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Hélène Barucq, Abderrahmane Bendali, Julien Diaz, Sébastien Tordeux. On the finite element solution of helmholtz problems in anisotropic media. Journées Ondes Sud-Ouest (JOSO), Mar 2019, Le Barp, France. hal-02138982

**HAL Id: hal-02138982**

**<https://inria.hal.science/hal-02138982v1>**

Submitted on 24 May 2019

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# On the finite element solution of helmholtz problems in anisotropic media

H. Barucq<sup>1</sup>, A. Bendali<sup>2</sup>, J. Diaz<sup>1</sup>, S. Tordeux<sup>1</sup>

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2. INSA of Toulouse

JOSO, 13 mars 2019



# The time harmonic anisotropic scalar wave equation

$$\begin{cases} \nabla \cdot (A(\mathbf{x})\nabla u(\mathbf{x})) + \mu(\mathbf{x})u(\mathbf{x}) = 0 & \text{in } \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \text{on } \partial\Omega, \end{cases}$$

- **Anisotropies** are handled inside the divergence operator
- The matrix  $A$  is **symmetric**
- $A = \lambda I$ , for **isotropic** media
- $A$  and  $\mu$  are often **piecewise constant**.
- The domain  $\Omega$  is 2D or 3D with boundary  $\partial\Omega$ .
- Other boundary conditions can be as well considered

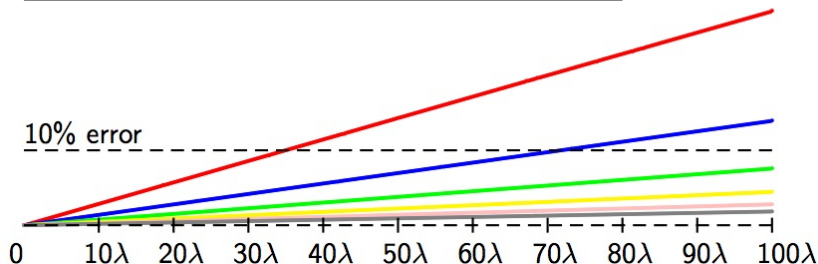
# Motivation: Computations on very large domains

- $\Omega$  is very large vs the wavelength
- Need to augment the density of nodes to maintain a given level of accuracy

*Babuska, SA Sauter, Is the **pollution effect** of the FEM avoidable for the Helmholtz equation considering high wave numbers? SIAM Journal on numerical analysis, 1997 (cited **476 times**)*

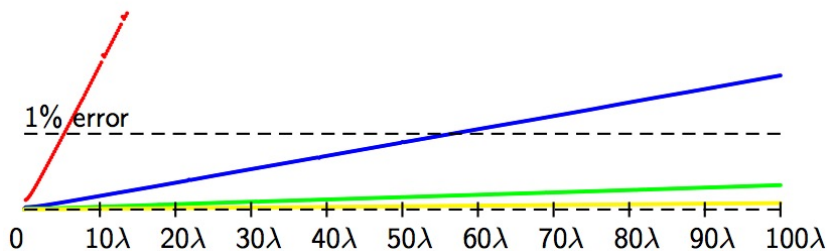
# Error at every wavelength in $\mathbb{P}_2$ for 10, 12, 14, 16, 18 segments per $\lambda$

- 20 segments per  $\lambda$ , 4800 dof per  $\lambda^2$
- 18 segments per  $\lambda$ , 3888 dof per  $\lambda^2$
- 16 segments per  $\lambda$ , 3072 dof per  $\lambda^2$
- 14 segments per  $\lambda$ , 2352 dof per  $\lambda^2$
- 12 segments per  $\lambda$ , 1728 dof per  $\lambda^2$
- 10 segments per  $\lambda$ , 1200 dof per  $\lambda^2$



# Error at every wavelength in $\mathbb{P}_3$ for 4, 6, 8, 10 segments per $\lambda$

- 10 segments per  $\lambda$ , 2000 dof per  $\lambda^2$
- 8 segments per  $\lambda$ , 1280 dof per  $\lambda^2$
- 6 segments per  $\lambda$ , 720 dof per  $\lambda^2$
- 4 segments per  $\lambda$ , 320 dof per  $\lambda^2$



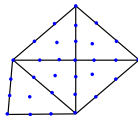
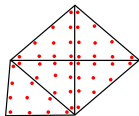
# Motivation: Computations on very large domains

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- Exceed the storage capacities.
- Discontinuous Galerkin methods do resist better to pollution effect

- Studies show that DG weak inter-element continuity contributes to fight the pollution effect
- But DG approximations imply to increase the number of nodes significantly



- Lead to Trefftz methods
- In particular, Ultra-Weak-Variational-Formulations proposed by B. Després and O. Cessenat.
- Trefftz method: shape functions are solutions to the problem
- Set on a single element  $T$ : Trefftz formulation reduces to the boundary of the element



# Rewrite the IPDG and UWVF formulation in a different context

- Classically, plane wave bases or Bessel functions inside each element.
- Here we use an **auxiliary method** to compute local solutions.

We call these methods

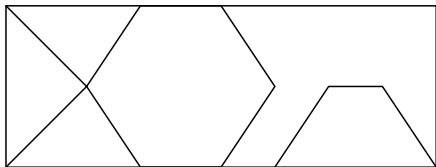
- **BEM-STDG**
- **FEM-STDG**
- **FD-STDG**
- **Plane-wave-STDG**
- **BEM-UWVF**
- **FEM-UWVF**
- **FD-UWVF**
- **Plane-wave-UWVF**

## For **isotropic media**

- Boundary Integral Equations (BIE) lead to less pollution effect than FEMs
- Recently, Hofreither et al. (2015) have proposed a FEM in which local shape functions are obtained on the basis of a BIE.
- In the same spirit, we propose a DG method using local shape solutions to the Helmholtz problem that are matched at the interface of the mesh thanks to the Dirichlet-to-Neumann (DtN) operator which is computed with a BIE.

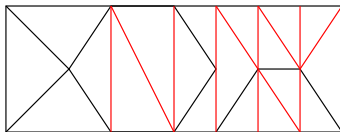
## For **anisotropic media**

- Very precise **Finite Element Method** to reproduce the BIE
- Can also handle strongly heterogeneous media

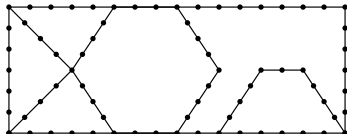


A smart finite element method

# The mesh



A classical finite element mesh



A smart finite element mesh

# Trial and test functions

Trial and test functions are solutions to the Helmholtz equation  
in each element

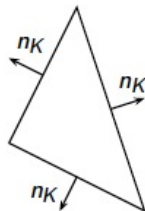
$$\operatorname{div}\left(A_K \nabla u_K(\mathbf{x})\right) + \mu_K u_K(\mathbf{x}) = 0 \text{ in } K$$

$u_K$  is uniquely defined by its Dirichlet trace if  $K$  is small enough (**geometrical criterion**)

$$u_K \in H^{1/2}(\partial K)$$

The discrete variational space is then obtained by considering a discrete trace space

$u_K$  is  $\mathbb{P}_r$ -continuous on  $\partial K$ .



# Trial and test functions

Trial and test functions are solutions to the Helmholtz equation  
in each element

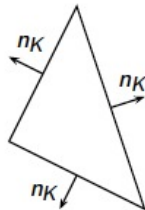
$$\operatorname{div}\left(A_K \nabla u_K(\mathbf{x})\right) + \mu_K u_K(\mathbf{x}) = 0 \text{ in } K$$

$u_K$  is uniquely defined by its Robin trace

$$A_k \nabla u_K \cdot n_K + i\eta u_K \in H^{-1/2}(\partial K)$$

The discrete variational space is then obtained by considering a discrete trace space

$A_k \nabla u_K \cdot n_K + i\eta u_K$  is  $\mathbb{P}_r$ -discontinuous on  $\partial K$ .



# The reciprocity principle for anisotropic media

Since  $u$  and  $v$  are solutions of the anisotropic Helmholtz equation on each element  $K$ .

$$\int_{\partial K} (A\nabla u) \cdot n v = \int_K (A\nabla u) \cdot \nabla v - \kappa^2 u v = \int_{\partial K} u \cdot n (A\nabla v) \quad (1)$$

The reciprocity principle is the main ingredient of any Trefftz method

$$\int_{\partial K} u q - p v = 0$$

Denoting by

$$p = (A\nabla u) \cdot n \quad q = (A\nabla v) \cdot n$$

This expression is the main ingredient of all the Trefftz formulation.

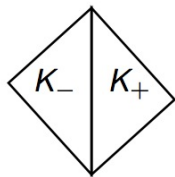
# Symmetric Trefftz Variational Formulation

Summing over all the elements

$$0 = \sum_K \int_{\partial K} uq - pv ds_x$$

with the collection of interior edges  $\Gamma$  and  $\partial\Omega$  the boundary of the domain

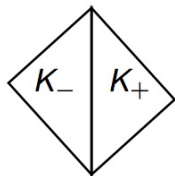
$$\left\{ \begin{array}{l} 0 = \int_{\Gamma} u_+ q_+ + u_- q_- - p_+ v_+ + p_- v_- ds_x \\ + \int_{\partial\Omega} uq - pv ds_x \end{array} \right.$$





# Symmetric Trefftz Variational Formulation

$$\begin{cases} 0 = \int_{\Gamma} u_+ q_+ + u_- q_- - p_+ v_+ + p_- v_- ds_x \\ + \int_{\partial\Omega} uq - pv ds_x \end{cases}$$



The exact solution satisfies  $u_+ = u_-$  and  $p_+ = -p_-$  on  $\Gamma$

$$\begin{cases} 0 = \int_{\Gamma} u_- q_+ + u_+ q_- + p_- v_+ + p_+ v_- ds_x \\ + \int_{\partial\Omega} uq - pv ds_x \end{cases}$$

and  $u = g$  on  $\partial\Omega$

$$\underbrace{\int_{\Gamma} u_- q_+ + u_+ q_- + p_- v_+ + p_+ v_- ds_x - \int_{\partial\Omega} pv + uq ds_x}_{a(u,p;v,q)} = \underbrace{-2 \int_{\partial\Omega} gq ds_x}_{\ell_2(v)}$$

# The symmetric variational formulation

Adding the penalization terms ( $[u] = 0$  on  $\Gamma$  and  $u = g$  on  $\partial\Omega$ ):

$$\underbrace{\int_{\Gamma} \alpha [u][v] + \int_{\partial\Omega} \alpha uv}_{b(u,p;v,q)} = \underbrace{\int_{\partial\Omega} \alpha g_D v}_{\ell_2(v)}$$

This leads to the Trefftz-DG formulation

$$a(u, p; v, q) + b(u, p; v, q) = \ell_1(q) + \ell_2(v).$$

Why **the symmetry** is important ?

- for the linear algebra solver: it needs less memory
- it has been observed that BIE methods are more stable.

Now, the unknowns are  $u$  and  $p = A\nabla u \cdot \mathbf{n}$  on each face of the mesh. One may be removed.

# The DtN operator

Let

$$u_K(\mathbf{x}) \text{ be given on } \partial K.$$

The Neumann trace

$$p_K = A_K \nabla u_K \cdot \mathbf{n}_K \text{ on } \partial K$$

may then be deduced thanks to the Dirichlet-to-Neumann operator

$$DtN : \begin{cases} H^{\frac{1}{2}}(\partial K) \longrightarrow H^{-\frac{1}{2}}(\partial K) \\ u_K \longmapsto p_K \end{cases}$$

and we end up with a system involving unknowns defined on the boundary of each element.

The problem to be addressed: compute the DtN operator

# The DtN operator approximation: include an auxiliary numerical method

We can think about different methods for **isotropic media** like:

- finite element/finite difference method based on the velocity/pressure formulation
- **Boundary element method**

**Why BEM? They do resist very well to pollution effect**

# The secondary numerical method: BEM

$$\frac{V_K p_K}{\lambda_K} = \frac{M_K u_K}{2} - N_K u_K$$

- $u_K$  is approximated by a  $\mathbb{P}_r$ -continuous function
- $p_K$  is approximated by a  $\mathbb{P}_{r'}$ -discontinuous function
- $V_K$  and  $N_K$  are the single layer and double layer operators.

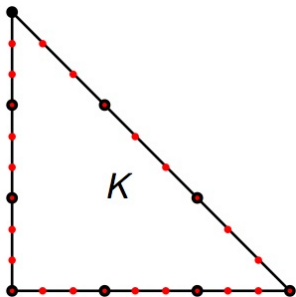
$$\left\{ \begin{array}{l} (M_K u_K, q_k)_{\partial K} = \int_{\partial K} u_K(\mathbf{x}) q_K(\mathbf{x}) ds_{\mathbf{x}}, \\ (V_K p_K, q_k)_{\partial K} = \int_{\partial K} \int_{\partial K} p_K(\mathbf{x}) G_K(\mathbf{x} - \mathbf{y}) q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}}, \\ (N_K u_K, q_k)_{\partial K} = \int_{\partial K} \int_{\partial K} p_K(\mathbf{x}) \frac{\partial G_K}{\partial \mathbf{n}_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}} \end{array} \right.$$

with

$$G(\mathbf{x}) = \frac{\exp(ik_K \|\mathbf{x}\|)}{4\pi \|\mathbf{x}\|} \text{ with } k_K = \sqrt{\frac{\mu_K}{\lambda_K}}$$

# The boundary element method

- $u_K$  is approximated by a  $\mathbb{P}_r$ -continuous function
- $p_K$  is approximated by a  $\mathbb{P}_{r'}$ -**dis**continuous function



- geometric nodes for  $p_K$
- geometric nodes for  $u_K$

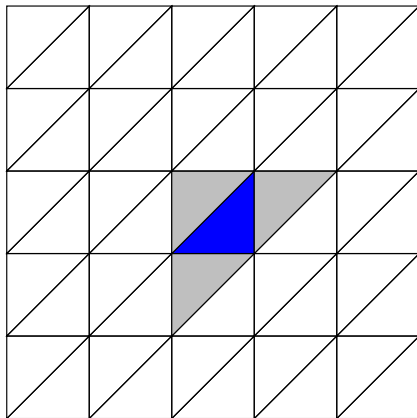
We can use different meshes for  $u_K$  and  $p_K$ .

**Idea:** the Neumann trace must be computed accurately

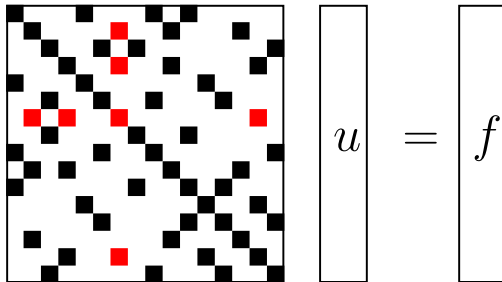
**Remark:**  $p_K$  is discontinuous, only at the geometric singularities.

# The skeleton of the matrix

Connection of the elements



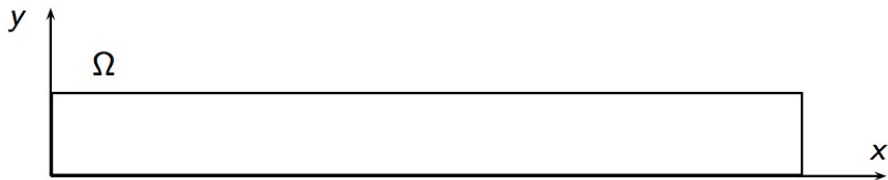
# The final formulation


$$u = f$$

- Symmetric block sparse matrix
- full small blocks
- really adapted to GMRES Solver



# A numerical simulation



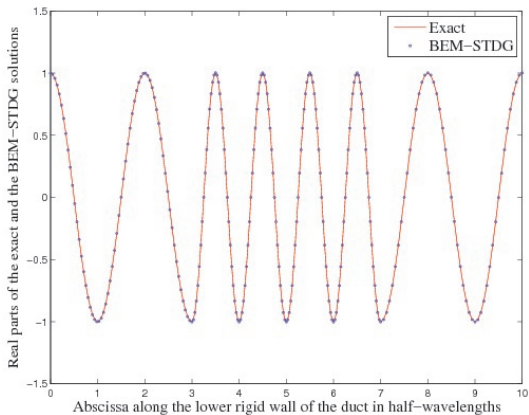
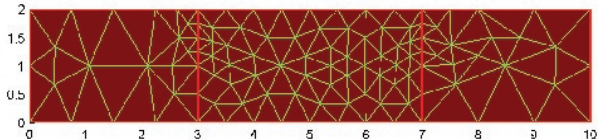
$$\left\{ \begin{array}{ll} \Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0 & \text{in } \Omega \\ u(\mathbf{x}) = 1 & \text{at } x = 0, \\ \frac{\partial u}{\partial n}(\mathbf{x}) = 0 & \text{at } \mathbf{x} = N\lambda \\ \frac{\partial u}{\partial n}(\mathbf{x}) + iku(\mathbf{x}) = 0 & \text{at } y = 0 \text{ and } N\lambda \end{array} \right. \quad (2)$$

Poly degree	nodes per $\lambda$	Method	Error at $175\lambda$ for $\mathbb{P}_2$ Error at $500\lambda$ for $\mathbb{P}_3$
$m = 2$	12	IPDG	72 %
		BEM-STDG	22 %
	16	IPDG	67 %
		BEM-STDG	5.6 %
	24	IPDG	13 %
		BEM-STDG	0.8 %
$m = 3$	12	IPDG	19 %
		BEM-STDG	1.6 %
	18	IPDG	1.7 %
		BEM-STDG	0.1 %
	24	IPDG	0.3 %
		BEM-STDG	0.02 %

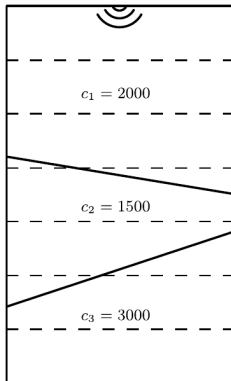
Degree	Density (nodes/ $\lambda$ )	Method	Condition number		CPU time
			50 $\lambda$	500 $\lambda$	500 $\lambda$
$m = 2$	12	IPDG	$8.8 \cdot 10^{10}$	$4.7 \cdot 10^{11}$	2.54
		BEM-STDG	$6.06 \cdot 10^7$	$1.00 \cdot 10^8$	4.76
	24	IPDG	$1.2 \cdot 10^{12}$	$2.76 \cdot 10^{12}$	19.03
		BEM-STDG	$5.96 \cdot 10^8$	$9.42 \cdot 10^9$	8.5
$m = 3$	12	IPDG	$4.2 \cdot 10^{11}$	$8.9 \cdot 10^{11}$	2.13
		BEM-STDG	$2.1 \cdot 10^8$	$1.5 \cdot 10^9$	4.75
	24	IPDG	$2.0 \cdot 10^{12}$	$6.2 \cdot 10^{13}$	20.81
		BEM-STDG	$9.52 \cdot 10^8$	$8.07 \cdot 10^{10}$	8.4
$m = 4$	8	IPDG	$1.43 \cdot 10^{11}$	$3.78 \cdot 10^{11}$	0.66
		BEM-STDG	$1.13 \cdot 10^8$	$1.08 \cdot 10^8$	3.89
	24	IPDG	$2.38 \cdot 10^{12}$	$2.41 \cdot 10^{14}$	17.91
		BEM-STDG	$1.7 \cdot 10^9$	$1.7 \cdot 10^{11}$	8.41

Condition number and CPU time for  $h$   $p$  refinements

# Case of an unstructured mesh

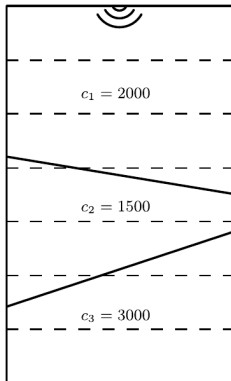


# Case of an unstructured mesh of an heterogenous medium

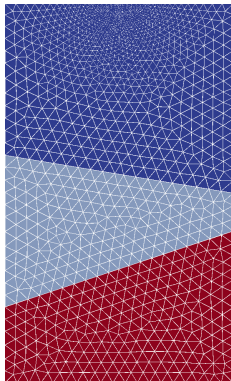


Configuration

# Case of an unstructured mesh of an heterogenous medium

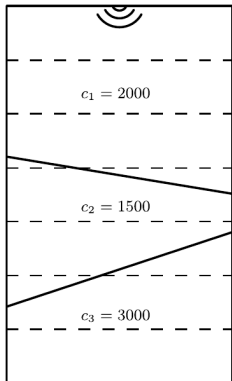


Configuration

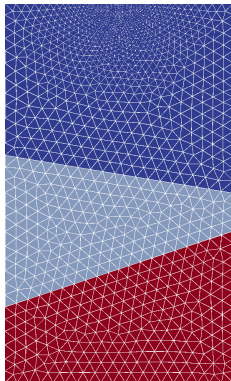


Mesh

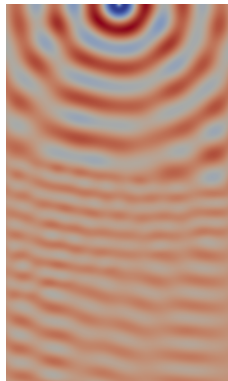
# Case of an unstructured mesh of an heterogenous medium



Configuration

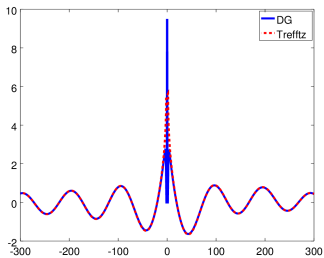


Mesh

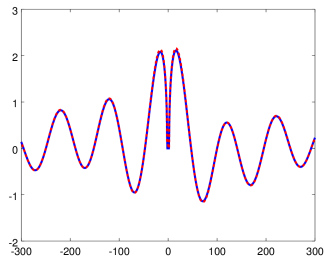


DG Solution  
(Imaginary part)

# Case of an unstructured mesh of an heterogenous medium



Real part



Imaginary part



# The anisotropic Ultra Weak Variational method

The basis of the reciprocity principle is the reciprocity principle

$$\sum_{\mathcal{K}} \int_{\partial\mathcal{K}} uq - pv = 0$$

and the identity

$$uq - pv = \frac{(p + i\eta u)(q - i\eta v) - (p - i\eta u)(q + i\eta v)}{2i\eta}$$

The choice of  $\eta$

- is well known for isotropic and homogenous media  $\eta = k$
- has not been defined correctly for heterogeneous media
- has not yet been defined for heterogeneous anisotropic media

# The trial and test functions

We denote by

$$\begin{cases} x_T = &= \frac{u_T}{2} + \frac{p_T}{2i\eta_T}, \\ \tilde{x}_T &= \frac{u_T}{2} - \frac{p_T}{2i\eta_T}, \\ y_T &= \frac{v_T}{2} + \frac{q_T}{2i\eta_T}, \\ \tilde{y}_T &= \frac{v_T}{2} - \frac{q_T}{2i\eta_T}. \end{cases}$$

The **reciprocity principle**:

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \int_{\partial K} y_T \tilde{x}_T \eta_T ds$$

- $x_T$  is a trial function

# Reflexion transmission between two media

$$\begin{array}{ccc} \nabla \cdot (A_T \nabla u_T) - \mu_T^2 u_T = 0 & & \nabla \cdot (A_L \nabla u_L) - \mu_L^2 u_L = 0 \\ & \begin{array}{c} \text{---} \\ p_T \neq -p_L \\ \leftarrow n_L \quad n_T \rightarrow \\ \text{---} \\ u_T \neq u_L \\ \text{---} \end{array} & \\ \text{media 1} & & \text{media 2} \end{array}$$

We consider a plane wave orthogonal to the boundary

$$\begin{cases} u_T(x) = I_T^{in} \exp(ik_T x_1) + I_T^{out} \exp(-ik_T x_1) & x_1 > 0, \\ u_L(x) = I_L^{in} \exp(-ik_L x_1) + I_L^{out} \exp(ik_L x_1) & x_1 < 0 \end{cases}$$

# Link with the choice of $\eta$ for the UWVF method

$$\left\{ \begin{array}{l} I_T^{in} = \frac{u_T}{2} + \frac{\rho_T}{2i\eta_T} \\ I_T^{out} = \frac{u_T}{2} - \frac{\rho_T}{2i\eta_T} \\ I_L^{in} = \frac{u_L}{2} + \frac{\rho_L}{2i\eta_L} \\ I_L^{out} = \frac{u_L}{2} - \frac{\rho_L}{2i\eta_L} \end{array} \right.$$

with  $\eta$  given by

$$\left\{ \begin{array}{l} \eta_T = k_T \sqrt{\mathbf{n} \cdot (A^T \mathbf{n})} \\ \eta_L = k_L \sqrt{\mathbf{n} \cdot (A^L \mathbf{n})} \end{array} \right.$$

This will be our choice of  $\eta$

# Transmission condition between two media

For the plane wave solution

$$I_T^{out} = R_{TL} I_T^{in} + T_{LT} I_L^{in},$$

$$I_L^{out} = R_{LT} I_L^{in} + T_{TL} I_T^{in}.$$

with

$$\left\{ \begin{array}{ll} R_{TL} = \frac{\eta_T - \eta_L}{\eta_L + \eta_T} & R_{LT} = \frac{\eta_L - \eta_T}{\eta_T + \eta_L} \\ T_{TL} = \frac{2\eta_T}{\eta_L + \eta_T} & T_{LT} = \frac{2\eta_L}{\eta_L + \eta_T} \end{array} \right.$$

# The Transmission Condition between two media

This last expression is rephrased in terms of the  $x_T$  variables

$$\begin{cases} \tilde{x}_T = R_T x_T + T_{LT} x_L, \\ \tilde{x}_L = R_L x_L + T_{TL} x_T. \end{cases}$$

and in terms of  $u$  and  $p$

$$\begin{cases} -p_T + i\eta_L u_T = p_L + i\eta_L u_L, \\ -p_L + i\eta_T u_L = p_T + i\eta_T u_T. \end{cases}$$

This is rather similar to an **upwind flux**.

# The boundary conditions

$$\begin{cases} u = g_D, & \text{(Dirichlet condition)} \\ p = g_N & \text{(Neumann condition)} \end{cases}$$

can be rewritten in terms of incoming and outgoing waves

$$-\frac{1}{2i\eta} (A\nabla u \cdot \mathbf{n} - i\eta u) = \frac{Q}{2i\eta} (A\nabla u \cdot \mathbf{n} + i\eta u) + g$$

with  $Q$  given here by

$$\begin{cases} Q = -1, & g = g_D, & \text{on } \partial\Omega_D, \\ Q = +1, & g = -g_N/i\eta, & \text{on } \partial\Omega_N, \end{cases}$$

# The boundary conditions

$$\begin{cases} u = g_D, & \text{(Dirichlet condition)} \\ p = g_N & \text{(Neumann condition)} \end{cases}$$

can be rewritten in terms of incoming and outgoing waves

$$\tilde{x}_T = Qx_T + g$$

with  $Q$  given here by

$$\begin{cases} Q = -1, & g = g_D, & \text{on } \partial\Omega_D, \\ Q = +1, & g = -g_N/i\eta, & \text{on } \partial\Omega_N, \end{cases}$$



The starting point is the reciprocity principle

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \int_{\partial T} \tilde{x}_T y_T \eta_T ds$$

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$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} \tilde{x}_T y_T \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} y_T \tilde{x}_T \eta_T ds$$

where we have decomposed the boundary into two parts

- the edges shared  $\mathcal{F}_I$  with another element  $L$
- the exterior edges  $\mathcal{F}_\partial$

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} \tilde{x}_T y_T \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} T_{TL} x_{LY_T} \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} R_{TL} x_T y_T \eta_T ds$$

For the edges of the boundary, we replace  $\tilde{x}_T$  by

$$\tilde{x}_T = Qx_T + g$$

# UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} (Q_T x_T + g) y_T \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} T_{TL} x_L y_T \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} R_{TL} x_T y_T \eta_T ds$$

For the edges of the boundary, we replace  $\tilde{x}_T$  by

$$\tilde{x}_T = Q x_T + g$$

For the interior edges, we replace  $\tilde{x}_T$  by

$$\tilde{x}_T = R_{TL} x_T + T_{LT} x_L.$$

It remains to understand what is the  $\tilde{\cdot}$  operator

# The incoming to outgoing operator

Given  $x_T \in L^2(\partial K)$ , we consider the solution of

$$\left\{ \begin{array}{l} \text{Find } u \in H^1(K) \text{ tel que } \nabla \cdot (A \nabla u) + \mu u = 0 \quad \text{dans } T \\ \frac{u_T}{2} - \frac{A \nabla u_T \cdot \mathbf{n}_T}{2i\eta_T} = x_T \quad \text{sur } \partial T \end{array} \right.$$

$$\tilde{x}_T = \frac{u_T}{2} + \frac{p_T}{2i\eta_T} = u_T - \left( \frac{u_T}{2} - \frac{p_T}{2i\eta_T} \right) = u_T - x_T.$$

No need to compute  $p_T$  to get  $\tilde{x}_T$ .

The incoming to outgoing operator is a **unitary operator**

$$\left\{ \begin{array}{l} \tilde{\cdot}_T : L^2(\partial T) \longrightarrow L^2(\partial T) \\ x_T \longmapsto \tilde{x}_T \end{array} \right.$$

# Conclusion and perspectives

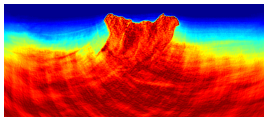
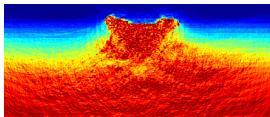
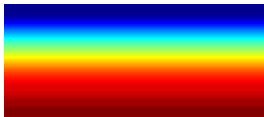
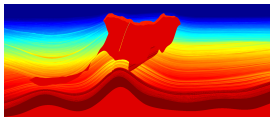
- Trefftz IPDG formulation combined with BEM reduces the pollution effect
- FEM UWVF shows the same properties. We have only very preliminary results that can not be yet presented
- The future: extension to elastic waves. Not that obvious when considering BEM or FEM...

## BEM and elasticity

- **Discontinuity of Neumann and Fourier data to be satisfied**
- For Elastic waves, it is still an open question because of the symmetry of the strain tensor. Mixed finite elements can be used but on structured meshes only.
- That is why we are considering DG approximation to compute the Neumann data.
- In the case of a source, a preprocessing is performed at the level of the element by using DG with homogeneous Dirichlet condition to come back to the case where the source is zero.

# FWI using DG or FD

Same acoustic model of size  $35 \times 15$  km. Exact same FWI algorithm (n iterations, frequencies, ...), **no initial information**.



Computations performed by Florian Faucher (Magique-3D)