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On the finite element solution of helmholtz problems in anisotropic media

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JOSO, 13 mars 2019



The time harmonic anisotropic scalar wave equation

$$\begin{cases} \nabla \cdot (A(\mathbf{x}) \nabla u(\mathbf{x})) + \mu(\mathbf{x}) u(\mathbf{x}) = 0 & \text{in } \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \text{on } \partial\Omega, \end{cases}$$

- **Anisotropies** are handled inside the divergence operator
- The matrix A is **symmetric**
- $A = \lambda I$, for **isotropic** media
- A and μ are often **piecewise constant**.
- The domain Ω is 2D or 3D with boundary $\partial\Omega$.
- Other boundary conditions can be as well considered

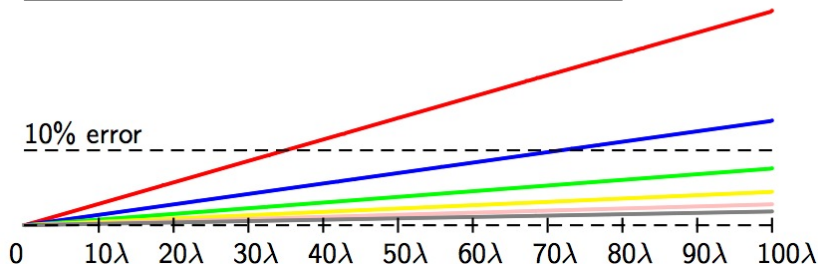
Motivation: Computations on very large domains

- Ω is very large vs the wavelength
- Need to augment the density of nodes to maintain a given level of accuracy

*Babuska, SA Sauter, Is the **pollution effect** of the FEM avoidable for the Helmholtz equation considering high wave numbers? SIAM Journal on numerical analysis, 1997 (**cited 476 times**)*

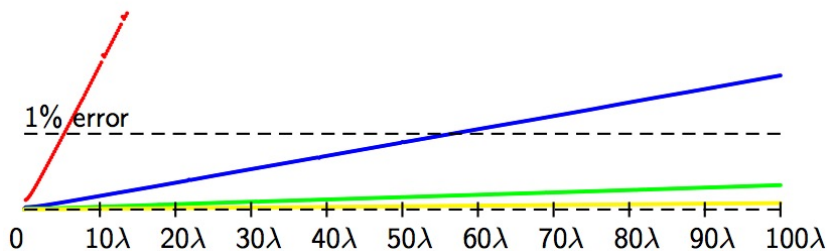
Error at every wavelength in \mathbb{P}_2 for 10, 12, 14, 16, 18 segments per λ

- 20 segments per λ , 4800 dof per λ^2
- 18 segments per λ , 3888 dof per λ^2
- 16 segments per λ , 3072 dof per λ^2
- 14 segments per λ , 2352 dof per λ^2
- 12 segments per λ , 1728 dof per λ^2
- 10 segments per λ , 1200 dof per λ^2



Error at every wavelength in \mathbb{P}_3 for 4, 6, 8, 10 segments per λ

- 10 segments per λ , 2000 dof per λ^2
- 8 segments per λ , 1280 dof per λ^2
- 6 segments per λ , 720 dof per λ^2
- 4 segments per λ , 320 dof per λ^2



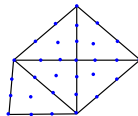
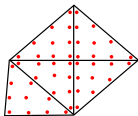
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- Exceed the storage capacities.
- Discontinuous Galerkin methods do resist better to pollution effect

- Studies show that DG weak inter-element continuity contributes to fight the pollution effect
- But DG approximations imply to increase the number of nodes significantly



- Lead to Trefftz methods
- In particular, Ultra-Weak-Variational-Formulations proposed by B. Després and O. Cessenat.
- Trefftz method: shape functions are solutions to the problem
- Set on a single element T : Trefftz formulation reduces to the boundary of the element

Rewrite the IPDG and UWVF formulation in a different context

- Classically, plane wave bases or Bessel functions inside each element.
- Here we use an **auxiliary method** to compute local solutions.

We call these methods

- | | |
|--------------------------|--------------------------|
| • BEM-STDG | • BEM-UWVF |
| • FEM-STDG | • FEM-UWVF |
| • FD-STDG | • FD-UWVF |
| • Plane-wave-STDG | • Plane-wave-UWVF |

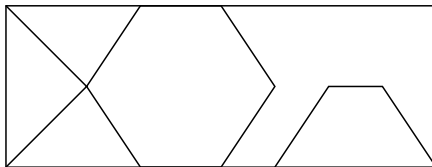
For **isotropic media**

- Boundary Integral Equations (BIE) lead to less pollution effect than FEMs
- Recently, Hofreither et al. (2015) have proposed a FEM in which local shape functions are obtained on the basis of a BIE.
- In the same spirit, we propose a DG method using local shape solutions to the Helmholtz problem that are matched at the interface of the mesh thanks to the Dirichlet-to-Neumann (DtN) operator which is computed with a BIE.

For **anisotropic media**

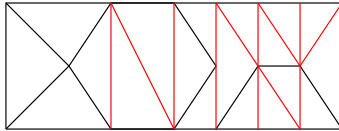
- Very precise **Finite Element Method** to reproduce the BIE
- Can also handle strongly heterogeneous media

The mesh

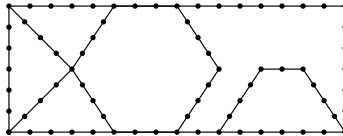


A smart finite element method

The mesh



A classical finite element mesh



A smart finite element mesh

Trial and test functions

Trial and test functions are solutions to the Helmholtz equation
in each element

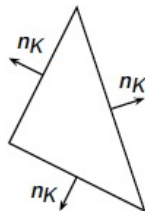
$$\operatorname{div}\left(A_K \nabla u_K(\mathbf{x})\right) + \mu_K u_K(\mathbf{x}) = 0 \text{ in } K$$

u_K is uniquely defined by its Dirichlet trace
if K is small enough (geometrical criterion)

$$u_K \in H^{1/2}(\partial K)$$

The discrete variational space is then obtained by considering a discrete trace space

u_K is \mathbb{P}_r -continuous on ∂K .



Trial and test functions

Trial and test functions are solutions to the Helmholtz equation
in each element

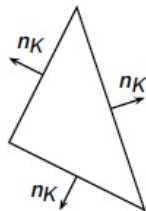
$$\operatorname{div}\left(A_K \nabla u_K(\mathbf{x})\right) + \mu_K u_K(\mathbf{x}) = 0 \text{ in } K$$

u_K is uniquely defined by its Robin trace

$$A_k \nabla u_K \cdot n_K + i\eta u_K \in H^{-1/2}(\partial K)$$

The discrete variational space is then obtained by considering a discrete trace space

$A_k \nabla u_K \cdot n_K + i\eta u_K$ is \mathbb{P}_r -discontinuous on ∂K .



The reciprocity principle for anisotropic media

Since u and v are solutions of the anisotropic Helmholtz equation on each element K .

$$\int_{\partial K} (A \nabla u) \cdot n v = \int_K (A \nabla u) \cdot \nabla v - \kappa^2 u v = \int_{\partial K} u \cdot n (A \nabla v) \quad (1)$$

The reciprocity principle is the main ingredient of any Trefftz method

$$\int_{\partial K} u q - p v = 0$$

Denoting by

$$p = (A \nabla u) \cdot n \quad q = (A \nabla v) \cdot n$$

This expression is the main ingredient of all the Trefftz formulation.

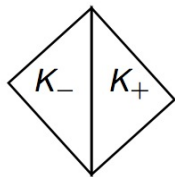
Symmetric Trefftz Variational Formulation

Summing over all the elements

$$0 = \sum_K \int_{\partial K} uq - pvd s_x$$

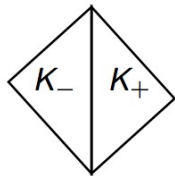
with the collection of interior edges Γ and $\partial\Omega$ the boundary of the domain

$$\left\{ \begin{array}{l} 0 = \int_{\Gamma} u_+ q_+ + u_- q_- - p_+ v_+ + p_- v_- ds_x \\ + \int_{\partial\Omega} uq - pvd s_x \end{array} \right.$$



Symmetric Trefftz Variational Formulation

$$\begin{cases} 0 &= \int_{\Gamma} u_+ q_+ + u_- q_- - p_+ v_+ + p_- v_- ds_x \\ &+ \int_{\partial\Omega} uq - pvd s_x \end{cases}$$



The exact solution satisfies $u_+ = u_-$ and $p_+ = -p_-$ on Γ

$$\begin{cases} 0 &= \int_{\Gamma} u_- q_+ + u_+ q_- + p_- v_+ + p_+ v_- ds_x \\ &+ \int_{\partial\Omega} uq - pvd s_x \end{cases}$$

and $u = g$ on $\partial\Omega$

$$\underbrace{\int_{\Gamma} u_- q_+ + u_+ q_- + p_- v_+ + p_+ v_- ds_x - \int_{\partial\Omega} pv + uq ds_x}_{a(u,p;v,q)} = \underbrace{-2 \int_{\partial\Omega} gq ds_x}_{\ell_2(v)}$$

The symmetric variational formulation

Adding the penalization terms ($[u] = 0$ on Γ and $u = g$ on $\partial\Omega$):

$$\underbrace{\int_{\Gamma} \alpha [u][v] + \int_{\partial\Omega} \alpha uv}_{b(u,p;v,q)} = \underbrace{\int_{\partial\Omega} \alpha g_D v}_{\ell_2(v)}$$

This leads to the Trefftz-DG formulation

$$a(u, p; v, q) + b(u, p; v, q) = \ell_1(q) + \ell_2(v).$$

Why **the symmetry** is important ?

- for the linear algebra solver: it needs less memory
- it has been observed that BIE methods are more stable.

Now, the unknowns are u and $p = A\nabla u \cdot \mathbf{n}$ on each face of the mesh. One may be removed.

The DtN operator

Let

$u_K(\mathbf{x})$ be given on ∂K .

The Neumann trace

$$p_K = A_K \nabla u_K \cdot \mathbf{n}_K \text{ on } \partial K$$

may then be deduced thanks to the Dirichlet-to-Neumann operator

$$DtN : \begin{cases} H^{\frac{1}{2}}(\partial K) \longrightarrow H^{-\frac{1}{2}}(\partial K) \\ u_K \longmapsto p_K \end{cases}$$

and we end up with a system involving unknowns defined on the boundary of each element.

The problem to be addressed: compute the DtN operator

The DtN operator approximation: include an auxiliary numerical method

We can think about different methods for **isotropic media** like:

- finite element/finite difference method based on the velocity/pressure formulation
- **Boundary element method**

Why BEM? They do resist very well to pollution effect

The secondary numerical method: BEM

$$\frac{V_K p_K}{\lambda_K} = \frac{M_K u_K}{2} - N_K u_K$$

- u_K is approximated by a \mathbb{P}_r -continuous function
- p_K is approximated by a $\mathbb{P}_{r'}$ -discontinuous function
- V_K and N_K are the single layer and double layer operators.

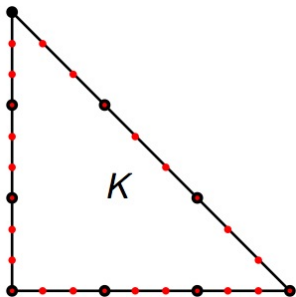
$$\begin{cases} (M_K u_K, q_k)_{\partial K} &= \int_{\partial K} u_K(\mathbf{x}) q_K(\mathbf{x}) ds_{\mathbf{x}}, \\ (V_K p_K, q_k)_{\partial K} &= \int_{\partial K} \int_{\partial K} p_K(\mathbf{x}) G_K(\mathbf{x} - \mathbf{y}) q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}}, \\ (N_K u_K, q_k)_{\partial K} &= \int_{\partial K} \int_{\partial K} p_K(\mathbf{x}) \frac{\partial G_K}{\partial \mathbf{n}_{\mathbf{y}}}(\mathbf{x} - \mathbf{y}) q_K(\mathbf{y}) ds_{\mathbf{x}} ds_{\mathbf{y}} \end{cases}$$

with

$$G(\mathbf{x}) = \frac{\exp(ik_K \|\mathbf{x}\|)}{4\pi \|\mathbf{x}\|} \text{ with } k_K = \sqrt{\frac{\mu_K}{\lambda_K}}$$

The boundary element method

- u_K is approximated by a \mathbb{P}_r -continuous function
- p_K is approximated by a $\mathbb{P}_{r'}\text{-discontinuous}$ function



- geometric nodes for p_K
- geometric nodes for u_K

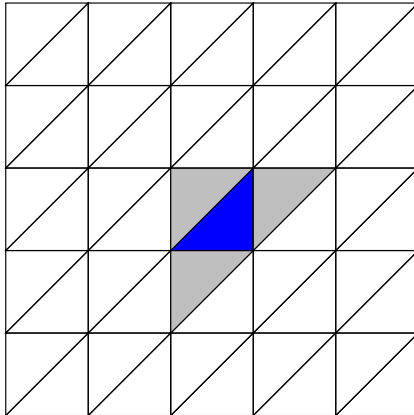
We can use different meshes for u_K and p_K .

Idea: the Neumann trace must be computed accurately

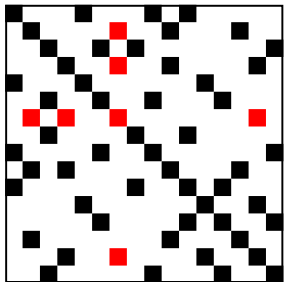
Remark: p_K is discontinuous, only at the geometric singularities.

The skeleton of the matrix

Connection of the elements

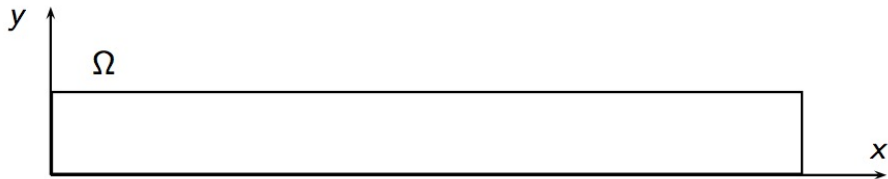


The final formulation


$$\boxed{u} = \boxed{f}$$

- Symmetric block sparse matrix
- full small blocks
- really adapted to GMRES Solver

A numerical simulation



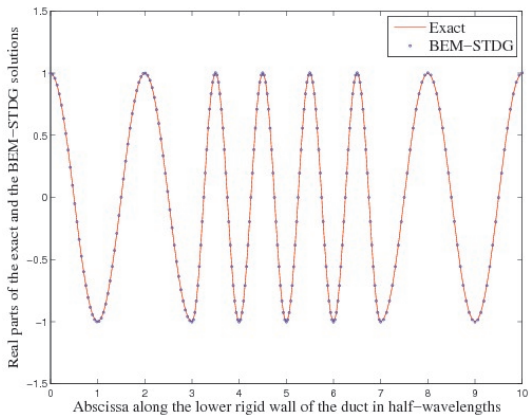
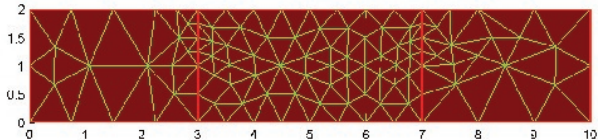
$$\left\{ \begin{array}{ll} \Delta u(\mathbf{x}) + k^2 u(\mathbf{x}) = 0 & \text{in } \Omega \\ u(\mathbf{x}) = 1 & \text{at } x = 0, \\ \frac{\partial u}{\partial n}(\mathbf{x}) = 0 & \text{at } x = N\lambda \\ \frac{\partial u}{\partial n}(\mathbf{x}) + iku(\mathbf{x}) = 0 & \text{at } y = 0 \text{ and } N\lambda \end{array} \right. \quad (2)$$

Poly degree	nodes per λ	Method	Error at 175λ for \mathbb{P}_2 Error at 500λ for \mathbb{P}_3
$m = 2$	12	IPDG	72 %
		BEM-STDG	22 %
	16	IPDG	67 %
		BEM-STDG	5.6 %
	24	IPDG	13 %
		BEM-STDG	0.8 %
$m = 3$	12	IPDG	19 %
		BEM-STDG	1.6 %
	18	IPDG	1.7 %
		BEM-STDG	0.1 %
	24	IPDG	0.3 %
		BEM-STDG	0.02 %

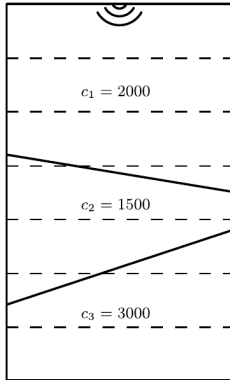
Degree	Density (nodes/ λ)	Method	Condition number 50 λ	500 λ	CPU time 500 λ
$m = 2$	12	IPDG	$8.8 \cdot 10^{10}$	$4.7 \cdot 10^{11}$	2.54
		BEM-STDG	$6.06 \cdot 10^7$	$1.00 \cdot 10^8$	4.76
	24	IPDG	$1.2 \cdot 10^{12}$	$2.76 \cdot 10^{12}$	19.03
		BEM-STDG	$5.96 \cdot 10^8$	$9.42 \cdot 10^9$	8.5
$m = 3$	12	IPDG	$4.2 \cdot 10^{11}$	$8.9 \cdot 10^{11}$	2.13
		BEM-STDG	$2.1 \cdot 10^8$	$1.5 \cdot 10^9$	4.75
	24	IPDG	$2.0 \cdot 10^{12}$	$6.2 \cdot 10^{13}$	20.81
		BEM-STDG	$9.52 \cdot 10^8$	$8.07 \cdot 10^{10}$	8.4
$m = 4$	8	IPDG	$1.43 \cdot 10^{11}$	$3.78 \cdot 10^{11}$	0.66
		BEM-STDG	$1.13 \cdot 10^8$	$1.08 \cdot 10^8$	3.89
	24	IPDG	$2.38 \cdot 10^{12}$	$2.41 \cdot 10^{14}$	17.91
		BEM-STDG	$1.7 \cdot 10^9$	$1.7 \cdot 10^{11}$	8.41

Condition number and CPU time for h p refinements

Case of an unstructured mesh

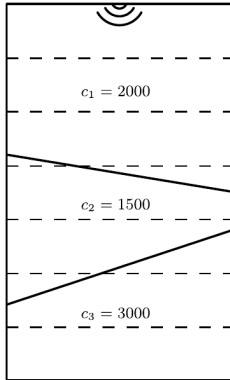


Case of an unstructured mesh of an heterogenous medium

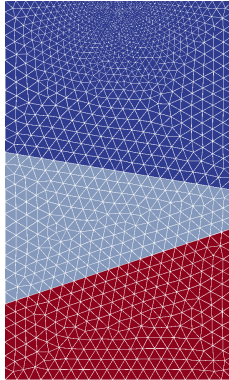


Configuration

Case of an unstructured mesh of an heterogenous medium

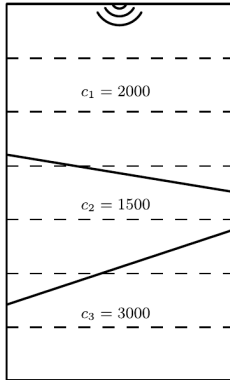


Configuration

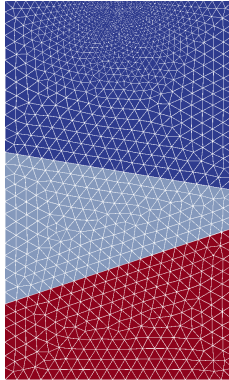


Mesh

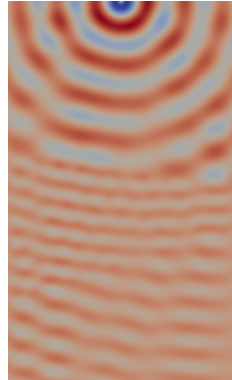
Case of an unstructured mesh of an heterogenous medium



Configuration

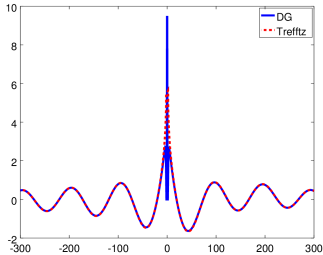


Mesh

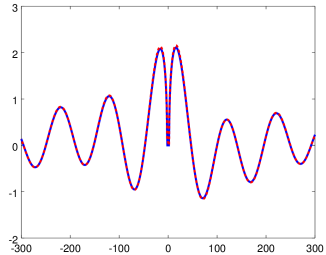


DG Solution
(Imaginary part)

Case of an unstructured mesh of an heterogenous medium



Real part



Imaginary part

The anisotropic Ultra Weak Variational method

The basis of the reciprocity principle is the reciprocity principle

$$\sum_K \int_{\partial K} uq - pv = 0$$

and the identity

$$uq - pv = \frac{(p + i\eta u)(q - i\eta v) - (p - i\eta u)(q + i\eta v)}{2i\eta}$$

The choice of η

- is well known for isotropic and homogenous media $\eta = k$
- has not been defined correctly for hetrogeneous media
- has not yet been defined for heterogeneous anisotropic media

The trial and test functions

We denote by

$$\left\{ \begin{array}{lcl} x_T & = & \frac{u_T}{2} + \frac{p_T}{2i\eta_T}, \\ \tilde{x}_T & = & \frac{u_T}{2} - \frac{p_T}{2i\eta_T}, \\ y_T & = & \frac{v_T}{2} + \frac{q_T}{2i\eta_T}, \\ \tilde{y}_T & = & \frac{v_T}{2} - \frac{q_T}{2i\eta_T}. \end{array} \right.$$

The **reciprocity principle**:

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \int_{\partial K} y_T \tilde{x}_T \eta_T ds$$

- x_T is a trial function

Reflexion transmission between two media

$$\begin{array}{ccc} \nabla \cdot (A_T \nabla u_T) - \mu_T^2 u_T = 0 & \begin{array}{c} p_T \neq -p_L \\ \leftarrow n_L \quad n_T \rightarrow \\ u_T \neq u_L \end{array} & \nabla \cdot (A_L \nabla u_L) - \mu_L^2 u_L = 0 \\ \text{media 1} & & \text{media 2} \end{array}$$

We consider a plane wave orthogonal to the boundary

$$\begin{cases} u_T(x) = I_T^{in} \exp(ik_T x_1) + I_T^{out} \exp(-ik_T x_1) & x_1 > 0, \\ u_L(x) = I_L^{in} \exp(-ik_L x_1) + I_L^{out} \exp(ik_L x_1) & x_1 < 0 \end{cases}$$

Link with the choice of η for the UWVF method

$$\left\{ \begin{array}{lcl} I_T^{in} & = & \frac{u_T}{2} + \frac{p_T}{2i\eta_T} \\ I_T^{out} & = & \frac{u_T}{2} - \frac{p_T}{2i\eta_T} \\ I_L^{in} & = & \frac{u_L}{2} + \frac{p_L}{2i\eta_L} \\ I_L^{out} & = & \frac{u_L}{2} - \frac{p_L}{2i\eta_L} \end{array} \right.$$

with η given by

$$\left\{ \begin{array}{lcl} \eta_T & = & k_T \sqrt{\mathbf{n} \cdot (A^T \mathbf{n})} \\ \eta_L & = & k_L \sqrt{\mathbf{n} \cdot (A^L \mathbf{n})} \end{array} \right.$$

This will be our choice of η

Transmission condition between two media

For the plane wave solution

$$I_T^{out} = R_{TL} I_T^{in} + T_{LT} I_L^{in},$$

$$I_L^{out} = R_{LT} I_L^{in} + T_{TL} I_T^{in}.$$

with

$$\left\{ \begin{array}{ll} R_{TL} = \frac{\eta_T - \eta_L}{\eta_L + \eta_T} & R_{LT} = \frac{\eta_L - \eta_T}{\eta_T + \eta_L} \\ T_{TL} = \frac{2\eta_T}{\eta_L + \eta_T} & T_{LT} = \frac{2\eta_L}{\eta_L + \eta_T} \end{array} \right.$$

The Transmission Condition between two media

This last expression is rephrased in terms of the x_T variables

$$\begin{cases} \tilde{x}_T = R_T x_T + T_{LT} x_L, \\ \tilde{x}_L = R_L x_L + T_{TL} x_T. \end{cases}$$

and in terms of u and p

$$\begin{cases} -p_T + i\eta_L u_T = p_L + i\eta_L u_L, \\ -p_L + i\eta_T u_L = p_T + i\eta_T u_T. \end{cases}$$

This is rather similar to an **upwind flux**.

The boundary conditions

$$\begin{cases} u = g_D, & \text{(Dirichlet condition)} \\ p = g_N & \text{(Neumann condition)} \end{cases}$$

can be rewritten in terms of incoming and outgoing waves

$$-\frac{1}{2i\eta} (A\nabla u \cdot \mathbf{n} - i\eta u) = \frac{Q}{2i\eta} (A\nabla u \cdot \mathbf{n} + i\eta u) + g$$

with Q given here by

$$\begin{cases} Q = -1, & g = g_D, & \text{on } \partial\Omega_D, \\ Q = +1, & g = -g_N/i\eta, & \text{on } \partial\Omega_N, \end{cases}$$

The boundary conditions

$$\begin{cases} u = g_D, & \text{(Dirichlet condition)} \\ p = g_N & \text{(Neumann condition)} \end{cases}$$

can be rewritten in terms of incoming and outgoing waves

$$\tilde{x}_T = Qx_T + g$$

with Q given here by

$$\begin{cases} Q = -1, & g = g_D, & \text{on } \partial\Omega_D, \\ Q = +1, & g = -g_N/i\eta, & \text{on } \partial\Omega_N, \end{cases}$$

UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \int_{\partial T} \tilde{x}_T y_T \eta_T ds$$

UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} \tilde{x}_T y_T \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_\mathcal{I}} \int_{F_{TL}} y_T \tilde{x}_T \eta_T ds$$

where we have decomposed the boundary into two parts

- the edges shared $\mathcal{F}_\mathcal{I}$ with another element L
- the exterior edges \mathcal{F}_∂

UWVF formulation with outgoing and incoming waves

The starting point is the reciprocity principle

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} \tilde{x}_T y_T \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} T_{TL} x_{LY_T} \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} R_{TL} x_T y_T \eta_T ds$$

For the edges of the boundary, we replace \tilde{x}_T by

$$\tilde{x}_T = Qx_T + g$$

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The starting point is the reciprocity principle

$$\int_{\partial T} x_T \tilde{y}_T \eta_T ds = \sum_{F_T \in \mathcal{F}_\partial} \int_{F_T} (Q_T x_T + g) y_T \eta_T ds + \\ \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} T_{TL} x_L y_T \eta_T ds + \sum_{F_{TL} \in \mathcal{F}_I} \int_{F_{TL}} R_{TL} x_T y_T \eta_T ds$$

For the edges of the boundary, we replace \tilde{x}_T by

$$\tilde{x}_T = Q x_T + g$$

For the interior edges, we replace \tilde{x}_T by

$$\tilde{x}_T = R_{TL} x_T + T_{LT} x_L.$$

It remains to understand what is the $\tilde{\cdot}$ operator

The incoming to outgoing operator

Given $x_T \in L^2(\partial K)$, we consider the solution of

$$\left\{ \begin{array}{ll} \text{Find } u \in H^1(K) \text{ tel que } \nabla \cdot (A \nabla u) + \mu u = 0 & \text{dans } T \\ \frac{u_T}{2} - \frac{A \nabla u_T \cdot \mathbf{n}_T}{2i\eta_T} = x_T & \text{sur } \partial T \end{array} \right.$$

$$\tilde{x}_T = \frac{u_T}{2} + \frac{p_T}{2i\eta_T} = u_T - \left(\frac{u_T}{2} - \frac{p_T}{2i\eta_T} \right) = u_T - x_T.$$

No need to compute p_T to get \tilde{x}_T .

The incoming to outgoing operator is a **unitary operator**

$$\left\{ \begin{array}{ll} \tilde{\cdot}_T : L^2(\partial T) & \longrightarrow L^2(\partial T) \\ x_T & \longmapsto \tilde{x}_T \end{array} \right.$$

Conclusion and perspectives

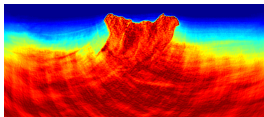
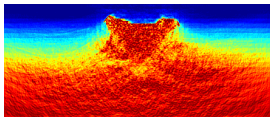
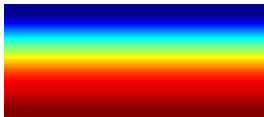
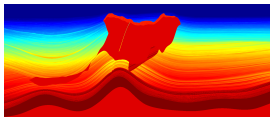
- Trefftz IPDG formulation combined with BEM reduces the pollution effect
- FEM UWVF shows the same properties. We have only very preliminary results that can not be yet presented
- The future: extension to elastic waves. Not that obvious when considering BEM or FEM...

BEM and elasticity

- **Discontinuity of Neumann and Fourier data to be satisfied**
- For Elastic waves, it is still an open question because of the symmetry of the strain tensor. Mixed finite elements can be used but on structured meshes only.
- That is why we are considering DG approximation to compute the Neumann data.
- In the case of a source, a preprocessing is performed at the level of the element by using DG with homogeneous Dirichlet condition to come back to the case where the source is zero.

FWI using DG or FD

Same acoustic model of size 35×15 km. Exact same FWI algorithm (n iterations, frequencies, ...), **no initial information**.



Computations performed by Florian Faucher (Magique-3D)