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# A forward-backward stochastic analysis of diffusion flows

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## Abstract

We present forward-backward stochastic semigroup formulae to analyse the difference of two diffusion flows driven by different drift and diffusion functions. These formulae are expressed in terms of the tangent (or gradient) and Hessian processes and can be interpreted as an extension of the Aleksev-Gröbner lemma to diffusion flows. We present two different approaches. The first one is based on a natural extension of the two-sided stochastic integration calculus introduced by Pardoux and Protter [38] to interpolation stochastic flows. The second one is based on a new backward version of the generalized Itô-Ventzell formula by Ocone and Pardoux [37], yielding what seems to be the first results of this type for this class of anticipative models.

We also present some natural spectral conditions that yield direct and simple proofs of time uniform estimates of the difference between the two flows when their drift and diffusion functions are not the same. We illustrate the impact of these results in the context of diffusion perturbation theory, interacting diffusions and discrete time approximations.

*Keywords* : Stochastic flows, variational equations, tangent and Hessian processes, perturbation semigroups, backward Itô-Ventzell formula, Aleksev-Gröbner lemma, Skorohod stochastic integral, Malliavin differential, Bismut-Elworthy-Li formulae.

*Mathematics Subject Classification* : 47D07, 93E15, 60H07.

## 1 Introduction

Let  $b_t(x)$  be a vector-valued function from  $\mathbb{R}^d$  into  $\mathbb{R}^d$  and  $\sigma_t(x) = [\sigma_{t,1}(x), \dots, \sigma_{t,r}(x)]$  be a matrix-valued function from  $\mathbb{R}^d$  into  $\mathbb{R}^{d \times r}$ , for some parameters  $d, r \geq 1$ . Both functions will be assumed to be differentiable. Let  $W_t$  be an  $r$ -dimensional Brownian motion and denote by  $\mathcal{W}_{s,t}$  the  $\sigma$ -field generated by the increments  $(W_u - W_v)$  of the Brownian motion, with  $u, v \in [s, t]$ .

For any time horizon  $s \geq 0$  we denote by  $X_{s,t}(x)$  be the stochastic flow defined for any  $t \in [s, \infty[$  and any starting point  $X_{s,s}(x) = x \in \mathbb{R}^d$  by the stochastic differential equation

$$dX_{s,t}(x) = b_t(X_{s,t}(x)) dt + \sigma_t(X_{s,t}(x)) dW_t \quad (1.1)$$

We assume that  $b_t(x)$  and  $\sigma_t(x)$  have continuous and uniformly bounded derivatives up to the third order. This condition is clearly met for linear Gaussian models as well as for the geometric Brownian motion. This condition ensures that the stochastic flow  $x \mapsto X_{s,t}(x)$  is a twice differentiable function

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of the initialisation  $x$ . In addition, all absolute moments of the flow and the ones of its first and second order derivatives exists for any time horizon. As it is well known, dynamical systems and hence stochastic models involving drift functions with quadratic growth require additional regularity conditions to ensure non explosion of the solution in finite time. It is also implicitly assumed that all functions  $(b_t, \sigma_t)$  are smooth functions w.r.t. the time parameter.

Let  $\overline{X}_{s,t}(x)$  be the stochastic flow associated with a stochastic differential equation defined as (1.1) by replacing  $(b_t, \sigma_t)$  by some drift and diffusion functions  $(\overline{b}_t, \overline{\sigma}_t)$  with the same regularity properties. Constant diffusion functions  $(\sigma_t, \overline{\sigma}_t)$  are defined by

$$\sigma_t(x) = \Sigma_t \quad \text{and} \quad \overline{\sigma}_t(x) = \overline{\Sigma}_t \quad \text{for some matrices } \Sigma_t \text{ and } \overline{\Sigma}_t. \quad (1.2)$$

In this context, we will assume that  $\Sigma_t$  and  $\overline{\Sigma}_t$  are uniformly bounded w.r.t. the time horizon.

The Markov transition semigroups associated with the flows  $X_{s,t}(x)$  and  $\overline{X}_{s,t}(x)$  are defined for any measurable function  $f$  on  $\mathbb{R}^d$  by the formula

$$P_{s,t}(f)(x) := \mathbb{E}(f(X_{s,t}(x))) \quad \text{and} \quad \overline{P}_{s,t}(f)(x) := \mathbb{E}(f(\overline{X}_{s,t}(x)))$$

In this paper we derive equations for the differences  $(X_{s,t} - \overline{X}_{s,t})$  and  $(P_{s,t} - \overline{P}_{s,t})$  in terms of the difference of their corresponding drifts and diffusion functions,

$$\Delta a_t := a_t - \overline{a}_t \quad \Delta b_t := b_t - \overline{b}_t \quad \text{and} \quad \Delta \sigma_t = \sigma_t - \overline{\sigma}_t \quad (1.3)$$

where  $a_t(x) := \sigma_t(x) \sigma_t(x)'$  and  $\overline{a}_t(x) := \overline{\sigma}_t(x) \overline{\sigma}_t(x)'$ . In some applications the functions  $\overline{b}_t = b_t - \Delta b_t$  and  $\overline{\sigma}_t = \sigma_t - \Delta \sigma_t$  can be interpreted as a local perturbation of the drift and the diffusion of the stochastic flow  $X_{s,t}$ .

We also address the problem of finding time-uniform estimates for the difference between the stochastic flows  $X_{s,t}$  and  $\overline{X}_{s,t}$  and their corresponding Markov transition kernels  $P_{s,t}$  and  $\overline{P}_{s,t}$ .

These important questions arise in a variety of domains including stochastic perturbation theory as well as in the stability and the qualitative theory of stochastic systems. Classical analytic estimates on the difference between the stochastic flows driven by different drift and diffusion functions are often much too large for most diffusion processes of practical interest. In some instances none of the diffusion flows are stable. In this context, any local perturbation of the stochastic model propagates so that any global error estimate eventually tends to  $\infty$  as the time horizon  $t \rightarrow \infty$ .

Whenever one of the stochastic flows is stable, classical perturbation bounds combining Lipschitz type inequalities with Gronwall lemma [8, 23] yield exceedingly pessimistic global estimates that grows exponentially fast w.r.t. the time horizon. Notice that an exponential type estimate of the form  $e^{\lambda t}$  for some parameter  $\lambda > 0$  and some time horizon  $t$  s.t.  $\lambda t \geq 199$  would induce an error bound larger than the estimated number  $10^{86}$  of elementary particles of matters in the visible universe. As mentioned in [27] in the context of Euler scheme type approximations of deterministic dynamical systems, one may encounter situations where  $\lambda = 10^8$  and  $t = 10^2$  and the resulting exponential bounds are clearly impractical from a numerical perspective.

The main contributions of the article are as follows.

- i. We present a forward-backward stochastic perturbation formula that expresses the difference between the stochastic flows  $X_{s,t}$  and  $\overline{X}_{s,t}$  and their transition probabilities in terms of first and second order derivatives of the flows, which we call the tangent and Hessian processes respectively, with respect to the space parameter.
- ii. We extend the two-sided stochastic calculus introduced by Pardoux and Protter in [38] to stochastic interpolation flows. We also present a novel backward version of the generalized Itô-Ventzell formula introduced by Ocone and Pardoux in [37].

- iii. We provide some natural spectral conditions on the gradients of  $b_t(x)$ ,  $\sigma_t(x)$ ,  $\bar{b}_t(x)$  and  $\bar{\sigma}_t(x)$  that allows us to derive in a direct way a series of realistic uniform estimates with respect to the time horizon.
- iv. We illustrate the impact of these results in the context of diffusion perturbation theory, interacting diffusions and discrete time approximations.

### 1.1 Statement of some main results

We represent the gradient of a real valued function of several variables as a column vector while the gradient and the Hessian of a (column) vector valued function as tensors of type  $(1, 1)$  and  $(2, 1)$ , see for instance (2.2) and (2.3); in more layman terms a  $(1, 1)$  tensor is a matrix while the  $(2, 1)$  tensor can be visualized as a “row of matrices”  $[A_1, \dots, A_n]$  where the entries  $A_i$  are matrices of a common dimension. We also use the tensor product and the transpose operator defined in (2.1), see also (2.4).

The diffusion flow (1.1) is defined in term of a column vector with twice continuously differentiable entries. For  $h \simeq 0$  we use the backward approximation:

$$\begin{aligned}
X_{s,t}(x) - X_{s-h,t}(x) &= X_{s,t}(x) - (X_{s,t} \circ X_{s-h,s})(x) \\
&\simeq X_{s,t}(x) - X_{s,t}(x + b_s(x) h + \sigma_s(x) (W_s - W_{s-h})) \\
&\simeq - \left[ \left( \nabla X_{s,t}(x)' b_s(x) + \frac{1}{2} \nabla^2 X_{s,t}(x)' a_s(x) \right) h + \nabla X_{s,t}(x)' \sigma_s(x) (W_s - W_{s-h}) \right]
\end{aligned} \tag{1.4}$$

In the above display,  $X_{s,t} \circ X_{s-h,s}$  stands for the composition of the mappings  $X_{s,t}$  and  $X_{s-h,s}$ . The above approximations are rigorously justified in section 4.2 and lead to the backward stochastic flow evolution equation:

$$d_s X_{s,t}(x) = - \left[ \left( \nabla X_{s,t}(x)' b_s(x) + \frac{1}{2} \nabla^2 X_{s,t}(x)' a_s(x) \right) ds + \nabla X_{s,t}(x)' \sigma_s(x) dW_s \right] \tag{1.5}$$

In the above display,  $d_s X_{s,t}^i(x)$  represents the change in  $X_{s,t}^i(x)$  w.r.t. the variable  $s$ , and  $\text{Tr}(A)$  stands for the trace of a matrix  $A$ .

In the same vein, for any  $s < u < t$  we have the interpolating semigroup decompositions

$$\begin{aligned}
&X_{u+h,t} \circ \bar{X}_{s,u+h} - X_{u,t} \circ \bar{X}_{s,u} \\
&= (X_{u+h,t} - X_{u,t}) \circ \bar{X}_{s,u} + (X_{u+h,t} \circ \bar{X}_{s,u+h} - X_{u+h,t} \circ \bar{X}_{s,u})
\end{aligned}$$

as well as the forward approximations

$$\begin{aligned}
&X_{u+h,t} (\bar{X}_{s,u}(x) + (\bar{X}_{s,u+h}(x) - \bar{X}_{s,u}(x))) - X_{u+h,t}(\bar{X}_{s,u}(x)) \\
&\simeq (\nabla X_{u+h,t}) (\bar{X}_{s,u}(x))' (\bar{X}_{s,u+h}(x) - \bar{X}_{s,u}(x)) + \frac{1}{2} (\nabla^2 X_{u+h,t}) (\bar{X}_{s,u}(x))' \bar{a}_u(\bar{X}_{s,u}(x)) h
\end{aligned} \tag{1.6}$$

The above approximations are rigorously justified in section 4.2 and lead to the forward-backward stochastic interpolation equation

$$\begin{aligned}
&d_u (X_{u,t} \circ \bar{X}_{s,u}) (x) \\
&= (d_u X_{u,t}) (\bar{X}_{s,u}(x)) + (\nabla X_{u,t}) (\bar{X}_{s,u}(x))' d_u \bar{X}_{s,u}(x) + \frac{1}{2} (\nabla^2 X_{u,t}) (\bar{X}_{s,u}(x))' \bar{a}_u(\bar{X}_{s,u}(x)) du
\end{aligned} \tag{1.7}$$

Notice that (1.5) can also be deduced formally from (1.7) by replacing  $\bar{X}_{s,u}$  by the stochastic flow  $X_{s,u}$  in (1.7), and then letting  $s = u$ .

This yields basically the following interpolation theorem.

**Theorem 1.1.** *We have the forward-backward stochastic interpolation formula*

$$X_{s,t}(x) - \bar{X}_{s,t}(x) = T_{s,t}(\Delta a, \Delta b)(x) + S_{s,t}(\Delta \sigma)(x) \quad (1.8)$$

with the stochastic process

$$\begin{aligned} & T_{s,t}(\Delta a, \Delta b)(x) \\ & := \int_s^t \left[ (\nabla X_{u,t})(\bar{X}_{s,u}(x))' \Delta b_u(\bar{X}_{s,u}(x)) + \frac{1}{2} (\nabla^2 X_{u,t})(\bar{X}_{s,u}(x))' \Delta a_u(\bar{X}_{s,u}(x)) \right] du \end{aligned} \quad (1.9)$$

and the fluctuation term given by the Skorohod stochastic integral

$$S_{s,t}(\Delta \sigma)(x) := \int_s^t (\nabla X_{u,t})(\bar{X}_{s,u}(x))' \Delta \sigma_u(\bar{X}_{s,u}(x)) dW_u \quad (1.10)$$

The proof of the interpolation formula (1.8) is provided in section 4.2. We will present two different approaches:

The first one is rather elementary and very intuitive. It combines the conventional Itô-type discrete time approximations of stochastic integrals discussed above with the two-sided stochastic integration calculus introduced in [38]. In this interpretation, the equation (1.8) can be seen as an extended version of the Itô-type change rule formula stated in theorem 6.1 in [38] to the interpolating flow

$$Z^{s,t} : u \in [s, t] \mapsto Z_u^{s,t} := X_{u,t} \circ \bar{X}_{s,u} \implies Z_s^{s,t} - Z_t^{s,t} = X_{s,t} - \bar{X}_{s,t} \quad (1.11)$$

Roughly speaking, the increments of the interpolating path are decomposed into two parts:

One comes from the backward increments of the flow  $u \mapsto X_{u,t}$  given the past values of the stochastic flow  $\bar{X}_{s,u}$ . The other one comes from the conventional Itô increments of  $u \mapsto \bar{X}_{s,u}$  given the future values of the stochastic flow  $X_{u,t}$ .

Next, we present a more direct and elegant interpretation of the stochastic interpolation equation (1.7) in terms of the backward Itô-Ventzell type formula presented below.

**Theorem 1.2.** *Let  $F$  be some function from  $\mathbb{R}^p$  into  $\mathbb{R}^q$ , and let  $y \in \mathbb{R}^p$  be some given state, for some  $p, q \geq 1$ . Suppose we are given a forward  $p$ -dimensional continuous semi-martingale  $Y_{s,t}$  and a backward random field  $F_{s,t}$  from  $\mathbb{R}^p$  into  $\mathbb{R}^q$  with a column-vector type canonical representation of the following form:*

$$\begin{cases} Y_{s,t} &= y + \int_s^t B_{s,u} du + \int_s^t \Sigma_{s,u} dW_u \\ F_{s,t}(x) &= F(x) + \int_s^t G_{u,t}(x) du + \int_s^t H_{u,t}(x) dW_u \end{cases} \quad (1.12)$$

for some  $\mathcal{W}_{s,t}$ -adapted functions  $B_{s,t}, G_{s,t}, H_{s,t}, \Sigma_{s,t}$  with appropriate dimensions. In this situation, under some regularity conditions for any  $s \leq u \leq v \leq t$  we have

$$\begin{aligned} F_{v,t}(Y_{s,v}) - F_{u,t}(Y_{s,u}) &= \int_u^v (\nabla F_{r,t}(Y_{s,r})' B_{s,r} + \frac{1}{2} \nabla^2 F_{r,t}(Y_{s,r})' \Sigma_{s,r} \Sigma'_{s,r} - G_{r,t}(Y_{s,r})) dr \\ &+ \int_u^v (\nabla F_{r,t}(Y_{s,r})' \Sigma_{s,r} - H_{r,t}(Y_{s,r})) dW_r \end{aligned} \quad (1.13)$$

For a more precise statement and the proof of the above theorem, we refer to section 4.2 and the appendix, on page 42. Up to a sign change, we recover (1.8) from (1.13) by choosing

$$\begin{aligned} (F_{s,t}(x), Y_{s,t}(y)) &= (X_{s,t}(x), \bar{X}_{s,t}(y)) & (B_{s,t}, \Sigma_{s,t}) &= (\bar{b}_t(\bar{X}_{s,t}(x)), \bar{\sigma}_t(\bar{X}_{s,t}(x))) \\ G_{u,t}(x) &= \nabla F_{u,t}(x)' b_u(x) + \frac{1}{2} \nabla^2 F_{u,t}(x)' a_u(x) & \text{and} & \quad H_{u,t}(x) = \nabla F_{u,t}(x)' \sigma_u(x) \end{aligned}$$

and letting  $(u, v) = (s, t)$  in (1.13). The above theorem can be seen as the backward version of the generalized Itô-Ventzell formula presented in [37] (see also theorem 3.2.11 in [33]).

The interpolation formula (1.8) with a fluctuation term given by the Skorohod stochastic integral (1.10) can be seen as a Aleksev-Gröbner formula of Skorohod type.

In this context, the integrability of the fluctuation term and any quantitative type estimates require a refined analysis of the Malliavin derivatives of the integrand. Under our regularity conditions the stochastic flows  $X_{s,t}(x)$  and  $\bar{X}_{s,t}(x)$  are Holder-continuous w.r.t. the time parameters as well as twice differentiable w.r.t. the space variables, with uniformly bounded first and second order derivatives. In addition, for any  $n \geq 1$  all the  $n$ -absolute moments of the stochastic flows are finite with at most linear growth w.r.t. the initial values. These properties ensure that the Skorohod stochastic integral (1.10) is well defined and they allow to derive several quantitative estimates. Section 6 provides a refined of the fluctuation term; see for instance theorem 6.2.

When  $\sigma_t = 0$  the flow  $X_{s,t}(x)$  is deterministic so that the Skorohod fluctuation term (1.10) reduces to the traditional Itô stochastic integral. In this context, quantitative estimates of the fluctuation term are obtained combining Burkholder-Davis-Gundy inequalities with the generalized Minkowski inequality. The resulting interpolation formula (1.8) can be seen as a Aleksev-Gröbner formula of Itô-type.

To distinguish these two classes of models, the interpolation formulae (1.8) associated with the case  $\sigma_t = 0$  will be called an Itô-Aleksev-Gröbner formula; the one associated with the case  $\Delta\sigma_t \neq 0$  will be called a Skorohod-Aleksev-Gröbner formula.

The second objective of this article is to derive uniform estimates w.r.t. the time parameter. Our methodology is mainly based on two different types of regularity conditions to be defined and discussed in detail in section 2.2:

- The first is a technical condition that ensures that the  $n$ -absolute moments of the flows  $X_{s,t}$  and  $\bar{X}_{s,t}$  are uniformly bounded w.r.t. the time horizon; we call this condition  $(M)_n$ .
- The second is a spectral condition on the gradient of the drift and diffusion matrices of the stochastic flows, which we call condition  $(T)_n$ . Without going into details, we state one usual case of interest: for constant diffusion functions (1.2) the spectral condition  $(T)_n$  is met for any  $n \geq 2$  as soon as the following log-norm conditions are met

$$\nabla b_t + (\nabla b_t)' \leq -2\lambda I \quad \text{and} \quad \nabla \bar{b}_t + (\nabla \bar{b}_t)' \leq -2\bar{\lambda} I \quad \text{for some } \lambda \wedge \bar{\lambda} > 0, \quad (1.14)$$

To motivate the above condition consider a linear drift function of the form  $b_t(x) = B_t x$  and  $\sigma = 0$ . In this case the tangent process  $\nabla X_{s,t}(x)$  satisfies a time-varying deterministic linear dynamical system

$$\partial_t \nabla X_{s,t}(x) = \nabla X_{s,t}(x) B_t'$$

The asymptotic behavior of this process cannot be characterized by the statistical properties of the spectral abscissa of the matrices  $B_t$ . Indeed, unstable semigroups associated with time-varying (deterministic) matrices  $B_t$  with negative eigenvalues are exemplified in [14, 43]. Conversely, stable semigroups with  $B_t$  having positive eigenvalues are given by Wu in [43]. In contrast, the uniform log-norm condition (1.14) provides a readily verifiable condition.

To describe with some precision the second main result of the article, we need to introduction some additional terminology. When there is no ambiguity, we denote by  $\|\cdot\|$  any (equivalent) norm on some finite dimensional vector space. For some multivariate function  $f_t(x)$ , for  $(t, x) \in [0, \infty) \times \mathbb{R}^d$ , let  $\|f(x)\| := \sup_t \|f_t(x)\|$  and the uniform norm be  $\|f\| := \sup_{t,x} \|f_t(x)\|$ . For any  $n \geq 1$  we also set

$$\|f(x)\|_n := \sup_{s \leq t} \mathbb{E} (\|f_t(\bar{X}_{s,t}(x))\|^n)^{1/n} \quad (1.15)$$

We denote by  $\kappa_n$  and  $\kappa_{\delta,n}$  some constants that depends on some parameters  $n$  and  $(\delta, n)$  but do not depend on the time horizon, nor on the space variable.

In this notation, the second main result of the article takes basically the following form.

**Theorem 1.3.** *Assume conditions  $(M)_{2n/\delta}$  and  $(T)_{2n/(1-\delta)}$  are satisfied for some parameters  $n \geq 2$  and  $\delta \in ]0, 1[$ . In this situation, we have the time-uniform estimates*

$$\begin{aligned} & \mathbb{E} [\|X_{s,t}(x) - \bar{X}_{s,t}(x)\|^n]^{1/n} \\ & \leq \kappa_{\delta,n} \left( \|\Delta a(x)\|_{2n/(1+\delta)} + \|\Delta b(x)\|_{2n/(1+\delta)} + \|\Delta \sigma(x)\|_{2n/\delta} (1 \vee \|x\|) \right) \end{aligned} \quad (1.16)$$

For constant diffusion functions (1.2), the estimate simplifies to

$$(1.14) \implies \forall n \geq 2 \quad \mathbb{E} [\|X_{s,t}(x) - \bar{X}_{s,t}(x)\|^n]^{1/n} \leq \kappa_n (\|\Delta b(x)\|_n + \|\Sigma - \bar{\Sigma}\|) \quad (1.17)$$

The estimates (1.16) come from (7.15) and (6.9). A more detailed proof is provided in the appendix, on page 50. The estimates (1.17) are direct consequences of (2.17) and (6.11).

When  $\sigma_t = \bar{\sigma}_t$  the Skorohod term is indeed absent and (1.8) reduces to

$$X_{s,t}(x) - \bar{X}_{s,t}(x) = \int_s^t (\nabla X_{u,t}) (\bar{X}_{s,u}(x))' \Delta b_u(\bar{X}_{s,u}(x)) du \quad (1.18)$$

We recover the interpolation formula for nonlinear stochastic flows presented in section 3.1 in the article [3]. In this context the analysis of  $\mathbb{L}_n$ -errors will proceed via two-step procedure. In section 3.1 we will derive the exponential bound

$$\sup_x \mathbb{E} (\|(\nabla X_{u,t})(x)\|_2^n)^{1/n} \leq \kappa_n \exp(-\lambda(n)(t-u)) \quad \text{for some } \lambda(n) > 0$$

Using the Minkowski integral inequality in (1.18) yields

$$\mathbb{E} [\|X_{s,t}(x) - \bar{X}_{s,t}(x)\|^n]^{1/n} \leq \int_s^t \mathbb{E} [\|(\nabla X_{u,t})(\bar{X}_{s,u}(x))\|^n \times \|\Delta b_u(\bar{X}_{s,u}(x))\|^n]^{1/n} du.$$

A further conditioning argument and the above exponential bound on the tangent process yields

$$\mathbb{E} [\|X_{s,t}(x) - \bar{X}_{s,t}(x)\|^n]^{1/n} \leq \kappa_n \int_s^t \exp(-\lambda(n)(t-u)) du \sup_{s \leq u} \mathbb{E} [\|\Delta b_u(\bar{X}_{s,u}(x))\|^n]^{1/n}.$$

Replacing the term outside the time integral with  $\|\Delta b(x)\|_n$  yields the stated result in (1.16) excluding the terms representing the difference in the diffusions.

We illustrate one use of theorem 1.1 in the context of analyzing the error in discretising the diffusion  $X_{s,t}(x)$  for some initial time point  $s \geq 0$ . Let  $h > 0$  denote the discretisation interval size and for any  $t \in [s + kh, s + (k+1)h[$  let

$$dX_{s,t}^h(x) = Y_{s,t}^h(x) dt + \Sigma dW_t \quad \text{with} \quad Y_{s,t}^h(x) := b \left( X_{s,s+kh}^h(x) \right)$$

for a fixed diffusion matrix  $\sigma_t(x) = \Sigma$ . Here  $X_{s,t}^h(x)$  is the discretisation of  $X_{s,t}(x)$  with resolution  $h$ . Note that the drift at time  $t$  is not a function of the instantaneous value of  $X_{s,t}^h(x)$ , at time  $t$ , but rather the value it took at the largest discrete time-point before  $t$ . In section 5 we discuss how the formula in (1.8) also applies in this context and establish that

$$X_{s,t}^h(x) - X_{s,t}(x) = \int_s^t (\nabla X_{u,t})(X_{s,u}^h(x))' \left[ Y_{s,u}^h(x) - b(X_{s,u}^h(x)) \right] du.$$

This comparison result when combined with the regularity assumptions (1.19) yields the moment bound below.

**Proposition 1.4.** *Assume that*

$$\nabla b + (\nabla b)' \leq -2\lambda I \quad \|\nabla b\| := \sup_x \|\nabla b(x)\| < \infty \quad \text{and} \quad \langle x, b(x) \rangle \leq -\beta \|x\|^2 \quad (1.19)$$

for some  $\lambda > 0$ ,  $\beta > 0$ . In this situation, for any  $n \geq 1$  we have the uniform estimates

$$\mathbb{E} \left( \|X_{s,t}^h(x) - X_{s,t}(x)\|^n \right)^{1/n} \leq \|\nabla b\| \left( [\|b(0)\| + \hat{m}_n(x) \|\nabla b\|] h + \sigma \sqrt{h} \right) / \lambda$$

where  $\hat{m}_n(x) \leq \kappa_n (1 + \|x\|)$ .

Proposition 1.4 is proved in section 7.3. To apply proposition 1.4 to a Langevin diffusion with a convex potential  $U(x)$ , the drift would be  $b_t(x) = -\nabla U(x)$  and the corresponding assumptions on  $U(x)$  are typical.

The rest of the article is organized as follows:

Section 3 provides some basic tools associated with the first and second variational equations associated with a diffusion flow. We also present some quantitative estimates of the tangent and the Hessian processes. For a more thorough discussion on stochastic flows and their differentiability properties we refer to [13, 30, 36].

Section 4 is mainly concerned with the forward-backward stochastic interpolation formula (1.8) stated in theorem 1.1. Section 4.1 present an extension of the two-sided stochastic calculus introduced by Pardoux and Protter in [38] to stochastic interpolation flows. Section 4.2 presents a new generalized backward version of the generalized Itô-Ventzell formula presented in [37], see for instance theorem 4.1. We illustrate this result with a multivariate Skorohod-Aleksev-Gröbner formula. Apart from more complex and sophisticated tensor notation, the quantitative stochastic analysis of these multivariate formulae follows the same arguments as the ones used in the proof of theorem 1.3. Thus, we have chosen to concentrate this introduction on stochastic flows.

Some extensions of the stochastic interpolation formula (1.8) are discussed in section 5, including non Markov perturbations and jump diffusion perturbation processes.

Section 6 is dedicated to the analysis of the Skorohod fluctuation process introduced in (1.10).

Section 7 presents some illustrations of the forward-backward interpolation formulae discussed in the present article in the context of diffusion perturbation theory, interacting diffusions and discrete time approximations.

The technical proofs of some results are housed in the appendix.

## 1.2 Comments and comparisons with existing literature

The forward-backward stochastic interpolation formula (1.8) can also be extended to more general classes of stochastic flows on abstract state spaces. For instance the recent article [28] provides a deterministic first order version of (1.8) on abstract Banach spaces.



Forward-backward interpolation formulae of the same form as (1.8) for stochastic matrix Riccati diffusion flows are also discussed in [11]. The articles [4, 5] also discuss similar interpolation formulae for mean field particle systems and deterministic nonlinear measure valued semigroups. In this context, the process  $X_{s,t}$  is given a deterministic measure-valued process and  $\overline{X}_{s,t}$  represents the evolution of the particle density profiles associated with an approximating mean field particle interpretation of  $X_{s,t}$ . The first referenced article [4] is concerned with interacting jumps models on path spaces, the second article [5] discusses the propagation of chaos properties of mean field type interacting diffusions.

The stability properties of these abstract models discussed above depend on the problem at hand. To focus on the main ideas without clouding the article with unnecessary technical details and sophisticated mathematical tools based on abstract ad hoc regularity conditions we have chosen to concentrate the article on diffusion flows on Euclidian spaces with simple and easily checked regularity conditions.

The forward-backward formula (1.8) can be seen as an extension of theorem 6.1 in [38] on two-sided stochastic integrals to diffusion flows. The terminology "two-sided" coined by the authors in [38] comes from the fact that the integrand of the Skorohod integral depend on the past as well as on the future of the history generated by the Brownian motion. This interpolation formula can also be interpreted as a backward version of the generalized Itô-Ventzell formula presented in [37] (see also theorem 3.2.11 in [33]).

The interpolation formula (1.8) can also be interpreted as an extension of Aleksev-Gröbner lemma [1, 22, 28] as well as an extended version of the variation-of-constant and related Gronwall type lemma [8, 23] to diffusion processes. In this connection we underline that the forward-backward formula (1.8) differs from the stochastic Gronwall lemma presented in [39] based on particular classes of stochastic linear inequalities that doesn't involve Skorohod type integrals.

It is worth to mention that the stochastic perturbations may come from auxiliary random sources, uncertainty propagations, as well as time discretization schemes and mean field type particle fluctuations. For instance, the forward-backward perturbation methodology discussed in the present article has been used in [3, 5] in the context of nonlinear diffusions and their mean field type interacting particle interpretations, see for instance section 2.3 in [5]. In this context, the random perturbations come from the fluctuations of a mean field particle interpretation of a class of nonlinear diffusions.

The extended version of the Itô-Aleksev-Gröbner formula (1.18) to nonlinear diffusions is also discussed in section 3.1 in the article [3]. In this context, the time varying drift and diffusion functions of the stochastic flows  $X_{s,t}$  and  $\overline{X}_{s,t}$  depend on some nonlinear measure valued semigroup which may start from two possibly different initial distributions. For a more thorough discussion on this class of nonlinear diffusions, we refer to the Itô-Aleksev-Gröbner formula (3.2) and corollary 3.2 in the article [3].

The discrete time version of the forward-backward perturbation semigroup methodology discussed in the present article can also be found in chapter 7 in [17], as well as in the series of articles [18, 19, 20]. In this context, the random perturbations come from the fluctuations of a genetic type particle interpretation of nonlinear Feynman-Kac semigroups.

The more recent articles [9, 10, 11] also provide a series of backward-forward interpolation formulae of the same form as (1.8) for stochastic matrix Riccati diffusion flows arising in data assimilation theory (cf. for instance theorem 1.3 in [11] as well as section 2.2 in [10] and the proof of theorem 2.3 in [9]). In this context, the random perturbations come from the fluctuations of a mean field particle interpretation of a class of nonlinear diffusions equipped with an interacting sample covariance matrix functional.

We underline that the Itô-Aleksev-Gröbner formula (4.7) discussed in [11] is an extension of the interpolation formula (1.8) to stochastic diffusion flows in matrix spaces. In this context the

unperturbed model is given by the flow of a deterministic matrix Riccati differential equation and the random perturbations are described by matrix-valued diffusion martingales.

These results were also used in [11] to quantify the fluctuation of the stochastic flow around the limiting deterministic Riccati equation, at any order. We will briefly discuss the analog of these Taylor type expansions in section 7.1 in the context of Euclidian diffusions.

In the series of articles discussed above, as in (1.7) the central common idea is to analyse the evolution of the interpolating process (1.11) between a given process  $X_{s,t}$  and some stochastic flow  $\bar{X}_{s,t}$  with an extra level of randomness. In discrete time settings, the differential interpolation formula (1.7) can also be recasted in terms of a telescoping sum reflecting the differences between a stochastic semigroup and its perturbations, see for instance chapter 7 in [17].

In all the application domains discussed above, this second order perturbation methodology has been developed to quantify uniformly w.r.t. the time horizon the propagations of some stochastic perturbations entering in *some deterministic and stable reference or unperturbed process*. In the context of Euclidian diffusions, this corresponds to the situation where the diffusion function  $\sigma = 0$ . In this interpretation, the present article can be seen as a natural extension of the second order perturbation methodology developed in the above referenced articles to diffusion type unperturbed processes; that is when  $\sigma \neq 0$ .

After most of this work was completed, we learned the existence of the recent work of Hude-Hutzenthaler-Jentzen-Mazonetto [25]. In this article the authors also discuss a multivariate Itô-Aleksev-Gröbner formula for abstract diffusion perturbation models of the form (5.1) using rather sophisticated discrete time approximations. The study [25] is also based on a series of particularly chosen and custom regularity conditions. For instance, the authors assume that the abstract diffusion perturbation models are chosen so that the Skorohod fluctuation term exists without providing any quantitative type estimate. We feel that our approach is more direct and intuitive as it relies on an extended version of Itô's change rule formula (1.7) to interpolating stochastic flows.

In section 6 in the present article, we will see that any quantitative analysis requires to estimate the absolute moments of the Malliavin derivatives of the stochastic integrands of the Brownian motion arising in the Skorohod fluctuation term. In our framework, these Malliavin derivatives depend on the gradient of both of the diffusion functions  $(\sigma, \bar{\sigma})$  as well as on the tangent process of the perturbed diffusion flow. The quantitative analysis developed in 6 can be extended without difficulties to abstract diffusion perturbation models satisfying appropriate differentiability and integrability conditions.

The article [25] also presents an application to tamed Euler type discrete time approximations of a stochastic van-der-Pol process introduced in [41], simplifying the analysis provided in an earlier work [26]. In this situation, we underline that the Skorohod fluctuation term is null so that the resulting Aleksev-Gröbner type formula resumes to the simple and elementary case discussed in (1.18) and in the article [3]. As expected for this class of "unstable processes", the authors recast a series of  $\mathbb{L}_2$ -estimates discussed in [26] into a series of estimates that grow exponentially fast with respect to the time horizon.

In contrast with the present work, the above article doesn't discuss any quantitative uniform estimates w.r.t. the time horizon. The analysis presented in [25] is mainly concerned with the proof of a Skorohod-Aleksev-Gröbner type formula for abstract diffusion perturbation models and it doesn't apply to derive any type of estimates to general diffusion perturbation models without adding regularity conditions.

Besides its elegance the forward-backward formula (1.8) is clearly of rather poor mathematical and numerical interest without a better understanding of the variational processes and the Skorohod fluctuation term (1.10). A crucial problem is to avoid exceedingly pessimistic exponential estimates that grow exponentially fast w.r.t. the time horizon.

One advantage of the second order perturbation methodology developed in the present article is that it allows to enter the stability properties of the tangent and the Hessian flow to the estimation of Skorohod fluctuation term and therefore the differences of the stochastic flows. Our main contribution is to develop a refined analysis of these variational processes and the Skorohod fluctuation terms. We also deduce several uniform perturbation propagation estimates with respect to the time horizon, yielding what seems to be the first results of this type for this class of models.

## 2 Preliminary results

### 2.1 Some basic notation

With a slight abuse of notation, we denote by  $I$  the identity  $(d \times d)$ -matrix, for any  $d \geq 1$ . We also denote by  $\|\cdot\|$  any (equivalent) norm on a finite dimensional vector space over  $\mathbb{R}$ . All vectors are column vectors by default.

We introduce some matrix notation needed from the onset.

We denote by  $\|A\|_2 := \lambda_{max}(AA')^{1/2} = \lambda_{max}(A'A)^{1/2}$ , resp.  $\|A\|_F = \text{Tr}(AA')^{1/2}$  and  $\rho(A) = \lambda_{max}((A+A')/2)$  the spectral norm, the Frobenius norm, and the logarithmic norm of some matrix  $A$ .  $A'$  is the transpose of  $A$  and  $\lambda_{max}(\cdot)$  the largest eigenvalue. The spectral norm is sub-multiplicative or  $\|AB\|_2 \leq \|A\|_2 \|B\|_2$  and compatible with the Euclidean norm for vectors, by that we mean for a vector  $x$  we have  $\|Ax\| \leq \|A\|_2 \|x\|$ .

Let  $[n]$  be the set of  $n$  multiple indexes  $i = (i_1, \dots, i_n) \in \mathcal{I}^n$  over some finite set  $\mathcal{I}$ . We denote by  $(A_{i,j})_{(i,j) \in [p] \times [q]}$  the entries of a  $(p, q)$ -tensor  $A$  with index set  $\mathcal{I}$  for  $[p]$  and  $\mathcal{J}$  for  $[q]$ . For the sake of brevity, the index sets will be implicitly defined through the context.

For a given  $(p_1, q)$ -tensor  $A$  and a given  $(q, p_2)$  tensor  $B$ ,  $AB$  and  $B'$  is a  $(p_1, p_2)$ -tensor resp. a  $(p_2, q)$ -tensor with entries given by

$$\forall (i, j) \in [p_1] \times [p_2] \quad (AB)_{i,j} = \sum_{k \in [q]} A_{i,k} B_{k,j} \quad \text{and} \quad B'_{j,k} := B_{k,j}. \quad (2.1)$$

The symmetric part  $A_{sym}$  of a  $(p, p)$ -tensor is the  $(p, p)$ -tensor  $A_{sym}$  with entries

$$\forall (i, j) \in [p] \times [p] \quad (A_{sym})_{i,j} = (A_{i,j} + A_{j,i})/2$$

We consider the Frobenius inner product given for any  $(p, q)$ -tensors  $A$  and  $B$  by

$$\langle A, B \rangle_F = \text{Tr}(AB') = \sum_i (AB')_{i,i} \quad \text{and the norm} \quad \|A\|_F = \sqrt{\text{Tr}(AA')}$$

For any  $(p, q)$ -tensors  $A$  and  $B$  we also check the Cauchy-Schwartz inequality

$$\langle A, B \rangle_F^2 \leq \|A\|_F \|B\|_F \quad \text{and} \quad \|A\|_2 \leq \|A\|_F \leq \text{Card}(\mathcal{I})^p \|A\|_2 \quad \text{with} \quad \|A\|_2 := \lambda_{max}(AA')^{1/2}$$

For any tensors  $A, B$  with appropriate dimensions we have the inequality

$$\|AB\|_F \leq \|A\|_F \|B\|_F$$

Given some tensor valued function  $T : (t, x) \mapsto T_t(x)$  we also set

$$\|T\|_F := \sup_{t,x} \|T_t(x)\|_F \quad \|T\|_2 := \sup_{t,x} \|T_t(x)\|_2 \quad \text{and} \quad \|T\| := \sup_{t,x} \|T_t(x)\|$$

Given some smooth function  $h(x)$  from  $\mathbb{R}^p$  into  $\mathbb{R}^q$  we denote by

$$\nabla h = [\nabla h^1, \dots, \nabla h^q] \quad \text{with} \quad \nabla h^i = \begin{bmatrix} \partial_{x_1} h^i \\ \vdots \\ \partial_{x_p} h^i \end{bmatrix} \quad (2.2)$$

the gradient  $(p, q)$ -matrix associated with the column vector-valued function  $h = (h^i)_{1 \leq i \leq q}$ . Building on this notation: let  $b : \mathbb{R}^n \rightarrow \mathbb{R}^p$  and let the mapping  $x \rightarrow G(x) = h(b(x))$ . Then  $\nabla G(x) = \nabla b(x) \times \nabla h(b(x))$ . Let

$$\nabla^2 h = [\nabla^2 h^1, \dots, \nabla^2 h^q] \quad \text{with} \quad \nabla^2 h^i = \begin{bmatrix} \partial_{x_1, x_1} h^i & \dots & \partial_{x_1, x_p} h^i \\ \vdots & \dots & \vdots \\ \partial_{x_p, x_1} h^i & \dots & \partial_{x_p, x_p} h^i \end{bmatrix} \quad (2.3)$$

The Hessian  $H = \nabla^2 h$  associated with the function  $h = (h^i)_{1 \leq i \leq q}$  is a  $(2, 1)$ -tensor where  $H_{(i,j),k} = (\nabla^2 h^k)_{i,j} = \partial_{x_i, x_j} h^k$ . In this notation we can compactly represent the second order term of the Taylor expansion of the the vector valued function  $h$ . For a vector  $y = (y_1, \dots, y_p)'$

$$\begin{bmatrix} y' \nabla^2 h^1(x) y \\ \vdots \\ y' \nabla^2 h^q(x) y \end{bmatrix} = \nabla^2 h(x)' y y'$$

where we have regarded the matrix  $yy'$  as the  $(2, 1)$ -tensor  $Y$  with  $Y_{(i,j),1} = y_i y_j$ .

In the same vein, in terms of the tensor product (2.1), for any pair of column vector-valued function  $h = (h^k)_{1 \leq k \leq q}$  and  $b = (b^i)_{1 \leq i \leq p}$  and any matrix function  $a = (a^{i,j})_{1 \leq i,j \leq p}$  from  $\mathbb{R}^p$  into  $\mathbb{R}^q$ , for any parameter  $1 \leq k \leq q$  we also have

$$\begin{aligned} (\nabla h(x)' b(x))^k &= \sum_{1 \leq i \leq p} (\nabla h(x))'_{k,i} b^i(x) = \sum_{1 \leq j \leq p} \partial_{x_i} h^k(x) b^i(x) = \langle \nabla h^k(x), b(x) \rangle \\ (\nabla^2 h(x)' a(x))^k &= \sum_{1 \leq i,j \leq p} (\nabla^2 h(x))'_{k,(i,j)} a^{i,j}(x) \\ &= \sum_{1 \leq i,j \leq p} \partial_{x_i, x_j} h^k(x) a^{i,j}(x) = \langle \nabla^2 h^k(x), a(x) \rangle_F \end{aligned}$$

In a more compact form, the above formula takes the form

$$\nabla h(x)' b(x) = \begin{bmatrix} \langle \nabla h^1(x), b(x) \rangle \\ \vdots \\ \langle \nabla h^q(x), b(x) \rangle \end{bmatrix} \quad \text{and} \quad \nabla^2 h(x)' a(x) = \begin{bmatrix} \langle \nabla^2 h^1(x), a(x) \rangle_F \\ \vdots \\ \langle \nabla^2 h^q(x), a(x) \rangle_F \end{bmatrix} \quad (2.4)$$

For any  $n \geq 1$  we let  $\mathcal{P}_n(\mathbb{R}^d)$  be the convex set of probability measures  $\mu_1, \mu_2$  on  $\mathbb{R}^d$  with absolute  $n$ -th moment and equipped with the Wasserstein distance of order  $n$  denoted by

$$\mathbb{W}_n(\mu_1, \mu_2) := \inf \mathbb{E}(\|X_1 - X_2\|^n)^{1/n}$$

In the above display the infimum is taken over all pair of random variables  $(X_1, X_2)$  with marginal distributions  $(\mu_1, \mu_2)$ . The stochastic transition semigroups associated with the flows  $X_{s,t}(x)$  and  $\overline{X}_{s,t}(x)$  are defined for any measurable function  $f$  on  $\mathbb{R}^d$  by the formulae

$$\mathbb{P}_{s,t}(f)(x) := f(X_{s,t}(x)) \quad \text{and} \quad \overline{\mathbb{P}}_{s,t}(f)(x) := f(\overline{X}_{s,t}(x))$$

Given some column vector-valued function  $f = (f^i)_{1 \leq i \leq p}$ , let  $\mathbb{P}_{s,t}(f)$  and  $P_{s,t}(f)$  denote the column vector-valued functions with entries  $\mathbb{P}_{s,t}(f^i)$  and  $P_{s,t}(f^i)$ . Building on the tensor notation, let  $\mathbb{P}_{s,t}(\nabla f)$  and  $\mathbb{P}_{s,t}(\nabla^2 f)$  respectively denote the (1,1) and (2,1)-tensor valued functions with entries

$$\mathbb{P}_{s,t}(\nabla f)(x)_{i,k} := \mathbb{P}_{s,t}(\partial_{x_i} f^k)(x) \quad \text{and} \quad \mathbb{P}_{s,t}(\nabla^2 f)(x)_{(i,j),k} := \mathbb{P}_{s,t}(\partial_{x_i, x_j} f^k)(x)$$

We also consider the random (2,1) and (2,2)-tensors given by

$$\begin{aligned} \nabla^2 X_{s,t}(x)_{(i,j),k} &= \partial_{x_i, x_j} X_{s,t}^k(x) = [\nabla^2 X_{s,t}(x)]'_{k,(i,j)} \\ [\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)]_{(i,j),(k,l)} &= \nabla X_{s,t}(x)_{i,k} \nabla X_{s,t}(x)_{j,l} = [\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)]'_{(k,l),(i,j)} \end{aligned}$$

Throughout the rest of the article, unless otherwise is stated we write  $\kappa, \kappa_\epsilon, \kappa_n, \kappa_{n,\epsilon}$  some constants whose values may vary from line to line but only depend on the parameters in their subscripts, i.e.  $n \geq 0$  and  $\epsilon > 0$ , as well as on the parameters of the model; that is, on the drift and diffusion functions. We also use the letters  $c, c_\epsilon, c_n, c_{n,\epsilon}$  to denote universal constants. Importantly these constants do not depend on the time horizon. We also consider the uniform log-norm parameters

$$\rho(\nabla \sigma)^2 := \sum_{1 \leq k \leq r} \rho(\nabla \sigma_k)^2 \quad \text{and} \quad \rho_\star(\nabla \sigma) := \sup_{1 \leq k \leq r} \rho(\nabla \sigma_k) \quad \text{with} \quad \rho(\nabla \sigma_k) := \sup_{t,x} \rho(\nabla \sigma_{t,k}(x)) \quad (2.5)$$

and the parameters  $\chi(b, \sigma)$  defined by

$$\chi(b, \sigma) := c + \|\nabla^2 b\| + \|\nabla^2 \sigma\|^2 + \rho_\star(\nabla \sigma)^2 \quad (2.6)$$

## 2.2 Regularity conditions and some preliminary results

We consider two different types of regularity conditions  $(\mathcal{M})_n$  and  $(\mathcal{T})_n$ , indexed by some parameter  $n \in [2, \infty[$ , for the diffusion  $(b_t, \sigma_t)$ .

$(\mathcal{M})_n$  There exists some parameters  $\kappa_n \geq 0$  such that for any  $x \in \mathbb{R}^d$  we have

$$m_n(x) := \sup_{s \leq t} \mathbb{E}(\|X_{s,t}(x)\|^n)^{1/n} \leq \kappa_n (1 \vee \|x\|)$$

$(\mathcal{T})_n$  There exists some parameter  $\lambda_A > 0$  such that

$$A_t := \nabla b_t + (\nabla b_t)' + \sum_{1 \leq k \leq r} \nabla \sigma_{k,t} (\nabla \sigma_{k,t})' \leq -2\lambda_A I \quad (2.7)$$

where  $\sigma_{k,t}$  denotes the  $k$ -th column of  $\sigma_t$ . In addition, the following condition is satisfied

$$\lambda_A(n) := \lambda_A - \frac{d(n-2)}{2} \rho_\star(\nabla \sigma)^2 > 0 \quad (2.8)$$

We now define the corresponding assumptions for the diffusion  $(\bar{b}_t, \bar{\sigma}_t)$ .

$(\bar{\mathcal{M}})_n$  The regularity condition defined as in  $(\mathcal{M})_n$  for the diffusion  $(\bar{b}_t, \bar{\sigma}_t)$ .

$(\bar{\mathcal{T}})_n$  Let  $\bar{A}_t$  be the symmetric matrix defined as  $A_t$  in (2.7) when  $(b_t, \sigma_t) = (\bar{b}_t, \bar{\sigma}_t)$ . Assume there exists some  $\lambda_{\bar{A}} > 0$  such that  $\bar{A}_t \leq -2\lambda_{\bar{A}} I$ . Furthermore, assume  $\lambda_{\bar{A}}(n) > 0$  where  $\lambda_{\bar{A}}(n)$  is defined as  $\lambda_A(n)$  when  $(\lambda_A, \sigma_t) = (\lambda_{\bar{A}}, \bar{\sigma}_t)$ .

$(M)_n$  We write  $(M)_n$  when both conditions  $(\mathcal{M})_n$  and  $(\overline{\mathcal{M}})_n$  are satisfied.

$(T)_n$  Both conditions  $(\mathcal{T})_n$  and  $(\overline{\mathcal{T}})_n$  are met, and let

$$\lambda_{A,\overline{A}}(n) := \lambda_A(n) \wedge \lambda_{\overline{A}}(n)$$

In practice, the uniform moment condition  $(\mathcal{M})_n$  is often checked using Lyapunov techniques. For example we can use the following polynomial growth condition.

$(\mathcal{P})_n$  There exists some parameters  $\alpha_i, \beta_i \geq 0$  with  $i = 0, 1, 2$  such that for any  $t \geq 0$  and any  $x \in \mathbb{R}^d$  we have

$$\|\sigma_t(x)\|_F^2 \leq \alpha_0 + \alpha_1 \|x\| + \alpha_2 \|x\|^2 \quad \text{and} \quad \langle x, b_t(x) \rangle \leq \beta_0 + \beta_1 \|x\| - \beta_2 \|x\|^2 \quad (2.9)$$

for some norm  $\|\sigma_t(x)\|$  of the matrix-valued diffusion function. In addition, we have

$$\beta_2(n) := \beta_2 - \frac{(n-1)}{2} \alpha_2 > 0$$

**Lemma 2.1.** *For any  $n \geq 2$  we have*

$$(\mathcal{P})_n \implies (\mathcal{M})_n \quad \text{with} \quad \kappa_n = 1 + \frac{(\gamma_1 + (n-2)\alpha_1) + (\gamma_0 + (n-2)\alpha_0)^{1/2}}{2\beta_2(n)^{1/2}} \quad (2.10)$$

The proof of the above assertion follows standard stochastic calculations, thus it is housed in the appendix, on page 46.

For one-dimensional geometric Brownian motions the condition  $(\mathcal{P})_n$  is a sufficient and necessary condition for the existence of uniformly bounded absolute  $n$ -moments. In this case  $(\mathcal{T})_n$  coincides with  $(\mathcal{P})_n$  by setting

$$\lambda_A = \beta_2 - \alpha_2/2 \quad \text{and} \quad \alpha_2 = \rho_\star (\nabla \sigma)^2$$

Whenever condition  $(M)_n$  is met for some  $n \geq 2$ , we also check the uniform estimates

$$\mathbb{E} (\| [X_{u,t} \circ \overline{X}_{s,u} ](x) \|^n)^{1/n} \leq \kappa_n (1 + \|x\|) \quad (2.11)$$

with the same parameter  $\kappa_n$  as the one associated with the condition  $(M)_n$ .

Recalling that the functions  $(b_t, \overline{b}_t)$  and  $(\sigma_t, \overline{\sigma}_t)$  have at most linear growth, with the  $\mathbb{L}_n$ -norms  $\|\cdot\|_n$  introduced in (1.15) we also have that

$$\|\|\Delta b(x)\|\|_n \leq \kappa_{1,n} (1 \vee \|x\|) \quad \text{and} \quad \|\|\Delta a(x)\|\|_{n/2} \leq \kappa_{2,n} (1 \vee \|x\|)^2 \quad (2.12)$$

To give more insight where these assumptions will be used, we now briefly state the stability results that stem from them. Condition  $(\mathcal{T})_n$  ensures that the exponential decays of the absolute and uniform  $n$ -moments of the tangent and the Hessian processes; that is, when  $(\mathcal{T})_n$  is met for some  $n \geq 2$  we have that

$$\mathbb{E} (\|\nabla X_{s,t}(x)\|^n)^{1/n} \vee \mathbb{E} (\|\nabla^2 X_{s,t}(x)\|^n)^{1/n} \leq \kappa_n e^{-\lambda(n)(t-s)} \quad \text{for some} \quad \lambda(n) > 0 \quad (2.13)$$

A more precise statement is provided in proposition 3.2 and proposition 3.10. These uniform estimates clearly imply, via a conditioning argument, that for any  $n \geq 2$  and  $s \leq u \leq t$  we have

$$\mathbb{E} (\|(\nabla X_{u,t})(\overline{X}_{s,u}(x))\|^n)^{1/n} \vee \mathbb{E} (\|(\nabla^2 X_{u,t})(\overline{X}_{s,u}(x))\|^n)^{1/n} \leq \kappa_n e^{-\lambda(n)(t-u)} \quad (2.14)$$

with the same parameters  $(\kappa_n, \lambda(n))$  as in (2.13).

The case  $\nabla\sigma = 0$  will also serve a useful purpose, for example in analysing the error of a numerical implementation as in proposition 1.4. For instance whenever  $(\mathcal{T})_2$  is met we have the almost sure and uniform gradient estimates

$$\|\nabla X_{s,t}\|_2 := \sup_x \|\nabla X_{s,t}(x)\|_2 \leq e^{-\lambda_A(t-s)} \quad (2.15)$$

In addition, we have the almost sure and uniform Hessian estimates

$$\|\nabla^2 X_{s,t}\|_F := \sup_x \|\nabla^2 X_{s,t}(x)\|_F \leq \frac{d}{\lambda_A} \|\nabla^2 b\|_F e^{-\lambda_A(t-s)} \quad (2.16)$$

A proof of the above estimates is provided in the beginning of section 3.1 and section 3.2. In this situation, whenever  $(\mathcal{T})_2$  is met we have

$$\mathbb{E} [\|T_{s,t}(\Delta a, \Delta b)(x)\|^n]^{1/n} \leq \kappa (\|\Delta b(x)\|_n + \|\Delta a(x)\|_n). \quad (2.17)$$

For instance, for a Langevin diffusion associated with some convex potential function  $U$  we have  $b = -\nabla U$  and  $\nabla\sigma = 0$ . Then assuming

$$\begin{aligned} \nabla^2 U \geq \lambda I &\implies (\mathcal{T})_2 \text{ is met} \\ \implies \|\nabla X_{s,t}\|_2 \leq e^{-\lambda(t-s)} &\text{ and } \|\nabla^2 X_{s,t}\|_F \leq \frac{d}{\lambda} \|\nabla^3 U\|_F e^{-\lambda(t-s)} \end{aligned} \quad (2.18)$$

where the almost sure tangent and Hessian bounds follow from (2.15) and (2.16) respectively.

In practice, it is often easier to work with  $a_t(x) = \sigma_t(x)\sigma_t(x)'$  than  $\sigma_t(x)$  and we now discuss some ways of estimating  $\Delta\sigma_t(x) = \sigma_t(x) - \bar{\sigma}_t(x)$  in terms of  $\Delta a_t(x) = a_t(x) - \bar{a}_t(x)$  and in the reverse direction. The latter is straightforward:

$$\|\Delta a_t(x)\| \leq \|\Delta\sigma_t(x)\| [\|\sigma_t(x)\| + \|\bar{\sigma}_t(x)\|].$$

To estimate  $\Delta\sigma_t$  in terms of  $\Delta a_t$ , assume the following ellipticity condition is satisfied

$$a_t(x) \geq v I \quad \text{and} \quad \bar{a}_t(x) \geq v I \quad \text{for some parameter } v > 0. \quad (2.19)$$

We recall the Ando-Hemmen inequality for any symmetric positive definite matrices  $Q_1, Q_2$

$$\|Q_1^{1/2} - Q_2^{1/2}\| \leq \left[ \lambda_{\min}^{1/2}(Q_1) + \lambda_{\min}^{1/2}(Q_2) \right]^{-1} \|Q_1 - Q_2\| \quad (2.20)$$

for any unitary invariant matrix norm  $\|\cdot\|$ . In the above display,  $\lambda_{\min}(\cdot)$  the minimal eigenvalue. We also have the square root inequality

$$Q_1 \geq Q_2 \implies Q_1^{1/2} \geq Q_2^{1/2} \quad (2.21)$$

See for instance theorem 6.2 on page 135 in [24], as well as proposition 3.2 in [2]. A proof of (2.21) can be found in [7]. In this situation, using (2.20) and (2.21) we check that

$$\|\Delta\sigma_t(x)\| \leq \frac{1}{\sqrt{v}} \|\Delta a_t(x)\| \quad \text{and} \quad \|\sigma_t(x)\| \leq \|\sigma_t(0)\| + \frac{1}{\sqrt{v}} [\|a_t(x)\| + \|a_t(0)\|] \quad (2.22)$$

This provides a way to estimate the growth of  $\sigma_t(x)$  in terms of the one of  $a_t(x)$ . For instance the estimate (1.16) combined with (2.22) implies that

$$\mathbb{E} [\|X_{s,t}(x) - \bar{X}_{s,t}(x)\|^n]^{1/n} \leq \kappa_{\delta,n} \left( \|\Delta b(x)\|_{2n/(1+\delta)} + \|\Delta a(x)\|_{2n/\delta} (1 \vee \|x\|) \right)$$

• Assume that  $(\bar{\mathcal{M}})_n$  is satisfied for some  $n \geq 1$ . Also let  $f_t(x)$  be some multivariate function such that

$$\|f(0)\| := \sup_t \|f_t(0)\| < \infty \quad \text{and} \quad \|\nabla f\| := \sup_{t,x} \|\nabla f_t(x)\| < \infty$$

In this situation, we have the estimates

$$\|f(x)\|_n \leq \|f(0)\| + \|\nabla f\| \bar{m}_n(x) \quad \text{and therefore} \quad \|f(x)\|_n \leq \kappa_n (\|f(0)\| + \|\nabla f\|)$$

### 2.3 Some results on anticipating stochastic calculus

In this section we review some results on Malliavin derivatives and Skorohod integration calculus which will be needed below. We restrict the presentation to unit time intervals. Let  $(\Omega, \mathcal{W})$  be the canonical space equipped with the Wiener measure  $\mathbb{P}$  associated with the  $r$ -dimensional Brownian motion  $W_t$  discussed in the introduction.

The Malliavin derivative  $D$  is a linear operator from some dense domain  $\mathbb{D}_{2,1} \subset \mathbb{L}_2(\Omega)$  into the space  $\mathbb{L}_2(\Omega \times [0, 1]; \mathbb{R}^r)$  of  $r$ -dimensional processes with square integrable states on the unit time interval. For multivariate  $d$ -column vector random variables  $F$  with entries  $F^i$ , we use the same rules as for the gradient and we set

$$D_t F = \left[ D_t F^1, \dots, D_t F^d \right] \quad \text{with} \quad D_t F^i = \begin{bmatrix} D_t^1 F^i \\ \vdots \\ D_t^r F^i \end{bmatrix}$$

For  $(p \times q)$ -matrices  $F$  with entries  $F_k^j$  we let  $D_t F$  be the tensor with entries

$$(D_t F)_{i,j,k} = D_t^i F_k^j$$

It is clearly out of the scope of this article to review the analytical construction of Malliavin differential calculus. For a more thorough discussion we refer the reader to the seminal book by Nualart [33], see also the more synthetic presentation in the articles [34, 37].

Formally, one can think the Malliavin derivatives  $D_t^i F$  of some  $F \in \mathbb{D}_{2,1}$  as way to extract from the random variable  $F$  the integrand of Brownian increment  $dW_t^i$ . For instance, when  $s \leq t$  we have

$$\begin{aligned} D_t^i X_{s,t}(x) &= \sigma_{t,i}(X_{s,t}(x)) \\ (D_t \nabla X_{s,t}(x))_{i,j,k} &= D_t^i (\nabla X_{s,t}(x))_{j,k} := (\nabla X_{s,t}(x) \nabla \sigma_{t,i}(X_{s,t}(x)))_{j,k} \end{aligned} \quad (2.23)$$

As conventional differentials, for any smooth function  $G$  from  $\mathbb{R}^d$  into  $\mathbb{R}^{p \times q}$ , Malliavin derivatives satisfy the chain rule properties

$$D_t^i (G_k^j \circ F) = \sum_{1 \leq l \leq d} \left( \partial_{x_l} G_k^j \right) (F) \times D_t^i F^l \quad \iff \quad D_t (G \circ F) = D_t F ((\nabla G) \circ F)$$

For instance, for any  $s \leq u \leq v$  we have

$$D_u (X_{u,t} \circ X_{s,u}) = (D_u X_{s,u}) [(\nabla X_{u,t}) \circ X_{s,u}] \quad \text{and} \quad D_u (\zeta_t \circ X_{s,t}) = (D_u X_{s,t}) [(\nabla \zeta_t) \circ X_{s,t}] \quad (2.24)$$



In the same vein, we have

$$\begin{aligned}
& D_u (\nabla X_{s,u} [(\nabla X_{u,t}) \circ X_{s,u}]) \\
&= (D_u \nabla X_{s,u}) [(\nabla X_{u,t}) \circ X_{s,u}] + (D_u X_{s,u} \otimes \nabla X_{s,u}) [(\nabla^2 X_{u,t}) \circ X_{s,u}]
\end{aligned} \tag{2.25}$$

Let  $\mathbb{L}_{2,1}(\mathbb{R}^r) \subset \mathbb{L}_2(\Omega \times [0, 1]; \mathbb{R}^r)$  be the Hilbert space of  $r$ -dimensional process  $U_t$  with Malliavin differentiable entries  $U_t^i \in \mathbb{D}_{2,1}$  equipped with the norm

$$\|U\| := \mathbb{E} \left( \int_{[0,1]} \|U_t\|^2 dt \right)^{1/2} + \mathbb{E} \left( \int_{[0,1]^2} \|D_s U_t\|^2 ds dt \right)^{1/2}$$

The Skorohod integral w.r.t. the Brownian motion  $W_t^i$  on the unit interval is defined a linear and continuous mapping from

$$V \in \mathbb{L}_{2,1}(\mathbb{R}) \mapsto \int_0^1 V_t dW_t^i \in \mathbb{L}_2(\Omega)$$

characterized by the two following properties

$$\begin{aligned}
\mathbb{E} \left( \int_0^1 V_t dW_t^i \right) &= 0 \\
\mathbb{E} \left( \left( \int_0^1 V_t dW_t^i \right)^2 \right) &= \mathbb{E} \left( \int_{[0,1]} V_t^2 dt \right) + \mathbb{E} \left( \int_{[0,1]^2} D_s^i V_t D_t^i V_s ds dt \right)
\end{aligned} \tag{2.26}$$

The above formula can be seen as an extended version of the Itô isometry to Skorohod integrals, for instance [35], as well as chapters 1.3 to 1.5 in the book by Nualart [33].

As for the Itô integral, the Skorohod integral w.r.t. the  $r$ -dimensional Brownian motion  $W_t$  of a matrix valued process with entries  $V_k^i \in \mathbb{L}_{2,1}(\mathbb{R})$  is defined by the column vector with entries

$$\left( \int_0^1 V_t dW_t \right)^i := \int_0^1 V_t^i dW_t := \sum_{1 \leq k \leq r} \int_0^1 V_{t,k}^i dW_t^k$$

### 3 Variational equations

#### 3.1 The tangent process

In terms of the tensor product (2.4), the gradient  $\nabla X_{s,t}(x)$  of the diffusion flow  $X_{s,t}(x)$  is given by the gradient  $(d \times d)$ -matrix

$$d \nabla X_{s,t}(x) = \nabla X_{s,t}(x) \left[ \nabla b_t(X_{s,t}(x)) dt + \sum_{1 \leq k \leq r} \nabla \sigma_{t,k}(X_{s,t}(x)) dW_t^k \right]$$

where  $W_t^k$  is the  $k$ -th component of the Brownian motion. After some calculations we check that

$$d \left[ \nabla X_{s,t}(x) \nabla X_{s,t}(x)' \right] = \nabla X_{s,t}(x) A_t(X_{s,t}(x)) \nabla X_{s,t}(x)' dt + dM_{s,t}(x) \tag{3.1}$$

with the matrix function  $A_t(x)$  defined in (2.7) and the symmetric matrix valued martingale

$$dM_{s,t}(x) := \sum_{1 \leq k \leq r} \nabla X_{s,t}(x) \left[ \nabla \sigma_{t,k}(X_{s,t}(x)) + \nabla \sigma_{t,k}(X_{s,t}(x))' \right] \nabla X_{s,t}(x)' dW_t^k$$

These expansions, when combined with condition  $(\mathcal{T})_2$ , yield the following estimates of the difference between  $X_t(x)$  and  $X_t(y)$ , i.e. the sample paths resulting from the different initialisations  $X_0 = x$  and  $X_0 = y$ .

**Proposition 3.1.** *Assume  $(\mathcal{T})_2$  is satisfied. Then*

$$\mathbb{E} (\|X_t(x) - X_t(y)\|^2)^{1/2} \leq \sqrt{d} e^{-\lambda_A(t-s)} \|x - y\|. \quad (3.2)$$

*In addition, we have the almost sure estimate*

$$\nabla\sigma = 0 \implies \|X_{s,t}(x) - X_{s,t}(y)\| \leq e^{-\lambda_A(t-s)} \|x - y\| \quad (3.3)$$

*Proof of Prop. 3.1.* Whenever  $(\mathcal{T})_2$  is met, we have the following uniform estimate from (3.1)

$$(\mathcal{T})_2 \implies \mathbb{E} (\|\nabla X_{s,t}(x)\|_2^2)^{1/2} \leq \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^2)^{1/2} \leq \sqrt{d} e^{-\lambda_A(t-s)} \quad (3.4)$$

where the  $\sqrt{d}$  term arises from imposing the initial condition  $\nabla X_{s,s}(x) = I$  on the resulting differential equation for  $\partial_t \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^2)^{1/2}$ . In addition, when  $\nabla\sigma = 0$  the martingale  $M_{s,t}(x) = 0$  is null and as a consequence of (3.1) we have the following almost sure estimate

$$\|\nabla X_{s,t}\|_2 := \sup_x \|\nabla X_{s,t}(x)\|_2 \leq e^{-\lambda_A(t-s)} \quad (3.5)$$

The Taylor expansion

$$\begin{aligned} X_{s,t}(x) - X_{s,t}(y) &= \int_0^1 \nabla X_{s,t}(\epsilon x + (1-\epsilon)y)'(x-y) d\epsilon \\ \implies \|X_{s,t}(x) - X_{s,t}(y)\|^2 &\leq \left[ \int_0^1 \|\nabla X_{s,t}(\epsilon x + (1-\epsilon)y)\|_2^2 d\epsilon \right] \|x - y\|^2 \end{aligned}$$

combined with (3.4) and (3.5) completes the proof.  $\square$

These contraction inequalities quantify the stability of the stochastic flow  $X_{s,t}(x)$  w.r.t. the initial state  $x$ . For instance, the estimate (3.2) ensures that the Markov transition semigroup is exponentially stable; that is, we have that

$$\mathbb{W}_2(\mu_0 P_{s,t}, \mu_1 P_{s,t}) \leq c \exp[-\lambda_A(t-s)] \mathbb{W}_2(\mu_0, \mu_1) \quad (3.6)$$

For the Langevin diffusions discussed in (2.18) the stochastic flow is time homogeneous; that is we have that  $X_{s,t} = X_{t-s} := X_{0,(t-s)}$  and  $P_{s,t} = P_{t-s} := P_{0,(t-s)}$ . In addition when  $\sigma(x) = \sigma I$ , the diffusion flow  $X_t(x)$  has a single invariant measure on  $\mathbb{R}^d$  given by the Boltzmann-Gibbs measure

$$\pi(dx) = \frac{1}{Z} \exp\left(-\frac{2}{\sigma^2} U(x)\right) dx \quad \text{with} \quad Z := \int e^{-\frac{2}{\sigma^2} U(x)} dx \quad (3.7)$$

From (2.18), it follows that

$$\nabla^2 U \geq \lambda I \implies \mathbb{W}_n(\mu P_{s,t}, \pi) \leq \exp[-\lambda(t-s)] \mathbb{W}_n(\mu, \pi)$$

for all  $n \geq 1$ .

Taking the trace in (3.1) we also find that

$$d \|\nabla X_{s,t}(x)\|_F^2 = \text{Tr} [\nabla X_{s,t}(x) A_t(X_{s,t}(x)) \nabla X_{s,t}(x)'] dt + dN_{s,t}(x)$$

with the martingale

$$dN_{s,t}(x) = \sum_{1 \leq k \leq r} \text{Tr} (\nabla X_{s,t}(x) [\nabla \sigma_{t,k}(X_{s,t}(x)) + \nabla \sigma_{t,k}(X_{s,t}(x))'] \nabla X_{s,t}(x)') dW_t^k$$

Observe that

$$\partial_t \langle N_{s,\cdot}(x) \rangle_t = \sum_k \text{Tr} (\nabla X_{s,t}(x) [\nabla \sigma_{t,k}(X_{s,t}(x)) + \nabla \sigma_{t,k}(X_{s,t}(x))'] \nabla X_{s,t}(x)')^2$$

This implies that

$$\begin{aligned} \partial_t \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^4) &= 2 \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^2 \text{Tr} [\nabla X_{s,t}(x) A_t(X_{s,t}(x)) \nabla X_{s,t}(x)']) \\ &\quad + \sum_{1 \leq k \leq r} \mathbb{E} \left( \text{Tr} (\nabla X_{s,t}(x) [\nabla \sigma_{t,k}(X_{s,t}(x)) + \nabla \sigma_{t,k}(X_{s,t}(x))'] \nabla X_{s,t}(x)')^2 \right) \end{aligned}$$

Whenever  $(\mathcal{T})_2$  is met, we have the estimate

$$\partial_t \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^4) \leq -4 [\lambda_A - \rho(\nabla \sigma)^2] \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^4)$$

with the uniform log-norm parameter  $\rho(\nabla \sigma)$  defined in (2.5). This yields the estimate

$$\partial_t \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^4)^{1/4} \leq -[\lambda_A - \rho(\nabla \sigma)^2] \mathbb{E} (\|\nabla X_{s,t}(x)\|_F^4)^{1/4}$$

More generally, we readily check the following result.

**Proposition 3.2.** *When condition  $(\mathcal{T})_n$  is met we have the following time-uniform bounds,*

$$\mathbb{E} (\|\nabla X_{s,t}(x)\|_F^n)^{1/n} \leq \sqrt{d} e^{-[\lambda_A - (n-2)\rho(\nabla \sigma)^2/2](t-s)} \quad (3.8)$$

### 3.2 The Hessian process

In terms of the tensor product (2.1), we have the matrix diffusion equation

$$\begin{aligned} d \nabla^2 X_{s,t}(x) &= [ [\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)] \nabla^2 b_t(X_{s,t}(x)) + \nabla^2 X_{s,t}(x) \nabla b_t(X_{s,t}(x))] dt + d\mathcal{M}_{s,t}(x) \end{aligned}$$

with the null matrix initial condition  $\nabla^2 X_{s,s}(x) = 0$  and the matrix-valued martingale

$$d\mathcal{M}_{s,t}(x) = \sum_{1 \leq k \leq r} ([\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)] \nabla^2 \sigma_{t,k}(X_{s,t}(x)) + \nabla^2 X_{s,t}(x) \nabla \sigma_{t,k}(X_{s,t}(x))) dW_t^k$$

Consider the tensor functions

$$v_t := \sum_{1 \leq k \leq d} (\nabla^2 \sigma_{t,k}) (\nabla^2 \sigma_{t,k})' \quad \text{and} \quad \tau_t := \nabla^2 b_t + \sum_{1 \leq k \leq d} (\nabla^2 \sigma_{t,k}) (\nabla \sigma_{t,k})' \quad (3.9)$$

After some computations, we check that

$$\begin{aligned} & d \left[ \nabla^2 X_{s,t}(x) \nabla^2 X_{s,t}(x)' \right] \\ &= \left\{ \left[ \nabla^2 X_{s,t}(x) A_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right] + 2 \left[ \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right] \tau_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right]_{sym} \right. \\ &\quad \left. + \left[ \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right] v_t(X_{s,t}(x)) \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right]' \right] \right\} dt + d\mathcal{N}_{s,t}(x) \end{aligned}$$

with the matrix function  $A_t(x)$  defined in (2.7) and the tensor-valued martingale

$$\begin{aligned} d\mathcal{N}_{s,t}(x) = & 2 \sum_{1 \leq k \leq r} \left\{ \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right] \nabla^2 \sigma_{t,k}(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right. \\ & \left. + \nabla^2 X_{s,t}(x) \nabla \sigma_{t,k}(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right\}_{sym} dW_t^k \end{aligned}$$

When  $\nabla \sigma = 0$  the above equation reduces to

$$\begin{aligned} & \partial_t \left[ \nabla^2 X_{s,t}(x) \nabla^2 X_{s,t}(x)' \right] \\ &= \left[ \nabla^2 X_{s,t}(x) A_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right] + 2 \left[ \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right] \nabla^2 b_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right]_{sym} \end{aligned}$$

Whenever  $(\mathcal{T})_2$  is met, taking the trace in the above display we check that

$$\partial_t \|\nabla^2 X_{s,t}(x)\|_F^2 \leq -2\lambda_A \|\nabla^2 X_{s,t}(x)\|_F^2 + 2\|\nabla^2 b\|_F \|\nabla X_{s,t}(x)\|_F^2 \|\nabla^2 X_{s,t}(x)\|_F$$

This yields the estimate

$$\partial_t \|\nabla^2 X_{s,t}(x)\|_F \leq -\lambda_A \|\nabla^2 X_{s,t}(x)\|_F + \|\nabla^2 b\|_F \|\nabla X_{s,t}(x)\|_F^2$$

Using (2.15) this implies that

$$\|\nabla^2 X_{s,t}(x)\|_F \leq \|\nabla^2 b\|_F e^{-\lambda_A(t-s)} \int_s^t e^{\lambda_A(u-s)} \|\nabla X_{s,u}(x)\|_F^2 du \leq \frac{d}{\lambda_A} \|\nabla^2 b\|_F e^{-\lambda_A(t-s)}$$

This ends the proof of the almost sure estimate (2.16).

For more general models, we have that

$$\begin{aligned} & d \|\nabla^2 X_{s,t}(x)\|_F^2 \\ &= \left\{ \text{Tr} \left[ \nabla^2 X_{s,t}(x) A_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right] + 2 \text{Tr} \left[ \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right] \tau_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right] \right. \\ &\quad \left. + \text{Tr} \left[ \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right] v_t(X_{s,t}(x)) \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right]' \right] \right\} dt + dM_{s,t}(x) \end{aligned}$$

with a continuous martingale  $M_{s,t}(x)$  with angle bracket

$$\begin{aligned} & \partial_t \langle M_{s,\cdot}(x) \rangle_t \\ &= 4 \sum_{1 \leq k \leq r} \text{Tr} \left\{ \left[ \nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x) \right] \nabla^2 \sigma_{t,k}(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right. \\ &\quad \left. + \nabla^2 X_{s,t}(x) \nabla \sigma_{t,k}(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \right\}^2 \end{aligned}$$

**Proposition 3.3.** *Assume  $(\mathcal{T})_n$  is met. In this situation, for any  $\epsilon > 0$  s.t.  $\lambda_A(n) > \epsilon$  we have*

$$\mathbb{E} \left( \|\nabla^2 X_{s,t}(x)\|_F^n \right)^{1/n} \leq n \epsilon^{-1} \chi(b, \sigma) \exp \left( -[\lambda_A(n) - \epsilon](t-s) \right) \quad (3.10)$$

with the parameters  $\chi(b, \sigma)$  and  $\lambda_A(n)$  defined in (2.6) and (2.8).

In the above display,  $\rho_\star(\nabla \sigma)$  is defined in (2.5). The proof of the above estimate is technical and thus housed in the appendix on page 47

### 3.3 Bismut-Elworthy-Li formulae

We further assume that ellipticity condition (2.19) is met. In this situation, we can extend gradient semigroup formulae to measurable functions using the Bismut-Elworthy-Li formula

$$\nabla P_{s,t}(f)(x) = \mathbb{E} \left( f(X_{s,t}(x)) \tau_{s,t}^\omega(x) \right) \quad (3.11)$$

with the stochastic process

$$\tau_{s,t}^\omega(x) := \int_s^t \partial_u \omega_{s,t}(u) \nabla X_{s,u}(x) a_u(X_{s,u}(x))^{-1/2} dW_u$$

The above formula is valid for any function  $\omega_{s,t} : u \in [s, t] \mapsto \omega_{s,t}(u) \in \mathbb{R}$  of the following form

$$\omega_{s,t}(u) = \varphi((u-s)/(t-s)) \implies \partial_u \omega_{s,t}(u) = \frac{1}{t-s} \partial \varphi((u-s)/(t-s)) \quad (3.12)$$

for some non decreasing differentiable function  $\varphi$  on  $[0, 1]$  with bounded continuous derivatives and such that

$$(\varphi(0), \varphi(1)) = (0, 1) \implies \omega_{s,t}(t) - \omega_{s,t}(s) = 1$$

Whenever  $(\mathcal{T})_2$  is met, combining (3.4) with (3.11), for any  $f$  s.t.  $\|f\| \leq 1$  we check that

$$\begin{aligned} \|\nabla P_{s,t}(f)\|^2 &\leq \mathbb{E} \left( \|\tau_{s,t}^\omega(x)\|^2 \right) \\ &\leq \kappa_1 \int_s^t e^{-2\lambda_A(u-s)} \|\partial_u \omega_{s,t}(u)\|^2 du = \frac{\kappa_1}{t-s} \int_0^1 e^{-2\lambda_A(t-s)v} (\partial \varphi(v))^2 dv \end{aligned}$$

Let  $\varphi_\epsilon$  with  $\epsilon \in ]0, 1[$  be some differentiable function on  $[0, 1]$  null on  $[0, 1 - \epsilon]$  and such that  $|\partial \varphi_\epsilon(u)| \leq c/\epsilon$  and  $(\varphi_\epsilon(1 - \epsilon), \varphi_\epsilon(1)) = (0, 1)$ . In this situation, we check that

$$\|\nabla P_{s,t}(f)\|^2 \leq \frac{\kappa_2}{\epsilon^2} \frac{1}{t-s} \int_{1-\epsilon}^1 e^{-2\lambda_A(t-s)v} dv$$

from which we find the rather crude uniform estimate

$$\|\nabla P_{s,t}(f)\| \leq \frac{\kappa}{\epsilon} \frac{1}{\sqrt{t-s}} e^{-\lambda_A(1-\epsilon)(t-s)} \quad (3.13)$$

In the same vein, for any  $s \leq u \leq t$  we have the formulae

$$\nabla^2 P_{s,t}(f)(x) = \mathbb{E} \left( f(X_{s,t}(x)) \tau_{s,t}^{[2],\omega}(x) + \nabla X_{s,t}(x) \nabla f(X_{s,t}(x)) \tau_{s,t}^\omega(x)' \right) \quad (3.14)$$

$$= \mathbb{E} \left( f(X_{s,t}(x)) \left[ \tau_{s,u}^{[2],\omega}(x) + \nabla X_{s,u}(x) \tau_{u,t}^\omega(X_{s,u}(x)) \tau_{s,u}^\omega(x)' \right] \right) \quad (3.15)$$

with the process

$$\begin{aligned} &\tau_{s,t}^{[2],\omega}(x) \\ &:= \int_s^t \partial_u \omega_{s,t}(u) \left[ \nabla^2 X_{s,u}(x) a_u(X_{s,u}(x))^{-1/2} + (\nabla X_{s,u}(x) \otimes \nabla X_{s,u}(x)) (\bar{\nabla} a_u^{-1/2})(X_{s,u}(x)) \right] dW_u \end{aligned}$$

In the above display  $\bar{\nabla} a_u^{-1/2}$  stands for the tensor function

$$(\bar{\nabla} a_u^{-1/2}(x))_{(i,j),k} := \partial_{x_i} a_u^{-1/2}(x)_{j,k} = - \left( a_u^{-1/2}(x) \left[ \partial_{x_i} a_u^{1/2}(x) \right] a_u^{-1/2}(x) \right)_{j,k}$$

Observe that

$$(2.19) \implies \sup_i \|\partial_{x_i} a_u^{-1/2}(x)\| \leq c \|\nabla \sigma\|/v$$

Whenever  $(\mathcal{T})_2$  is met, using the estimate (3.3) for any  $\epsilon \in ]0, 1[$

$$\|\nabla^2 P_{s,t}(f)\| \leq \frac{\kappa}{\epsilon} \frac{1}{\sqrt{t-s}} e^{-\lambda_A(t-s)(1-\epsilon)} (\|f\| + \|\nabla f\|) \quad (3.16)$$

In the same vein, using (3.15) for any  $u \in ]s, t[$  and any bounded measurable function  $f$  s.t.  $\|f\| \leq 1$  we also check the rather crude uniform estimate

$$\|\nabla^2 P_{s,t}(f)\| \leq \frac{\kappa_1}{\epsilon} \frac{1}{\sqrt{u-s}} e^{-\lambda_A(u-s)(1-\epsilon)} + \frac{\kappa_2}{\epsilon^2} \frac{1}{\sqrt{(t-u)(u-s)}} e^{-\lambda_A(u-s)} e^{-\lambda_A(t-s)(1-\epsilon)}$$

Choosing  $u = s + (1-\epsilon)(t-s)$  in the above display we check that for any  $\epsilon \in ]0, 1[$  we obtain the uniform estimate

$$\|\nabla^2 P_{s,t}(f)\| \leq \frac{\kappa}{\epsilon} \frac{1}{\sqrt{t-s}} e^{-\lambda_A(t-s)(1-\epsilon)} \left(1 + \frac{1}{\epsilon} \frac{1}{\sqrt{t-s}} e^{-\lambda_A(t-s)(1-\epsilon)}\right) \|f\| \quad (3.17)$$

The extended versions of the above formulae in the context of diffusions on differentiable manifolds can be found in the series of articles [6, 12, 21, 32, 40].

## 4 Backward semigroup analysis

### 4.1 The two-sided stochastic integration

For any given time horizon  $s \leq t$  we have the rather well known backward stochastic flow equation

$$X_{s,t}(x) = x + \int_s^t \left[ \nabla X_{u,t}(x)' b_u(x) + \frac{1}{2} \nabla^2 X_{u,t}(x)' a_u(x) \right] du + \int_s^t \nabla X_{u,t}(x)' \sigma_u(x) dW_u \quad (4.1)$$

In a more synthetic form, the above backward formula reduces to (1.5).

An elementary proof of the above formula based on Taylor expansions is presented in [16], different approaches can also be found in [29] and [31]. Extensions of the backward Itô formula (4.1) to jump type diffusion models as well as nonlinear diffusion flows can also be found in [15] and in the appendix of [3].

Consider the discrete time interval  $[s, t]_h := \{u_0, \dots, u_{n-1}\}$  associated with some refining time mesh  $u_{i+1} = u_i + h$  from  $u_0 = s$  to  $u_n = t$ , for some time step  $h > 0$ . In this notation, combining (1.4) with (1.6) for any  $u \in [s, t]_h$  we have the Taylor type approximation

$$\begin{aligned} & X_{u+h,t} \circ \bar{X}_{s,u+h} - X_{u,t} \circ \bar{X}_{s,u} \\ & \simeq - \left( (\nabla X_{u+h,t}) (\bar{X}_{s,u}(x))' \Delta b_u(\bar{X}_{s,u}(x)) + \frac{1}{2} (\nabla^2 X_{u+h,t}) (\bar{X}_{s,u}(x))' \Delta a_u(\bar{X}_{s,u}(x)) \right) h \\ & \quad - (\nabla X_{u+h,t}) (\bar{X}_{s,u}(x))' \Delta \sigma_u(\bar{X}_{s,u}(x)) (W_{u+h} - W_u) \end{aligned}$$

We obtain formally (1.8) by summing the above terms and passing to the limit  $h \downarrow 0$ .

To be more precise, we follow the two-sided stochastic integration calculus introduced by Pardoux and Protter in [38]. As mentioned by the authors this methodology can be seen as a variation of Itô

original construction of the stochastic integral. In this framework, the Skorohod stochastic integral (1.10) arising in (1.7) is defined by the  $\mathbb{L}_2$ -convergence

$$\begin{aligned} & S_{s,t}(\varsigma)(x) \\ & := \lim_{h \rightarrow 0} \sum_{u \in [s,t]_h} (\nabla X_{u+h,t})(\bar{X}_{s,u}(x))' \varsigma_u(\bar{X}_{s,u}(x)) (W_{u+h} - W_u) \quad \text{with} \quad \varsigma_u = \Delta \sigma_u \end{aligned} \quad (4.2)$$

The proof of the above assertion is based on a slight extension of proposition 3.3 in [38] to Skorohod integrals of the form (1.10). For the convenience of the reader, a detailed proof of the above assertion for one dimensional models is provided in the appendix A.

Using (4.2), the complete proof of (1.7) now follows the same line of arguments as the ones used in the proof of Itô-type change rule formula stated in theorem 6.1 in [38], thus it is skipped.

## 4.2 A generalized backward Itô-Ventzell formula

This section is mainly concerned with a more precise statement of theorem 1.2.

Let  $F_{s,t}(x)$  and  $Y_{s,t}$  be the forward and backward random field model defined as in (1.12) for some functions satisfying the following conditions:

(H1): *The functions  $G_{u,t}$ ,  $\nabla H_{u,t}$ ,  $\nabla^2 F_{u,t}$  and the Malliavin derivatives  $D_v \nabla F_{u,t}(x)$  and  $D_v H_{u,t}(x)$  are continuous w.r.t.  $x$  for any given  $u, v \in [s, t]$  and  $\omega \in \Omega$ .*

(H2) *The function  $G_{u,t}$ ,  $\nabla^2 F_{u,t}$ ,  $\nabla H_{u,t}$ , and the Malliavin derivatives  $D_v H_{u,t}$ ,  $D_v \nabla F_{u,t}$  have at most polynomial growth w.r.t. the state variable, uniformly with respect to  $\omega \in \Omega$ .*

(H3) *The random processes  $B_{s,u}$ ,  $\Sigma_{s,u}$  as well as the Malliavin derivatives  $D_v \Sigma_{s,u}$  have moments of any order.*

**Theorem 4.1** (Generalized backward Itô-Ventzell type formula). *Assume that. the conditions (H1), (H2) and (H3) stated above are met. In this situation, we have. the generalized backward Itô-Ventzell type formula (1.13).*

The proof of the above theorem is provided in Appendix A (see theorem 7.4).

In terms of the tensor product (2.1), for any  $p \geq 1$  and any twice differentiable function  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}^p$  with at most polynomial growth the function  $F_{s,t} := \mathbb{P}_{s,t}(f)$  satisfies the backward formula (1.12) with the random fields

$$G_{u,t}(x) := \nabla F_{u,t}(x)' b_u(x) + \frac{1}{2} \nabla^2 F_{u,t}(x)' a_u(x) \quad \text{and} \quad H_{u,t}(x) := \nabla F_{u,t}(x)' \sigma_u(x)$$

Using the quantitative estimates presented in section 6.2, we checked that conditions (H1), (H2) and (H3) stated above are satisfied. Rewritten in terms of the stochastic semigroups  $\mathbb{P}_{s,t}$  and  $\bar{\mathbb{P}}_{s,t}$  we obtain the forward-backward multivariate interpolation formula

$$\mathbb{P}_{s,t}(f)(x) - \bar{\mathbb{P}}_{s,t}(f)(x) = \mathbb{T}_{s,t}(f, \Delta a, \Delta b)(x) + \mathbb{S}_{s,t}(f, \Delta \sigma)(x) \quad (4.3)$$

with the stochastic integro-differential operator

$$\begin{aligned} & \mathbb{T}_{s,t}(f, \Delta a, \Delta b)(x) \\ & := \int_s^t \left[ \nabla \mathbb{P}_{u,t}(f)(\bar{X}_{s,u}(x))' \Delta b_u(\bar{X}_{s,u}(x)) + \frac{1}{2} \nabla^2 \mathbb{P}_{u,t}(f)(\bar{X}_{s,u}(x))' \Delta a_u(\bar{X}_{s,u}(x)) \right] du \end{aligned} \quad (4.4)$$

and the Skorohod stochastic integral term given by

$$\mathbb{S}_{s,t}(f, \Delta\sigma)(x) := \int_s^t \nabla \mathbb{P}_{u,t}(f)(\bar{X}_{s,u}(x))' \Delta\sigma_u(\bar{X}_{s,u}(x)) dW_u \quad (4.5)$$

Using elementary differential calculus, for twice differentiable (column vector-valued) function  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}^p$  we readily check the gradient and the Hessian formulae

$$\begin{aligned} \nabla \mathbb{P}_{s,t}(f)(x) &= \nabla X_{s,t}(x) \mathbb{P}_{s,t}(\nabla f)(x) \\ \nabla^2 \mathbb{P}_{s,t}(f)(x) &= [\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)] \mathbb{P}_{s,t}(\nabla^2 f)(x) + \nabla^2 X_{s,t}(x) \mathbb{P}_{s,t}(\nabla f)(x) \end{aligned} \quad (4.6)$$

Observe that (4.3) coincides with (1.8) for the identity function; that is, we have that

$$f(x) = x \implies \mathbb{T}_{s,t}(f, \Delta a, \Delta b) = T_{s,t}(\Delta a, \Delta b) \quad \text{and} \quad \mathbb{S}_{s,t}(f, \Delta\sigma) = S_{s,t}(\Delta\sigma)$$

The above discussion shows that the analysis of the differences of the stochastic semigroups ( $\mathbb{P}_{s,t} - \bar{\mathbb{P}}_{s,t}$ ) in terms of the tangent and the Hessian processes is essentially the same as the one of the difference of the stochastic flows ( $X_{s,t} - \bar{X}_{s,t}$ ). For instance, the estimates stated in theorem 1.1 can be easily extended at the level of the stochastic semigroups using the discussion provided section 6.3.

The  $\mathbb{L}_2$ -norm of the Skorohod stochastic integrals in (1.8) and (4.3) are uniformly estimated as soon as the pair of drift and diffusion functions  $(b_t, \sigma_t)$  and  $(\bar{b}, \bar{\sigma}_t)$  satisfy condition  $(\mathcal{T})_2$ . For a more thorough discussion we refer to section 6.1, see for instance the  $\mathbb{L}_n$ -norm estimates presented in theorem 6.2.

### 4.3 Semigroup perturbation formulae

Besides the fact that the Skorohod integral in the r.h.s. of (4.3) is not a martingale (w.r.t. the Brownian motion filtration) it is centered. Thus, taking the expectation in the univariate version of (4.3) we obtain the following interpolation semigroup decomposition.

**Corollary 4.2.** *For any twice differentiable function  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}$  with bounded derivatives we have the forward-backward semigroup interpolation formula*

$$\begin{aligned} P_{s,t}(f)(x) - \bar{P}_{s,t}(f)(x) &= \int_s^t \mathbb{E} \left( \langle \nabla P_{u,t}(f)(\bar{X}_{s,u}(x)), \Delta b_u(\bar{X}_{s,u}(x)) \rangle \right) du \\ &\quad + \frac{1}{2} \int_s^t \mathbb{E} \left( \text{Tr} \left[ \nabla^2 P_{u,t}(f)(\bar{X}_{s,u}(x)) \Delta a_u(\bar{X}_{s,u}(x)) \right] \right) du \end{aligned} \quad (4.7)$$

*In addition, under some appropriate regularity conditions for any differentiable function  $f$  such that  $\|f\| \leq 1$  and  $\|\nabla f\| \leq 1$  we have the uniform estimate*

$$|P_{s,t}(f)(x) - \bar{P}_{s,t}(f)(x)| \leq \kappa \left[ \|\Delta a(x)\|_1 + \|\Delta b(x)\|_1 \right] \quad (4.8)$$

Rewritten in terms of the infinitesimal generators  $(L_t, \bar{L}_t)$  of the stochastic flows  $(X_{s,t}, \bar{X}_{s,t})$  we recover the rather well known semigroup perturbation formula

$$P_{s,t} = \bar{P}_{s,t} + \int_s^t \bar{P}_{s,u}(L_u - \bar{L}_u)P_{u,t} du$$



The above formula can be readily checked using the interpolating formula given for any  $s \leq u < t$  by the evolution equation

$$\partial_u(\overline{P}_{s,u}P_{u,t}) = (\partial_u\overline{P}_{s,u})P_{u,t} + \overline{P}_{s,u}(\partial_uP_{u,t}) = \overline{P}_{s,u}\overline{L}_uP_{u,t} - \overline{P}_{s,u}L_uP_{u,t}$$

Now we come to the proof of (4.8). Whenever  $(\mathcal{T})_2$  is met, combining (3.13) with (3.16) for any differentiable function  $f$  s.t.  $\|f\| \leq 1$  and  $\|\nabla f\| \leq 1$  and for any  $\epsilon \in ]0, 1[$  we check that

$$|P_{s,s+t}(f)(x) - \overline{P}_{s,s+t}(f)(x)| \leq \frac{\kappa}{\epsilon} [\|\Delta a(x)\|_1 + \|\Delta b(x)\|_1] \int_0^t \frac{1}{\sqrt{u}} e^{-\lambda_A(1-\epsilon)u} du$$

This ends the proof of (4.8). ■

After some elementary manipulations the forward-backward interpolation formula (4.7) yields the following corollary.

**Corollary 4.3.** *Let  $X_t$  and  $\overline{X}_t$  be some ergodic diffusions associated with some time homogeneous drift and diffusion functions  $(b, \sigma)$  and  $(\overline{b}, \overline{\sigma})$ . The invariant probability measures  $\pi$  and  $\overline{\pi}$  of  $X_t$  and  $\overline{X}_t$  are connected for any twice differentiable function  $f$  from  $\mathbb{R}^d$  into  $\mathbb{R}$  with bounded derivatives by the following interpolation formula*

$$(\pi - \overline{\pi})(f) = \int_0^\infty \mathbb{E} \left( \langle \nabla P_t(f)(\overline{Y}), \Delta b(\overline{Y}) \rangle + \frac{1}{2} \text{Tr} [\nabla^2 P_t(f)(\overline{Y}) \Delta a(\overline{Y})] \right) dt \quad (4.9)$$

In the above display  $\overline{Y}$  stands for a random variable with distribution  $\overline{\pi}$  and  $P_t$  stands for the Markov transition semigroup of the process  $X_t$ .

The formula (4.9) can be used to estimate the invariant measure of a stochastic flow associated with some perturbations of the drift and the diffusion function.

For instance, for homogeneous Langevin diffusions  $X_t$  associated with some convex potential function  $U$  we have

$$b = -\nabla U \quad \text{and} \quad \sigma = I \quad \implies \quad \pi(dx) \propto \exp(-2U(x)) dx$$

In the above display,  $dx$  stands for the Lebesgue measure on  $\mathbb{R}^d$ . In this situation, using (4.9), for any ergodic diffusion flow  $\overline{X}_t$  with some drift  $\overline{b}$  and an unit diffusion matrix we have

$$\overline{\pi}(f) = \pi(f) + \int_0^\infty \mathbb{E} (\langle (\overline{b} + \nabla U)(\overline{Y}), \nabla P_t(f)(\overline{Y}) \rangle) dt$$

Notice that the above formula is implicit as the r.h.s. term depends on  $\overline{\pi}$ . By symmetry arguments, we also have the following more explicit perturbation formula

$$\overline{\pi}(f) = \pi(f) + \int_0^\infty \mathbb{E} (\langle (\overline{b} + \nabla U)(Y), \nabla \overline{P}_t(f)(Y) \rangle) dt$$

In the above display  $Y$  stands for a random variable with distribution  $\pi$  and  $\overline{P}_t$  stands for the Markov transition semigroup of the process  $\overline{X}_t$ .

## 5 Some extensions

Several extensions of the forward-backward stochastic interpolation formula (1.8) to more general stochastic perturbation processes can be developed. For instance, suppose we are given some stochastic processes  $\bar{Y}_{s,t}(x) \in \mathbb{R}^d$  and  $\bar{Z}_{s,t}(x) \in \mathbb{R}^{d \times r}$  adapted to the filtration of the Brownian motion  $W_t$ , and let  $\bar{X}_{s,t}(x)$  be the stochastic flow defined by the stochastic differential equation

$$d\bar{X}_{s,t}(x) = \bar{Y}_{s,t}(x) dt + \bar{Z}_{s,t}(x) dW_t \quad (5.1)$$

In this situation, the interpolation formula (1.7) remains valid when  $\bar{a}_u(\bar{X}_{s,u}(x))$  is replaced by the stochastic matrices  $\bar{Z}_{s,t}(x)\bar{Z}_{s,t}(x)'$ . This yields without further work the forward-backward stochastic interpolation formula (1.8) with the local perturbations

$$\begin{aligned} \Delta b_u(\bar{X}_{s,u}(x)) &:= b_u(\bar{X}_{s,u}(x)) - \bar{Y}_{s,u}(x) \\ \Delta \sigma_u(\bar{X}_{s,u}(x)) &:= \sigma_u(\bar{X}_{s,u}(x)) - \bar{Z}_{s,u}(x) \quad \text{and} \quad \Delta a_u(\bar{X}_{s,u}(x)) := a_u(\bar{X}_{s,u}(x)) - \bar{Z}_{s,u}(x)\bar{Z}_{s,u}(x)' \end{aligned}$$

The corresponding interpolation formula should be used with some caution as the  $\mathbb{L}_2$ -norm of the Skorohod integral (1.10) depends on the Malliavin differential of the integrand process of the Brownian motion; see for instance the variance formula provided in lemma 6.1.

### 5.1 Non Markovian perturbations

Assume that  $\sigma = I$  and the regularity condition  $(\mathcal{T})_2$  is met. Also suppose  $\bar{X}_{s,t}(x)$  is given by a stochastic differential equation of the form (5.1) with  $r = d$  and  $\bar{Z}_{s,t}(x) = I$ . Arguing as above, in terms of the tensor product (2.1) we have

$$X_{s,t}(x) - \bar{X}_{s,t}(x) = \int_s^t (\nabla X_{u,t})(\bar{X}_{s,u}(x))' (b_u(\bar{X}_{s,u}(x)) - \bar{Y}_{s,u}(x)) du \quad (5.2)$$

Combining (2.15) with the generalized Minkowski inequality, we check the following proposition.

**Proposition 5.1.** *Assume that  $(\mathcal{T})_2$  is met for some  $\lambda_A > 0$ . In this situation, for any  $1 \leq n \leq \infty$  we have the estimates*

$$\mathbb{E} [\|X_{s,t}(x) - \bar{X}_{s,t}(x)\|^n]^{1/n} \leq \int_s^t e^{-\lambda_A(t-u)} \mathbb{E} [\|b_u(\bar{X}_{s,u}(x)) - \bar{Y}_{s,u}(x)\|^n]^{1/n} du \quad (5.3)$$

In the same vein, we have

$$P_{s,t}(f)(x) - \bar{P}_{s,t}(f)(x) = \int_s^t \mathbb{E} (\langle \nabla P_{u,t}(f)(\bar{X}_{s,u}(x)), b_u(\bar{X}_{s,u}(x)) - \bar{Y}_{s,u}(x) \rangle) du \quad (5.4)$$

For instance, for the Langevin diffusion discussed in (2.18) and (3.7) the weak expansion (5.4) implies that

$$[\pi \bar{P}_{s,t} - \pi](f) = \int_s^t \int \pi(dx) \mathbb{E} (\langle \nabla P_{t-u}(f)(\bar{X}_{s,u}(x)), \nabla U(\bar{X}_{s,u}(x)) + \bar{Y}_{s,u}(x) \rangle) du \quad (5.5)$$

This yields the  $\mathbb{W}_1$ -Wasserstein estimate

$$\mathbb{W}_1(\pi \bar{P}_{s,t}, \pi) \leq \int_s^t e^{-\lambda_A(t-u)} \int \pi(dx) \mathbb{E} (\|\nabla U(\bar{X}_{s,u}(x)) + \bar{Y}_{s,u}(x)\|) du$$

Combining (3.13) with (5.5), for any  $\epsilon \in ]0, 1[$  we also have the total variation norm estimate

$$\|\pi \bar{P}_{s,t} - \pi\|_{tv} \leq \frac{c}{\epsilon} \int_s^t \frac{1}{\sqrt{t-u}} e^{-\lambda_A(1-\epsilon)(t-u)} \left[ \int \pi(dx) \mathbb{E} (\|\nabla U(\bar{X}_{s,u}(x)) + \bar{Y}_{s,u}(x)\|) \right] du \quad (5.6)$$

## 5.2 Jump diffusion perturbations

Assume that  $\sigma = 0$  and  $b = \bar{b}$ . In this situation, the Skorohod fluctuation term (1.10) reduces to a conventional Itô stochastic integral. In addition, in terms of the tensor product (2.1), formula (1.8) reduces to the interpolation decomposition

$$\bar{X}_{s,t}(x) - X_{s,t}(x) = \int_s^t \left[ \frac{1}{2} (\nabla^2 X_{u,t}) (\bar{X}_{s,u}(x))' \bar{a}_u(\bar{X}_{s,u}(x)) du + dM_u^{s,t}(x) \right]$$

In the above display,  $u \in [s, t] \mapsto M_u^{s,t}(x)$  stands for the multivariate martingale given by the formulae

$$dM_u^{s,t}(x) := (\nabla X_{u,t}) (\bar{X}_{s,u}(x))' d\bar{M}_{s,u}(x) \quad \text{with} \quad d\bar{M}_{s,u}(x) := \bar{\sigma}_u(X_{s,u}(x)) dW_u$$

We further assume that  $\bar{X}_{s,t}(x)$  is the stochastic flow associated with a jump diffusion process adapted to some filtration  $\mathcal{F}_t$  and given by an stochastic differential equation of the following form

$$d\bar{X}_{s,t}(x) = b_t(X_{s,t}(x)) dt + d\bar{M}_{s,t}(x) + d\bar{N}_{s,t}(x)$$

In the above display,  $\bar{N}_{s,t}(x)$  stands for some martingale associated with the jump of the process; that is, we have that

$$d\bar{N}_{s,t}(x) = \Delta_t \bar{X}_{s,t}(x) - \mathbb{E}(\Delta_t \bar{X}_{s,t}(x) | \mathcal{F}_{t-}) \quad \text{with} \quad \Delta_t \bar{X}_{s,t}(x) := \bar{X}_{s,t}(x) - \bar{X}_{s,t-}(x)$$

At some given jump rate  $\iota_t(\bar{X}_{s,t}(x))$  the càdlàg stochastic flow  $\bar{X}_{s,t-}(x)$  jumps to a new location  $\bar{X}_{s,t}(x) = y$  randomly chosen with some distribution denoted by  $J_t(\bar{X}_{s,t}(x), dy)$ .

In this context, the jumps of the interpolating process  $Z_u^{s,t}$  are given by

$$\Delta_u Z_u^{s,t}(x) = X_{u,t}(\bar{X}_{s,u-}(x) + \Delta_u \bar{X}_{s,u}(x)) - X_{u,t}(\bar{X}_{s,u-}(x))$$

We denote by  $u \in [s, t] \mapsto N_u^{s,t}(x)$  the multivariate martingale given by the formulae

$$dN_u^{s,t}(x) := \Delta_u Z_u^{s,t}(x) - \mathbb{E}(\Delta_u Z_u^{s,t}(x) | \mathcal{F}_{u-})$$

In this notation, following the same lines of arguments as in (1.7) and using the tensor product (2.1), we check the following proposition.

**Proposition 5.2.** *For any  $s \leq t$  and  $x \in \mathbb{R}^d$  we have backward-forward interpolation formula*

$$\begin{aligned} \bar{X}_{s,t}(x) - X_{s,t}(x) &= \int_s^t \left[ \frac{1}{2} (\nabla^2 X_{u,t}) (\bar{X}_{s,u}(x))' \bar{a}_u(\bar{X}_{s,u}(x)) du + dM_u^{s,t}(x) \right] \\ &\quad + \int_s^t \left[ \partial_u \mathbb{E}(\Delta_u Z_u^{s,t}(x) - (\nabla X_{u,t}) (\bar{X}_{s,u}(x))' \Delta_u \bar{X}_{s,u}(x) | \mathcal{F}_{u-}) du + dN_u^{s,t}(x) \right] \end{aligned}$$

Using the first order Taylor expansion

$$\begin{aligned} &\Delta_u Z_u^{s,t}(x) - (\nabla X_{u,t})(\bar{X}_{s,u-}(x))' \Delta_u \bar{X}_{s,u}(x) \\ &= \int_0^1 (1 - \epsilon) (\nabla^2 X_{u,t})(\bar{X}_{s,u-}(x) + \epsilon \Delta_u \bar{X}_{s,u}(x))' (\Delta_u \bar{X}_{s,u}(x)) \otimes^2 d\epsilon \end{aligned}$$

we also check the decomposition

$$\begin{aligned}\bar{X}_{s,t}(x) - X_{s,t}(x) &= (M_t^{s,t}(x) - M_s^{s,t}(x)) + (N_t^{s,t}(x) - N_s^{s,t}(x)) \\ &+ \frac{1}{2} \int_s^t (\nabla^2 X_{u,t}) (\bar{X}_{s,u}(x))' \bar{a}_u(\bar{X}_{s,u}(x)) du \\ &+ \int_s^t \int_0^1 (1-\epsilon) \partial_u \mathbb{E} \left( (\nabla^2 X_{u,t}) (\bar{X}_{s,u-}(x) + \epsilon \Delta_u \bar{X}_{s,u}(x))' (\Delta_u \bar{X}_{s,u}(x))^{\otimes 2} \mid \mathcal{F}_{u-} \right) d\epsilon du\end{aligned}$$

We further assume that  $(\mathcal{T})_2$  is met. In this situation, using (2.16) the norm of the last term in the above display is almost surely bounded by

$$\kappa \int_s^t e^{-\lambda_A(t-u)} \iota_{u-}(\bar{X}_{s,u-}(x)) \mathbb{E} (\|\Delta_u \bar{X}_{s,u}(x)\|^2 \mid \mathcal{F}_{u-}) du$$

In the same vein, using (2.15) we have

$$\begin{aligned}\mathbb{E}(\|N_t^{s,t}(x) - N_s^{s,t}(x)\|^2) &= \int_s^t \mathbb{E} [\iota_{u-}(\bar{X}_{s,u-}(x)) \|X_{u,t}(\bar{X}_{s,u-}(x) + \Delta_u \bar{X}_{s,u}(x)) - X_{u,t}(\bar{X}_{s,u-}(x))\|^2] du \\ &\leq \int_s^t e^{-2\lambda_A(t-u)} \mathbb{E} [\iota_{u-}(\bar{X}_{s,u-}(x)) \|\Delta_u \bar{X}_{s,u}(x)\|^2] du\end{aligned}$$

This yields the estimate

$$\begin{aligned}\mathbb{E}(\|\bar{X}_{s,t}(x) - X_{s,t}(x)\|^2)^{1/2} &\leq \kappa_1 (\|\bar{\sigma}(x)\|_2 + \|\bar{a}(x)\|_2) \\ &+ \kappa_2 \left( \int_s^t e^{-2\lambda_A(t-u)} \mathbb{E} (\iota_{u-}(\bar{X}_{s,u-}(x)) \|\Delta_u \bar{X}_{s,u}(x)\|^2) du \right)^{1/2} \\ &+ \kappa_3 \int_s^t e^{-\lambda_A(t-u)} \mathbb{E} \left( \iota_{u-}(\bar{X}_{s,u-}(x))^2 \mathbb{E} (\|\Delta_u \bar{X}_{s,u}(x)\|^2 \mid \mathcal{F}_{u-})^2 \right)^{1/2} du\end{aligned}$$

For instance, whenever the jump rates and amplitudes are uniformly bounded

$$\iota_{u-}(\bar{X}_{s,u-}(x)) \leq \kappa_0 \quad \text{and} \quad \mathbb{E} (\|\Delta_u \bar{X}_{s,u}(x)\|^2 \mid \mathcal{F}_{u-}) \leq \delta^2$$

for some parameters  $\kappa_0$  and  $\delta$  we have the uniform estimates

$$\mathbb{E}(\|\bar{X}_{s,t}(x) - X_{s,t}(x)\|^2)^{1/2} \leq \kappa (\delta (1 + \delta) + \|\bar{\sigma}(x)\|_2 + \|\bar{a}(x)\|_2)$$

## 6 Skorohod fluctuation processes

### 6.1 A variance formula

Let  $\varsigma_t(x)$  be some differentiable  $(d \times r)$ -matrix valued function on  $\mathbb{R}^d$  such that

$$\|\nabla \varsigma\| < \infty \quad \text{and} \quad \|\varsigma(0)\| := \sup_t \|\varsigma_t(0)\| < \infty \tag{6.1}$$

Observe that  $(W_{u+h} - W_u)$  is independent of the flows  $\bar{X}_{s,u}$  and  $\nabla X_{u+h,t}$ . This already implies that Skorohod stochastic integral is centered; that is, we have that  $\mathbb{E}(S_{s,t}(\varsigma)(x)) = 0$ .

Following (4.2), the variance can be computed using the following approximation formula

$$\begin{aligned} \mathbb{E} [\|S_{s,t}(\varsigma)(x)\|^2] &= \lim_{h \rightarrow 0} \sum_{u,v \in [s,t]_h} \sum_{1 \leq i \leq d} \sum_{1 \leq j,k \leq r} \\ &\mathbb{E} \left\{ [(\nabla X_{u+h,t}) (\bar{X}_{s,u}(x))' \varsigma_u(\bar{X}_{s,u}(x))]_{i,j} [(\nabla X_{v+h,t}) (\bar{X}_{s,v}(x))' \varsigma_v(\bar{X}_{s,v}(x))]_{i,k} \right. \\ &\quad \left. (W_{u+h}^j - W_u^j)(W_{v+h}^k - W_v^k) \right\} \end{aligned} \quad (6.2)$$

The proof of the above assertion is provided in the appendix, see for instance proposition 7.3.

Consider the matrix valued function

$$\Sigma_{s,u,t}(x) := [(\nabla X_{u,t})' \circ \bar{X}_{s,u}] (x) \varsigma_u(\bar{X}_{s,u}(x)) \quad (6.3)$$

In this notation, the limiting diagonal term  $u = v$  in the r.h.s. of (6.2) is clearly equal to

$$\int_s^t \mathbb{E} \left[ \sum_{i,j} \Sigma_{s,u,t}(x)_{i,j} \Sigma_{s,u,t}(x)_{i,j} \right] du = \int_s^t \mathbb{E} [\|\Sigma_{s,u,t}(x)\|_F^2] du$$

In addition, whenever condition  $(\mathcal{T})_2$  is met and  $\varsigma$  is bounded, (3.4) readily yields the estimate

$$\left[ \int_s^t \mathbb{E} [\|\Sigma_{s,u,t}(x)\|_F^2] du \right]^{1/2} \leq \|\varsigma\|_2 \sqrt{d/(2\lambda_A)} \quad (6.4)$$

More generally, using (3.8) whenever  $(\bar{\mathcal{M}})_{2/\delta}$  and  $(\mathcal{T})_{2/(1-\delta)}$  are met for some  $\delta \in ]0, 1[$  we have the estimate

$$\mathbb{E} [\|\Sigma_{s,u,t}(x)\|^2] \leq c_{1,\delta} [\|\varsigma(0)\|^2 + \|\nabla \varsigma\|^2 (1 + \|x\|)^2] e^{-2\lambda_A(2/(1-\delta))(t-u)}$$

This implies that

$$\left[ \int_s^t \mathbb{E} [\|\Sigma_{s,u,t}(x)\|_F^2] du \right]^{1/2} \leq c_{2,\delta} [\|\varsigma(0)\| + \|\nabla \varsigma\| (1 + \|x\|)] / \sqrt{\lambda_A} \quad (6.5)$$

The non-diagonal term can be computed in a more direct way using Malliavin derivatives of the functions  $\Sigma_{s,u,t}$ . For any  $s \leq u \leq v \leq t$  we have

$$D_v \{ [(\nabla X_{u,t})' \circ \bar{X}_{s,u}] [\varsigma_u \circ \bar{X}_{s,u}] \} = [(D_v (\nabla X_{u,t})' \circ \bar{X}_{s,u}) [\varsigma_u \circ \bar{X}_{s,u}]] \quad (6.6)$$

As expected, observe that

$$\nabla \sigma = 0 \implies D_v \Sigma_{s,u,t}(x) = 0$$

In the reverse angle, whenever  $s \leq v \leq u \leq t$  we have the chain rule formula

$$\begin{aligned} &D_v ([\varsigma_u \circ \bar{X}_{s,u}] [(\nabla X_{u,t}) \circ \bar{X}_{s,u}]) \\ &:= [D_v (\varsigma_u \circ \bar{X}_{s,u})] [(\nabla X_{u,t}) \circ \bar{X}_{s,u}] + [D_v \bar{X}_{s,u} \otimes (\varsigma_u \circ \bar{X}_{s,u})] [(\nabla^2 X_{u,t}) \circ \bar{X}_{s,u}] \end{aligned} \quad (6.7)$$

As above, Malliavin differentials  $D_v(\zeta_u \circ \bar{X}_{s,u})$  and  $D_v \bar{X}_{s,u}$  can be computed using the chain rule formulae (2.24).

A more detailed analysis of the chain rules formulae (2.24), (2.25) and (6.7) for one dimensional models is provided in Appendix A (cf. lemma 7.2).

Observe that

$$\nabla \zeta = 0 \implies D_v [\Sigma'_{s,u,t}] = [D_v \bar{X}_{s,u} \otimes (\zeta_u \circ \bar{X}_{s,u})] [(\nabla^2 X_{u,t}) \circ \bar{X}_{s,u}]$$

We consider the inner product

$$\langle D_u \Sigma_{s,v,t}(x), D_v \Sigma_{s,u,t}(x) \rangle := \sum_{i,j,k} (D_v \Sigma_{s,u,t}(x))_{k,i,j} (D_u \Sigma_{s,v,t}(x))_{j,i,k}$$

In this notation, an explicit description of the  $\mathbb{L}_2$ -norm of the Skorohod integral in terms of Malliavin derivatives is given below.

**Lemma 6.1.** *The  $\mathbb{L}_2$ -norm of the Skorohod integral  $S_{s,t}(\zeta)(x)$  introduced in (4.2) is given for any  $x \in \mathbb{R}^d$  and  $s \leq t$  by the formulae*

$$\mathbb{E} [\|S_{s,t}(\zeta)(x)\|^2] = \int_{[s,t]} \mathbb{E} [\|\Sigma_{s,u,t}(x)\|_F^2] du + \int_{[s,t]^2} \mathbb{E} [\langle D_v \Sigma_{s,u,t}(x), D_u \Sigma_{s,v,t}(x) \rangle] du dv$$

with the random matrix function  $\Sigma_{s,u,t}$  defined in (6.3) and the Malliavin derivative  $D_v \Sigma_{s,u,t}$  given in formulae (6.6) and (6.7). In addition, we have

$$\nabla \sigma = 0 \implies \mathbb{E} [\|S_{s,t}(\zeta)(x)\|^2] = \int_{[s,t]} \mathbb{E} [\|\Sigma_{s,u,t}(x)\|_F^2] du$$

The above lemma can be interpreted as a matrix version of the isometry property (2.26). A proof of the above lemma based on the  $\mathbb{L}_2$ -approximation of two-sided stochastic integrals is provided in Appendix A on page 37.

## 6.2 Quantitative estimates

For any  $p > 1$  and any tensor norms we also quote the rather well known  $\mathbb{L}_p$ -norm estimates

$$\begin{aligned} & \mathbb{E} [\|S_{s,t}(\zeta)(x)\|^p]^{2/p} \\ & \leq c_{1,p} \int_{[s,t]} \mathbb{E} [\|\Sigma_{s,u,t}(x)\|^2] du + c_{2,p} \mathbb{E} \left[ \left( \int_{[s,t]^2} \|D_v \Sigma_{s,u,t}(x)\|^2 du dv \right)^{p/2} \right]^{2/p} \end{aligned}$$

for some finite constants  $c_{i,p}$  whose values only depends on  $p$ . A proof of these estimates can be found in [34, 42], see also [35] for multiple Skorohod integrals. By the generalized Minkowski inequality, for any  $n \geq 2$  we also have the estimate

$$\begin{aligned} & \mathbb{E} [\|S_{s,t}(\zeta)(x)\|^n]^{2/n} \\ & \leq c_{1,n} \int_{[s,t]} \mathbb{E} [\|\Sigma_{s,u,t}(x)\|^2] du + c_{2,n} \int_{[s,t]^2} \mathbb{E} [\|D_v \Sigma_{s,u,t}(x)\|^n]^{2/n} du dv \end{aligned} \tag{6.8}$$

Observe that for any  $n \geq 2$  we have

$$(\overline{\mathcal{M}})_n \implies \|\zeta(x)\|_n \leq \kappa_n (\|\zeta(0)\| + \|\nabla \zeta\|) (1 \vee \|x\|)$$

The main objective of this section is to prove the following theorem.

**Theorem 6.2.** *Assume that  $(M)_{2n/\delta}$  and  $(T)_{2n/(1-\delta)}$  are satisfied for some parameter  $n \geq 2$  and some  $\delta \in ]0, 1[$ . In this situation, we have the uniform estimate*

$$\mathbb{E} [\|S_{s,t}(\varsigma)(x)\|^n]^{1/n} \leq \kappa_{\delta,n} \|\varsigma(x)\|_{2n/\delta} (1 \vee \|x\|) \quad (6.9)$$

For uniformly bounded diffusion functions  $(\varsigma, \sigma, \bar{\sigma})$  whenever  $(T)_{2n}$  is met for some  $n \geq 2$  we have

$$\mathbb{E} [\|S_{s,t}(\varsigma)(x)\|^n]^{1/n} \leq \kappa_n (\|\varsigma\| + \|\nabla \varsigma\|) \quad (6.10)$$

In addition, for constant diffusion functions  $(\varsigma, \sigma, \bar{\sigma})$  whenever  $(T)_2$  is met, for any  $n \geq 2$  we have the uniform estimate

$$\mathbb{E} [\|S_{s,t}(\varsigma)(x)\|^n]^{1/n} \leq \kappa_n \|\varsigma\| \quad (6.11)$$

The proof of the above theorem, including a more detailed description of the parameters  $\kappa_{\delta,n}$  and  $\kappa_n$  is provided below.

Next, we estimate the  $\mathbb{L}_n$ -norm of the Malliavin differential  $D_v \Sigma_{s,u,t}(x)$  in the two cases ( $s \leq u \leq v \leq t$ ) and ( $s \leq v \leq u \leq t$ ).

**Case ( $s \leq u \leq v \leq t$ ):**

Using (6.6) we have

$$\|D_v \Sigma_{s,u,t}(x)\| \leq c \|\varsigma_u(\bar{X}_{s,u}(x))\| \|(D_v \nabla X_{u,t})(\bar{X}_{s,u}(x))\|$$

Using (2.24) and (2.25) this yields the estimate

$$\|D_v \Sigma_{s,u,t}(x)\| \leq c_1 \mathbb{I}_{s,u,t}(x) + c_2 \mathbb{J}_{s,u,t}(x)$$

with the functions

$$\mathbb{I}_{s,u,t}(x) := \|\nabla \sigma\| \|\varsigma_u(\bar{X}_{s,u}(x))\| \|(\nabla X_{u,v})(\bar{X}_{s,u}(x))\| \|(\nabla X_{v,t})(Z_u^{s,v}(x))\|$$

$$\mathbb{J}_{s,u,t}(x) := \|\sigma_v(Z_u^{s,v}(x))\| \|\varsigma_u(\bar{X}_{s,u}(x))\| \|(\nabla X_{u,v})(\bar{X}_{s,u}(x))\| \|(\nabla^2 X_{v,t})(Z_u^{s,v}(x))\|$$

- Firstly assume that  $\|\varsigma\| \vee \|\sigma\| < \infty$  and  $(\mathcal{T})_{2n}$  is satisfied for some parameter  $n \geq 1$ . In this situation, applying proposition 3.2 and proposition 3.3, for any  $\epsilon \in ]0, 1[$  we have the uniform estimates

$$\mathbb{E} (\|D_v \Sigma_{s,u,t}(x)\|^n)^{1/n} \leq \|\varsigma\| \chi_{n,\epsilon}(b, \sigma) \exp(-(1-\epsilon)\lambda_A(2n)(t-u))$$

with the parameter  $\chi_{n,\epsilon}(b, \sigma)$  given by

$$\chi_{n,\epsilon}(b, \sigma) := c [\|\sigma\| \vee \|\nabla \sigma\|] \left[ 1 + \frac{1}{\epsilon} \frac{n}{\lambda_A(2n)} \chi(b, \sigma) \right] \quad \text{with } \chi(b, \sigma) \text{ given in (2.6).}$$

- More generally, when  $\|\nabla \varsigma\| \vee \|\nabla \sigma\| < \infty$  the functions  $\varsigma_t(x)$  and  $\sigma_t(x)$  may grow at the most linearly with respect to  $\|x\|$ . Assume that conditions  $(M)_{2n/\delta}$  and condition  $(\mathcal{T})_{2n/(1-\delta)}$  are satisfied for some parameters  $n \geq 1$  and  $\delta \in ]0, 1[$ . In this situation, applying Hölder inequality we check that

$$\begin{aligned} \mathbb{E} (\|\mathbb{I}_{s,u,t}(x)\|^n)^{1/n} &\leq c \|\nabla \sigma\| \mathbb{E} \left( \|\varsigma_u(\bar{X}_{s,u}(x))\|^{n/\delta} \right)^{\delta/n} \\ &\times \mathbb{E} \left( \|(\nabla X_{u,v})(\bar{X}_{s,u}(x))\|^{2n/(1-\delta)} \right)^{(1-\delta)/(2n)} \mathbb{E} \left( \|(\nabla X_{v,t})(Z_u^{s,v}(x))\|^{2n/(1-\delta)} \right)^{(1-\delta)/(2n)} \end{aligned}$$

Applying proposition 3.2 we check that

$$\mathbb{E} (\|\mathbb{I}_{s,u,t}(x)\|^n)^{1/n} \leq c_{n,\delta} \|\nabla\sigma\| \|\zeta(x)\|_{n/\delta} e^{-\lambda_A(2n/(1-\delta))(t-u)}$$

In the same vein, combining proposition 3.2 and proposition 3.3 with the uniform moment estimates (2.11) we check that

$$\begin{aligned} \mathbb{E} (\|\mathbb{J}_{s,u,t}(x)\|^n)^{1/n} &\leq c_{n,\delta} [\|\sigma(0)\| + \|\nabla\sigma\|] \frac{1}{\epsilon} \frac{\chi(b,\sigma)}{\lambda_A(2n/(1-\delta))} \\ &\quad \times \|\zeta(x)\|_{2n/\delta} [1 + \|x\|] e^{-(1-\epsilon)\lambda_A(2n/(1-\delta))(t-u)} \end{aligned}$$

We conclude that

$$\mathbb{E} (\|D_v \Sigma_{s,u,t}(x)\|^n)^{1/n} \leq \chi_{n,\delta,\epsilon}(b,\sigma) \|\zeta(x)\|_{2n/\delta} [1 + \|x\|] e^{-(1-\epsilon)\lambda_A(2n/(1-\delta))(t-u)}$$

with the parameter

$$\chi_{n,\delta,\epsilon}(b,\sigma) := c_{n,\delta} [\|\sigma(0)\| + \|\nabla\sigma\|] \left(1 + \frac{1}{\epsilon} \frac{\chi(b,\sigma)}{\lambda_A(2n/(1-\delta))}\right)$$

**Case** ( $s \leq v \leq u \leq t$ ):

We use (6.7) to check that

$$\begin{aligned} \|D_v \Sigma_{s,u,t}(x)\| &\leq \| [D_v (\varsigma_u \circ \bar{X}_{s,u})](x) \| \| (\nabla X_{u,t}) (\bar{X}_{s,u}(x)) \| \\ &\quad + \| [D_v \bar{X}_{s,u}](x) \otimes \varsigma_u(\bar{X}_{s,u}(x)) \| \| (\nabla^2 X_{u,t}) (\bar{X}_{s,u}(x)) \| \end{aligned}$$

On the other hand, using the chain rules (2.24) we have

$$\begin{aligned} D_v \bar{X}_{s,u} &:= (D_v \bar{X}_{s,v}) [(\nabla \bar{X}_{v,u}) \circ \bar{X}_{s,v}] \\ D_v (\varsigma_u \circ \bar{X}_{s,u}) &= (D_v \bar{X}_{s,u}) [(\nabla \varsigma_u) \circ \bar{X}_{s,u}] \end{aligned}$$

This yields the estimate

$$\begin{aligned} \|D_v \Sigma_{s,u,t}(x)\| &\leq c_1 \|\bar{\sigma}_v(\bar{X}_{s,v}(x))\| \|\nabla\zeta\| \|(\nabla \bar{X}_{v,u})(\bar{X}_{s,v}(x))\| \|(\nabla X_{u,t})(\bar{X}_{s,u}(x))\| \\ &\quad + c_2 \|\bar{\sigma}_v(\bar{X}_{s,v}(x))\| \|\varsigma_u(\bar{X}_{s,u}(x))\| \|(\nabla \bar{X}_{v,u})(\bar{X}_{s,v}(x))\| \|(\nabla^2 X_{u,t})(\bar{X}_{s,u}(x))\| \end{aligned}$$

- Firstly assume that  $\|\zeta\| \vee \|\bar{\sigma}\| < \infty$  and condition  $(T)_{2n}$  is satisfied for some  $n \geq 1$ . In this situation, arguing as above for any  $\epsilon \in ]0, 1[$  we have the uniform estimates

$$\mathbb{E} (\|D_v \Sigma_{s,u,t}(x)\|^n)^{1/n} \leq (\|\zeta\| + \|\nabla\zeta\|) \bar{\chi}_{n,\epsilon}(b,\sigma) \exp\left(- (1-\epsilon)\lambda_{A,\bar{A}}(2n)(t-v)\right)$$

for some universal constant  $c$  and the parameter  $\bar{\chi}_{n,\epsilon}(b,\sigma)$  given by

$$\bar{\chi}_{n,\epsilon}(b,\sigma) := c \|\bar{\sigma}\| \left[1 + \frac{1}{\epsilon} \frac{n}{\lambda_{A,\bar{A}}(2n)} \chi(b,\sigma)\right] \quad \text{with } \chi(b,\sigma) \text{ given in (2.6).}$$



- More generally assume that  $\|\nabla\varsigma\| \vee \|\nabla\bar{\sigma}\| < \infty$ . Also assume that conditions  $(M)_{2n/\delta}$  and  $(T)_{2n/(1-\delta)}$  are satisfied for some parameters  $n \geq 1$  and  $\delta \in ]0, 1[$ . In this situation, we have

$$\begin{aligned} & \mathbb{E} (\|D_v \Sigma_{s,u,t}(x)\|^n)^{1/n} \\ & \leq \chi_{n,\delta,\epsilon}(b, \sigma, \bar{\sigma}) \|\varsigma(x)\|_{2n/\delta} [1 + \|x\|] e^{-(1-\epsilon)\lambda_{A,\bar{A}}(2n/(1-\delta))(t-v)} \end{aligned}$$

with the parameter

$$\chi_{n,\delta,\epsilon}(b, \sigma, \bar{\sigma}) := c_{n,\delta} [\|\bar{\sigma}(0)\| + \|\nabla\bar{\sigma}\|] \left( 1 + \frac{1}{\epsilon} \frac{\chi(b, \sigma)}{\lambda_{A,\bar{A}}(2n/(1-\delta))} \right)$$

The end of the proof of theorem 6.2 is a direct consequence of the estimates discussed above combined with (6.8) and the diagonal estimates presented in (6.4).  $\blacksquare$

### 6.3 Some extensions

This section is concerned with the Skorohod stochastic integral (4.5). Using the gradient formula in (4.6) the Skorohod stochastic integral in (4.5) takes the form

$$\mathbb{S}_{s,t}(f, \Delta\sigma)(x) = \int_s^t \Sigma_{s,u,t}(f)(x) dW_u$$

with the Skorohod integrands

$$\Sigma_{s,u,t}(f)(x) := \nabla f(Z_u^{s,t}(x))' \Sigma_{s,u,t}(x) \quad \text{and} \quad \Sigma_{s,u,t}(x) := [(\nabla X_{u,t})' \circ \bar{X}_{s,u}] [\Delta\sigma_u \circ \bar{X}_{s,u}]$$

As in (2.25), using the chain rule properties of Malliavin derivatives we check that

$$D_v^i \Sigma_{s,u,t}(f) = (D_v^i \nabla f(Z_u^{s,t})') \Sigma_{s,u,t} + \nabla f(Z_u^{s,t})' D_v^i \Sigma_{s,u,t}$$

as well as

$$D_v^i \nabla f(Z_u^{s,t})' = \nabla^2 f(Z_u^{s,t})' D_v^i Z_u^{s,t}$$

This yields the differential formula

$$D_v^i \Sigma_{s,u,t}(f) = \nabla f(Z_u^{s,t})' D_v^i \Sigma_{s,u,t} + \nabla^2 f(Z_u^{s,t})' (D_v^i Z_u^{s,t}) \Sigma_{s,u,t}$$

The Malliavin derivatives  $D_v^i \Sigma_{s,u,t}$  are computed using formulae (6.6) and (6.7); thus, it remains to compute the Malliavin derivatives  $D_v Z_u^{s,t}$  of the interpolating path.

- When  $u \leq v$  we have

$$Z_u^{s,t} = (X_{v,t} \circ X_{u,v}) \circ \bar{X}_{s,u} = X_{v,t} \circ Z_u^{s,v}$$

In this situation, as in (2.24) using the chain rule properties of Malliavin derivatives we check that

$$D_v Z_u^{s,t} = D_v Z_u^{s,v} ((\nabla X_{v,t}) \circ Z_u^{s,v}) = ((D_v X_{u,v}) \circ \bar{X}_{s,u}) ((\nabla X_{v,t}) \circ Z_u^{s,v})$$

By (2.23) we conclude that

$$D_v Z_u^{s,t} = (\sigma_v \circ Z_u^{s,v}) ((\nabla X_{v,t}) \circ Z_u^{s,v})$$

- When  $v \leq u$  we have

$$Z_u^{s,t} = X_{u,t} \circ (\bar{X}_{v,u} \circ \bar{X}_{s,v}) = Z_u^{v,t} \circ \bar{X}_{s,v}$$

In this situation, arguing as above we check that

$$D_v Z_u^{s,t} = D_v \bar{X}_{s,v} ((\nabla Z_u^{v,t}) \circ \bar{X}_{s,v}) = D_v \bar{X}_{s,v} ((\nabla \bar{X}_{v,u}) \circ \bar{X}_{s,v}) ((\nabla X_{u,t}) \circ \bar{X}_{s,u})$$

By (2.23) we conclude that

$$D_v Z_u^{s,t} = (\bar{\sigma}_v \circ \bar{X}_{s,v}) ((\nabla \bar{X}_{v,u}) \circ \bar{X}_{s,v}) ((\nabla X_{u,t}) \circ \bar{X}_{s,u})$$

## 7 Illustrations

### 7.1 Perturbation analysis

Assume that  $\bar{\sigma} = \sigma$  and the drift function  $\bar{b}_t$  is given by a first order expansion

$$\bar{b}_t(x) = b_{\delta,t}(x) := b_t(x) + \delta b_{\delta,t}^{(1)}(x) \quad \text{with} \quad b_{\delta,t}^{(1)}(x) = b_t^{(1)}(x) + \frac{\delta}{2} b_{\delta,t}^{(2)}(x)$$

for some perturbation parameter  $\delta \in [0, 1]$  and some functions  $b_{\delta,t}^{(i)}(x)$  with  $i = 1, 2$ .

In this context, the stochastic flow  $\bar{X}_{s,t}(x) := X_{s,t}^\delta(x)$  can be seen as a  $\delta$ -perturbation of  $X_{s,t}(x) := X_{s,t}^0(x)$ .

We further assume that the unperturbed diffusion satisfies condition  $(\mathcal{T})_2$ .

To avoid unnecessary technical discussions on the existence of absolute moments of the flows we also assume that  $b_{\delta,t}^{(i)}(x)$  are uniformly bounded w.r.t. the parameters  $(\delta, t, x)$ . In addition,  $b_t^{(1)}(x)$  is differentiable w.r.t. the coordinate  $x$  and it has uniformly bounded gradients. In this situation, we set

$$\|b^{(i)}\| := \sup_{\delta,t,x} \|b_{\delta,t}^{(i)}(x)\| \quad \text{and} \quad \|\nabla b^{(1)}\| := \sup_{t,x} \|\nabla b_t^{(1)}(x)\|$$

With some additional work to estimate the absolute moments of the flows, the perturbation analysis presented below allows to handle more general models. The methodology described in this section can also be extended to expand the flow  $X_{s,t}^\delta(x)$  at any order as soon as  $\delta \mapsto b_{\delta,t}(x)$  is sufficiently smooth.

The first order approximation is given by the following theorem.

**Theorem 7.1.** *For any  $s \leq t$ ,  $x \in \mathbb{R}^d$  and  $\delta \geq 0$  we have the first order expansion*

$$X_{s,t}^\delta(x) = X_{s,t}(x) + \delta \partial X_{s,t}(x) + \frac{\delta^2}{2} \partial_\delta^2 X_{s,t}(x) \tag{7.1}$$

with the first order stochastic flow

$$\partial X_{s,t}(x) := \int_s^t (\nabla X_{u,t})(X_{s,u}(x))' b_u^{(1)}(X_{s,u}(x)) du$$

The remainder second order term  $\partial_\delta^2 X_{s,t}(x)$  in the above display is such that for any  $n \geq 2$  s.t.  $\lambda_A(n) > 0$  we have the uniform estimate

$$\sup_{s,t,x} \mathbb{E}[\|\partial_\delta^2 X_{s,t}(x)\|^n]^{1/n} \leq c_n$$

Observe that the first order stochastic flow satisfies the diffusion equation

$$d \partial X_{s,t}(x) = \left[ b_t^{(1)}(X_{s,t}(x)) + \nabla b_t(X_{s,t}(x))' \partial X_{s,t}(x) \right] dt + \sum_{1 \leq k \leq r} \nabla \sigma_{t,k}(X_{s,t}(x))' \partial X_{s,t}(x) dW_t^k$$

The rest of this section is dedicated to the proof of the above theorem.

Using (5.2) we readily check that

$$DX_{s,t}^\delta(x) := \delta^{-1}[X_{s,t}^\delta(x) - X_{s,t}(x)] = \int_s^t (\nabla X_{u,t})(X_{s,u}^\delta(x))' b_{\delta,u}^{(1)}(X_{s,u}^\delta(x)) du$$

By proposition 3.2 for any  $n \geq 2$  we have

$$\lambda_A^+(n) := \lambda_A - (n-2)\rho(\nabla\sigma)^2/2 > 0 \implies \mathbb{E} \left( \|DX_{s,t}^\delta(x)\|^n \right)^{1/n} \leq c \|b^{(1)}\|/\lambda_A^+(n) \quad (7.2)$$

This yields the first order Taylor expansion (7.1) with

$$\partial_\delta^2 X_{s,t}(x) := R_{\delta,s,t}^{(2,1)}(x) + R_{\delta,s,t}^{(2,2)}(x)$$

and the second order remainder terms

$$\begin{aligned} \partial_\delta^{(2,2)} X_{s,t}(x) &:= \int_s^t (\nabla X_{u,t})(X_{s,u}^\delta(x))' b_{\delta,t}^{(2)}(X_{s,u}^\delta(x)) du \\ \partial_\delta^{(2,1)} X_{s,t}(x) &:= 2\delta^{-1} \int_s^t \left[ (\nabla X_{u,t})(X_{s,u}^\delta(x)) - (\nabla X_{u,t})(X_{s,u}(x)) \right]' b_u^{(1)}(X_{s,u}^\delta(x)) du \\ &\quad + 2\delta^{-1} \int_s^t (\nabla X_{u,t})(X_{s,u}(x))' [b_u^{(1)}(X_{s,u}^\delta(x)) - b_u^{(1)}(X_{s,u}(x))] du \end{aligned}$$

Arguing as above, for any  $n \geq 2$  s.t.  $\lambda_A^+(n) > 0$  we have the uniform estimate

$$\mathbb{E} \left( \|\partial_\delta^{(2,2)} X_{s,t}(x)\|^n \right)^{1/n} \leq c \|b^{(2)}\|/\lambda_A^+(n)$$

To estimate  $\partial_\delta^{(2,1)} X_{s,t}(x)$  we need to consider the second order decompositions

$$\begin{aligned} &2^{-1} \partial_\delta^{(2,1)} X_{s,t}(x) \\ &= \int_0^1 \int_s^t [\nabla^2 X_{u,t}] \left( X_{s,u}(x) + \epsilon(X_{s,u}^\delta(y) - X_{s,u}(x)) \right)' \left[ b_u^{(1)}(X_{s,u}^\delta(x)) \otimes DX_{s,u}^\delta(x) \right] du d\epsilon \\ &\quad + \int_0^1 \int_s^t (\nabla X_{u,t})(X_{s,u}(x))' \nabla b_u^{(1)} \left( X_{s,u}(x) + \epsilon(X_{s,u}^\delta(x) - X_{s,u}(x)), y \right)' DX_{s,u}^\delta(x) du d\epsilon \end{aligned}$$

Combining proposition 3.3 with the estimate (7.2) for any  $n \geq 2$  s.t.  $\lambda_A(n) > 0$  we check that

$$\mathbb{E}[\|\partial_\delta^{(2,1)} X_{s,t}(x)\|^n]^{1/n} \leq c (1 + n \chi(b, \sigma)/\lambda_A(n)) \left( \|b^{(1)}\|/\lambda_A(n) \right)^2$$

for some universal constant  $c < \infty$  and the parameter  $\chi(b, \sigma)$  introduced in (2.6). This ends the proof of (7.1).  $\blacksquare$

## 7.2 Interacting diffusions

Consider a system of  $N$  interacting and  $\mathbb{R}^d$ -valued diffusion flows  $X_{s,t}^i(x)$ , with  $1 \leq i \leq N$  given by a stochastic differential equation of the form

$$dX_{s,t}^i(x) = B_t \left( X_{s,t}^i(x), \frac{1}{N} \sum_{1 \leq i \leq N} X_{s,t}^j(x) \right) dt + \sigma_t \left( \frac{1}{N} \sum_{1 \leq i \leq N} X_{s,t}^j(x) \right) dW_t^i$$

for some Lipschitz functions  $B_t(x, y)$  and  $\sigma_t(y)$  with appropriate dimensions. In the above display,  $W_t^i$  stands for a collection of independent copies of  $d$ -dimensional Brownian motion  $W_t$ . Assume that  $B_t(x, y)$  linear w.r.t. the first coordinate.

In this situation, up to a change of probability space, the empirical mean of the process

$$\bar{X}_{s,t}(x) := \frac{1}{N} \sum_{1 \leq i \leq N} Y_{s,t}^j(x)$$

satisfies the stochastic differential equation

$$d\bar{X}_{s,t}(x) = b_t(\bar{X}_{s,t}(x)) dt + \frac{1}{\sqrt{N}} \sigma_t(\bar{X}_{s,t}(x)) dW_t \quad \text{with} \quad b_t(x) := B_t(x, x)$$

Formally, the above diffusion converges as  $N \rightarrow \infty$  to the flow of the dynamical system

$$\partial_t X_{s,t}(x) = b_t(X_{s,t}(x))$$

Without further work, the forward-backward interpolation formula (1.8) yields the bias-variance error decomposition

$$\begin{aligned} \bar{X}_{s,t}(x) - X_{s,t}(x) &= \frac{1}{2N} \int_s^t (\nabla^2 X_{u,t}) (\bar{X}_{s,u}(x))' a_u(\bar{X}_{s,u}(x)) du \\ &\quad + \frac{1}{\sqrt{N}} \int_s^t (\nabla X_{u,t}) (\bar{X}_{s,u}(x))' \sigma_u(\bar{X}_{s,u}(x)) dW_u \end{aligned}$$

After some elementary manipulations we check that

$$\lim_{N \rightarrow \infty} N [\mathbb{E}(\bar{X}_{s,t}(x)) - X_{s,t}(x)] = \frac{1}{2} \int_s^t (\nabla^2 X_{u,t}) (X_{s,u}(x))' a_u(X_{s,u}(x)) du$$

We also have the almost sure fluctuation theorem

$$\lim_{N \rightarrow \infty} \sqrt{N} [\bar{X}_{s,t}(x) - X_{s,t}(x)] = \int_s^t (\nabla X_{u,t}) (X_{s,u}(x))' \sigma_u(X_{s,u}(x)) dW_u$$

## 7.3 Time discretization schemes

This section is mainly concerned with the proof of proposition 1.4. We fix some parameter  $h > 0$  and some  $s \geq 0$  and for any  $t \in [s + kh, s + (k + 1)h[$  we set

$$dX_{s,t}^h(x) = Y_{s,t}^h(x) dt + \sigma dW_t \quad \text{with} \quad Y_{s,t}^h(x) := b \left( X_{s,s+kh}^h(x) \right)$$

for some fluctuation parameter  $\sigma \geq 0$ . For any  $s + kh \leq u < s + (k + 1)h$  we have

$$X_{s,u}^h(x) - X_{s,s+kh}^h(x) = Y_{s,u}^h(x) (u - (s + kh)) + \sigma (W_u - W_{s+kh})$$

Using (5.2), in terms of the tensor product (2.1) we readily check that

$$X_{s,t}^h(x) - X_{s,t}(x) = \int_s^t (\nabla X_{u,t}) (X_{s,u}^h(x))' \left[ Y_{s,u}^h(x) - b(X_{s,u}^h(x)) \right] du$$

Combining (3.5) with the Minkowski integral inequality we check that

$$\begin{aligned} \mathbb{E} \left( \|X_{s,t}^h(x) - X_{s,t}(x)\|^n \right)^{1/n} &= \int_s^t \mathbb{E} \left( \|(\nabla X_{u,t}) (X_{s,u}^h(x))' \left[ Y_{s,u}^h(x) - b(X_{s,u}^h(x)) \right]\|^n \right)^{1/n} du \\ &= \int_s^t e^{-\lambda(t-u)} \mathbb{E} \left( \|Y_{s,u}^h(x) - b(X_{s,u}^h(x))\|^n \right)^{1/n} du \end{aligned}$$

where the second line follows from the exponential estimate of the tangent process from proposition 3.1. The integrand will be bounded as follows: for any  $s + kh \leq u < s + (k+1)h$  and any  $n \geq 1$  we have

$$\mathbb{E} \left( \|b(X_{s,u}^h(x)) - Y_{s,u}^h(x)\|^n \right)^{1/n} \leq \|\nabla b\| \left( [\|b(0)\| + \hat{m}_n(x) \|\nabla b\|] h + \sigma \sqrt{h} \right)$$

which then yields the stated result of the proposition. We now prove the stated bound on the difference of the drift processes. For any  $s + kh \leq u < s + (k+1)h$  we have

$$\begin{aligned} &b(X_{s,u}^h(x)) - Y_{s,u}^h(x) \\ &= \left[ \int_0^1 \nabla b \left( X_{s,s+kh}^h(x) + \epsilon(X_{s,u}^h(x) - X_{s,s+kh}^h(x)) \right)' b \left( X_{s,s+kh}^h(x) \right) d\epsilon \right] (u - (s + kh)) \\ &\quad + \left[ \int_0^1 \nabla b \left( X_{s,s+kh}^h(x) + \epsilon(X_{s,u}^h(x) - X_{s,s+kh}^h(x)) \right)' d\epsilon \right] \sigma (W_u - W_{s+kh}) \end{aligned} \quad (7.3)$$

The  $\mathbb{L}_n$ -norm of the second integral term is bounded by  $\|\nabla b\| \sigma \sqrt{h}$ .

The assumption  $\langle x, b(x) \rangle \leq -\beta \|x\|^2$ , for some  $\beta > 0$ , implies the stochastic flows  $X_{s,t}(x)$  has uniform absolute moments of any order  $n \geq 1$  w.r.t. the time horizon, that is, we have that

$$m_n(x) \leq \kappa_n (1 + \|x\|) \quad \text{with } m_n(x) \text{ defined in (2.10).}$$

The stochastic flows  $X_{s,t}^h(x)$  also obey a similar moment bound: observe that for any  $t \in [s + kh, s + (k+1)h[$  we have

$$\begin{aligned} &d\|X_{s,t}^h(x)\|^2 \\ &\leq \left[ -2\lambda_0 \|X_{s,t}^h(x)\|^2 + 2 \langle X_{s,t}^h(x), b(X_{s,s+kh}^h(x)) - b(X_{s,t}^h(x)) \rangle + \sigma^2 d \right] dt + 2\sigma X_{s,t}^h(x)' dW_t \end{aligned}$$

Thus, for any  $\epsilon > 0$  we have

$$d\|X_{s,t}^h(x)\|^2 \leq \left[ (-2\lambda_0 + \epsilon) \|X_{s,t}^h(x)\|^2 + \epsilon^{-1} \|\nabla b\| + \sigma^2 d \right] dt + 2\sigma X_{s,t}^h(x)' dW_t$$

We can check that the stochastic flows  $X_{s,t}^h(x)$  also has uniform moments w.r.t. the time horizon; that is, for any  $n \geq 1$  we have that

$$\hat{m}_n(x) := \sup_{h \geq 0} \sup_{t \geq s} \mathbb{E} \left[ \|X_{s,t}^h(x)\|^n \right]^{1/n} \leq c_n (1 + \|x\|)$$

Using this bounds, we check that

$$\mathbb{E}(\|b(X_{s,s+kh}^h(x))\|^n)^{1/n} = \|b(0)\| + \hat{m}_n(x) \|\nabla b\|$$

The end of the proof now follows elementary manipulations, thus it is skipped. The proof of proposition 1.4 is now completed.  $\blacksquare$

## Appendix A

For clarity and to avoid unnecessary sophisticated multi-index notation, we only consider one dimensional model. The proof of the results presented in this appendix in the general case can be reproduced word-for-word for multidimensional models.

To simplify the presentation, we write  $\partial^n f$  the derivative of order  $n \geq 1$  of a smooth function  $f$ . We also set  $Y_{s,t}(x) := \bar{X}_{s,t}(x)$ . We also reduce the analysis to the unit interval. In this context, for any  $t \in [0, 1]$  we set

$$Y_t := Y_{0,t} \quad \text{and} \quad X^t := X_{t,1} \quad (7.4)$$

### Two-sided stochastic integration

The aim of this section is to extend the two-sided stochastic integration introduced in [38] to Skorohod integrals of the form (4.2), for some time homogeneous function  $\varsigma_u = \varsigma$  satisfying (6.1). For any  $t \in [0, 1]$  we set

$$\Phi(X^t, Y_t(x)) := \partial X^t(Y_t(x)) \varsigma(Y_t(x)) \quad (7.5)$$

In this notation (4.2) takes the form

$$S_{0,1}(\varsigma)(x) = \int_0^1 \Phi(X^t, Y_t(x)) dW_t$$

To simplify the presentation, we fix the state variable  $x$  and we write  $Y_t$  and  $\Phi(X^t, Y_t)$  instead of  $Y_t(x)$  and  $\Phi(X^t, Y_t(x))$ . Next technical lemma provided a more explicit description of the Malliavin derivatives of the processes  $\Phi(X^t, Y_t)$ .

**Lemma 7.2.** *For any  $s < t$  we have*

$$D_s \Phi(X^t, Y_t) = [\partial((\partial X^t) \circ Y_{s,t})(Y_s) (\varsigma \circ Y_{s,t})(Y_s) + ((\partial X^t) \circ Y_{s,t})(Y_s) \times \partial(\varsigma \circ Y_{s,t})(Y_s)] \bar{\sigma}(Y_s)$$

*In addition, we have*

$$D_t \Phi(X^s, Y_s) = [\partial((\partial X^t) \circ X_{s,t})(Y_s) (\sigma \circ X_{s,t})(Y_s) + \partial(X^t \circ X_{s,t})(Y_s) \partial\sigma(X_{s,t}(Y_s))] \varsigma(Y_s)$$

*Proof.* Using the chain rules properties, for any  $s < t$  we have

$$\begin{aligned} D_s \Phi(X^t, Y_t) &= D_s ((\partial X^t)(Y_{s,t}(Y_s)) (\varsigma \circ Y_{s,t})(Y_s)) \\ &= D_s ((\partial X^t) \circ Y_{s,t})(Y_s) (\varsigma \circ Y_{s,t})(Y_s) + (\partial X^t) \circ Y_{s,t}(Y_s) D_s(\varsigma \circ Y_{s,t})(Y_s) \end{aligned}$$

The end of the proof of the first assertion comes from the fact that

$$D_s ((\partial X^t) \circ Y_{s,t})(Y_s) = \partial((\partial X^t) \circ Y_{s,t})(Y_s) D_s Y_s \quad \text{with} \quad D_s Y_s = \bar{\sigma}(Y_s)$$

In the same vein, we have

$$D_s(\varsigma \circ Y_{s,t})(Y_s) = \partial(\varsigma \circ Y_{s,t})(Y_s) \bar{\sigma}(Y_s)$$

We also have that

$$\begin{aligned} D_t \Phi(X^s, Y_s) &= D_t ((\partial X^s)(Y_s) \varsigma(Y_s)) \\ &= D_t (\partial(X^t \circ X_{s,t})(Y_s)) \varsigma(Y_s) = D_t (((\partial X^t) \circ X_{s,t})(Y_s) (\partial X_{s,t})(Y_s)) \varsigma(Y_s) \end{aligned}$$

The last assertion comes from the fact that

$$\begin{aligned} & D_t \left( ((\partial X^t) \circ X_{s,t})(Y_s) (\partial X_{s,t})(Y_s) \right) \\ &= D_t \left( (\partial X^t) \circ X_{s,t}(Y_s) (\partial X_{s,t})(Y_s) + ((\partial X^t) \circ X_{s,t})(Y_s) D_t(\partial X_{s,t})(Y_s) \right) \end{aligned}$$

The r.h.s. term in the above display can be rewritten as follows

$$\begin{aligned} & D_t(\partial X_{s,t})(Y_s) = \partial \sigma(X_{s,t}(Y_s)) (\partial X_{s,t})(Y_s) \\ & \implies ((\partial X^t) \circ X_{s,t})(Y_s) D_t(\partial X_{s,t})(Y_s) = \partial(X^t \circ X_{s,t})(Y_s) \partial \sigma(X_{s,t}(Y_s)) \end{aligned}$$

In the same vein, we have

$$\begin{aligned} & D_t \left( (\partial X^t) \circ X_{s,t}(Y_s) \right) = ((\partial^2 X^t) \circ X_{s,t})(Y_s) D_t X_{s,t}(Y_s) = ((\partial^2 X^t) \circ X_{s,t})(Y_s) \sigma(X_{s,t}(Y_s)) \\ & \implies D_t \left( (\partial X^t) \circ X_{s,t}(Y_s) (\partial X_{s,t})(Y_s) \right) = \partial \left( (\partial X^t) \circ X_{s,t}(Y_s) \right) \sigma(X_{s,t}(Y_s)) \end{aligned}$$

This ends the proof of the second assertion. The proof of the lemma is now completed.  $\blacksquare$

From the above lemma, we also check that all the  $n$ -absolute moments of the Malliavin derivatives  $D_s \Phi(X^t, Y_t)$  are finite with at most quadratic growth w.r.t. the initial values.

Next proposition extends proposition 3.3 in [38] to stochastic processes of the form (7.5).

**Proposition 7.3.** *Let  $[0, 1]_h$  be any refining sequence of partitions of the unit interval. For any  $h > 0$  we define*

$$S^h(\Phi) := \sum_{t \in [0, 1]_h} \Phi(X^{t+h}, Y_t) (W_{t+h} - W_t)$$

*Then  $S^h(\Phi)$  is a Cauchy sequence in  $\mathbb{L}_2(\Omega)$ . In addition, for any decreasing sequence of time steps  $h_1 > h_2$  we have the formula*

$$\lim_{h_1 \rightarrow 0} \mathbb{E} \left( S^{h_1}(\Phi) S^{h_2}(\Phi) \right) = \mathbb{E} \left( \int_0^1 \Phi(X^t, Y_t)^2 dt + \int_{[0, 1]^2} D_s \Phi(X^t, Y_t) D_t \Phi(X^s, Y_s) ds dt \right) \quad (7.6)$$

Before entering into the details of the proof of the proposition, we give a couple of comments. The hypothesis that  $[0, 1]_h$  is a refining sequence indexed by  $h$  is not essential but it simplify the proof of the proposition, see for instance lemma 3.1.1 in [33]. Arguing as in the proof of theorem 3.3 and theorem 7.1 in [38] the above proposition ensures that the two-sided integral defined by the  $\mathbb{L}_2(\Omega)$ -limit coincides with the Skorohod integral of the process  $\Phi(X^t, Y_t)$  over the unit interval; that is, we have that

$$\int_0^1 \Phi(X^t, Y_t) dW_t := \mathbb{L}_2 - \lim_{h \rightarrow 0} \sum_{t \in [0, 1]_h} \Phi(X^{t+h}, Y_t) (W_{t+h} - W_t)$$

**Proof of the proposition:**

We fix  $h_1 > h_2$  and we assume that  $[0, 1]_{h_2}$  is a refinement of  $[0, 1]_{h_1}$ . For any  $(s, t) \in ([0, 1]_{h_1} \times [0, 1]_{h_2})$  we also set

$$\Pi_{s,t}^{h_1, h_2} := \Phi(X^{s+h_1}, Y_s) \Phi(X^{t+h_2}, Y_t) (W_{s+h_1} - W_s) (W_{t+h_2} - W_t)$$

With a slight abuse of notation we set

$$\Delta W_s := (W_{s+h_1} - W_s) \quad \text{and} \quad \Delta W_t := (W_{t+h_2} - W_t)$$

- For any overlapping pair  $s < t < t + h_2 < s + h_1$  using the decomposition

$$\Delta W_s = (W_{s+h_1} - W_{t+h_2}) + \Delta W_t + (W_t - W_s)$$

we have

$$\mathbb{E} \left( \Phi(X^{s+h_1}, Y_s) \Phi(X^{t+h_2}, Y_t) \Delta W_t \Delta W_s \mid \mathcal{W}_t \vee \mathcal{W}^{t+h_2} \right) = \Phi(X^{s+h_1}, Y_s) \Phi(X^{t+h_2}, Y_t) h_2$$

It follows from the continuity properties of the processes that

$$\mathbb{E} \left( \sum_{s < t < t+h_2 < s+h_1} \Pi_{s,t}^{h_1, h_2} \right) \xrightarrow{h_1 \rightarrow 0} \mathbb{E} \left( \int_0^1 \Phi(X^t, Y_t)^2 dt \right)$$

- When  $s + h_1 < t$  we have

$$\begin{aligned} \partial X^{s+h_1} &= \partial(X^t \circ X_{s+h_1, t}) \\ &= \partial(X^{t+h_2} \circ X_{s+h_1, t}) + \left( (\partial X^t - \partial X^{t+h_2}) \circ X_{s+h_1, t} \right) \times \partial X_{s+h_1, t} \end{aligned}$$

On the other hand, we have the decomposition

$$\begin{aligned} \partial X^t - \partial X^{t+h_2} &= \left( (\partial X^{t+h_2}) \circ X_{t, t+h_2} \right) \times \partial X_{t, t+h_2} - \partial X^{t+h_2} \\ &= \left( (\partial X^{t+h_2}) \circ (I + \Delta X_t) - \partial X^{t+h_2} \right) + \partial X^{t+h_2} \times \Delta X'_t \\ &\quad + \left( (\partial X^{t+h_2}) \circ (I + \Delta X_t) - \partial X^{t+h_2} \right) \times \Delta X'_t \end{aligned}$$

with the increment functions

$$\Delta X'_t := \partial X_{t, t+h_2} - 1 \quad \text{and} \quad \Delta X_t := X_{t, t+h_2} - I$$

With a slight abuse of notation, we shall denote by  $O(h^p)$  some possible random variable with any  $n$ -absolute moment of order  $h^p$ , for some  $p > 0$  with  $0 < h < 1$ . In this notation, we have

$$\begin{aligned} \Delta X'_t(x) &= \int_t^{t+h_2} \partial \sigma(X_{t,u}(x)) \partial X_{t,u}(x) dW_u + O(h_2) = O(h_2^{1/2}) \\ \Delta X_t(x) &= \int_t^{t+h_2} \sigma(X_{t,u}(x)) dW_u + O(h_2) = O(h_2^{1/2}) \end{aligned}$$

Given a smooth function  $\theta$  we set

$$\partial^n \theta(x, y) := \int_0^1 \frac{(1-\epsilon)^{n-1}}{(n-1)!} \theta''(x + \epsilon y)$$

In this notation, we have the first and second order decompositions

$$\begin{aligned} & \left( (\partial X^{t+h_2}) \circ (I + \Delta X_t) - \partial X^{t+h_2} \right) (x) \\ &= (\partial^2 X^{t+h_2})(x, \Delta X_t(x)) \Delta X_t(x) = (\partial^2 X^{t+h_2})(x) \Delta X_t(x) + (\partial^3 X^{t+h_2})(x, \Delta X_t(x)) \Delta X_t(x)^2 \end{aligned}$$



This implies that

$$\begin{aligned}
& (\partial X^t - \partial X^{t+h_2})(x) \\
&= (\partial^2 X^{t+h_2})(x) \Delta X_t(x) + \partial X^{t+h_2}(x) \times \Delta X'_t(x) \\
&\quad + (\partial^3 X^{t+h_2})(x, \Delta X_t(x)) \Delta X_t(x)^2 + (\partial^2 X^{t+h_2})(x, \Delta X_t(x)) \Delta X_t(x) \times \Delta X'_t(x)
\end{aligned}$$

from which we conclude that

$$\begin{aligned}
& \partial X^{s+h_1} = \partial(X^{t+h_2} \circ X_{s+h_1,t}) \\
&+ \left[ \partial((\partial X^{t+h_2}) \circ X_{s+h_1,t}) \times ((\Delta X_t) \circ X_{s+h_1,t}) + \partial(X^{t+h_2} \circ X_{s+h_1,t}) \times ((\Delta X'_t) \circ X_{s+h_1,t}) \right] + O(h_2)
\end{aligned}$$

This yields the first order decomposition

$$\begin{aligned}
& \Phi(X^{s+h_1}, Y_s) \\
&= \psi_{s,t}^0(Y_s) + \psi_{s,t}^1(Y_s) ((\Delta X_t) \circ X_{s+h_1,t})(Y_s) + \psi_{s,t}^2(Y_s) ((\Delta X'_t) \circ X_{s+h_1,t})(Y_s) + O(h_2)
\end{aligned}$$

with the functions

$$\begin{aligned}
& \psi_{s,t}^0(Y_s) := \partial(X^{t+h_2} \circ X_{s+h_1,t})(Y_s) \varsigma(Y_s) \\
& \psi_{s,t}^1(Y_s) := \partial((\partial X^{t+h_2}) \circ X_{s+h_1,t})(Y_s) \varsigma(Y_s) \quad \text{and} \quad \psi_{s,t}^2(Y_s) := \partial(X^{t+h_2} \circ X_{s+h_1,t})(Y_s) \varsigma(Y_s)
\end{aligned}$$

Notice that none of the functions but the increment functions  $(\Delta X_t)$  and  $(\Delta X'_t)$  depend on  $\mathcal{W}_{t,t+h_2}$ , nor on  $\mathcal{W}_{s,s+h_1}$ .

In the reverse angle, we have

$$\begin{aligned}
& (\partial X^{t+h_2}) \circ Y_t \\
&= (\partial X^{t+h_2}) \circ (Y_{s+h_1,t} \circ Y_s) + \left[ ((\partial X^{t+h_2}) \circ Y_{s+h_1,t}) \circ (I + \Delta Y_s) - ((\partial X^{t+h_2}) \circ Y_{s+h_1,t}) \right] \circ Y_s
\end{aligned}$$

with

$$\Delta Y_s := (Y_{s,s+h_1} - I) \implies Y_{s+h_1} = (I + \Delta Y_s) \circ Y_s$$

Arguing as above, we have

$$\begin{aligned}
& \left[ ((\partial X^{t+h_2}) \circ Y_{s+h_1,t}) \circ (y + \Delta Y_s(y)) - ((\partial X^{t+h_2}) \circ Y_{s+h_1,t})(y) \right] \\
&= \partial((\partial X^{t+h_2}) \circ Y_{s+h_1,t})(y) \Delta Y_s(y) + \partial^2((\partial X^{t+h_2}) \circ Y_{s+h_1,t})(y, \Delta Y_s(y)) (\Delta Y_s(y))^2
\end{aligned}$$

We conclude that

$$\begin{aligned}
& (\partial X^{t+h_2}) \circ Y_t \\
&= (\partial X^{t+h_2}) \circ (Y_{s+h_1,t} \circ Y_s) + \partial((\partial X^{t+h_2}) \circ Y_{s+h_1,t})(Y_s) ((\Delta Y_s) \circ Y_s) + O(h_1)
\end{aligned}$$

In the same vein, we have

$$\varsigma \circ Y_t = (\varsigma \circ Y_{s+h_1,t} \circ Y_s) + \partial(\varsigma \circ Y_{s+h_1,t})(Y_s) ((\Delta Y_s) \circ Y_s) + O(h_1)$$

Multiplying these terms, we check that

$$\Phi(X^{t+h_2}, Y_t) = \Psi_{s,t}^0(Y_s) + \Psi_{s,t}^1(Y_s) ((\Delta Y_s) \circ Y_s) + O(h_1)$$

with the functions

$$\begin{aligned} \Psi_{s,t}^0(Y_s) &:= ((\partial X^{t+h_2}) \circ Y_{s+h_1,t})(Y_s) \times (\varsigma \circ Y_{s+h_1,t})(Y_s) \\ \Psi_{s,t}^1(Y_s) &:= [\partial((\partial X^{t+h_2}) \circ Y_{s+h_1,t})(Y_s) \times (\varsigma \circ Y_{s+h_1,t})(Y_s) \\ &\quad + ((\partial X^{t+h_2}) \circ Y_{s+h_1,t})(Y_s) \times \partial(\varsigma \circ Y_{s+h_1,t})(Y_s)] \end{aligned}$$

None of the functions but the increment  $\Delta Y_s$  depend on  $\mathcal{W}_{s,s+h_1}$ , nor on  $\mathcal{W}_{t,t+h_2}$ .

Recall that the functions  $\Phi(X^{t+h_2}, Y_t)$  and  $\psi_{s,t}^0(Y_s)$  don't depend on  $\Delta W_t$ . In addition, the functions  $\Phi(X^{s+h_1}, Y_s)$  and  $\Psi_{s,t}^0(Y_s)$  don't depend on  $\Delta W_s$ . This yields the formula

$$\begin{aligned} &\mathbb{E} \left( \Phi(X^{s+h_1}, Y_s) \Phi(X^{t+h_2}, Y_t) \Delta W_s \Delta W_t \right) \\ &= \mathbb{E} \left( \left[ \Phi(X^{s+h_1}, Y_s) - \psi_{s,t}^0(Y_s) \right] \left[ \Phi(X^{t+h_2}, Y_t) - \Psi_{s,t}^0(Y_s) \right] \Delta W_s \Delta W_t \right) \\ &= \mathbb{E} \left( \Psi_{s,t}^1(Y_s) \psi_{s,t}^1(Y_s) [((\Delta Y_s) \circ Y_s) \Delta W_s] [((\Delta X_t) \circ X_{s+h_1,t})(Y_s) \Delta W_t] \right) \\ &\quad + \mathbb{E} \left( \Psi_{s,t}^1(Y_s) \psi_{s,t}^2(Y_s) [((\Delta Y_s) \circ Y_s) \Delta W_s] [((\Delta X'_t) \circ X_{s+h_1,t})(Y_s) \Delta W_t] \right) + O \left( h_1^{2+1/2} \right) \end{aligned}$$

To take the final step, observe that

$$\begin{aligned} &\mathbb{E} (\Delta Y_s(y) \Delta W_s) \\ &= \mathbb{E} \left( \int_s^{s+h_1} \bar{b}(Y_{s,u}(y)) (W_{s+h_1} - W_u) du \right) + \mathbb{E} \left( \int_s^{s+h_1} \bar{\sigma}(Y_{s,u}(y)) du \right) \\ &= \mathbb{E} \left( \int_s^{s+h_1} \bar{\sigma}(Y_{s,u}(y)) du \right) + O \left( h_1^{1+1/2} \right) \end{aligned}$$

In the same vein, we have

$$\begin{aligned} &\mathbb{E} ((\Delta X_t)(X_{s+h_1,t}(y)) \Delta W_t \mid \mathcal{W}_{s+h_1,t}) \\ &= \mathbb{E} \left( \int_t^{t+h_2} \sigma(X_{s+h_1,u}(y)) du \mid \mathcal{W}_{s+h_1,t} \right) + O \left( h_2^{1+1/2} \right) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} ((\Delta X'_t)(X_{s+h_1,t}(y)) \Delta W_t \mid \mathcal{W}_{s+h_1,t}) \\ &= \mathbb{E} \left( \int_t^{t+h_2} \partial \sigma(X_{s+h_1,u}(y)) (\partial X_{t,u})(X_{s+h_1,t}(y)) du \right) + O \left( h_2^{1+1/2} \right) \\ &= \mathbb{E} \left( \int_t^{t+h_2} \partial (\sigma \circ X_{t,u})(X_{s+h_1,t}(y)) du \right) + O \left( h_2^{1+1/2} \right) \end{aligned}$$

This shows that

$$\begin{aligned}
& h_1^{-1} h_2^{-1} \mathbb{E} \left( \Phi(X^{s+h_1}, Y_s) \Phi(X^{t+h_2}, Y_t) \Delta W_s \Delta W_t \right) \\
&= \mathbb{E} \left( \Psi_{s,t}^1(Y_s) \psi_{s,t}^1(Y_s) h_1^{-1} \left[ \int_s^{s+h_1} \bar{\sigma}(Y_u) du \right] h_2^{-1} \left[ \int_t^{t+h_2} \sigma(X_{s+h_1,u}(Y_s)) du \right] \right) \\
&+ \mathbb{E} \left( \Psi_{s,t}^1(Y_s) \psi_{s,t}^2(Y_s) h_1^{-1} \left[ \int_s^{s+h_1} \bar{\sigma}(Y_u) du \right] h_2^{-1} \left[ \int_t^{t+h_2} \partial(\sigma \circ X_{t,u})(X_{s+h_1,t}(Y_s)) du \right] \right) + O(h_1^{1/2})
\end{aligned}$$

It follows that

$$\begin{aligned}
& \lim_{h_1 \rightarrow 0} \mathbb{E} \left( \sum_{s+h_1 < t} \Pi_{s,t}^{h_1, h_2} \right) \\
&= \mathbb{E} \left( \int_0^1 \int_0^t \left[ \partial((\partial X^t) \circ Y_{s,t})(Y_s) (\varsigma \circ Y_{s,t})(Y_s) + ((\partial X^t) \circ Y_{s,t})(Y_s) \times \partial(\varsigma \circ Y_{s,t})(Y_s) \right] \bar{\sigma}(Y_s) \right. \\
&\quad \left. \times \left[ \partial((\partial X^t) \circ X_{s,t})(Y_s) \sigma(X_{s,t}(Y_s)) + \partial(X^t \circ X_{s,t})(Y_s) \partial\sigma(X_{s,t}(Y_s)) \right] \varsigma(Y_s) ds dt \right)
\end{aligned}$$

We end the proof of (7.6) using lemma 7.2 and symmetry arguments. This ends the proof of the proposition.  $\blacksquare$

## A backward Itô-Ventzell type formula

This section is mainly concerned with the proof of theorem 4.1. Before entering into the details of the proof we discuss how it applies to the process  $(X^t, Y_t)$  introduced in (7.4).

Consider the random fields

$$\begin{aligned}
F_t(x) &:= X^t(x) \implies \partial F_t = \partial X^t \quad \text{and} \quad \partial^2 F_t = \partial^2 X^t \\
G_t(x) &:= \partial X^t(x) b(x) + \frac{1}{2} \partial^2 X^t(x) a(x) \quad \text{and} \quad H_t(x) := \partial X^t(x) \sigma(x)
\end{aligned} \tag{7.7}$$

In this notation, the backward random field formula (4.1) with  $t \in [0, 1]$  takes the form

$$F_t(x) := F_1(x) + \int_t^1 G_s(x) ds + \int_t^1 H_s(x) dW_s \quad \text{with} \quad F_1(x) = x \tag{7.8}$$

We fix some given  $Y_0 = y \in \mathbb{R}$  and we write  $Y_t$  instead of  $Y_t(y)$  and set

$$(A_u, B_u, \Sigma_u) := (\bar{a}(Y_u), \bar{b}(Y_u), \bar{\sigma}(Y_u))$$

In this notation, we have

$$Y_t = y + \int_0^t B_u du + \int_0^t \Sigma_u dW_u \tag{7.9}$$

Observe that  $B_u, \Sigma_u$  as well as the Malliavin derivatives  $D_v \Sigma_u = \partial \bar{\sigma}(Y_u) D_v Y_u$  have moments of any order. Consider the processes

$$\begin{aligned}
U_t &:= \partial F_t(Y_t) B_t + \frac{1}{2} \partial^2 F_t(Y_t) A_t - G_t(Y_t) = \partial X^t(Y_t) (\bar{b} - b)(Y_t) + \frac{1}{2} \partial^2 X^t(Y_t) (\bar{a} - a)(Y_t) \\
V_t &:= \partial F_t(Y_t) \Sigma_t - H_t(Y_t) = \partial X^t(Y_t) (\bar{\sigma} - \sigma)(Y_t)
\end{aligned}$$

In this notation, up to a change of sign and replacing  $x$  by  $Y_0$  in (1.8) the stochastic interpolation formula stated in theorem 1.1 on the unit interval takes the following form

$$F_1(Y_1) - F_0(Y_0) = \int_0^1 U_s ds + \int_0^1 V_s dW_s$$

More generally, suppose we are given a forward real valued continuous semi-martingale  $Y_t$  of the form (7.9) for some  $\mathcal{W}_{0,t}$ -adapted functions  $B_t$  and  $\Sigma_t$ , and a backward random field models of the form (7.8) for some  $\mathcal{W}_{t,1}$ -adapted functions  $F_t(x), G_t(x), H_t(x)$ .

We consider the following conditions:

(H1)': *The functions  $F_t(x), G_t(x), H_t(x)$  and the differentials  $\partial F_t(x), \partial^2 F_t(x)$  and  $\partial H_t(x)$  are continuous w.r.t.  $x$  for any given  $t \in [0, 1]$  and  $\omega \in \Omega$ . In addition, for any  $n \geq 1$  we have*

$$\begin{aligned} \sup_{|y| \leq n} (|F_t(Y_t + y)| \vee |G_t(Y_t + x)|) &\leq g_n(t) \quad \text{with} \quad \mathbb{E} \left( \int_0^1 g_n(t) dt \right) < \infty \\ \sup_{|y| \leq n} (|H_t(Y_t + y)| \vee |\partial H_t(Y_t + x)|) &\leq h_n(t) \quad \text{with} \quad \mathbb{E} \left( \int_0^1 h_n^2(t) dt \right) < \infty \\ \sup_{|y| \leq n} (|\partial F_t(Y_t + y)| \vee |\partial^2 F_t(Y_t + x)|) &\leq f_n(t) \quad \text{with} \quad \mathbb{E} \left( \int_0^1 f_n^4(t) dt \right) < \infty \end{aligned} \quad (7.10)$$

(H2)': *The Malliavin derivatives  $D_s \partial F_t(x)$  and  $D_s H_t(x)$  are continuous w.r.t.  $x$  for any given  $(s, t) \in [0, 1]^2$  and  $\omega \in \Omega$ . In addition, for any  $n \geq 1$  we have*

$$\begin{aligned} \sup_{|y| \leq n} (|(D_s H_t)(Y_t + y)|) &\leq \alpha_n(s, t) \quad \text{with} \quad \mathbb{E} \left( \int_{[0,1]^2} \alpha_n^2(s, t) ds dt \right) < \infty \\ \sup_{|y| \leq n} (|(D_s \partial F_t)(Y_t + y)|) &\leq \beta_n(s, t) \quad \text{with} \quad \mathbb{E} \left( \int_{[0,1]^2} \beta_n^4(s, t) ds dt \right) < \infty \end{aligned} \quad (7.11)$$

(H3): *The random processes  $B_u, \Sigma_u$  as well as  $D_v \Sigma_u$  have moments of any order.*

The next theorem is a slight extension of theorem 4.1 applied to the semi-martingale and the random fields models discussed in (7.9) and (7.7).

**Theorem 7.4.** *Consider a backward random field models of the form (7.8) for some functions  $F_t(x), G_t(x), H_t(x)$  satisfying (H1)' and (H2)'. Also let  $Y_t$  be a continuous semi-martingale of the form (7.9) functions  $B_t$  and  $\Sigma_t$  satisfying (H3)*

*In this situation, for any  $t \in [0, 1]$  we have*

$$\begin{aligned} &F_t(Y_t) - F_0(Y_0) \\ &= \int_0^t \left( \partial F_s(Y_s) B_s + \frac{1}{2} \partial^2 F_s(Y_s) A_s - G_s(Y_s) \right) ds + \int_0^t (\partial F_s(Y_s) \Sigma_s - H_s(Y_s)) dW_s \end{aligned} \quad (7.12)$$

**Proof of theorem 7.4:**

Consider a mollifier type approximation of the identify given for any  $\epsilon > 0$  by the function

$$\varphi_\epsilon(x) := \varphi(x/\epsilon)/\epsilon \quad \text{for some smooth compactly supported function } \varphi \text{ s.t. } \int_{-\infty}^{\infty} \varphi(x) dx = 1.$$

For any  $x$ , applying the Itô-type change rule formula stated in proposition 8.2 in [34] to the product function

$$\Gamma(X^t(x), \varphi_\epsilon(Y_t - x)) := X^t(x) \varphi_\epsilon(Y_t - x)$$

we check that

$$\begin{aligned} & (F_t(x) \varphi_\epsilon(Y_t - x)) - (F_0(x) \varphi_\epsilon(Y_0 - x)) \\ &= \int_0^t \left( F_s(x) \partial \varphi_\epsilon(Y_s - x) B_s - \varphi_\epsilon(Y_s - x) G_s(x) + \frac{1}{2} F_s(x) \partial^2 \varphi_\epsilon(Y_s - x) A_s \right) ds \\ &+ \int_0^t (F_s(x) \partial \varphi_\epsilon(Y_s - x) \Sigma_s - \varphi_\epsilon(Y_s - x) H_s(x)) dW_s \end{aligned}$$

Integrating by parts where derivatives of  $\varphi_\epsilon$  appear, we check that

$$F_t^\epsilon(Y_t) := \int F_t(x) \varphi_\epsilon(Y_t - x) dx = \int F_0(x) \varphi_\epsilon(Y_0 - x) dx + \int_0^t U_s^\epsilon ds = \int_0^t V_s^\epsilon dW_s$$

with the approximating processes

$$\begin{aligned} U_s^\epsilon &:= \int \left( \partial F_s(x) B_s + \frac{1}{2} \partial^2 F_s(x) A_s - G_s(x) \right) \varphi_\epsilon(Y_s - x) dx \\ V_s^\epsilon &:= \int (\partial F_s(x) \Sigma_s - H_s(x)) \varphi_\epsilon(Y_s - x) dx \end{aligned}$$

The functions  $F_t(x)$ ,  $\partial F_t(x)$ ,  $\partial^2 F_t(x)$  and  $G_t(x)$  are almost surely continuous w.r.t.  $x$  and uniformly locally bounded. In addition, the random variables  $\partial F_t(Y_t)$ , and  $G_t(Y_t)$  as well as  $\bar{a}(Y_t)$  and  $\bar{b}(Y_t)$  are integrable at any order. Thus, by the dominated convergence theorem on  $(\Omega \times [0, 1])$  equipped with the product measure  $(\mathbb{P}(d\omega) \otimes dt)$  we have

$$\int_0^t U_s^\epsilon ds \xrightarrow{\epsilon \rightarrow 0} \int_0^t U_s ds \quad \text{as well as} \quad F_t^\epsilon(Y_t) \xrightarrow{\epsilon \rightarrow 0} F_t(Y_t)$$

It remains to check that

$$\mathbb{E} \left( \int_0^t (V_s^\epsilon - V_s)^2 ds \right) + \mathbb{E} \left( \int_{[0, t]^2} (D_r V_s^\epsilon - D_r V_s) (D_s V_r^\epsilon - D_s V_r) dr ds \right) \xrightarrow{\epsilon \rightarrow 0} 0 \quad (7.13)$$

Observe that

$$\begin{aligned} & \int_0^t (V_s^\epsilon - V_s)^2 ds \\ & \leq 2 \int_0^t \int (\partial F_s(x) - \partial F_s(Y_s))^2 \varphi_\epsilon(Y_s - x) dx ds + 2 \int_0^t (H_s(x) - H_s(Y_s))^2 \varphi_\epsilon(Y_s - x) dx ds \end{aligned}$$

Using the chain rule property we have

$$\begin{aligned} & D_t V_s^\epsilon \\ & := \int D_t (\partial F_s(x) \Sigma_s - H_s(x)) \varphi_\epsilon(Y_s - x) dx + \int (\partial F_s(x) \Sigma_s - H_s(x)) D_t \varphi_\epsilon(Y_s - x) dx \end{aligned}$$

Recalling that

$$D_t \varphi_\epsilon(Y_s - x) = D_t Y_s \partial \varphi_\epsilon(Y_s - x)$$

and integrating by parts, we check that

$$D_t V_s^\epsilon = \int [D_t (\partial F_s(x) \Sigma_s - H_s(x)) + (\partial^2 F_s(x) \Sigma_s - \partial H_s(x)) D_t Y_s] \varphi_\epsilon(Y_s - x) dx$$

Observe that

$$\begin{aligned} & D_t (\partial F_s(x) \Sigma_s - H_s(x)) + (\partial^2 F_s(x) \Sigma_s - \partial H_s(x)) D_t Y_s \\ &= ((D_t \partial F_s)(x) + \partial^2 F_s(x) D_t Y_s) \Sigma_s + \partial F_s(x) D_t \Sigma_s - ((D_t H_s)(x) + \partial H_s(x) D_t Y_s) \end{aligned}$$

On the other hand, we have

$$\begin{aligned} D_t V_s &= D_t (\partial F_s(Y_s)) \Sigma_s + \partial F_s(Y_s) D_t \Sigma_s - D_t (H_s(Y_s)) \\ &= ((D_t \partial F_s)(Y_s) + \partial^2 F_s(Y_s) D_t Y_s) \Sigma_s + \partial F_s(Y_s) D_t \Sigma_s - ((D_t H_s)(Y_s) + \partial H_s(Y_s) D_t Y_s) \end{aligned}$$

Arguing as above, we have the estimate

$$\begin{aligned} & \mathbb{E} \left( \int_{[0,1]^2} (D_r V_s^\epsilon - D_r V_s) (D_s V_r^\epsilon - D_s V_r) dr ds \right) \\ & \leq 2 \mathbb{E} \left( \int_{[0,1]^2} (D_t V_s^\epsilon - D_t V_s)^2 ds dt \right) \leq 2^4 \sum_{1 \leq i \leq 5} J_i(\epsilon) \end{aligned}$$

In the above display,  $J_i(\epsilon)$  stands for the sequences

$$\begin{aligned} J_1(\epsilon) &:= \mathbb{E} \left( \int_{[0,1]^2 \times \mathbb{R}} (\partial F_s(x) - \partial F_s(Y_s))^2 (D_t \Sigma_s)^2 \varphi_\epsilon(Y_s - x) ds dt dx \right) \\ J_2(\epsilon) &:= \mathbb{E} \left( \int_{[0,1]^2 \times \mathbb{R}} (\partial H_s(x) - \partial H_s(Y_s))^2 (D_t Y_s)^2 \varphi_\epsilon(Y_s - x) ds dt dx \right) \\ J_3(\epsilon) &:= \mathbb{E} \left( \int_{[0,1]^2 \times \mathbb{R}} (\partial^2 F_s(x) - \partial^2 F_s(Y_s))^2 (D_t Y_s)^2 A_s \varphi_\epsilon(Y_s - x) ds dt dx \right) \end{aligned}$$

The last two terms depends on the Malliavin derivatives of  $\partial F_s$  and  $H_s$  are they are given by

$$\begin{aligned} J_4(\epsilon) &:= \mathbb{E} \left( \int_{[0,1]^2 \times \mathbb{R}} ((D_t \partial F_s)(x) - (D_t \partial F_s)(Y_s))^2 A_s \varphi_\epsilon(Y_s - x) ds dt dx \right) \\ J_5(\epsilon) &:= \mathbb{E} \left( \int_{[0,1]^2 \times \mathbb{R}} ((D_t H_s)(x) - (D_t H_s)(Y_s))^2 \varphi_\epsilon(Y_s - x) ds dt dx \right) \end{aligned}$$

Arguing as above, by the dominated convergence theorem we conclude that the Skorohod integral

$$\int_0^t V_s^\epsilon dW_s \text{ converges in } \mathbb{L}_2(\Omega) \text{ as } \epsilon \rightarrow 0 \text{ to the Skorohod integral } \int_0^t V_s dW_s$$

This ends the proof of (7.13), and the proof of the theorem is now easily completed.  $\blacksquare$

We end this appendix with some comments.

**Remark 7.5.** *Recalling that the diffusion flow  $Y_t$  introduced in (7.4) has finite absolute moments of any order, the integrability conditions stated in (7.10) and (7.11) are satisfied as soon as the functions  $F_t, G_t, H_t$ , the differentials  $\partial F_t, \partial^2 F_t, \partial H_t$ , and the Malliavin derivatives  $D_s H_t, D_s \partial F_t$  have at most polynomial growth w.r.t. the state variable.*

*It is now readily check that (H1)' and (H2)' are met for the random fields introduced in (7.7). The proof can be also be extended without difficulties to multivariate models.*

## Appendix B

In this appendix we prove the estimates (1.16) and (2.10) and proposition 3.3.

### Proof of (2.10)

Whenever  $(\mathcal{M})_n$  is satisfied, we have

$$2\langle x, b_t(x) \rangle + \|\sigma_t(x)\|_F^2 \leq \gamma_0 + \gamma_1 \|x\| - \gamma_2 \|x\|^2$$

with the parameters

$$\gamma_0 = \alpha_0 + 2\beta_0 \quad \gamma_1 = \alpha_1 + 2\beta_1 \quad \text{and} \quad \gamma_2 = 2\beta_2 - \alpha_2$$

Observe that

$$\begin{aligned} & d\|X_{s,t}(x)\|^2 \\ &= \left[ 2\langle X_{s,t}(x), b_t(X_{s,t}(x)) \rangle + \|\sigma_t(X_{s,t}(x))\|_F^2 \right] dt + 2 \sum_k \langle X_{s,t}(x), \sigma_{k,t}(X_{s,t}(x)) \rangle dW_t^k \end{aligned}$$

After some elementary computations, for any  $n \geq 1$  we check that

$$\begin{aligned} n^{-1} \partial_t \mathbb{E} [\|X_{s,t}(x)\|^{2n}] &\leq -[\gamma_2 - 2(n-1)\alpha_2] \mathbb{E} [\|X_{s,t}(x)\|^{2n}] \\ &\quad + [\gamma_1 + 2(n-1)\alpha_1] \mathbb{E} [\|X_{s,t}(x)\|^{2n-1}] + [\gamma_0 + 2(n-1)\alpha_0] \mathbb{E} [\|X_{s,t}(x)\|^{2(n-1)}] \end{aligned}$$

This implies that

$$\begin{aligned} \partial_t \mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/n} &\leq -[\gamma_2 - 2(n-1)\alpha_2] \mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/n} \\ &\quad + [\gamma_1 + 2(n-1)\alpha_1] \mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/(2n)} + [\gamma_0 + 2(n-1)\alpha_0] \end{aligned}$$

from which we check that for any  $\epsilon > 0$  we have

$$\begin{aligned} & \partial_t \mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/n} \\ & \leq -[\gamma_2 - 2(n-1)\alpha_2 - 2\epsilon] \mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/n} + \frac{1}{8\epsilon} [\gamma_1 + 2(n-1)\alpha_1]^2 + [\gamma_0 + 2(n-1)\alpha_0] \end{aligned}$$

This implies that

$$\begin{aligned} & \partial_t \mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/n} \\ & \leq -2[\beta_2 - (n-1/2)\alpha_2 - \epsilon] \mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/n} + \frac{1}{8\epsilon} [\gamma_1 + 2(n-1)\alpha_1]^2 + [\gamma_0 + 2(n-1)\alpha_0] \end{aligned}$$

from which we check that

$$\mathbb{E} [\|X_{s,t}(x)\|^{2n}]^{1/n} \leq e^{-2[\beta_2 - (n-1/2)\alpha_2 - \epsilon](t-s)} \|x\|^2 + \frac{1}{8\epsilon} \frac{[\gamma_1 + 2(n-1)\alpha_1]^2 + [\gamma_0 + 2(n-1)\alpha_0]}{2[\beta_2 - (n-1/2)\alpha_2 - \epsilon]}$$

as soon as  $\epsilon < \beta_2 - (n-1/2)\alpha_2$  and  $n \geq 1$ . Replacing  $\epsilon$  by  $\epsilon(\beta_2 - (n-1/2)\alpha_2)$  and then  $(2n)$  by  $n$  we check that

$$\begin{aligned} & \mathbb{E} [\|X_{s,t}(x)\|^n]^{1/n} \\ & \leq e^{-(1-\epsilon)\beta_2(n)(t-s)} \|x\| + \frac{1}{4\sqrt{\epsilon(1-\epsilon)}} \frac{\gamma_1(n) + \gamma_0(n)^{1/2}}{\beta_2(n)^{1/2}} \quad \text{with} \quad \gamma_i(n) := \gamma_i + (n-2)\alpha_i \end{aligned}$$

This ends the proof of (2.10). ■

### Proof of proposition 3.3

The proof of the estimate (3.10) is mainly based on the following technical lemma of its own interest.

**Lemma 7.6.** *Let  $Z_t$  be a non negative diffusion process satisfying an inequality of the form*

$$dZ_t \leq (-\lambda Z_t + \alpha_t \sqrt{Z_t} + \beta_t) dt + dM_t \quad \text{with} \quad \partial_t \langle M \rangle_t \leq (u_t \sqrt{Z_t} + v_t Z_t)^2$$

for some parameters  $\lambda > 0$  and  $v_t \geq 0$ , and some non negative processes  $(\alpha_t, \beta_t, u_t)$ . In this situation, for any  $\epsilon > 0$  we have

$$\mathbb{E}(Z_t^n)^{1/n} \leq e^{\int_0^t \lambda_{n,s}(\epsilon) ds} \mathbb{E}(Z_0^n)^{1/n} + \int_0^t e^{\int_s^t \lambda_{n,u}(\epsilon) du} z_s^n(\epsilon) ds \quad (7.14)$$

with the parameters

$$\begin{aligned} \lambda_{n,t}(\epsilon) & := -\lambda + \frac{n-1}{2} v_t^2 + \frac{\epsilon}{2} \\ z_t^n(\epsilon) & := \mathbb{E} [\beta_t^n]^{1/n} + \frac{n-1}{2} \mathbb{E} [u_t^{2n}]^{1/n} + \frac{1}{\epsilon} \left( \mathbb{E} [\alpha_t^{2n}]^{1/n} + (n-1)^2 \mathbb{E} [(u_t v_t)^{2n}]^{1/n} \right) \end{aligned}$$

*Proof.* Applying Itô's formula, for any  $n \geq 2$ , we have

$$\begin{aligned} & n^{-1} \partial_t \mathbb{E}(Z_t^n) \\ & \leq \mathbb{E} \left[ Z_t^{n-1} (-\lambda Z_t + \alpha_t \sqrt{Z_t} + \beta_t) + \frac{n-1}{2} (u_t \sqrt{Z_t} + v_t Z_t)^2 Z_t^{n-2} \right] \\ & = \left( -\lambda + \frac{n-1}{2} v_t^2 \right) \mathbb{E}(Z_t^n) + \mathbb{E} \left[ \left( \beta_t + \frac{n-1}{2} u_t^2 \right) Z_t^{n-1} \right] + \mathbb{E} \left[ [\alpha_t + (n-1)u_t v_t] Z_t^{n-1/2} \right] \end{aligned}$$

On the other hand, for any  $\epsilon > 0$  we have the almost sure inequality

$$[\alpha_t + (n-1)u_t v_t] Z_t^{(n-1)/2} Z_t^{n/2} \leq \frac{1}{2\epsilon} [\alpha_t + (n-1)u_t v_t]^2 Z_t^{n-1} + \frac{\epsilon}{2} Z_t^n$$



This implies that

$$\begin{aligned} & n^{-1} \partial_t \mathbb{E}(Z_t^n) \\ & \leq \lambda_{n,t}(\epsilon) \mathbb{E}(Z_t^n) + \mathbb{E} \left[ \left( \beta_t + \frac{n-1}{2} u_t^2 + \frac{1}{2\epsilon} [\alpha_t + (n-1)u_t v_t]^2 \right) Z_t^{n-1} \right] \end{aligned}$$

Applying Hölder inequality we check that

$$\begin{aligned} & \mathbb{E} \left[ \left( \beta_t + \frac{n-1}{2} u_t^2 + \frac{1}{2\epsilon} [\alpha_t + (n-1)u_t v_t]^2 \right) Z_t^{n-1} \right] \\ & \leq \mathbb{E} \left[ \left( \beta_t + \frac{n-1}{2} u_t^2 + \frac{1}{2\epsilon} [\alpha_t + (n-1)u_t v_t]^2 \right)^n \right]^{1/n} \mathbb{E}(Z_t^n)^{1-1/n} \leq z_t^n \mathbb{E}(Z_t^n)^{1-1/n} \end{aligned}$$

This yields the estimate

$$\partial_t \mathbb{E}(Z_t^n)^{1/n} = \mathbb{E}(Z_t^n)^{-(1-1/n)} n^{-1} \partial_t \mathbb{E}(Z_t^n) \leq \lambda_{n,t}(\epsilon) \mathbb{E}(Z_t^n)^{1/n} + z_t^n$$

This ends the proof of the lemma. ■

We set

$$Y_{s,t}(x) := \|\nabla^2 X_{s,t}(x)\|_F^2 \quad \text{and} \quad T_{s,t}(x) := \|\nabla X_{s,t}(x)\|_F$$

and we also consider the collection of parameters

$$\|\tau\|_F := \sup_{t,x} \|\tau_t(x)\|_F \quad \rho(v) := \sup_{t,x} \lambda_{\max}(v_t(x))$$

with the tensor functions  $(\tau_t, v_t)$  introduced in (3.9). Observe that

$$\|\tau\|_F \leq \|\nabla^2 b\|_F + d \|\nabla^2 \sigma\|_F^2 \quad \text{and} \quad \rho(v) \leq d \|\nabla^2 \sigma\|_2^2$$

Whenever  $(\mathcal{T})_2$  is met we have

$$\text{Tr} [\nabla^2 X_{s,t}(x) A_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)'] \leq -2\lambda_A Y_{s,t}(x)$$

Also observe that

$$|\text{Tr} [[\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)] \tau_t(X_{s,t}(x)) \nabla^2 X_{s,t}(x)']| \leq \|\tau\|_F Y_{s,t}(x)^{1/2} T_{s,t}(x)^2$$

and

$$\text{Tr} [[\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)] v_t(X_{s,t}(x)) [\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)']'] \leq \rho(v) T_{s,t}(x)^4$$

In the same vein, we have

$$\begin{aligned} & |\text{Tr} \{ [\nabla X_{s,t}(x) \otimes \nabla X_{s,t}(x)] \nabla^2 \sigma_{t,k}(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \\ & \quad + \nabla^2 X_{s,t}(x) \nabla \sigma_{t,k}(X_{s,t}(x)) \nabla^2 X_{s,t}(x)' \} | \\ & \leq \|\nabla^2 \sigma_k\|_F T_{s,t}(x)^2 Y_{s,t}(x)^{1/2} + \rho(\nabla \sigma_k) Y_{s,t}(x) \end{aligned}$$

We are now in position to prove proposition 3.3.

**Proof of proposition 3.3:**

Applying the above lemma to the processes

$$Z_t = Y_{s,t}(x) \quad \lambda = 2\lambda_A \quad \alpha_t = 2\|\tau\|_F T_{s,t}(x)^2 \quad \beta_t = \rho(v) T_{s,t}(x)^4$$

and the parameters

$$u_t = 2\sqrt{d} \|\nabla^2 \sigma\|_F T_{s,t}(x)^2 \quad \text{and} \quad v_t = 2\sqrt{d} \rho_*(\nabla \sigma)$$

we obtain the estimate (7.14) with the parameters

$$\begin{aligned} \lambda_{n,t}(\epsilon) &:= -2 \left[ \lambda_A - d(n-1)\rho_*(\nabla \sigma)^2 - \frac{\epsilon}{4} \right] \\ z_t^n(\epsilon) &:= \left\{ \rho(v) + 2d(n-1) \|\nabla^2 \sigma\|_F^2 + \frac{4}{\epsilon} \left( \|\tau\|_F^2 + 4d^2(n-1)^2 \rho_*(\nabla \sigma)^2 \|\nabla^2 \sigma\|_F^2 \right) \right\} \\ &\quad \times \mathbb{E} \left[ \|\nabla X_{s,t}(x)\|_F^{4n} \right]^{1/n} \end{aligned}$$

Observe that

$$z_t^n(\epsilon) \leq cn^2 (1 \vee \epsilon^{-1}) \chi(b, \sigma)^2 \mathbb{E} \left[ \|\nabla X_{s,t}(x)\|_F^{4n} \right]^{1/n}$$

for some universal constant  $c < \infty$  and the parameter  $\chi(b, \sigma)$  defined in (2.6). Using (3.8) we check that

$$\begin{aligned} &\mathbb{E} \left( \|\nabla^2 X_{s,t}(x)\|_F^{2n} \right)^{1/n} \\ &\leq cn^2 (1 \vee \epsilon^{-1}) \chi(b, \sigma)^2 \int_s^t e^{-2[\lambda_A - d(n-1)\rho_*(\nabla \sigma)^2 - \frac{\epsilon}{4}](t-u)} e^{-4[\lambda_A - (n-1)\rho(\nabla \sigma)^2](u-s)} du \\ &= cn^2 (1 \vee \epsilon^{-1}) \chi(b, \sigma)^2 e^{-2[\lambda_A - d(n-1)\rho_*(\nabla \sigma)^2 - \frac{\epsilon}{4}](t-s)} \\ &\quad \int_s^t e^{-2[\lambda_A - (n-1)\rho(\nabla \sigma)^2 + (n-1)[d\rho_*(\nabla \sigma)^2 - \rho(\nabla \sigma)^2] + \frac{\epsilon}{8}](u-s)} du \end{aligned}$$

Assume that

$$\lambda_A > d(n-1)\rho_*(\nabla \sigma)^2$$

In this case there exists some  $0 < \epsilon_n \leq 1$  such that for any  $0 < \epsilon \leq \epsilon_n$  we have

$$\lambda_A - d(n-1)\rho_*(\nabla \sigma)^2 > \epsilon$$

and therefore

$$\mathbb{E} \left( \|\nabla^2 X_{s,t}(x)\|_F^{2n} \right)^{1/(2n)} \leq cn \epsilon^{-1} \chi(b, \sigma) \exp \left( - [\lambda_A - d(n-1)\rho_*(\nabla \sigma)^2 - \epsilon] (t-s) \right)$$

This ends the proof of the proposition. ■

## Proof of (1.16)

Using (2.14), the generalized Minkowski inequality applied to (1.8) whenever  $(\mathcal{T})_{n/\delta}$  is met for some  $\delta \in ]0, 1[$  and  $n \geq 2$  gives

$$\begin{aligned} & \mathbb{E} [\|T_{s,t}(\Delta a, \Delta b)(x)\|^n]^{1/n} \\ & \leq \frac{\kappa_{n/\delta}}{\lambda(n/\delta)} \left( \|\Delta b(x)\|_{n/(1-\delta)} + \|\Delta a(x)\|_{n/(1-\delta)} \right) \quad \text{with } (\kappa_n, \lambda(n)) \text{ given in (2.13)}. \end{aligned} \tag{7.15}$$

Using (7.15) and (6.9) we check that

$$\begin{aligned} & \mathbb{E} [\|X_{s,t}(x) - \overline{X}_{s,t}(x)\|^n]^{1/n} \\ & \leq \kappa_{(\delta_1, \delta_2), n} \left( \|\Delta a(x)\|_{n/(1-\delta_1)} + \|\Delta b(x)\|_{n/(1-\delta_1)} + \|\Delta \sigma(x)\|_{2n/\delta_2} (1 \vee \|x\|) \right) \end{aligned}$$

as soon as the regularity conditions  $(\mathcal{T})_{n/\delta_1}$ ,  $(M)_{2n/\delta_2}$  and  $(T)_{2n/(1-\delta_2)}$  are satisfied for some parameter  $n \geq 2$  and some  $\delta_1, \delta_2 \in ]0, 1[$ . Choosing  $\delta_1 = (1 - \delta_2)/2$  and setting  $\delta = \delta_2$  we check that

$$\begin{aligned} & \mathbb{E} [\|X_{s,t}(x) - \overline{X}_{s,t}(x)\|^n]^{1/n} \\ & \leq \kappa_{\delta, n} \left( \|\Delta a(x)\|_{2n/(1+\delta)} + \|\Delta b(x)\|_{2n/(1+\delta)} + \|\Delta \sigma(x)\|_{2n/\delta} (1 \vee \|x\|) \right) \end{aligned}$$

as soon as  $(M)_{2n/\delta}$  and  $(T)_{2n/(1-\delta)}$  are satisfied for some parameter  $n \geq 2$  and some  $\delta \in ]0, 1[$ . For instance,  $(\mathcal{M})_{2n/\delta}$  and  $(\mathcal{T})_{2n/(1-\delta)}$  are satisfied as soon as

$$\beta_2 - \alpha_2/2 > (n/\delta - 1) \alpha_2 \quad \text{and} \quad \lambda_A > d(n/(1 - \delta) - 1) \rho_\star(\nabla \sigma)^2$$

This ends the proof of (1.16). ■

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