O_7 \cup O_9 \cup O_{11} \\
\mathbf{o}_z & = & O_7 \cup O_9 & & \mathbf{o}_{y\vee t} & = & O_6 \cup O_8 \cup O_9 \cup O_{12} \\
\mathbf{o}_t & = & O_8 \cup O_9 & & \mathbf{o}_{z\vee t} & = & O_7 \cup O_8 \cup O_9 \cup O_{13}
\end{array}$$

Besides, using Preliminary 2 (see Appendix E),

$$\begin{aligned}
\mathbf{o}_{(x\vee y)\wedge(x\vee z)} &= \mathbf{o}_{x\vee y} \cap \mathbf{o}_{x\vee z} \\
&= O_5 \cup O_9 \\
&= \mathbf{o}_x \,.
\end{aligned}$$

From Preliminary 3, we then get $x = (x \vee y) \wedge (x \vee z) = x'$. By a similar way, we also derive $y = y'$, $z = z'$ and $t = t'$. Thus, $(x' : y' \; WAP \; z' : t')$ is complete through attributes and assertion (b) is true. □

Comment. The other implication given by Proposition 21, that is

> *If* $(x : x \; WAP \; z : t)$ *is complete through objects and is a bi-complete WAP, then* $((x \wedge y) : (x \wedge z) \; WAP \; (y \wedge t) : (z \wedge t))$ *is complete through attributes and is a bi-complete WAP.*

can be derived by a similar reasoning as the one above.

Let us now consider the first item of the proof of Proposition 22. In Section 6.2, we define the following tables from $(x : y \; WAP \; z : t)$

i	$\mathbf{o}_i$	$\mathbf{a}_i$
1	$\widetilde{\mathbf{o}_x}$	$(\mathbf{a}_z \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap$
2	$\widetilde{\mathbf{o}_y}$	$(\mathbf{a}_y \cap \mathbf{a}_t) \setminus \mathbf{a}_\cap$
3	$\widetilde{\mathbf{o}_z}$	$(\mathbf{a}_x \cap \mathbf{a}_z) \setminus \mathbf{a}_\cap$
4	$\widetilde{\mathbf{o}_t}$	$(\mathbf{a}_x \cap \mathbf{a}_y) \setminus \mathbf{a}_\cap$

i	$\mathbf{o}'_i$	$\mathbf{a}'_i$
1	$(\mathbf{o}_z \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_x}$
2	$(\mathbf{o}_y \cap \mathbf{o}_t) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_y}$
3	$(\mathbf{o}_x \cap \mathbf{o}_z) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_z}$
4	$(\mathbf{o}_x \cap \mathbf{o}_y) \setminus \mathbf{o}_\cap$	$\widetilde{\mathbf{a}_t}$

If $(x : y \; WAP \; z : t)$ is bi-complete, such that it is complete through attributes, the previous proof permits to derive that

i	$\mathbf{o}_i$	$\mathbf{a}_i$
1	O_5	A_4
2	O_6	A_3
3	O_7	A_2
4	O_8	A_1

i	$\mathbf{o}'_i$	$\mathbf{a}'_i$
1	$\varnothing$	A_5
2	$\varnothing$	A_6
3	$\varnothing$	A_7
4	$\varnothing$	A_8

with the same notations as above. Since A_1, A_2, A_3 and A_4, O_5, O_6, O_7 and O_8 are non empty, the first table has no empty subset.

If $(x : y \; WAP \; z : t)$ is bi-complete, such that it is complete through objects, we can prove that the table on right has no empty sets by a similar way.

In the context composed with only the attributes and objects present in concepts x, y, z, t, $x \vee y$, $x \vee z$, $y \vee t$, $z \vee t$, the first table is a complete analogical complex (see Section 6.1, in particular Definition 12). Consequently, using the second postprocessing of the algorithm `WAPtoCPLX`, the derivation of at least one complete analogical complex from a bi-complete WAP is guaranteed.