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A Small Dice Game for the Kingman Formula

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Abstract. There are three main factors influencing the waiting time of a single-arrival single-process system: the utilization, the fluctuation of the arrival, and the fluctuation of the process time. The influence of these is not linear, and the combination of these effects is worse than the individual sums. Different approximations exist for this relation, the most popular one being probably the Kingman equation. Now it is one thing to understand this in theory, but experiencing this in practice makes it much easier to understand and will prepare practitioners much better for its effect. This paper describes a quick and easy game to have the practitioners experience the individual and combined effects of both utilization and fluctuation.

Keywords: Kingman equation, utilization, variation, waiting time

1 Introduction

In manufacturing, or actually in most processing systems, the waiting time for the objects to be processed is often significant. In a supermarket, the waiting time of the customer is relevant for customer satisfaction [1]. In manufacturing, the waiting time of parts is a major contributor to the lead time. Most real live systems have a network of multiple processes and parts with varying utilizations and fluctuations. Often, the distribution of the inter-arrival time is the result of the output behavior of the preceding processes. Such complex systems can be simulated, but their relations are often difficult to grasp by humans and usually also difficult to analyze in queueing theory.

However, to understand the principles behind it, it is helpful to look in more detail at single-arrival single-departure processes, also known as single-server queue. This is often abbreviated as G/G/1 queue in Kendal's notation, where the G stands for a generally random distribution of the inter-arrival and service time, and 1 stands for a single process or server [2]. Such a system is visualized in Fig. 1.

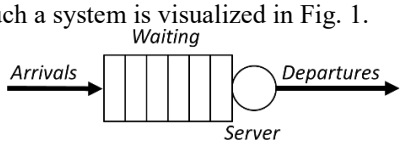

Fig. 1. Illustration of the general G/G/1 queue with arrivals, waiting objects, and server

Exact solutions of the relation between arrivals, departures, and waiting time are available for selected random distributions. Best known is the M/M/1 queue with exponentially distributed inter-arrival and service times. The average waiting time *E(W)* for a steady state system is a function of the parameter of the arrival distribution p and the service distribution μ as shown in equation (1) if the mean inter-arrival time exceeds the mean process time [3].

$$E(W) = \frac{\frac{p}{\mu}}{1-p} \tag{1}$$

For the general G/G/1 queue, there exist different approximations. The most common one is the Kingman approximation as shown in equation (2). Here, p stands for the utilization of the server (i.e., the mean arrival time μA divided by the mean service time μS). CA and CS are the coefficients of variation (standard deviation divided by the mean) of the arrival distribution and the service distribution [4].

$$E(W) = \frac{p}{1-p} \cdot \left(\frac{C_A^2 + C_S^2}{2}\right) \cdot \mu_s \tag{2}$$

Please note that this is only an approximation, with the restrictions that it is only valid for higher utilization, that the utilization is below 100%, and that the arrival and service times need to be independently distributed. Other approximations exist like [5] or [6], but for our purposes, the Kingman equation will suffice, especially since it shows the main effects clearly.

1.1 Effect of Utilization

The effect of utilization on the waiting time is represented by the first part of equation (2). As the utilization p approaches 100%, the waiting time approaches infinity. Please note that this is not a linear effect. This is also graphically visualized in Fig. 2. Please note that the y-axis has intentionally no labels, as this would depend on a specific situation. Please also note that the Kingman equation is not very accurate for low utilization.

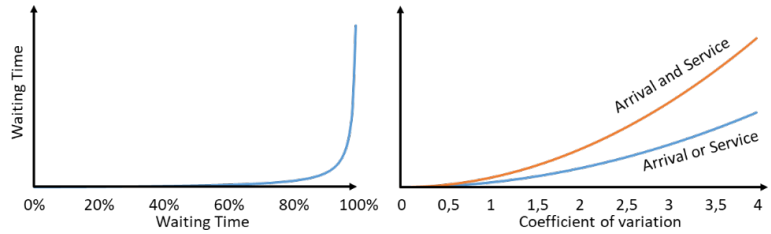

Fig. 2. Illustration of the general behavior of the effect of utilization and coefficient of variation on the waiting time

1.2 Effect of Variance

The middle part of equation (2) represents the effect of the variance of the arrival and service times. The increase of the waiting time is also not linear but squared to the coefficient of variation. Fig. 2 shows the effect of increasing the coefficient of variation of either the arrival or departures in the lower line. The effect of increasing both arrival

and departure variation simultaneously is shown in the upper line, which is simply double of the lower line.

1.3 Joint Effect of Utilization and Variance

Please note that the joint effect of utilization and variance is not the sum but the product of the two individual effects. Hence, if both utilization and variance increase, the waiting time will increase significantly more.

2 Game Objectives

It is one thing to know the Kingman equation in theory. It is something else to experience this relation in practice. The exponential behavior of increasing either the variation or the utilization, and especially the multiplicative behavior of a joint increase, is hard to internalize for practitioners.

The target group is people who are working or will be working with process systems. This includes shop floor supervisors, production planning staff, managers, and students of engineering or management. The game aims to teach its participants the severity of the problem of having both high utilization and high waiting time. During the game, the participants will guess the magnitude of the change due to an increase in utilization or variation. The resulting actual outcome usually far exceeds these estimates, improving the learning experience.

3 Game Outline

The game is an extended version of the number game found in [7], which shows the effect of utilization. In this original game, the effect of increasing utilization onto the system is experienced using normal six-sided (D6) dice. This game extends the original game to also show the effect of variation and the combined effect.

The game can be performed within thirty minutes and is hence a good warm-up for, as an example, a full-day or multi-day training. The game can handle a wide range of participants but ideally has around six to thirty participants split into groups of two. In each group one person represents the arrival, the other represents the service. Depending on the industry, these can be renamed as supplier and customer for manufacturing, customer and check out for supermarket cash registers, etc. as needed. The randomness is represented by dice throws. To model different variations, dice with different numbers are needed as shown in Fig. 3. The game will be explained using four-sided, twelve-sided, and twenty-sided dice, abbreviated ad D4, D12, and D20, but this can easily be adapted to other dice sizes. For each number, multiple dice are needed throughout the game. Ideally, there is one dice available in each size per person, but if there are not enough dice, one dice per team is also sufficient.

Fig. 3. Dice with different numbers. Top row from left to right D4, D6, D8, bottom row D10, D12, and D20. D30 dice are also commercially available.

3.1 First Game: Baseline System

The game is played in a total of six games, each with twenty rounds between the arrival and the service. In the first game, the player uses D4 dice. Since not everybody may be familiar with the unusual shape of the dice, inform the participants that the number on top is the number of the throw. In Fig. 3, this would be a 4. Both the arrival and the service process throw the dice. The arrival adds 8 to the throw, and the service adds 10. This is a Δ of 2. Since the average throw of a D4 is 2.5, this means that in each round, 10.5 items are arriving and 12.5 items can be processed. This gives an average utilization of 84%.

If by chance more parts are arriving than can be serviced, the remaining parts are the queue. This is written down on a data sheet as shown in Fig. 4 on the left. In the first round, the service exceeded the arrival. In the second round, arrival threw a 4 and service a 1, hence one item remained in the queue to be processed at the next round. Since arrival and service threw 4 and 2 respectively, the queue remained at length 1. Only in the fourth round was the service able to reduce the queue to zero again. This is repeated for twenty rounds, and the sum, as well as the average of the queue length, is calculated.

The expected outcome based on two hundred simulations is around 0.06 with a standard deviation of also 0.06. The results of your game will, of course, be a different number for each team, somewhere in that range. In the game, however, you do not know the exact numbers but merely get a slightly different result from every team. The results of every team are marked on a chart to convey an idea of the range of expected results, and an estimated mean of the results is highlighted. Please be aware that the numbers will become quite high as indicated in Fig. 4. Yet, adding a scale from the beginning gives participants clues on the expected result and diminishes the learning experience.

3.2 Second Game: Increase Utilization

In the next round, we keep the D4 dice but reduce the Δ to 1. Arrival adds 8, but service only adds 9. The expected utilization is now 91.3%. Repeating the twenty rounds will result in an average queue length of 0.5 with a standard deviation of 0.5. Again the results of the different teams are added to the chart.

3.3 Third Game: Increase Utilization to 100%

In the next game, we keep the D4 dice but reduce the Δ to 0. Both arrival and service add 8. The expected utilization is now 100%. Before playing this game, the participants should guess the expected outcome. Most will assume some linear relationship and guess around 1, vastly underestimating the true outcome. In the long term this would result in an infinite queue, but since in this game we play only twenty rounds, we expect an average queue length of 3 with a standard deviation of 2.2. Emphasize the nonlinear relationship by adding a curve through the tree means.

3.4 Fourth Game: Increase Variation by using D12

In this game, we now use a D12 dice and add 4 to the arrival and 6 to the service. The Δ is now again 2 as in the first game. The expected mean value of a D12 is 6.5, hence there will be in average 10.5 parts arriving and 12.5 serviced. Please note that these averages are identical with the first game, and the utilization is therefore also 84%. Only the variation around the mean has increased. After twenty rounds again, the results of the teams are added to a chart. The results are expected to have a mean of 3.2 with a standard deviation of 3.2.

Round	1	2	3	4	5	6
Dice	D4					
+Arriving	8					
+Check Out	10					
1	0					
2	1					
3	1					
4	0					
5	0					
6	0					
7	0					
8	0					
9	0					
10	1					
11	1					
12	0					
13	0					
14	0					
15	0					
16	1					
17	0					
18	0					
19	0					
20	0					
Sum	5					
Ø	0-25					

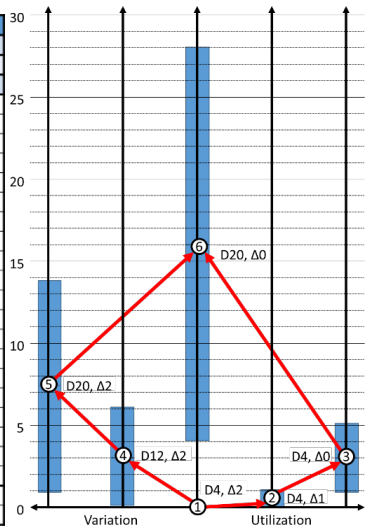

Fig. 4. Datasheet for the dice game with the first column filled out on the left and expected mean results for the six games with boxes for ± 1 standard deviations on the right

3.5 Fifth Game: Increase Variation by using D20

The fifth game uses a D20. To get the same average arrivals of 10.5 and service of 12.5 with a utilization of 84% as in game 1 and 4, we add nothing to the arrivals and 2 to the service dice throw. Before playing the game, have the participants guess the expected

outcome. Most of them again would assume a linear relationship and vastly underestimate the actual results, which are expected to have a mean 7.6 with a standard deviation of 6.06. Again, highlight the nonlinear relationship in the graph.

3.6 Sixth Game: Increase Variation to D20 and Utilization to 100%

In the last game, we combine the high variance with a high utilization. Both the arrival and the service get a D20 dice and cannot add anything to their dice throw, giving a Δ of 0 and hence a utilization of 100%. The participants should guess the expected outcome. Again, the participants vastly underestimate the outcome due to the multiplicative relationship of the effects. The results are expected to have a mean of 15.9 with a standard deviation of 11.9.

4 Discussion

To emphasize the key learnings of the nonlinear effect of both the utilization and the fluctuations and especially the multiplicative effect of both the game is closed with a discussion and review round. The moderator should emphasize and point out this nonlinear effect, and how the continuously larger numbers surprised the participants. Depending on the mathematical skills of the group, the Kingman equation can be introduced. Understanding this behavior is important for practitioners to define a production system. Often, inexperienced planners plan for a utilization of 100% and ignore the effect of variance. As a result, waiting times increase and therefore costs go up.

5 Participants Experience and Outcomes

This quick exercise is usually a surprising eye-opener for the participants, as shown by comments like "*I never though it would get so high*". After the initial first game, participants consistently underestimate the effect of the changes. The participants usually assume a linear relation and underestimate the effect of the third game by estimating a queue length of less than 1 instead of the actual 3. The fifth game is usually only slightly underestimated with a predicted value of around 6 instead of the actual 7.6. For the combined effect of the sixth game, participants merely add the two effects and predict a queue of around 10, whereas the actual result is almost 16.

6 Summary

The game lets the participants experience the nonlinear effects of utilization, variation, and the nonlinear combination of both. Depending on the qualification of the participants, the Kingman equation may be introduced, but this is optional. We have played this game with different groups on different continents and have obtained consistent results of the participants being surprised by the magnitude of the effect. This game is

a good warm-up for many trainings in the field of process optimization or lean manufacturing. Datasheets and more statistics can be downloaded at http://www.allaboutlean.com/dice-game-kingman-formula/.

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