



HAL
open science

On global existence of strong and classical solutions of Navier-Stokes equations

Andrey Polyakov

► **To cite this version:**

Andrey Polyakov. On global existence of strong and classical solutions of Navier-Stokes equations. 2019. hal-02096215v1

HAL Id: hal-02096215

<https://inria.hal.science/hal-02096215v1>

Preprint submitted on 11 Apr 2019 (v1), last revised 23 Apr 2019 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On global existence of strong and classical solutions of Navier-Stokes equations

Andrey Polyakov

the date of receipt and acceptance should be inserted later

Abstract The paper presents a simple proof of global existence of strong and classical solutions of Navier-Stokes equations in \mathbb{R}^3 accepting that their existence for small initial data is already proven, for example, in [2] and [4].

Keywords Navier-Stokes-Equations · Strong Solutions · Dilation Symetry

1 Introduction

In this short note we exploit a very simple idea to prove global existence of the strong solutions of the Navier-Stokes equation, which is explained by means of the following trivial example in \mathbb{R}^n .

Let a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be symmetric with respect to a dilation of the argument, i.e. $\exists \alpha \in \mathbb{R}$ such that

$$f(e^s u) = e^{(\alpha+1)s} f(u), \quad \forall u \in \mathbb{R}^n, \forall s \in \mathbb{R}$$

If $u(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$ is a classical solution of

$$\frac{du}{dt} = f(u), \quad t > 0$$

with the initial condition $u(0) = u_0$ then

$$u_s(t) := e^s u(e^{\alpha s} t)$$

is defined on $[0, +\infty)$ and, due to symmetry, we derive

$$\frac{du_s}{dt} = e^{(\alpha+1)s} f(u(e^{\alpha s} t)) = f(u_s(t)), \quad t > 0,$$

A. Polyakov
Inria Lille, Univ. Lille, CNRS, UMR 9189 - CRISTAL, (F-59000 Lille, France),
Tel.: +33-359577802
E-mail: andrey.polykov@inria.fr

i.e. u_s is a classical solution of the same differential equation with the initial condition $u_s(0) = e^s x_0$, where $s \in \mathbb{R}$.

Let $\exists \varepsilon > 0$ such that a classical solution of the differential equation exists on $[0, +\infty)$ for any initial value $u(0) = u_0 \in B_\varepsilon := \{u \in \mathbb{R}^n : |u| < \varepsilon\}$. To construct a solution for $u_0 \notin B_\varepsilon$ we first need to scale $u_0 \rightarrow e^{s_0} u_0$, where $s_0 \in \mathbb{R}$ is such that

$$|e^{s_0} u_0| < \varepsilon.$$

If $u(\cdot, e^{s_0} u_0)$ is a solution with the initial condition $u(0) = \lambda_0 u_0$ then $\tilde{u}(t) = e^{-s_0} u(e^{-\alpha s_0}, e^{s_0} u_0)$ is also a solution of the considered system and, obviously,

$$\tilde{u}(0) = e^{s_0} u(0, e^{s_0} u_0) = e^{-s_0} e^{s_0} u_0 = u_0.$$

In this paper we use the well-known dilation symmetry (see, [1, formula (1.5)]) to prove global existence of strong solutions of the Navier-Stokes equations in \mathbb{R}^3

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla u)u - \nabla p, \\ \mathbf{0} &= \operatorname{div} u \end{aligned}$$

where u denotes the velocity of a fluid, p denotes the scalar pressure and $\nu > 0$ denotes viscosity of the fluid. The main assumption is that a global-in-time existence of strong or classical solutions of the Navier-Stokes equations for small initial data is already proven, as for example, in [2] and [4]

The standard *notation* is utilized through the paper, e.g. \mathbb{R} is the field of real numbers; $L^1_{loc}((0, T) \times \mathbb{R}^n, \mathbb{R})$ denotes the space of locally integrable functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}$; $L^p(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p \leq +\infty$ is a Lebesgue space of function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ the norm $\|\cdot\|_p$; $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ with compact support and $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions which vanish at infinity, where $0 < T \leq \infty$. For composition of operators A, B we also use the notation $A \circ B$.

2 Preliminaries: Dilations in functional spaces

The following lemmas deal with the most common dilation groups in functional spaces.

Lemma 1 *The operator $\mathbf{d}(s)$ given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\gamma s} t, e^{\beta s} x), \quad (1)$$

where $s \in \mathbb{R}$, z is a function $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $0 < T \leq +\infty$, $x \in \mathbb{R}^n$ and $\alpha, \beta, \gamma \in \mathbb{R}$ are constant parameters,

- maps $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ onto $C_c^\infty((0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$;
- maps $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ onto $C_0^\infty([0, e^{-\gamma s} T) \times \mathbb{R}^n, \mathbb{R}^m)$.

The inverse operator is given by $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

Proof 1) Since the linear function $(t, x) \rightarrow (e^{\gamma s t}, e^{\beta s x})$ maps a compact in $(0, T) \times \mathbb{R}^n$ to a compact in $(0, e^{-\gamma s T}) \times \mathbb{R}^n$ then $\mathbf{d}(s)$ defined on whole $C_c^\infty((0, T), \mathbb{R}^n)$ and if $z \in C_c^\infty((0, T), \mathbb{R}^n)$ (i.e. z is smooth and has a compact support in $(0, T) \times \mathbb{R}^n$) then $\mathbf{d}(s)z \in C_c^\infty((0, e^{-\gamma s T}), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth, but it has a compact support in $(0, e^{-\gamma s T}) \times \mathbb{R}^n$). Obviously, $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z = z$ for any $z \in C_c^\infty((0, T), \mathbb{R}^n)$ and any $s \in \mathbb{R}$.

Let us show that $\mathbf{d}(s)$ maps $C_c^\infty((0, T), \mathbb{R}^n)$ onto $C_c^\infty((0, e^{-\gamma s T}), \mathbb{R}^n)$. Suppose the opposite: $\exists z^* \in C_c^\infty((0, e^{-\gamma s T}), \mathbb{R}^n)$ such that $z^* \neq \mathbf{d}(s)y, \forall y \in C_c^\infty((0, T), \mathbb{R}^n)$. This is impossible, since $\mathbf{d}(-s)z^* \in C_c^\infty((0, T), \mathbb{R}^n)$ and $z^* = (\mathbf{d}(s) \circ \mathbf{d}(-s))z^* \in \mathbf{d}(s)C_c^\infty((0, T), \mathbb{R}^n)$.

2) The proof for C_0^∞ is almost identical. Obviously, if $z \in C_0^\infty([0, T), \mathbb{R}^n)$ (i.e. z is smooth and vanishing at infinity) then $\mathbf{d}(s)z \in C_0^\infty([0, e^{-\gamma s T}), \mathbb{R}^n)$ (i.e. $\mathbf{d}(s)z$ is also smooth and vanishing at infinity) and $\mathbf{d}(s) \circ \mathbf{d}(-s)z = \mathbf{d}(-s) \circ \mathbf{d}(s)z = z$ for any $z \in C_0^\infty([0, T), \mathbb{R}^n)$.

Let us show that $\mathbf{d}(s)$ maps $C_0^\infty([0, T), \mathbb{R}^n)$ onto $C_0^\infty([0, e^{-\gamma s T}), \mathbb{R}^n)$. Suppose the opposite: there exists $z^* \in C_0^\infty([0, e^{-\gamma s T}), \mathbb{R}^n)$ such that $\mathbf{d}(s)y \neq z^*, \forall y \in C_0^\infty([0, T), \mathbb{R}^n)$. This is impossible since $\mathbf{d}(-s)z^* \in C_0^\infty([0, T), \mathbb{R}^n)$ and $z^* = \mathbf{d}(s) \circ \mathbf{d}(-s)z^* \in \mathbf{d}(s)C_0^\infty([0, T), \mathbb{R}^n)$.

Lemma 2 *The operator $\mathbf{d}(s)$ given by*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x), \quad (2)$$

where $s \in \mathbb{R}$, z is a function $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ are constant parameters, is a linear bounded invertible operator on $L^p(\mathbb{R}^n, \mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha - \beta/p)s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R},$$

where $1 \leq p \leq \infty$. The inverse operator is given by $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

Proof Let $1 < p < \infty$. If $z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n} |z(x)|^p dx < +\infty$$

and using the change-of-variable theorem for the Lebesgue integral (see e.g. [3]) we derive

$$\int_{\mathbb{R}^n} |z(x)|^p dx = e^{\beta s} \int_{\tilde{\mathbb{R}}^n} |z(e^{\beta s} x)|^p dx = e^{(\beta - \alpha p)s} \int_{\mathbb{R}^n} |(\mathbf{d}(s)z)(x)|^p dx < +\infty.$$

Since $e^{(\beta - \alpha p)s} > 0$ for any $\alpha, \beta, p, s \in \mathbb{R}$ then $\mathbf{d}(s)z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$ for any $s \in \mathbb{R}$. Obviously, $\mathbf{d}(s)$ is a linear operator on , i.e. $\mathbf{d}(s)(\mu_1 z_1 + z_2) = \mu_1 \mathbf{d}(s)z_1 +$

$\mu_2 \mathbf{d}(s)z_2$, for any $\mu_1, \mu_2 \in \mathbb{R}$ and $z_1, z_2 \in L^p(\mathbb{R}^n, \mathbb{R}^m)$. Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-\beta/p)s} \|z\|_p, \quad \|z\|_p := \left(\int_{\mathbb{R}^n} |z(x)|^p dx \right)^{1/p}.$$

Hence, the operator $\mathbf{d}(s) : L^p(\mathbb{R}^n, \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^m)$ is bounded for any $s \in \mathbb{R}$.

Let $p = \infty$. If $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\text{ess sup}|z(x)| = \text{ess sup}|z(e^{\beta s}x)| < +\infty$$

for any $\beta, s \in \mathbb{R}$ and $\|\mathbf{d}(s)z\|_\infty = e^{\alpha s} \|z\|_\infty$ for any $s \in \mathbb{R}$. Therefore, $\mathbf{d}(s)$ is also a linear bounded operator on $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

Obviously, $(\mathbf{d}(s) \circ \mathbf{d}(-s))z = (\mathbf{d}(-s) \circ \mathbf{d}(s))z$ for any $z : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and any $s \in \mathbb{R}$ and we derive $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$.

3 Global Existence of Strong Solutions of Navier-Stokes Equations

Below for shortness we omit \mathbb{R}^3 in the notations of $\int_{\mathbb{R}^3}$, L^p , C_0^∞ and C_c^∞ if the context is clear. Without loss of generality (see e.g. [4, page 4]) we also assume that $\nu = 1$, where ν is a viscosity coefficient.

Let us consider the weak form of the Navier-Stokes equations

$$\int u(0) \cdot \xi(0) + \int_0^T \int u \cdot (\partial_t \xi + \Delta \xi) + p \operatorname{div} \xi = \int_0^T \int u \cdot (u \cdot \nabla) \xi, \quad \forall \xi \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3) \quad (3)$$

with $u(t) \in V$ for $t \in (0, T)$, where V is a set of the so-called weakly divergence free velocity fields:

$$V := \left\{ u \in L^2 : \int u \cdot \nabla \phi = 0, \forall \phi \in C_0^\infty \right\}.$$

The classical idea of analysis is to prove existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this analysis has been initiated by Jean Leray in 1938 (see [2]). We use the recent review [4] of his results.

Definition 1 ([4], Definition 3.7) A pair (u, p) is said to be a strong solution of the Navier-Stokes equations on $[0, T)$ if $u(t) \in V$ for $t \in (0, T)$, $p \in L_{loc}^1((0, T) \times \mathbb{R}^3, \mathbb{R})$,

$$u \in C([0, T), L^2) \cap C((0, T), L^\infty),$$

$\|u(t)\|_\infty$ is bounded as $t \rightarrow 0^+$ and (3) is satisfied.

A possible way for expansion of regularity to a larger set of initial conditions is to use the dilation symmetry as explained in the introduction. Several symmetries for Navier-Stokes equations are known (see e.g. [1] and references therein). Below we use just one of them (see [1, formula (1.5)]).

Lemma 3 *If (u, p) is a strong solution of the Navier-Stokes equations with the initial data $u(0) = u_0 \in V \cap L^\infty$ defined on $[0, T)$ then for any $s \in \mathbb{R}$ the pair (u_s, p_s) given by*

$$u_s(t, x) = e^s u(e^{2s}t, e^s x), \quad p_s(t, x) = e^{2s} p(e^{2s}t, e^s x), \quad t \geq 0, x \in \mathbb{R}^n, s \in \mathbb{R}.$$

is a strong solution of the Navier-Stokes equations defined on $[0, e^{-2s}T)$ with the initial value $u_s(0) = \mathbf{d}(s)u_0$.

Proof 1) Let $\mathbf{d}_1(s)$ be defined by the formula (2) with $\alpha = 1, \beta = 1$.

Since u is a strong solution then the function $t \rightarrow u(t)$ is continuous in L^2 and in L^∞ . According Lemma 2, $\mathbf{d}_1(s)$ is a linear bounded invertible operator on L^2 and on L^∞ . Hence, for any fixed $s \in \mathbb{R}$ the function $t \rightarrow \mathbf{d}_1(s)u(t)$ is also continuous in L^2 and L^∞ , consequently, $u_s \in C([0, e^{-2s}T), L^2)$ and $u_s \in C((0, e^{-2s}T), L^\infty)$.

Notice $\|\mathbf{d}_1(s)z\|_\infty = e^s \|z\|_\infty$ for any $z \in L^\infty$ and any $s \in \mathbb{R}$ (see Lemma 2). Since $\|u(t)\|_\infty$ is bounded as $t \rightarrow 0^+$ then $\|u_s(t)\|_\infty = e^s \|u(e^s t)\|_\infty$ is also bounded as $t \rightarrow 0^+$.

If $u_0 \in V$ then $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Indeed,

$$\int u_0 \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty$$

and using the change-of-variable theorem in the Lebesgue integral we derive

$$0 = \int u_0(x) \cdot \nabla \phi(x) dx = e^s \int u_0(e^s x) \cdot (\nabla \phi)(e^s x) dx = e^{-2s} \int (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi},$$

where $\tilde{\phi} = \mathbf{d}_1(s)\phi$. Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ (see Lemma 1) then

$$0 = \int (\mathbf{d}_1(s)u_0) \cdot \nabla \tilde{\phi}, \quad \forall \tilde{\phi} \in C_0^\infty,$$

i.e. $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$. Hence, the inclusion $u(t) \in V, t \in [0, T)$ implies $u_s(t) \in V, t \in [0, e^{-2s}T)$.

2) Let $\mathbf{d}_2(s)$ be given by the formula (1) with $\alpha = 2, \beta = 1$ and $\gamma = 2$. Since $p \in L_{loc}^1((0, T) \times \mathbb{R}^n, \mathbb{R})$ then

$$\int_0^T \int |p(t, x)\xi(t, x)| dx dt < +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$$

hence using the change-of-variable theorem in the Lebesgue integral for the functions $t \rightarrow e^{2s}t$ and $x \rightarrow e^s x$ we derive

$$\begin{aligned} \int_0^T \int |p(t, x)\xi(t, x)| dx dt &= e^{3s} \int_0^{e^{-2s}T} \int |p(e^{2s}t, e^s x)\xi(e^{2s}t, e^s x)| dx dt = \\ e^{-s} \int_0^{e^{-2s}T} \int |p_s \cdot \mathbf{d}_2(s)\xi| &< +\infty, \quad \forall \xi \in C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}). \end{aligned}$$

Since the operator $\mathbf{d}_2(s)$ maps $C_c^\infty((0, T) \times \mathbb{R}^3, \mathbb{R})$ onto $C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$ then

$$\int_0^{e^{-2s}T} \int |p_s \cdot \tilde{\xi}| < +\infty, \quad \forall \tilde{\xi} \in C_c^\infty((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R}).$$

Therefore, $p_s \in L^1_{loc}((0, e^{-2s}T) \times \mathbb{R}^3, \mathbb{R})$.

3) Let us show that (u_s, p_s) satisfies the equation (3). Since (u, p) is a solution defined on $[0, +\infty)$ then $\forall \xi \in C_0^\infty([0, T) \times \mathbb{R}^3, \mathbb{R}^3)$ we have

$$\begin{aligned} \int u(0, x) \cdot \xi(0, x) dx + \int_0^T \int u(t, x) \cdot (\partial_t \xi(t, x) + (\Delta \xi)(t, x) + p(t, x)(\operatorname{div} \xi)(t, x)) dx dt = \\ \int_0^T \int u(t, x) \cdot (u(t, x) \cdot \nabla) \xi(t, x) dx dt \end{aligned}$$

Using the change-of-variable theorem in the Lebesgue integral for the functions $t \rightarrow e^{2s}t$ and $x \rightarrow e^s x$ we derive

$$\begin{aligned} e^s \int u(0, e^s x) \cdot \xi(0, e^s x) dx + e^{3s} \int_0^{e^{-2s}T} \int u(e^{2s}t, e^s x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + \\ (\Delta \xi)(e^{2s}t, e^s x)) + p(e^{2s}t, e^{2s}x)(\operatorname{div} \xi)(e^{2s}t, e^s x) dx dt = \\ e^{3s} \int_0^{e^{-2s}T} \int u(e^{2s}t, e^s x) \cdot (u(e^{2s}t, e^s x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} \int u_s(0, x) \cdot \xi(0, e^s x) dx + \int_0^{e^{-2s}T} \int e^{2s} u_s(t, x) \cdot (\partial_t \xi(e^{2s}t, e^s x) + (\Delta \xi)(e^{2s}t, e^s x)) \\ + e^s p_s(t, x)(\operatorname{div} \xi)(e^{2s}t, e^s x) dx dt = \\ e^s \int_0^{e^{-2s}T} \int u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi(e^{2s}t, e^s x) dx dt \end{aligned}$$

Let us denote $\xi_s(t, x) = \xi(e^{2st}, e^s x)$. Hence,

$$(\Delta \xi_s)(t, x) = e^{2s} (\Delta \xi)(e^{2st}, e^s x),$$

$$(\nabla \xi_s)(t, x) = e^s (\nabla \xi)(e^{2st}, e^s x),$$

$$(\partial_t \xi_s)(t, x) = e^{2s} (\partial_t \xi)(e^{2st}, e^s x),$$

$$(\operatorname{div} \xi_s)(t, x) = e^s (\operatorname{div} \xi)(e^{2st}, e^s x),$$

and

$$\int u_s(0) \cdot \xi_s(0) + \int_0^{e^{-2s}T} \int u_s \cdot (\partial_t \xi_s + \Delta \xi_s + p_s(\operatorname{div} \xi_s)) = \int_0^{e^{-2s}T} \int u_s \cdot (u_s \cdot \nabla) \xi_s,$$

where $\xi_s \in C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$. Since for any $s > 0$ the operator $\mathbf{d}_3(s)$ defined as

$$(\mathbf{d}_3(s)\xi)(t, x) = \xi(e^{2st}, e^s x), \quad t > 0, x \in \mathbb{R}^3$$

maps $C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$ onto $C_0^\infty([0, e^{-2s}T], \times \mathbb{R}^3, \mathbb{R}^3)$ (see Lemma 1), then

$$\int_0^{e^{-2s}T} \int u_s(t, x) \cdot (\partial_t \xi_s + \Delta \xi_s + p_s(\operatorname{div} \xi_s)) = \int_0^{e^{-s}T} \int u_s(t, x) \cdot (u_s(t, x) \cdot \nabla) \xi_s.$$

holds for all $\xi_s \in C_0^\infty([0, e^{-\mu s}T] \times \mathbb{R}^3, \mathbb{R}^3)$. Therefore, (u_s, p_s) is a strong solution of the Navier-Stokes equations on $[0, e^{-2s}T)$ and $u_s(0) = \mathbf{d}(s)u_0$.

The proven lemma implies the following result.

Theorem 1 *Let there exist $\varepsilon > 0$ such that for any*

$$u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$$

a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$.

Then a strong solution of the Navier-Stokes equations with the initial condition $u(0) = u_0$ exists on $[0, +\infty)$ for any $u_0 \in V \cap L^\infty$.

Proof Let \mathbf{d}_1 be defined as in the proof of Lemma 3 and let an initial value $u_0 \in V \cap L^\infty$ be such that $\|u_0\|_2^2 \|u_0\|_\infty \geq \varepsilon$. Let us show that in this case the Navier-Stokes equations also have a strong solution on $[0, +\infty)$.

In the proof of Lemma 3 we have shown that $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$ and $\mathbf{d}_1(s)u_0 \in L^\infty$ for any $s \in \mathbb{R}$.

By Lemma 2 we derive $\|\mathbf{d}_1(s)u_0\|_2 = e^{s/2} \|u_0\|_2$ and $\|\mathbf{d}_1(s)u_0\|_\infty = e^s \|u_0\|_\infty$.

$$\|\mathbf{d}_1(s)u_0\|_2^2 \|\mathbf{d}_1(s)u_0\|_\infty = e^{2s} \|u_0\|_2^2 \|u_0\|_\infty.$$

Let us select

$$-\infty < s_0 < \frac{1}{2} \ln \frac{\varepsilon}{\|u_0\|_2^2 \|u_0\|_\infty}.$$

In this case, we, obviously, have

$$\|\mathbf{d}_1(s_0)u_0\|_2^2 \|\mathbf{d}_1(s_0)u_0\|_\infty < \varepsilon \text{ and } \mathbf{d}_1(s_0)u_0 \in V \cap L^\infty.$$

Hence, by assumption of the theorem, a strong solution (u, p) with $u(0) = \mathbf{d}_1(s_0)u_0$ exists on $[0, +\infty)$.

By Lemma 3 the pair (\tilde{u}, \tilde{p}) given by

$$\tilde{u}(t, x) = e^{-s_0 t} u(e^{-2s_0 t}, e^{-s_0 x}), \tilde{p}(t, x) = e^{-2s_0 t} p(e^{-2s_0 t}, e^{-s_0 x}), t \in [0, +\infty), x \in \mathbb{R}^3$$

is also a strong solution of the Navier-Stokes equation. Since $u(0) = \mathbf{d}_1(s_0)u_0$ means that

$$u(0, x) = e^{s_0} u_0(e^{s_0} x), \quad x \in \mathbb{R}^3$$

then

$$\tilde{u}(0, x) = e^{-s_0} u(0, e^{-s_0} x) = u_0(x),$$

i.e. $\tilde{u}(0) = u_0$.

Finally, let us mention the following properties of the strong solutions introduced by Definition 1 proven before:

- *Global-in-time existence for small initial data* [4, Corollary 3.13 and Lemma 3.10]

There exist $\varepsilon > 0$ and $C > 0$ such that for any

$$u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$$

a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$ and $\|u(t)\|_\infty \leq C \|u_0\|_\infty$.

- *Uniqueness of strong solutions* [4, Theorem 3.9]
For any $u_0 \in V \cap L^\infty$ a strong solution with $u(0) = u_0$ is unique.
- *Smoothness* [4, Corollary 3.3].

If (u, p) is a strong solution of the Navier-Stokes equation then

$$\partial_t^k \nabla^m u, \partial_t^k \nabla^m p \in C((0, T), L^2) \cap C((0, T), L^\infty), \quad \forall m, k \geq 0$$

and, in particular, $u, p \in C^\infty(\mathbb{R}^3 \times (0, T))$ constitute a classical solution of the Navier-Stokes equations on $(0, T) \times \mathbb{R}^3$.

Theorem 2 *For any initial condition on $u(0) = u_0 \in V \cap L^\infty$ the Navier-Stokes equations have a unique strong solution on $[0, +\infty) \times \mathbb{R}^3$ (that is, in fact, a classical C^∞ solution on $(0, +\infty) \times \mathbb{R}^3$) and there exists $C > 0$ such that for any solution (u, p) we have*

$$\|u(t)\|_\infty \leq C \|u_0\|_\infty, \quad t \geq 0.$$

Proof The global existence and uniqueness of the strong and classical solution is the straightforward combination of Theorem 1 with [4, Corollary 3.13, Theorem 3.9 and Corollary 3.3].

The existence of a constant $C > 0$ such that $\|u(t)\|_\infty \leq C\|u_0\|_\infty$ for all $t \geq 0$ is proven by [4, Lemma 3.10] for any $u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$.

Let $\mathbf{d}_1(s)$ be defined as in the proof of Theorem 1 and (u, p) (with $u_0 \in \{u \in V \cap L^\infty : \|u\|_2^2 \|u\|_\infty < \varepsilon\}$) and (u_s, p_s) be two pairs of solutions given by the statement of Lemma 3. Since $u_s(t) = \mathbf{d}_1(s)u(e^{2st})$ for all $t \geq 0$ and $\|\mathbf{d}_1(s)z\|_\infty = e^s \|z\|_\infty$ for all $z \in L^\infty$ then

$$\|u_s(t)\|_\infty = e^s \|u(e^{2st})\|_\infty \leq C e^s \|u(0)\|_\infty = C \|\mathbf{d}_1(s)u(0)\|_\infty = C \|u_s(0)\|.$$

References

1. Fushchych, W., Popovych, R.: Symmetry reduction and exact solutions of the Navier-Stokes equations. *J. Nonlinear Math. Phys.* **1**, 75–113, 158–188 (1994)
2. Leray, J.: Sur le mouvement d'un liquide visqueux qui limite des parois. *Acta Math* **63**, 193–248 (1938)
3. Netuka, I.: The change-of-variables theorem for the Lebesgue integral. *ACTA UNIVERSITATIS MATTHIAE BELII, series MATHEMATICS* **19**, 37–42 (2011)
4. Ozanski, W., Pooley, B.: Leray's fundamental work on the Navier-Stokes equations: a modern review of "Sur le mouvement d'un liquide visqueux emplissant l'espace", pp. 11–203. *London Mathematical Society Lecture Note Series: Partial Differential Equations in Fluid Mechanics, Series Number 452*. Cambridge University Press (2018). URL (see also arXiv:1708.09787[math.AP])