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Randomized incremental construction of Delaunay triangulations of nice point sets

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Abstract

Randomized incremental construction (RIC) is one of the most important paradigms for building geometric data structures. Clarkson and Shor developed a general theory that led to numerous algorithms that are both simple and efficient in theory and in practice.

Randomized incremental constructions are most of the time space and time optimal in the worst-case, as exemplified by the construction of convex hulls, Delaunay triangulations and arrangements of line segments. However, the worst-case scenario occurs rarely in practice and we would like to understand how RIC behaves when the input is nice in the sense that the associated output is significantly smaller than in the worst-case. For example, it is known that the Delaunay triangulations of nicely distributed points on polyhedral surfaces in E^3 has linear complexity, as opposed to a worst-case quadratic complexity. The standard analysis does not provide accurate bounds on the complexity of such cases and we aim at establishing such bounds in this paper. More precisely, we will show that, in the case of nicely distributed points on polyhedral surfaces, the complexity of the usual RIC is $O(n \log n)$, which is optimal. In other words, without any modification, RIC nicely adapts to good cases of practical value.

Our proofs also work for some other notions of nicely distributed point sets, such as (ε, κ) -samples. Along the way, we prove a probabilistic lemma for sampling without replacement, which may be of independent interest.

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1 Introduction

The *randomized incremental construction* (RIC) is an algorithmic paradigm introduced by Clarkson and Shor [10], which has since found immense applicability in computational geometry, e.g., [21, 20]. The general idea is to process the input points sequentially in a random order, and to analyze the expected complexity of the resulting procedure. The theory developed by Clarkson and Shor is quite general and has led to numerous algorithms that are simple and efficient, both in theory and in practice. On the theory side, randomized incremental constructions are most of the time space and time optimal in the worst-case, as exemplified by the construction of convex hulls, Delaunay triangulations and arrangements of line segments. Randomized incremental constructions appear also to be very efficient in practice, which, together with their simplicity, make them the most popular candidates for implementations. Not surprisingly, the CGAL library includes several randomized incremental algorithms, e.g., for computing Delaunay triangulations [22].

This paper aims at extending the analysis of RIC to the case of *nice-case* complexity. More precisely, our goal is to understand how randomized incremental constructions behave when the input is nice in the sense that the associated construction is significantly smaller than in the worst-case.

In this paper, we shall consider the case where the underlying space is a polyhedral surface in \mathbb{E}^3 . This is a commonly-occurring practical scenario in e.g., surface reconstruction [1, 6], and has been studied by several authors [2, 4, 5, 17]. Further, we need a model of good point sets to describe the input data and analyze the algorithm. This will be done through the notion of ε -nets. When we enforce such a hypothesis of “nice” distribution of the points in space, a result of Attali and Boissonnat [4] ensures that the complexity of the Delaunay triangulation is linear in the number of points. Unfortunately, to be able to control the complexity of the usual randomized incremental algorithms [3, 9, 10, 12], it is not enough to control the final complexity of the Delaunay triangulation. We need to control also the complexity of the triangulation of random subsets. One might expect that a random subsample of size k of an ε -net is also an ε' -net for $\varepsilon' = \varepsilon\sqrt{\frac{n}{k}}$. Actually this is not quite true, it may happen with reasonable probability that a ball of radius $O(\varepsilon')$ contains $\Omega(\log k / \log \log k)$ points or that a ball of radius $\Omega(\varepsilon'\sqrt{\log k})$ does not contain any point. However, it can only be shown that such a subsample is an $\left(\frac{\varepsilon'}{\log(1/\varepsilon)}\right)$ -covering and an $(\varepsilon' \log(1/\varepsilon))$ -packing, with high probability. Thus this approach can transfer the complexity of an ε -net to the one of a random subsample of an ε -net but with an extra multiplicative factor of $\Omega(\log 1/\varepsilon) = \Omega(\log n)$. It follows that, in the case we consider, the standard analysis does not provide accurate bounds on the complexity of the (standard) randomized incremental construction. Our results are based on proving that the above bad scenarios occur rarely, and the algorithm achieves optimal run-time complexity, in expectation.

Related Work

The Delaunay triangulations of nicely-distributed points have been studied since the 50's, e.g., by Meijering [18], and later by Møller [19], Dwyer [13, 14], and others. Erickson [15, 16] proved upper and lower bounds for point samples with bounded *spread* (the ratio between the maximum to minimum distance between any two points) in \mathbb{E}^3 . For polyhedral surfaces, Golin and Na [17] gave an $O(n \log^4 n)$ bound for Poisson-distributed points. Attali and Boissonnat [4] showed that for (ε, κ) -samples, the complexity of the Delaunay triangulation is linear. Under some extra assumptions, this was extended by Amenta, Attali, and Devillers [2] to higher-dimensional polyhedral surfaces. Attali, Boissonnat, and Lieutier [5] proved an

$O(n \log n)$ bound for (ε, κ) -samples on *smooth* surfaces in \mathbb{E}^3 .

Except for a few authors such as Dwyer [14] and Erickson [16], most of the above results discuss only the combinatorial aspects and not the algorithmic ones. For Poisson and uniformly distributed point samples, we observe that the standard analysis of the RIC procedure immediately implies an optimal bound on the expected run-time. However, for deterministic notions of nice distributions such as ε -nets, (ε, κ) -samples, and bounded spread point sets, the standard RIC analysis is not optimal, since, as we observed, it gives at least an extra logarithmic factor for (ε, κ) -samples and even worse for bounded spread point-sets, as stated in an open problem by Erickson [16].

Our Contribution

For ε -nets on polyhedral surfaces in \mathbb{E}^3 , we establish tight bounds on the complexity of random subsamples of any given size. Using this, we show that the complexity of the usual RIC is $O(n \log n)$, which is optimal. Hence, without any modification, the standard RIC nicely adapts to polyhedral surfaces in \mathbb{E}^3 .

Our technical developments rely on a general bound for the probability of certain non-monotone events in sampling without replacement, which may be of independent interest. We use this together with a geometric construction that, given a point p on a plane P , and a threshold radius r , allows us to bound the probability of existence of any empty disk in P with radius at least r , having p on its boundary. Lastly, the boundary effects need to be explicitly controlled, which requires a careful handling along the lines of the result of Attali and Boissonnat [4], along with some new ideas which we develop. (For a more detailed outline of the ideas, see the discussion in Section 3).

We remark that though we focus on polyhedral surfaces in \mathbb{E}^3 in this paper, our techniques are more general, and can be extended to e.g., ε -nets on d -dimensional flat torii, etc., which we do in the full version of this paper.

Outline

The rest of the paper is as follows. In Section 2, we define the basic concepts of Delaunay triangulation, ε -nets and random samples. We state our theorems and their proofs in Section 3. In Section 4, we give the proofs of some technical lemmas needed for the proofs of our theorems. Proofs missing from the main sections can be found in the full version of this paper [7].

2 Background

2.1 Notations

We shall use $\|\cdot\|$ to denote the Euclidean ℓ_2 norm. We denote by $\Sigma(p, r)$, $B(p, r)$ and $B[p, r]$, the sphere, the open ball, and the closed ball of center p and radius r respectively. For $x \in \mathbb{E}^2$, $y \geq 0$, $D(x, r)$ denotes the disk with center x and radius r , i.e. the set of points $\{y \in \mathbb{E}^2 : \|y - x\| < r\}$, and similarly $D[x, r]$ denotes the corresponding closed disk.

For an event \mathcal{E} in some probability space Ω , we use $\mathbb{1}_{[\mathcal{E}]}$ to denote the indicator variable $\mathbb{1}_{[\mathcal{E}]} = \mathbb{1}_{[\mathcal{E}]}(\omega)$ which is 1 whenever $\omega \in \mathcal{E}$, and zero otherwise. We use $[n]$ to mean the set $\{1, 2, \dots, n\}$. Given a discrete set A , $\#(A)$ denotes its cardinality and, for $k \in \mathbb{Z}^+$, $\binom{A}{k}$ denotes the set of k -sized subsets of A . Given an event A in some probability space, $\mathbb{P}[A]$ denotes the probability of A occurring. For a random variable Z in a probability space, $\mathbb{E}[Z]$

denotes the expected value of Z . Lastly, $e = 2.7182\dots$ denotes the base of the natural logarithm.

2.2 ε -nets

A set \mathcal{X} of n points in a metric space \mathcal{M} , is an ε -packing if any pair of points in \mathcal{X} are at least distance ε apart, and an ε -cover if each point in \mathcal{M} is at distance at most ε from some point of \mathcal{X} . \mathcal{X} is an ε -net if it is an ε -cover and an ε -packing simultaneously.

The definition of an ε -net applies for any metric space. In the case of the Euclidean metric, we can prove some additional properties, which will be given in Section 3.

2.3 Delaunay Triangulation

For simplicity of exposition and no real loss of generality, all finite point sets considered in this paper will be assumed to be in *general position*, i.e. there are no 5 points lying on a sphere in \mathbb{E}^3 , and no plane has a set of 4 points lying on a circle. Given a set \mathcal{X} in some ambient topological space, the *Delaunay complex* of \mathcal{X} is the (abstract) simplicial complex with vertex set \mathcal{X} which is the nerve of the Voronoi diagram of \mathcal{X} , that is, a simplex σ (of arbitrary dimension) belongs to $Del(\mathcal{X})$ iff the Voronoi cells of its vertices have a non empty common intersection. Equivalently, σ can be circumscribed by an empty ball, i.e. a ball whose bounding sphere contains the vertices of σ and whose interior contains no points of \mathcal{X} .

For point sets in \mathbb{E}^3 in general position, the Delaunay complex embeds in \mathbb{E}^3 and is a triangulation of the space.

2.4 Polyhedral Surfaces in \mathbb{E}^3

A *polyhedral surface* \mathcal{S} in \mathbb{E}^3 is a collection of a finite number of polygons $F \subset \mathcal{S}$, called *facets*, which are pairwise disjoint or meet along an edge. In this paper, \mathcal{S} will denote an arbitrary but fixed polyhedral surface, with C facets, and having total length of the boundaries of its faces L and total area of its faces A . Any non-convex polygonal facet $F \in \mathcal{S}$ can be triangulated and replaced in \mathcal{S} by the collection of triangular facets obtained. This will only change the total length L of the boundaries, which, for a given triangulation, still depends only on the original surface \mathcal{S} . Thus without any real loss of generality, we can (and shall) assume the facets of \mathcal{S} are convex.

2.5 Randomized Incremental Construction and Random Subsamples

For the algorithmic complexity aspects, we state a version of a standard theorem for the RIC procedure, (see e.g., [11]). We first need a necessary condition for the theorem. When a new point p is added to an existing triangulation, a *conflict* is defined to be a previously existing simplex whose circumball contains p .

▷ **Condition 1.** At each step of the RIC, the set of simplices in conflict can be removed and the set of newly introduced conflicts can be computed in time proportional to the number of conflicts.

We now come to the general theorem on the algorithmic complexity of RIC using the Clarkson-Shor technique (see e.g., Devillers [11] Theorem 5(1,2)).

► **Theorem 2.** Let $F(s)$ denote the expected number of simplices that appear in the Delaunay triangulation of a uniform random sample of size s , from a given point set P . Then, if Condition 1 holds and $F(s) = O(s)$, we have

- (i) The expected space complexity of computing the Delaunay triangulation is $O(n)$.
- (ii) The expected time complexity of computing the Delaunay triangulation is $O(n \log n)$.

A subset \mathcal{Y} of set \mathcal{X} is a *uniform random sample* of \mathcal{X} of size s if \mathcal{Y} is any possible subset of \mathcal{X} of size s with equal probability.

In order to work with uniform random samples, we shall prove a lemma on the uniformly random sampling distribution or *sampling without replacement*, which is stated below, and will be a key probabilistic component of our proofs. The lemma provides a bound on the probability of a non-monotone compound event, that is, if the event holds true for a fixed set of k points, there could exist supersets as well as subsets of the chosen set for which the event does not hold. This may well be of general interest, as most natural contiguity results with Bernoulli (i.e. independent) sampling, are for monotone events.

► **Lemma 3.** *Given $a, b, c \in \mathbb{Z}^+$, with $2b \leq a \leq c$, $t \leq c$. Let C be a set, and B and T two disjoint subsets of C . If A is a random subset of C , chosen uniformly from all subsets of C having size a , the probability that A contains B and is disjoint from T , is at most $\left(\frac{a}{c}\right)^b \left(1 - \frac{t}{c-b}\right)^{a-b} \leq \left(\frac{a}{c}\right)^b \cdot \exp\left(-\frac{at}{2c}\right)$, where a, b, c are the cardinalities of A, B , and C respectively, and the cardinality of T is at least t .*

Proof. The total number of ways of choosing the random sample A is $\binom{c}{a}$. The number of ways of choosing A such that $B \subset A$ and $T \cap A = \emptyset$, is $\binom{c-b-t}{a-b}$. Therefore the required probability is

$$\begin{aligned} \mathbb{P}[B \subset A, T \cap A = \emptyset] &= \frac{\binom{c-b-t}{a-b}}{\binom{c}{a}} \\ &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=b}^{a-1} (a-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=b}^{a-1} (c-i)} \cdot \frac{\prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{a-b-1} (a-b-i)} \\ &= \frac{\prod_{i=0}^{b-1} (a-i) \prod_{i=0}^{a-b-1} (c-b-t-i)}{\prod_{i=0}^{b-1} (c-i) \prod_{i=0}^{a-b-1} (c-b-i)} \\ &= (a/c)^b \frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)} \left(1 - \frac{t}{c-b}\right)^{a-b} \left(\frac{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b-t})}{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b})}\right) \\ &\leq (a/c)^b \left(1 - \frac{t}{c-b}\right)^{a-b}, \end{aligned}$$

where in the last step, observe that for the product $\frac{\prod_{i=0}^{b-1} (1-i/a)}{\prod_{i=0}^{b-1} (1-i/c)}$ for each i , the term $(1-i/a)$ in the numerator is smaller than the corresponding term $(1-i/c)$ in the denominator, since $a \leq c$. A similar observation holds for the product $\left(\frac{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b-t})}{\prod_{i=0}^{a-b-1} (1 - \frac{i}{c-b})}\right)$.

Now, observe that $\left(1 - \frac{t}{c-b}\right)^{a-b} \leq \exp\left(-\left(\frac{t(a-b)}{c-b}\right)\right) \leq \exp\left(-\left(\frac{at}{2c}\right)\right)$, if $b \leq a/2$ and $b < c$. ◀

3 Results and Main Proofs

We show that the expected complexity of the Delaunay triangulation of a uniformly random subsample of an ε -net on a polyhedral surface is linear in the size of the subsample:

► **Theorem 4.** *Let $\varepsilon \in (0, 1]$, \mathcal{X} be an ε -net on a polyhedral surface \mathcal{S} , having n points and let $\mathcal{Y} \subset \mathcal{X}$ be a random sub-sample of \mathcal{X} having size s . Then, in expectation, the Delaunay triangulation $\text{Del}(\mathcal{Y})$ of \mathcal{Y} on \mathcal{S} has $O(s)$ simplices.*

Algorithmic Bounds: We next use the above combinatorial bound to get the space and time complexity of the randomized incremental construction of the Delaunay triangulation of an ε -net on a polyhedral surface in \mathbb{E}^3 .

► **Theorem 5 (Randomized incremental construction).** *Let $\varepsilon \in (0, 1]$, and let \mathcal{X} be an ε -net in general position over a fixed polyhedral surface $\mathcal{S} \subset \mathbb{E}^3$, then the randomized incremental construction of the Delaunay triangulation takes $O(n \log n)$ expected time and $O(n)$ expected space, where $n = \#(\mathcal{X})$ and the constant in the big O depends on (and only on) \mathcal{S} .*

► **Remark 6.** Theorem 4 also works for the case when the random sample is a Bernoulli sample of parameter $q := \frac{s}{n}$.

► **Remark 7.** Our results can be extended to other types of good samples, e.g., the weaker notion of (ε, κ) -samples for which any ball of radius ε contains at least one point and at most κ points. If we fix $\kappa = \kappa_0 = 2^{O(d)}$, we get exactly the same result. The bounds can be straightforwardly adapted to accommodate other values of κ .

Before presenting the proof of Theorem 4, we briefly discuss the outline of the proof.

Main Ideas

Our overall strategy will be to mesh the proof of Attali and Boissonnat [4] with some new ideas which are needed for random subsamples of ε -nets. Briefly, Attali and Boissonnat reduce the problem to counting the Delaunay edges of the point sample, which they do by distinguishing between *boundary* points, which lie in a strip of width ε near the boundaries of the facets of the polyhedral surface, and the other points, called *interior* points. For boundary points, they allow all possible edges. For interior points, the case of edges with endpoints on the same facet is easy to handle, while geometric constructions are required to handle the case of endpoints on different facets, or that of edges with one endpoint in the interior and another on the boundary.

However, we shall need to introduce a couple of new ideas. Firstly, an edge can have multiple balls passing through its endpoints and, as soon as one of these balls is empty, the edge is in the triangulation. To bound therefore, the probability of a potential edge appearing in the triangulation, we need to simultaneously bound the probability of any of these balls being empty. To ensure this, we use a geometric construction (see Lemma 17). Basically, the idea is to build a constant-sized packing of a sphere centered on a given point, using large balls, such that any sphere of a sufficiently large radius which passes through the point, must contain a ball from the packing.

Secondly, since we have randomly spaced points at the boundaries, boundary effects are no longer necessarily contained in the fixed strip of width ε around the boundary, and could potentially penetrate deep into the interior. To handle this, we generalize the fixed-width strip using the notion of *levels* of a facet. We then use a probabilistic, rather than deterministic, classification of boundary and interior points. The new classification is based on the level of a point and the radius of the largest empty disk passing through it.

Recall the definitions of \mathcal{X} , \mathcal{Y} and \mathcal{S} from Theorem 4. We shall use κ to denote the maximum number of points of a given point set in a disk of radius 2ε . When \mathcal{X} is an ε -net, κ is at most 25, using a packing argument (the maximum number of disjoint discs of radius $\varepsilon/2$ that can be packed in a disc of radius $2\varepsilon + \varepsilon/2$, is $\frac{\pi(5\varepsilon/2)^2}{\pi(\varepsilon/2)^2} \leq 25$). We define $q := \frac{s}{n}$, and

$\delta := \varepsilon/\sqrt{q}$. For a curve Γ , $l(\Gamma)$ denotes its length. For a subset of a surface $R \subset \mathcal{S}$, $a(R)$ denotes the area of R . For sets $A, B \subset \mathbb{E}^3$, $A \oplus B$ denotes the Minkowski sum of A and B , i.e. the set $\{x + y : x \in A, y \in B\}$. For convenience, the special case $A \oplus B(0, r)$ shall be denoted by $A \oplus r$.

We next present some general lemmas, which will be needed in the proofs of the main lemmas.

Level Sets, Boundary Points and Interior Points

We now introduce some definitions which will play a central role in the analysis. First we define the notion of levels. Given facet $F \in \mathcal{S}$ and $k \geq 0$, define the *level set* $L_{\leq k} := F \cap (\partial F \oplus 2^k \delta)$. $L_{=k} := L_{\leq k} \setminus L_{\leq k-1}$. For $x \in \mathcal{X}$, the *level* of x , denoted $Lev(x)$, is k such that $x \in L_{=k}$. Let $L_{<k}(\mathcal{X}), L_{=k}(\mathcal{X})$ denote $L_{<k} \cap \mathcal{X}, L_{=k} \cap \mathcal{X}$ respectively. Note that for $x \in L_{=k}, k \geq 1$, the distance $d(x, \partial F) \in (2^{k-1}\delta, 2^k\delta]$. Hence, if $Lev(x) = k, D(x, 2^{k-1}\delta) \subset F$. For $k = 0, d(x, \partial F) \in [0, \delta]$.

Given $x \in F$ having $Lev(x) = k, x$ is a *boundary point* or $x \in Bd_F(\mathcal{Y})$ if $k = 0$ or if there exists an empty disk (w.r.t. \mathcal{Y}) of radius greater than $2^{k-1}\delta$, whose boundary passes through x . x is an *interior point* or $x \in Int_F(\mathcal{Y})$ if and only if $x \in \mathcal{Y} \setminus Bd_F(\mathcal{Y})$. In general, $x \in Bd_S(\mathcal{Y})$ if $x \in Bd_F(\mathcal{Y})$ for some $F \in \mathcal{S}$, and $x \in Int_S(\mathcal{Y})$ is defined similarly.

The above bi-partition induces a classification of potential edges of $Del(\mathcal{Y})$, depending on whether the end-points are boundary or interior points. Let E_1 denote the set of edges whose end points are two boundary points. Let E_2 denote the set of edges having as end-points, two interior points of the same facet of \mathcal{S} . Let E_3 denote the set of edges having as end-points, two interior points of different facets of \mathcal{S} . Let E_4 denote the set of edges having an interior point and a boundary point as end-points.

We have the following lemmas, to be proved in section 4.2.

► **Lemma 8.** $\mathbb{E} [\#(E_1)] \leq O(1) \cdot (\kappa^2 L^2 / A) \cdot s.$

► **Lemma 9.** $\mathbb{E} [\#(E_2)] \leq O(1) \cdot \kappa s.$

► **Lemma 10.** $\mathbb{E} [\#(E_3)] \leq O(1) \cdot (C - 1) \cdot \kappa s.$

► **Lemma 11.** $\mathbb{E} [\#(E_4)] \leq O(1) \cdot \frac{\kappa^2 L^2}{A} s.$

Given the above lemmas, the proof of Theorem 4 follows easily.

Proof of Theorem 4. As in [4] (Section 4), by Euler's formula, the number of tetrahedra $t(Del(\mathcal{Y}))$ in the Delaunay triangulation of \mathcal{S} , is at most $e(Del(\mathcal{Y})) - \#(\mathcal{Y}) = e(Del(\mathcal{Y})) - s$, where $e(Del(\mathcal{Y}))$ is the number of edges in the Delaunay triangulation. Therefore, it suffices to count the edges of $Del(\mathcal{Y})$. Next, observe that any point $x \in \mathcal{Y}$ is either a boundary or an interior point, that is $Bd_S(\mathcal{Y}) \sqcup Int_S(\mathcal{Y}) = \mathcal{Y}$. An edge in $Del(\mathcal{Y})$, therefore, can be either between two points in $Bd_S(\mathcal{Y})$, or two points in $Int_S(\mathcal{Y})$, or between a point in $Bd_S(\mathcal{Y})$ and another in $Int_S(\mathcal{Y})$. The case of a pair of points in $Int_S(\mathcal{Y})$ is further split based on whether the points belong to the same facet of \mathcal{S} or different facets. Thus using the above exhaustive case analysis, the proof follows simply by summing the bounds of Lemmas 8 to 11. ◀

Next, we show how Theorem 4 implies bounds on the computational complexity of constructing Delaunay triangulations of ε -nets. Our main tool shall be Theorem 2. However, we need to show first that Condition 1 holds. The standard proof of this (see e.g., [10], [9], also the discussion in [8](Section 2.2 D)) is sketched below.

Now we come to the proof of Theorem 5.

Proof of Theorem 5. To verify that Condition 1 indeed holds in the case of polyhedral surfaces, observe first that the union \mathcal{C}_p of the simplices in conflict with a new point p is a connected set. Therefore, walking on the adjacency graph of the simplices by rotating around the edge or triangle shared between two adjacent faces on the boundary of \mathcal{C}_p , is enough to yield the set of new conflicts. Now Theorem 2 can be applied to get the claimed result. ◀

4 Proofs of Lemmas 8- 11

Before proving Lemmas 8-11, we need a few technical lemmas.

4.1 Some Technical Lemmas

The following geometric and probabilistic lemmas prove certain properties of ε -nets on polyhedral surfaces and random subsets, as well as exploit the notion of boundary and interior points to get an exponential decay for boundary effects penetrating into the interior.

► **Lemma 12.** *Given $a > 0$, $b \in (0, 1)$, the sum $\sum_{n \in \mathbb{Z}_+} 2^{an} \cdot \exp(-b \cdot 2^{an})$ is at most $\frac{2 \log_2(1/b)}{eab}$.*

► **Proposition 13** ([4]). *Let F be a (convex)¹ facet of \mathcal{S} . For any convex Borel set $R \subset F$, we have*

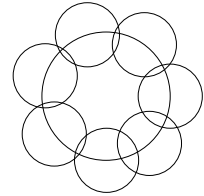
$$\left(\frac{a(R)}{4\pi\varepsilon^2} \right) \leq \#(R \cap \mathcal{X}) \leq \left(\frac{\kappa \cdot a(R \oplus \varepsilon)}{\pi\varepsilon^2} \right), \text{ and therefore,} \quad (1)$$

$$\left(\frac{A}{4\pi\varepsilon^2} \right) \leq \#(\mathcal{S} \cap \mathcal{X}) = n. \quad (2)$$

► **Proposition 14** ([4]). *Let F be a facet of \mathcal{S} , let $\Gamma \subset F$ be a curve contained in F , and $k \in \mathbb{N}$. Then*

$$\#((\Gamma \oplus k\varepsilon) \cap \mathcal{X}) \leq \kappa \frac{l(\Gamma)(2k+2)\varepsilon}{\pi\varepsilon^2} \leq \left(2k\kappa \frac{l(\Gamma)}{\varepsilon} \right), \text{ when } k \geq 1. \quad (3)$$

► **Lemma 15.** *Given a circle $\Sigma_1 \subset \mathbb{E}^2$ of unit radius centered at the origin, seven disks having centers in Σ_1 and radius $1/2$, are necessary and sufficient to cover Σ_1 .*



► **Lemma 16** (Level Size). $\#(L_{=k} \cap \mathcal{X}) \leq \#(L_{\leq k} \cap \mathcal{X}) \leq 9\kappa L \left(\frac{2^k \delta}{\varepsilon^2} \right)$.

► **Lemma 17.** *Let F be a facet of \mathcal{S} with supporting plane P , and $x \in F$ with $\text{Lev}(x) > 0$. Then given any $k \in [0, \text{Lev}(x))$, $k \in \mathbb{N}$, there exists a collection \mathcal{D}_x of at most $c_B = 7$ disks in F , such that*

- (i) Each $D \in \mathcal{D}_x$ is contained in F ,
- (ii) Each $D \in \mathcal{D}_x$ has radius $r_0/4$, where $r_0 = 2^k \delta$ and $k \in \mathbb{N}$ such that $0 \leq k < \text{Lev}(x)$, and
- (iii) Any disk $D \subset P$ of radius at least r_0 , such that $x \in \partial D$, contains at least one disk in \mathcal{D}_x .

► **Lemma 18** (Decay Lemma). *Given $x_1, \dots, x_t \in \mathcal{X}$, with x_i contained in the facet F_i with supporting plane P_i , such that $\text{Lev}(x_i) > 0$, $1 \leq i \leq t$, then for all $0 \leq k_i < \text{Lev}(x_i)$, with $r_i^* := 2^{k_i} \delta$, the probability of the event*

$$E := \{\forall i \in [t] : \exists D_i = D(y_i, r_i) \subset P_i : r_i \geq r_i^*, x_i \in \mathcal{Y}, x_i \in \partial D_i \text{ and } \text{int}(D_i) \cap \mathcal{Y} = \emptyset\},$$

¹ Recall our assumption in 2.4.

is given by

$$\mathbb{P}[E] \leq \begin{cases} q^t, & \text{if } k_{\max} = 0, \\ c_1 \cdot q^t \cdot \exp(-c_2 \cdot 2^{2k_{\max}}), & \text{if } k_{\max} > 0, \end{cases}$$

where $c_1 = c_B^t$, $c_2 \geq 2^{-7}$, and $k_{\max} := \max_i \{k_i\}$. Thus

$$\mathbb{P}[E] \leq c_1 \cdot q^t \cdot \exp(-c_2 \cdot 2^{2k_{\max}}), \quad k_{\max} \geq 0.$$

► **Lemma 19** (Growth Lemma). *Given any point $x \in \mathcal{S}$ in a facet F , and $0 \leq k < \text{Lev}(x)$, we have*

- (i) $2^{2k-2}/q \leq \#(D(x, 2^k \delta) \cap \mathcal{X}) \leq 4 \cdot (2^{2k}/q)$.
- (ii) $2^{2k-2} \leq \mathbb{E}[\#(D(x, 2^k \delta) \cap \mathcal{Y})] \leq 4 \cdot (2^{2k})$.

4.2 Proofs of Lemmas 8-11

The proofs of Lemmas 8 and 9 now follow by adapting the analysis of [4] to random subsamples of ε -nets, using the Decay and Growth lemmas.

Proof of Lemma 8. To bound the expected number of edges in E_1 , we simply bound the number of pairs $(x_1, x_2) \in \text{Bd}_{\mathcal{S}}(\mathcal{Y}) \times \text{Bd}_{\mathcal{S}}(\mathcal{Y})$. Consider a pair of points $x_1, x_2 \in \mathcal{X}$. Let $l_1 := \text{Lev}(x_1)$ and $l_2 := \text{Lev}(x_2)$, and let $l := \max\{l_1, l_2\}$. By definition, if $l = 0$, then $x_1, x_2 \in \text{Bd}_{\mathcal{S}}(\mathcal{Y})$. For $l \geq 1$, we get that $x_1 \in \text{Bd}_{\mathcal{S}}(\mathcal{Y})$ and $x_2 \in \text{Bd}_{\mathcal{S}}(\mathcal{Y})$ only if there exists a disk of radius at least $2^{l-1}\delta$ passing through x_1 or x_2 , and containing no points of \mathcal{Y} . Therefore to bound the probability that $(x_1, x_2) \in (\text{Bd}_{\mathcal{S}}(\mathcal{Y}))^2$, we can apply the Decay Lemma 18, with $t = 2$, for $i \in \{1, 2\}$. We get

$$\begin{aligned} \mathbb{P}[(x_1, x_2) \in E_1] &\leq \mathbb{P}[(x_1, x_2) \in (\text{Bd}_{\mathcal{S}}(\mathcal{Y}))^2] \\ &\leq c_1 q^2 \cdot \exp(-c_2 \cdot 2^{2l-2}) \leq c_1 q^2 \cdot \exp(-c'_2 \cdot 2^{2l}), \end{aligned} \quad (4)$$

where $c'_2 = c_2/4 = 2^{-9}$. Summing over all choices of levels of x_1 and x_2 , we have

$$\mathbb{E}[\#(E_1)] \leq \sum_{l_1 \geq 0} \#(L_{=l_1} \cap \mathcal{X}) \sum_{l_2 \geq 0} \#(L_{=l_2} \cap \mathcal{X}) \mathbb{P}[(x_1, x_2) \in (\text{Bd}_{\mathcal{S}}(\mathcal{Y}))^2].$$

By symmetry, it is enough to assume without loss of generality that $l_1 \geq l_2$, i.e. $l = l_1$. Thus,

$$\mathbb{E}[\#(E_1)] \leq 2 \sum_{l_1 \geq 0} \#(L_{=l_1} \cap \mathcal{X}) \sum_{l_2=0}^{l_1} \#(L_{=l_2} \cap \mathcal{X}) \mathbb{P}[(x_1, x_2) \in (\text{Bd}_{\mathcal{S}}(\mathcal{Y}))^2].$$

Applying equation (4) and the Level Size Lemma 16, we get

$$\begin{aligned} \mathbb{E}[\#(E_1)] &\leq 2 \sum_{l_1 \geq 0} \#(L_{\leq l_1} \cap \mathcal{X}) \sum_{l_2=0}^{l_1} \#(L_{\leq l_2} \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 \cdot 2^{2l_1}) \\ &\leq 2c_1 q^2 \sum_{l_1 \geq 0} (9\kappa L \cdot (2^{l_1} \delta / \varepsilon^2)) \sum_{l_2=0}^{l_1} (9\kappa L \cdot (2^{l_2} \delta / \varepsilon^2)) \cdot \exp(-c'_2 \cdot 2^{2l_1}) \\ &\leq 2c_1 q^2 (9\kappa L \left(\frac{\delta}{\varepsilon^2}\right))^2 \sum_{l_1 \geq 0} 2^{l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}) \sum_{l_2=0}^{l_1} 2^{l_2}. \end{aligned}$$

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Using the definitions of q and δ , together with Proposition 13, and writing the terms outside the summation as N_1 , we get $N_1 := 2 \cdot c_1 q^2 (9\kappa L (\frac{\delta}{\varepsilon^2}))^2 = 2c_1 \cdot (9\kappa L)^2 (\frac{s}{n\varepsilon^2}) \leq 4c_1 \cdot (\frac{4\pi(9\kappa L)^2}{A}) \cdot s$. We get

$$\mathbb{E} [\#(E_1)] \leq N_1 \sum_{l_1 \geq 0} 2^{l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}) \cdot 2 \cdot 2^{l_1} \leq 2N_1 \sum_{l_1 \geq 0} 2^{2l_1} \cdot \exp(-c'_2 \cdot 2^{2l_1}).$$

The summation can be bounded using Lemma 12, to get

$$\mathbb{E} [\#(E_1)] \leq 2N_1 \cdot \left(2 \cdot \frac{\log 1/c'_2}{2ec'_2}\right) = 2N_1 \cdot \frac{\log 1/c'_2}{e \cdot c'_2}.$$

Now substituting $c'_2 = 2^{-9}$ gives $\mathbb{E} [\#(E_1)] \leq 2 \cdot 10^4 \cdot c_1 \cdot (\frac{4\pi(9\kappa L)^2}{A}) \cdot s$. \blacktriangleleft

The proof of Lemma 9 follows simply from the fact that for any given face, the Delaunay graph formed by the points in \mathcal{Y} is planar, and therefore the number of edges is at most 3 times the number of points. The total number of such edges, summed over all faces of \mathcal{S} , is at most $3s$. The proof of Lemma 10 is based on combining a construction of Attali-Boissonnat with Lemma 9, and is omitted here. For the proofs of Lemmas 10 and 11, we need some more geometric ideas of [4]. Before proving Lemma 11, we briefly describe a construction, which will be central to our analysis.

\triangleright **Construction 20** (Attali-Boissonnat [4]). Let P be a plane and Z be a finite set of points. To each point $x \in Z$, assign the region $V(x) = V_x(Z) \subset P$ of points $y \in P$ such that the sphere tangent to P at y and passing through x encloses no point of Z . Let $\mathcal{V} := \{V(x) : x \in Z\}$.

We summarize some conclusions of Attali-Boissonnat regarding the construction. The proofs of these propositions can be found in [4].

- \blacktriangleright **Proposition 21.** (i) \mathcal{V} is a partition of P .
- (ii) For each $x \in Z$, $V(x)$ is an intersection of regions that are either disks or complements of disks.
- (iii) The total length of the boundary curves in \mathcal{V} is equal to the total length of the convex boundaries.

For the rest of this subsection, we shall apply Construction 20 on the plane P , and the points in $Bd_{\mathcal{S}}(\mathcal{Y})$ as Z . Let $\mathcal{T} := \text{Int}_F(\mathcal{Y})$ for some facet $F \in \mathcal{S}$. Given $x \in Z$, $y \in P \setminus V(x)$, let $k_y = k_y(x)$ denote the least $k \geq 0$ such that $y \in \partial V(x) \oplus 2^k \delta$.

\blacktriangleright **Proposition 22** (Attali-Boissonnat [4]). Suppose there exists a ball $B \subset \mathbb{E}^3$ and $y \in P$, such that $y, x \in \partial B$, and $B \cap \mathcal{T} = \emptyset$. Then the disk $D_y = P \cap B$ satisfies $D_y \cap \mathcal{T} = \emptyset$, $y \in \partial D_y$ and $D_y \cap V_x \neq \emptyset$.

\blacktriangleright **Lemma 23.** If $\{x, y\} \in E_4$ with $x \in Bd_{\mathcal{S}}(\mathcal{Y})$, $y \in \text{Int}(F)$, then $k_y \leq \text{Lev}(y)$.

Proof. Suppose $\{x, y\} \in E_4$. Then there exists a ball $B \in \mathbb{E}^3$ with $x, y \in \partial B$, and $\text{int}(B) \cap \mathcal{Y} = \emptyset$. Therefore $D_y := B \cap P$ also satisfies $\text{int}(D_y) \cap \mathcal{Y} = \emptyset$. By Proposition 22 we have that $D_y \cap V(x) \neq \emptyset$. Therefore, $y \in V(x) \oplus 2r_y$, where r_y is the radius of D_y . But since $y \in \text{Int}(F)$, we have that any disk having y on its boundary and containing no point of \mathcal{Y} in its interior can have radius at most $2^{\text{Lev}(y)-1} \delta$. Therefore $r_y \leq 2^{\text{Lev}(y)-1} \delta$. Now taking k_y such that $2^{k_y} \delta = 2r_y$, we get that $k_y \leq \text{Lev}(y)$. \blacktriangleleft

Now we partition the pairs of vertices $\{x, y\} \in E_4$ with $x \in Bd_S(\mathcal{Y})$, depending on whether $y \in V_F(x)$ or $y \in \partial V_F(x) \oplus 2^{k_y}\delta$. That is, given a facet $F \in \mathcal{S}$, let $E_4(Int(F))$ denote the set of edges $\{x, y\} \in E_4$ with $y \in int(V_F(x))$, and $E_4(Bd(F))$ denote the set of edges in E_4 with $y \in \partial(V_F(x)) \oplus 2^k\delta$, for $k \in [0, k_y]$. Define $E_4(Int) := \bigcup_{F \in \mathcal{S}} E_4(Int(F))$ and $E_4(Bd) := \bigcup_{F \in \mathcal{S}} E_4(Bd(F))$ respectively.

Lemma 11. The proof follows from Lemmas 24 and 25, which bound the expected number of edges in $E_4(Int)$ and $E_4(Bd)$ respectively. \blacktriangleleft

► **Lemma 24.** Given a facet $F \in \mathcal{S}$, $\mathbb{E}[E_4(Int(F))] \leq q \cdot \sharp(\mathcal{X} \cap F)$. As a consequence, $\mathbb{E}[E_4(Int)] \leq s$.

Proof. Let $x \in \mathcal{X}$ and $y \in \mathcal{X} \cap F$. Let $\mathcal{E}_{x,y}$ denote the event $\{x, y\} \in E_4(Int(F))$. Then $\mathcal{E}_{x,y}$ can occur only if (i) $x \in Bd_S(\mathcal{Y})$ and, (ii) $y \in Int_S(Y) \cap V_F(x)$. Fix a choice of \mathcal{Y} , say $Y \in \binom{\mathcal{X}}{s}$. Conditioning on this choice of \mathcal{Y} , $Bd_S(\mathcal{Y})$ is a fixed set of points. The number of pairs contributing to $E_4(Int(F))$ is at most $\sharp(\{(x, y) \in Y \times Y \mid x \in Bd_S(\mathcal{Y}), y \in V_F(x)\})$. The main observation is now that since \mathcal{V} restricted to F is a sub-division of F , for each $y \in \mathcal{X} \cap F$, there is a unique $x = x_y \in Bd_S(\mathcal{Y})$ such that $y \in V_F(x)$. Therefore we get

$$E_4(Int(F)) \leq \sum_{V_F(x) \in \mathcal{V}: x \in Bd_S(\mathcal{Y})} \sharp(V_F(x) \cap Y) \leq \sharp(Y \cap F).$$

Since the last bound holds for any choice of Y , taking expectation over all choices we get

$$\mathbb{E}[E_4(Int(F))] \leq \mathbb{E}[\sharp(\mathcal{Y} \cap F)] = q \cdot \sharp(\mathcal{X} \cap F).$$

Now summing over all faces gives $\mathbb{E}[E_4(Int)] \leq \mathbb{E}[\sharp(\mathcal{Y})] = s$. \blacktriangleleft

► **Lemma 25.** Given a facet $F \in \mathcal{S}$, $\mathbb{E}[E_4(Bd(F))] \leq \left(O(1) \cdot \frac{\kappa^2 L \cdot l(\partial F) s}{A}\right)$. As a consequence, $\mathbb{E}[E_4(Bd(\mathcal{S}))] \leq \left(O(1) \cdot \frac{\kappa^2 L^2 s}{A}\right)$.

Proof. To compute the expected value of $E_4(Bd(\mathcal{S}))$, fix a face $F \in \mathcal{S}$. Consider a pair of points $x, y \in \mathcal{X}$, such that $y \in F$. Let $\mathcal{E}_{x,y}$ denote the event $\{x, y\} \in E_4(Bd(F))$.

The value of $E_4(Bd)$ is the number of $x, y \in \mathcal{X}$, such that $\mathcal{E}_{x,y}$ occurs. Taking expectations,

$$\mathbb{E}[E_4(Bd(F))] \leq \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{X} \cap F} \mathbb{P}[\mathcal{E}_{x,y}]. \quad (5)$$

Observe that $\mathcal{E}_{x,y}$ occurs only if (i) $x \in Bd_S(\mathcal{Y})$ and (ii) $k_y(x) \leq Lev(y)$, by applying Construction 20, on the plane P , $Z = Bd_S(\mathcal{Y})$, and $\mathcal{T} = \mathcal{Y} \cap P$, and using Proposition 22. By Lemma 23, $k_y(x) \in [0, Lev(y)]$.

Let P_{l_1, l_2} denote the probability that $\{x, y\} \in E_4(Bd(F))$, with $Lev(x) = l_1$, and $k_y(x) = l_2$. Equation (5) can be rewritten in terms of l_1 and l_2 as

$$\mathbb{E}[E_4(Bd(F))] \leq \sum_{l_1 \geq 0} \sharp(L_{=l_1} \cap \mathcal{X}) \sum_{l_2 \geq 0} \sum_{V_F \in \mathcal{V}} \sharp((\partial V_F \oplus 2^{l_2}\delta) \cap \mathcal{X}) \cdot P_{l_1, l_2}.$$

Applying the Decay Lemma 18 with $t = 2$, $x_1 = x$, $x_2 = y$, $k_1 = \max\{0, l_1 - 1\}$ (since $x \in Bd_S(\mathcal{Y})$), and $k_2 = \max\{0, l_2 - 1\}$, we get

$$P_{l_1, l_2} \leq c_1 q^2 \cdot \exp(-f(l^*)),$$

where $l^* := \max\{0, l_1 - 1, l_2 - 1\}$, and $f(l^*) = 0$ if $l^* = 0$, and $c'_2 \cdot 2^{2l^*}$ otherwise, with $c'_2 = c_2/4$. As in the proof of Lemma 8, we shall use symmetry to combine the cases $l_1 \geq l_2$ and $l_2 > l_1$ together.

$$\mathbb{E}[E_4(Bd(F))] \leq 2 \sum_{l_1 \geq 0} \#(L_{=l_1} \cap \mathcal{X}) \sum_{l_2 \leq l_1} \sum_{V(x) \in \mathcal{V}} \#((\partial V(x) \oplus 2^{l_2} \delta) \cap \mathcal{X}) \cdot c_1 q^2 \cdot \exp(-c'_2 2^{2l_1}).$$

By the Level Size Lemma 16, we get that $\#(L_{=l_1} \cap \mathcal{X}) \leq \frac{2\kappa L 2^{l_1} \delta}{\varepsilon^2}$. Using Proposition 14, we get that $\#(\{\partial V(x) \oplus 2^{l_2} \delta\} \cap \mathcal{Y}) \leq \frac{2\kappa \cdot l(\partial V(x)) 2^{l_2} \delta}{\varepsilon^2}$. By Proposition 21 (iii), each boundary in the partition \mathcal{V} is convex for some $x \in Bd_{\mathcal{S}}(\mathcal{Y})$, and so we need to sum $l(\partial V(x))$ only over the convex curves in $\partial V(x)$, $x \in Bd_{\mathcal{S}}(\mathcal{Y})$, whose length we observe is at most $l(\partial F)$. Thus,

$$\mathbb{E}[E_4(Bd(F))] \leq 2L \cdot l(\partial F) \cdot \left(\frac{2\kappa \delta q}{\varepsilon^2}\right)^2 \sum_{l_1 \geq 0} 2^{l_1} \sum_{l_2 \leq l_1} 2^{l_2} c_1 \cdot \exp(-c'_2 \cdot 2^{2l_1}).$$

Using Lemma 12, the above summation is bounded by a constant. This comes to $c_1 \cdot O(1) \left(\frac{(2\kappa)^2 L \cdot l(\partial F) \delta^2 q^2}{\varepsilon^4}\right) = O\left(\frac{\kappa^2 L \cdot l(\partial F) s}{A}\right)$, where the last step followed from the lower bound on n in Proposition 13 (2), and the identities $q = s/n = \delta^2/\varepsilon^2$. Summing y over all facets F in \mathcal{S} , we get $\mathbb{E}[E_4(Bd(\mathcal{S}))] = \left(O(1) \cdot \frac{\kappa^2 L^2 s}{A}\right)$. \blacktriangleleft

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