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► **To cite this version:**

Andrey Polyakov. On global existence of strong and classical solutions of Navier-Stokes equations. 2019. hal-02096215v2

HAL Id: hal-02096215

<https://inria.hal.science/hal-02096215v2>

Preprint submitted on 12 Apr 2019 (v2), last revised 23 Apr 2019 (v3)

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On global existence of strong and classical solutions of Navier-Stokes equations

Andrey Polyakov

the date of receipt and acceptance should be inserted later

Abstract The paper presents a simple proof of global existence of strong and classical solutions of Navier-Stokes equations in \mathbb{R}^3 accepting that their existence for small initial data is already proven, for example, in [2] and [4].

Keywords Navier-Stokes-Equations · Strong Solutions · Dilation Symetry

1 Introduction

In this short note we exploit a very simple idea to prove global existence of the strong solutions of the Navier-Stokes equation, which is explained by means of the following trivial example in \mathbb{R}^n .

Let a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be symmetric with respect to a dilation of the argument, i.e. $\exists \alpha \in \mathbb{R}$ such that

$$f(e^s u) = e^{(\alpha+1)s} f(u), \quad \forall u \in \mathbb{R}^n, \forall s \in \mathbb{R}$$

If $u(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$ is a classical solution of

$$\frac{du}{dt} = f(u), \quad t > 0$$

with the initial condition $u(0) = u_0$ then

$$u_s(t) := e^s u(e^{\alpha s} t)$$

is defined on $[0, +\infty)$ and, due to symmetry, we derive

$$\frac{du_s}{dt} = e^{(\alpha+1)s} f(u(e^{\alpha s} t)) = f(u_s(t)), \quad t > 0,$$

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i.e. u_s is a classical solution of the same differential equation with the initial condition $u_s(0) = e^s x_0$, where $s \in \mathbb{R}$.

Let $\exists \varepsilon > 0$ such that a classical solution of the differential equation exists on $[0, +\infty)$ for any initial value $u(0) = u_0 \in B_\varepsilon := \{u \in \mathbb{R}^n : |u| < \varepsilon\}$. To construct a solution for $u_0 \notin B_\varepsilon$ we first need to scale $u_0 \rightarrow e^{s_0} u_0$, where $s_0 \in \mathbb{R}$ is such that

$$|e^{s_0} u_0| < \varepsilon.$$

If $u(\cdot, e^{s_0} u_0)$ is a solution with the initial condition $u(0) = \lambda_0 u_0$ then $\tilde{u}(t) = e^{-s_0} u(e^{-\alpha s_0}, e^{s_0} u_0)$ is also a solution of the considered system and, obviously,

$$\tilde{u}(0) = e^{s_0} u(0, e^{s_0} u_0) = e^{-s_0} e^{s_0} u_0 = u_0.$$

In this paper we use the well-known dilation symmetry (see, [1, formula (1.5)]) to prove global existence of strong solutions of the Navier-Stokes equations in \mathbb{R}^3

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla) u - \nabla p, \\ \mathbf{0} &= \operatorname{div} u \end{aligned}$$

where u denotes the velocity of a fluid, p denotes the scalar pressure and $\nu > 0$ denotes viscosity of the fluid. The main assumption is that a global-in-time existence of strong or classical solutions of the Navier-Stokes equations for small initial data is already proven, as for example, in [2] and [4]

The standard *notation* is utilized through the paper, e.g. \mathbb{R} is the field of real numbers; $L^1_{loc}((0, T) \times \mathbb{R}^n, \mathbb{R})$ denotes the space of locally integrable functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}$; $L^p(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p \leq +\infty$ is a Lebesgue space of function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ with the norm $\|\cdot\|_p$; $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ with compact support and $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions which vanish at infinity, where $0 < T \leq \infty$. For composition of operators A, B we also use the notation $A \circ B$.

2 Preliminaries: Dilations in functional spaces

The following lemmas deal with the most common dilation groups in functional spaces.

Lemma 1 *The operator $\mathbf{d}(s)$ given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\gamma s} t, e^{\beta s} x), \quad (1)$$

where $s \in \mathbb{R}$, z is a function $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $0 < T \leq +\infty$, $x \in \mathbb{R}^n$ and $\alpha, \beta, \gamma \in \mathbb{R}$ are constant parameters,

- maps $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ onto $C_c^\infty((0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$;
- maps $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ onto $C_0^\infty([0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$.

The inverse operator is given by $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

Proof 1) Since the linear function $(t, x) \rightarrow (e^{\gamma s t}, e^{\beta s x})$ maps a compact in $(0, T) \times \mathbb{R}^n$ to a compact in $(0, e^{-\gamma s T}) \times \mathbb{R}^n$ then $\mathbf{d}(s)$ defined on whole $C_c^\infty((0, T), \mathbb{R}^n)$ and if $z \in C_c^\infty((0, T), \mathbb{R}^n)$ (i.e. z is smooth and has a compact support in $(0, T) \times \mathbb{R}^n$) then $\mathbf{d}(s)z \in C_c^\infty((0, e^{-\gamma s T}), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth, but it has a compact support in $(0, e^{-\gamma s T}) \times \mathbb{R}^n$). Obviously, $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z = z$ for any $z \in C_c^\infty((0, T), \mathbb{R}^n)$ and any $s \in \mathbb{R}$.

Let us show that $\mathbf{d}(s)$ maps $C_c^\infty((0, T), \mathbb{R}^n)$ onto $C_c^\infty((0, e^{-\gamma s T}), \mathbb{R}^n)$. Suppose the opposite: $\exists z^* \in C_c^\infty((0, e^{-\gamma s T}), \mathbb{R}^n)$ such that $z^* \neq \mathbf{d}(s)y, \forall y \in C_c^\infty((0, T), \mathbb{R}^n)$. This is impossible, since $\mathbf{d}(-s)z^* \in C_c^\infty((0, T), \mathbb{R}^n)$ and $z^* = (\mathbf{d}(s) \circ \mathbf{d}(-s))z^* \in \mathbf{d}(s)C_c^\infty((0, T), \mathbb{R}^n)$.

2) The proof for C_0^∞ is almost identical. Obviously, if $z \in C_0^\infty([0, T), \mathbb{R}^n)$ (i.e. z is smooth and vanishing at infinity) then $\mathbf{d}(s)z \in C_0^\infty([0, e^{-\gamma s T}), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth and vanishing at infinity) and $\mathbf{d}(s) \circ \mathbf{d}(-s)z = \mathbf{d}(-s) \circ \mathbf{d}(s)z = z$ for any $z \in C_0^\infty([0, T), \mathbb{R}^n)$.

Let us show that $\mathbf{d}(s)$ maps $C_0^\infty([0, T), \mathbb{R}^n)$ onto $C_0^\infty([0, e^{-\gamma s T}), \mathbb{R}^n)$. Suppose the opposite: there exists $z^* \in C_0^\infty([0, e^{-\gamma s T}), \mathbb{R}^n)$ such that $\mathbf{d}(s)y \neq z^*, \forall y \in C_0^\infty([0, T), \mathbb{R}^n)$. This is impossible since $\mathbf{d}(-s)z^* \in C_0^\infty([0, T), \mathbb{R}^n)$ and $z^* = \mathbf{d}(s) \circ \mathbf{d}(-s)z^* \in \mathbf{d}(s)C_0^\infty([0, T), \mathbb{R}^n)$.

Lemma 2 *The operator $\mathbf{d}(s)$ given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x), \quad (2)$$

where $s \in \mathbb{R}$, z is a function $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ are constant parameters, is a linear bounded invertible operator on $L^p(\mathbb{R}^n, \mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha - \beta/p)s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R},$$

where $1 \leq p \leq \infty$. The inverse operator is given by $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

Proof Let $1 < p < \infty$. If $z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n} |z(x)|^p dx < +\infty$$

and using the change-of-variable theorem for the Lebesgue integral (see e.g. [3]) we derive

$$\int_{\mathbb{R}^n} |z(x)|^p dx = e^{\beta s} \int_{\tilde{\mathbb{R}}^n} |z(e^{\beta s} x)|^p dx = e^{(\beta - \alpha p)s} \int_{\mathbb{R}^n} |(\mathbf{d}(s)z)(x)|^p dx < +\infty.$$

Since $e^{(\beta - \alpha p)s} > 0$ for any $\alpha, \beta, p, s \in \mathbb{R}$ then $\mathbf{d}(s)z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$ for any $s \in \mathbb{R}$. Obviously, $\mathbf{d}(s)$ is a linear operator on L^p , i.e. $\mathbf{d}(s)(\mu_1 z_1 + z_2) =$

$\mu_1 \mathbf{d}(s)z_1 + \mu_2 \mathbf{d}(s)z_2$, for any $\mu_1, \mu_2 \in \mathbb{R}$ and $z_1, z_2 \in L^p(\mathbb{R}^n, \mathbb{R}^m)$. Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-\beta/p)s} \|z\|_p, \quad \|z\|_p := \left(\int_{\mathbb{R}^n} |z(x)|^p dx \right)^{1/p}.$$

Hence, the operator $\mathbf{d}(s) : L^p(\mathbb{R}^n, \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^m)$ is bounded for any $s \in \mathbb{R}$.

Let $p = \infty$. If $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\text{ess sup}|z(x)| = \text{ess sup}|z(e^{\beta s}x)| < +\infty$$

for any $\beta, s \in \mathbb{R}$ and $\|\mathbf{d}(s)z\|_\infty = e^{\alpha s} \|z\|_\infty$ for any $s \in \mathbb{R}$. Therefore, $\mathbf{d}(s)$ is also a linear bounded operator on $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Obviously, $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z$ for any $z : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and any $s \in \mathbb{R}$ and we derive $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

3 Global Existence of Strong Solutions of Navier-Stokes Equations

Below for shortness we omit \mathbb{R}^3 in the notations of $\int_{\mathbb{R}^3}$, L^p , C_0^∞ and C_c^∞ if the context is clear. Without loss of generality (see e.g. [4, page 4]) we also assume that $\nu = 1$, where ν is a viscosity coefficient.

Let us consider the weak form of the Navier-Stokes equations

$$\int u(0) \cdot \xi(0) + \int_0^T \int u \cdot (\partial_t \xi + \Delta \xi) + p \operatorname{div} \xi = \int_0^T \int u \cdot (u \cdot \nabla) \xi, \quad \forall \xi \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3) \quad (3)$$

with $u(t) \in V$ for $t \in (0, T)$, where V is a set of the so-called weakly divergence free velocity fields:

$$V := \left\{ u \in L^2 : \int u \cdot \nabla \phi = 0, \forall \phi \in C_0^\infty \right\}.$$

The classical idea of analysis is to prove existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this analysis has been initiated by Jean Leray in 1938 (see [2]). We use the recent review [4] of his results.

Definition 1 ([4], Definition 3.7) A pair (u, p) is said to be a strong solution of the Navier-Stokes equations on $[0, T)$ if $u(t) \in V$ for $t \in (0, T)$, $p \in L_{loc}^1((0, T) \times \mathbb{R}^3, \mathbb{R})$,

$$u \in C([0, T), L^2) \cap C((0, T), L^\infty),$$

$\|u(t)\|_\infty$ is bounded as $t \rightarrow 0^+$ and (3) is satisfied.

A possible way for expansion of regularity to a larger set of initial conditions is to use the dilation symmetry as explained in the introduction. Several symmetries for Navier-Stokes equations are known (see e.g. [1] and references therein). Below we use just one of them (see [1, formula (1.5)]).

Lemma 3 *If (u, p) is a strong solution of the Navier-Stokes equations with the initial data $u(0) = u_0 \in V \cap L^\infty$ defined on $[0, T)$ then for any $s \in \mathbb{R}$ the pair (u_s, p_s) given by*

$$u_s(t, x) = e^s u(e^{2s}t, e^s x), \quad p_s(t, x) = e^{2s} p(e^{2s}t, e^s x), \quad t \geq 0, x \in \mathbb{R}^n, s \in \mathbb{R}.$$

is a strong solution of the Navier-Stokes equations defined on $[0, e^{-2s}T)$ with the initial value $u_s(0) = \mathbf{d}(s)u_0$.

Proof 1) Let $\mathbf{d}_1(s)$ be defined by the formula (2) with $\alpha = 1, \beta = 1$.

Since u is a strong solution then the function $t \rightarrow u(t)$ is continuous in L^2 and in L^∞ . According to Lemma 2, $\mathbf{d}_1(s)$ is a linear bounded invertible operator on L^2 and on L^∞ . Hence, for any fixed $s \in \mathbb{R}$ the function $t \rightarrow \mathbf{d}_1(s)u(t)$ is also continuous in L^2 and L^∞ , consequently, $u_s \in C([0, e^{-2s}T), L^2)$ and $u_s \in C((0, e^{-2s}T), L^\infty)$.

Notice $\|\mathbf{d}_1(s)z\|_\infty = e^s \|z\|_\infty$ for any $z \in L^\infty$ and any $s \in \mathbb{R}$ (see Lemma 2). Since $\|u(t)\|_\infty$ is bounded as $t \rightarrow 0^+$ then $\|u_s(t)\|_\infty = e^s \|u(e^s t)\|_\infty$ is also bounded as $t \rightarrow 0^+$.

If $u_0 \in V$ then $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Indeed,

$$\int u_0 \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty$$

and using the change-of-variable theorem in the Lebesgue integral we derive

$$0 = \int u_0(x) \cdot \nabla \phi(x) dx = e^s \int u_0(e^s x) \cdot (\nabla \phi)(e^s x) dx = e^{-2s} \int (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi},$$

where $\tilde{\phi} = \mathbf{d}_1(s)\phi$. Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ (see Lemma 1) then

$$0 = \int (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi}, \quad \forall \tilde{\phi} \in C_0^\infty,$$

i.e. $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Hence, the inclusion $u(t) \in V, t \in [0, T)$ implies $u_s(t) \in V, t \in [0, e^{-2s}T)$.

2) Let $\mathbf{d}_2(s)$ be given by the formula (1) with $\alpha = 2, \beta = 1$ and $\gamma = 2$. Since $p \in L_{loc}^1((0, T) \times \mathbb{R}^n, \mathbb{R})$ then

$$\int_0^T \int |p(t, x)\xi(t, x)| dx dt < +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$$

hence using the change-of-variable theorem in the Lebesgue integral for the functions $t \rightarrow e^{2s}t$ and $x \rightarrow e^s x$ we derive

$$\begin{aligned} \int_0^T \int |p(t, x)\xi(t, x)| dx dt &= e^{3s} \int_0^{e^{-2s}T} \int |p(e^{2s}t, e^s x)\xi(e^{2s}t, e^s x)| dx dt = \\ e^{-s} \int_0^{e^{-2s}T} \int |p_s \cdot \mathbf{d}_2(s)\xi| &< +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}). \end{aligned}$$

Since the operator $\mathbf{d}_2(s)$ maps $C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$ onto $C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$ then

$$\int_0^{e^{-2s}T} \int |p_s \cdot \tilde{\xi}| < +\infty, \quad \forall \tilde{\xi} \in C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R}).$$

Therefore, $p_s \in L^1_{loc}((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$.

3) Let us show that (u_s, p_s) satisfies the equation (3). Since (u, p) is a solution defined on $[0, +\infty)$ then $\forall \xi \in C_0^\infty([0, T) \times \mathbb{R}^3, \mathbb{R}^3)$ we have

$$\begin{aligned} \int u(0, x) \cdot \xi(0, x) dx + \int_0^T \int u(t, x) \cdot (\partial_t \xi(t, x) + (\Delta \xi)(t, x) + p(t, x)(\operatorname{div} \xi)(t, x)) dx dt = \\ \int_0^T \int u(t, x) \cdot (u(t, x) \cdot \nabla) \xi(t, x) dx dt \end{aligned}$$

Using the change-of-variable theorem in the Lebesgue integral for the functions $t \rightarrow e^{2s}t$ and $x \rightarrow e^s x$ we derive

$$\begin{aligned} e^s \int u(0, e^s x) \cdot \xi(0, e^s x) dx + e^{3s} \int_0^{e^{-2s}T} \int u(e^{2s}t, e^s x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + \\ (\Delta \xi)(e^{2s}t, e^s x) + p(e^{2s}t, e^{2s}x)(\operatorname{div} \xi)(e^{2s}t, e^s x)) dx dt = \\ e^{3s} \int_0^{e^{-2s}T} \int u(e^{2s}t, e^s x) \cdot (u(e^{2s}t, e^s x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} \int u_s(0, x) \cdot \xi(0, e^s x) dx + \int_0^{e^{-2s}T} \int e^{2s} u_s(t, x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + (\Delta \xi)(e^{2s}t, e^s x)) \\ + e^s p_s(t, x)(\operatorname{div} \xi)(e^{2s}t, e^s x) dx dt = \\ e^s \int_0^{e^{-2s}T} \int u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

Let us denote $\xi_s(t, x) = \xi(e^{2st}, e^s x)$. Hence,

$$(\Delta \xi_s)(t, x) = e^{2s} (\Delta \xi)(e^{2st}, e^s x),$$

$$(\nabla \xi_s)(t, x) = e^s (\nabla \xi)(e^{2st}, e^s x),$$

$$(\partial_t \xi_s)(t, x) = e^{2s} (\partial_t \xi)(e^{2st}, e^s x),$$

$$(\operatorname{div} \xi_s)(t, x) = e^s (\operatorname{div} \xi)(e^{2st}, e^s x),$$

and

$$\int u_s(0) \cdot \xi_s(0) + \int_0^{e^{-2s}T} \int u_s \cdot (\partial_t \xi_s + \Delta \xi_s + p_s(\operatorname{div} \xi_s)) = \int_0^{e^{-2s}T} \int u_s \cdot (u_s \cdot \nabla) \xi_s,$$

where $\xi_s \in C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$. Since for any $s > 0$ the operator $\mathbf{d}_3(s)$ defined as

$$(\mathbf{d}_3(s)\xi)(t, x) = \xi(e^{2st}, e^s x), \quad t > 0, x \in \mathbb{R}^3$$

maps $C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$ onto $C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$ (see Lemma 1), then

$$\int u_s(0) \cdot \xi_s(0) + \int_0^{e^{-2s}T} \int u_s(t, x) \cdot (\partial_t \xi_s + \Delta \xi_s + p_s(\operatorname{div} \xi_s)) = \int_0^{e^{-2s}T} \int u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi_s.$$

holds for all $\xi_s \in C_0^\infty([0, e^{-\mu s}T] \times \mathbb{R}^3, \mathbb{R}^3)$. Therefore, (u_s, p_s) is a strong solution of the Navier-Stokes equations on $[0, e^{-2s}T)$ and $u_s(0) = \mathbf{d}(s)u_0$.

The proven lemma implies the following result.

Theorem 1 *Let there exist $\varepsilon > 0$ such that for any*

$$u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$$

a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$.

Then a strong solution of the Navier-Stokes equations with the initial condition $u(0) = u_0$ exists on $[0, +\infty)$ for any $u_0 \in V \cap L^\infty$.

Proof Let \mathbf{d}_1 be defined as in the proof of Lemma 3 and let an initial value $u_0 \in V \cap L^\infty$ be such that $\|u_0\|_2^2 \|u_0\|_\infty \geq \varepsilon$. Let us show that in this case the Navier-Stokes equations also have a strong solution on $[0, +\infty)$.

In the proof of Lemma 3 we have shown that $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$ and $\mathbf{d}_1(s)u_0 \in L^\infty$ for any $s \in \mathbb{R}$.

By Lemma 2 we derive $\|\mathbf{d}_1(s)u_0\|_2 = e^{s/2} \|u_0\|_2$ and $\|\mathbf{d}_1(s)u_0\|_\infty = e^s \|u_0\|_\infty$.

$$\|\mathbf{d}_1(s)u_0\|_2^2 \|\mathbf{d}_1(s)u_0\|_\infty = e^{2s} \|u_0\|_2^2 \|u_0\|_\infty.$$

Let us select

$$-\infty < s_0 < \frac{1}{2} \ln \frac{\varepsilon}{\|u_0\|_2^2 \|u_0\|_\infty}.$$

In this case, we, obviously, have

$$\|\mathbf{d}_1(s_0)u_0\|_2^2 \|\mathbf{d}_1(s_0)u_0\|_\infty < \varepsilon \text{ and } \mathbf{d}_1(s_0)u_0 \in V \cap L^\infty.$$

Hence, by assumption of the theorem, a strong solution (u, p) with $u(0) = \mathbf{d}_1(s_0)u_0$ exists on $[0, +\infty)$.

By Lemma 3 the pair (\tilde{u}, \tilde{p}) given by

$$\tilde{u}(t, x) = e^{-s_0} u(e^{-2s_0} t, e^{-s_0} x), \tilde{p}(t, x) = e^{-2s_0} p(e^{-2s_0} t, e^{-s_0} x), t \in [0, +\infty), x \in \mathbb{R}^3$$

is also a strong solution of the Navier-Stokes equation. Since $u(0) = \mathbf{d}_1(s_0)u_0$ means that

$$u(0, x) = e^{s_0} u_0(e^{s_0} x), \quad x \in \mathbb{R}^3$$

then

$$\tilde{u}(0, x) = e^{-s_0} u(0, e^{-s_0} x) = u_0(x),$$

i.e. $\tilde{u}(0) = u_0$ and the proof is complete.

Let us mention the following properties of the strong solutions (see Definition 1) proven before:

- *Global-in-time existence for small initial data* [4, Corollary 3.13 and Lemma 3.10]

There exist $\varepsilon > 0$ and $C > 0$ such that for any

$$u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$$

a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$ and $\|u(t)\|_\infty \leq C \|u_0\|_\infty$.

- *Uniqueness of strong solutions* [4, Theorem 3.9]

For any $u_0 \in V \cap L^\infty$ a strong solution with $u(0) = u_0$ is unique.

- *Smoothness* [4, Corollary 3.3].

If (u, p) is a strong solution of the Navier-Stokes equation then

$$\partial_t^k \nabla^m u, \partial_t^k \nabla^m p \in C((0, T), L^2) \cap C((0, T), L^\infty), \quad \forall m, k \geq 0$$

and, in particular, $u, p \in C^\infty(\mathbb{R}^3 \times (0, T))$ constitute a classical solution of the Navier-Stokes equations on $(0, T) \times \mathbb{R}^3$.

Theorem 2 *For any initial condition on $u(0) = u_0 \in V \cap L^\infty$ the Navier-Stokes equations have a unique strong solution on $[0, +\infty) \times \mathbb{R}^3$ (that is, in fact, a classical C^∞ solution on $(0, +\infty) \times \mathbb{R}^3$) and there exists $C > 0$ such that for any solution (u, p) we have*

$$\|u(t)\|_\infty \leq C \|u_0\|_\infty, \quad t \geq 0.$$

Proof The global existence and uniqueness of the strong and classical solution is the straightforward combination of Theorem 1 with [4, Corollary 3.13, Theorem 3.9 and Corollary 3.3].

The existence of a constant $C > 0$ such that $\|u(t)\|_\infty \leq C\|u_0\|_\infty$ for all $t \geq 0$ is proven by [4, Lemma 3.10] for any $u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$.

Let $\mathbf{d}_1(s)$ be defined as in the proof of Theorem 1 and (u, p) (with $u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$) and (u_s, p_s) be two pairs of solutions given by the statement of Lemma 3. Since $u_s(t) = \mathbf{d}_1(s)u(e^{2st})$ for all $t \geq 0$ and $\|\mathbf{d}_1(s)z\|_\infty = e^s \|z\|_\infty$ for all $z \in L^\infty$ then

$$\|u_s(t)\|_\infty = e^s \|u(e^{2st})\|_\infty \leq C e^s \|u(0)\|_\infty = C \|\mathbf{d}_1(s)u(0)\|_\infty = C \|u_s(0)\|.$$

Remark 1 The same conclusion may be made accepting that a regularity and global-in-time existence of solutions is already proven for another set of initial conditions. For example, the case $u_0 \in C^1 \cap H^1$ is studied in [2], where global-in-time existence of solutions for small initial data is announced. Since $\|\mathbf{d}_1(s)u_0\|_C = e^s \|u_0\|_C$, $\|\nabla \mathbf{d}_1(s)u_0\|_C = e^{2s} \|\nabla u_0\|_C$, $\|\mathbf{d}_1(s)u_0\|_2 = e^{s/2} \|u_0\|_2$ and $\|\mathbf{d}_1(s)\nabla u_0\|_2 = e^{3/2s} \|\nabla u_0\|_2$ then repeating the proof of Theorem 1 we conclude that a regular global-in-time solution exists for any $u_0 \in C^1 \cap H^1$.

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