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# On Expansion of Regularity of Nonlinear Evolution Equations by Means of Dilation Symmetry

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**Abstract** The paper present a dilation symmetry based approach to expansion of regularity of nonlinear evolution equations. In particular, it is shown that a symmetry of an operator in a Banach, which describes a non-linear evolution equation, is inherited by solutions of this equation. In the case of dilation symmetry, the latter implies that global-in-time existence of solutions for small initial data always imply global-in-time existence of solutions for large initial data. As an example, an impossibility of a finite-time blow up of solutions of the Navier-Stokes equation in  $\mathbb{R}^3$  is proven accepting that the existence of global-in-time solutions for small initial data is already proven, for example, in [5] and [7].

**Keywords** Dilation Symmetry · Nonlinear Evolution Equations

## 1 Introduction

According to the classical concept of homogeneity introduced by Leonhard Euler in 18th century, homogeneity is a sort of symmetry of an object (e.g. a function or a set) with respect to a group of transformations known today as dilations. For example, a function  $f$  in  $\mathbb{R}^n$  is homogeneous in the classical (standard) sense if it is symmetric with respect to a uniform dilation of an argument, i.e. there exists  $\nu \in \mathbb{R}$  such that

$$f(e^s u) = e^{(\nu+1)s} f(u), \quad u \in \mathbb{R}^n, s \in \mathbb{R}.$$

Homogeneity of a function is inherited by other objects induced by this function. For example, the Euler's Homogeneous Function Theorem implies that

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any derivative of  $f$  is homogeneous too. Similarly, if  $u(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$  is a classical solution of

$$\frac{du}{dt} = f(u), \quad t > 0$$

with the initial condition  $u(0) = u_0$  then  $u_s(t) := e^s u(e^{\alpha s} t)$  is defined on  $[0, +\infty)$ , then, due to symmetry, we derive  $\frac{du_s}{dt} = e^{(\alpha+1)s} f(u(e^{\alpha s} t)) = f(u_s(t))$ ,  $t > 0$ , i.e.  $u_s$  is a classical solution of the same differential equation with the initial condition  $u_s(0) = e^s u_0$ , where  $s \in \mathbb{R}$ .

Let  $\exists \varepsilon > 0$  such that a classical solution of the differential equation exists on  $[0, +\infty)$  for any initial value  $u(0) = u_0 \in B_\varepsilon := \{u \in \mathbb{R}^n : \|u\| < \varepsilon\}$ . Hence, exploiting the symmetry we derive existence of a classical solution for any initial condition  $u(0) = u_0 \in \bigcup_{s \in \mathbb{R}} \mathbf{d}(s)B_\varepsilon = \mathbb{R}^n$ .

In this paper we show that the same trick can be utilized for non-linear evolution equations in a Banach space  $\mathbb{B}$  provided that a dilation group is properly introduced in  $\mathbb{B}$ . As an example, we prove global existence and uniqueness of classical solutions of Navier-Stokes equations in  $\mathbb{R}^3$  accepting that the existence of global-in-time solutions for small initial data is already proven, for example, in [5] and [7].

The standard *notation* is utilized through the paper, e.g.  $\mathbb{R}$  is the field of real numbers;  $\mathbb{B}$  denotes a Banach space;  $\mathbb{X}$  is a linear vector space;  $L^1_{loc}((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$  denotes the space of locally integrable functions  $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ ;  $L^p(\mathbb{R}^n, \mathbb{R}^m)$ ,  $1 \leq p \leq +\infty$  is a Lebesgue space of function  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  with the norm  $\|\cdot\|_p$ ;  $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$  is a space of smooth functions  $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  with compact support and  $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$  is a space of smooth functions which vanish at infinity, where  $0 < T \leq \infty$ . The notation  $\stackrel{a.e.}{\in}$  and  $\stackrel{a.e.}{=}$  is utilized in order to indicate that an inclusion or identity is fulfilled almost everywhere on a domain.

## 2 Dilation symmetry of nonlinear operators

### 2.1 Dilation group

Let  $\mathbb{X}$  be a (linear) vector space and  $\mathbf{d} := \{\mathbf{d}(s)\}_{s \in \mathbb{R}}$  be a family of operators  $\mathbf{d}(s) : \mathbb{X} \rightarrow \mathbb{X}$ . If

- $\mathbf{d}(0) = I$ , where  $I$  is an identity operator (i.e.  $Iz = z$  for all  $z \in \mathbb{X}$ );
- $\mathbf{d}(t+s)z = (\mathbf{d}(t) \circ \mathbf{d}(s))z = (\mathbf{d}(s) \circ \mathbf{d}(t))z$  for all  $t, s \in \mathbb{R}, z \in \mathbb{X}$

then, by definition,  $\mathbf{d}$  is a *group*. Using the group properties for  $t = -s$  we derive  $(\mathbf{d}(-s) \circ \mathbf{d}(s))z = (\mathbf{d}(s) \circ \mathbf{d}(-s))z = z$ , the operator  $\mathbf{d}(s)$  is invertible and  $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$ . Moreover,  $\mathbf{d}(s)$  maps  $\mathbb{X}$  onto  $\mathbb{X}$  for any  $s \in \mathbb{R}$ . Indeed, suppose the contrary, i.e.  $\exists z^* \in \mathbb{X}$  and such  $s^* \in \mathbb{R}$  such that  $z^* \notin \mathbf{d}(s^*)\mathbb{X}$ . Since  $u^* := \mathbf{d}(-s^*)z^* \in \mathbb{X}$  then  $z^* = \mathbf{d}(s)\mathbf{d}(-s^*)z^* = \mathbf{d}(s)u^* \in \mathbf{d}(s)\mathbb{X}$ .

**Definition 1** A group  $\mathbf{d}$  of operators on a normed vector space  $\mathbb{X}$  is said to be a *dilation group* (or simply *dilation*) on  $\mathbb{X}$  if  $\mathbf{d}(s)\mathbf{0} = \mathbf{0}$  for any  $s \in \mathbb{R}$  and the following *limit property* holds

$$\liminf_{s \rightarrow -\infty} \|\mathbf{d}(s)z\| = 0 \quad \text{and} \quad \limsup_{s \rightarrow +\infty} \|\mathbf{d}(s)z\| = \infty \quad \text{for} \quad z \neq \mathbf{0}.$$

The limit property given above specifies a group being a dilation in an abstract space. We refer the reader to [2] for more details about topological characterization of dilations.

*Example 1* Let us recall a few well-know dilation groups in  $\mathbb{R}^n$ :

1) *Uniform dilation* (L. Euler, 18th century):

$$\mathbf{d}(s) = e^s I, \quad s \in \mathbb{R}$$

where  $I$  is the identity matrix  $\mathbb{R}^{n \times n}$ .

2) *Weighted dilation* [13]:

$$\mathbf{d}(s) = \begin{pmatrix} e^{r_1 s} & 0 & \dots & 0 \\ 0 & e^{r_2 s} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & e^{r_n s} \end{pmatrix},$$

where  $r_i > 0$ ,  $i = 1, 2, \dots, n$ .

3) *Geometric dilation* (see e.g. [4], [11], [3]) is a flow generated by an Euler vector field <sup>1</sup>.

The uniform dilation  $\mathbf{d}(s) = e^s I \in \mathbb{B}$ ,  $s \in \mathbb{R}$  is the simplest example of a dilation in any normed vector space. Let us consider a few other examples.

*Example 2* Let  $\mathbb{B}$  be a space of bounded uniformly continuous functions  $\mathbb{R}^n \rightarrow \mathbb{R}^m$  with the supremum norm. A dilation group  $\mathbf{d}$  in  $\mathbb{B}$  can be defined as follows

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(x + \beta s),$$

where  $s \in \mathbb{R}$  is the group parameter,  $z \in \mathbb{X}$ ,  $x \in \mathbb{R}$  and  $\alpha > 0$  and  $\beta \in \mathbb{R}$  are a constant parameters. Indeed,  $\mathbf{d}(s)z \in \mathbb{X}$  if  $z \in \mathbb{B}$ ,  $s \in \mathbb{R}$  and for  $v = \mathbf{d}(s)z$  we have

$$(\mathbf{d}(t) \circ \mathbf{d}(s)z)(x) = (\mathbf{d}(t)v)(x) = e^{\alpha t} v(x + \beta t) = e^{\alpha t} e^{\alpha s} z(x + \beta s + \beta t) = (\mathbf{d}(s+t)z)(x).$$

The limit property also holds since

$$\|\mathbf{d}(s)z\| = \sup_{x \in \mathbb{R}^n} |e^{\alpha s} z(x + \beta s)| = e^{\alpha s} \sup_{x \in \mathbb{R}^n} |z(x + \beta s)| = e^{\alpha s} \|z\|.$$

<sup>1</sup> A  $C^1$  vector field  $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called Euler if it is complete and  $-\nu$  is globally asymptotically stable.

The next lemma introduces the most common dilation in functional spaces.

**Lemma 1** *The family  $\mathbf{d}$  of the operators  $\mathbf{d}(s)$  given by,*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x), \quad (1)$$

where  $s \in \tilde{\mathbb{R}}$ ,  $z$  is a function  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $x \in \mathbb{R}^n$  and  $\alpha, \beta \in \mathbb{R}$  are constant parameters, is

- 1) a group of linear invertible operators on  $L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ ,  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$  and on  $C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)$ ;
- 2) a dilation group of bounded linear invertible operators on  $L^p(\mathbb{R}^n, \mathbb{R}^m)$ ,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-\beta/p)s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R}$$

provided that  $\beta < p\alpha$ ;

where  $0 < p \leq \infty$ .

*Proof 1*) If  $z \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ , i.e.  $z$  is smooth and has a compact support, then  $\mathbf{d}(s)z \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ , i.e.  $\mathbf{d}(s)z$  is also smooth and has a compact support, and

$$\begin{aligned} (\mathbf{d}(s+t)\phi)(x) &= e^{\alpha(t+s)} z(e^{\beta(t+s)} x) = e^{\alpha s} e^{\alpha t} \phi(e^{\beta s} e^{\beta t} x) = \\ &e^{\alpha s} (\mathbf{d}(t)\phi)(e^{\beta s} x) = (\mathbf{d}(s) \circ \mathbf{d}(t)\phi)(x) \end{aligned}$$

for all  $s, t \in \mathbb{R}$ ,  $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ ,  $x \in \mathbb{R}^n$ , i.e.  $\mathbf{d}$  is a group of linear invertible operators on  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$  and  $\mathbf{d}(s)$  maps  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$  onto  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$  for any  $s \in \mathbb{R}$  (see considerations in the beginning of this section). Considerations for  $C_0^\infty$  (smooth functions vanishing at infinity) are identically the same.

If  $z \in L^1_{loc}$  then

$$\int_{\mathbb{R}^n} |z \cdot \phi| < +\infty, \quad \forall \phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m).$$

Hence, using the change-of-variable theorem (see e.g. [6]) for the Lebesgue integral we derive

$$\int_{\mathbb{R}^n} |z(x) \cdot \phi(x)| dx = e^{\beta s} \int_{\mathbb{R}^n} |z(e^{s\beta} x) \cdot \phi(e^{\beta s} x)| dx = e^{(\beta-2\alpha)s} \int_{\mathbb{R}^n} |\mathbf{d}(s)z \cdot \tilde{\phi}| dx,$$

where  $\tilde{\phi} = \mathbf{d}(s)\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ . Since  $e^{(\beta-2\alpha)s} > 0$  and  $\mathbf{d}(s)$  maps  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$  onto  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$  then

$$\int_{\mathbb{R}^n} |\mathbf{d}(s)z \cdot \tilde{\phi}| dx < +\infty, \quad \forall \tilde{\phi} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m).$$

Similarly, we derive that the group property  $\mathbf{d}(s+t)z = (\mathbf{d}(s) \circ \mathbf{d}(t))z$  also holds for any  $s, t \in \mathbb{R}$  and  $z \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$  in the weak sense

$$\int_{\mathbb{R}^n} \mathbf{d}(s+t)z \cdot \phi = \int_{\mathbb{R}^n} (\mathbf{d}(s) \circ \mathbf{d}(t))z \cdot \phi.$$

Hence,  $\mathbf{d}(-s) \circ \mathbf{d}(s)z = \mathbf{d}(s) \circ \mathbf{d}(-s)z = z$  for any  $z \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$  and  $\mathbf{d}(s) = [\mathbf{d}(-s)]^{-1}$ .

The operator  $\mathbf{d}(s)$  is, obviously, linear

$$\mathbf{d}(s)(\mu_1 z_1 + \mu_2 z_2) = \mu_1 \mathbf{d}(s)z_1 + \mu_2 \mathbf{d}(s)z_2, \quad \mu_1, \mu_2 \in \mathbb{R}, z_1, z_2 \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m).$$

2) Similarly, if  $z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$  then

$$\int_{\mathbb{R}^n} |z(x)|^p dx < +\infty$$

and the change-of-variable theorem (see e.g. [6]) for the Lebesgue integral we derive

$$\int_{\mathbb{R}^n} |z(x)|^p dx = e^{\beta s} \int_{\mathbb{R}^n} |z(e^{\beta s} x)|^p dx = e^{(\beta - \alpha)p s} \int_{\mathbb{R}^n} |(\mathbf{d}(s)z)(x)|^p dx < +\infty$$

Since  $e^{(\beta - \alpha)p s} > 0$  for any  $\alpha, \beta, p, s \in \mathbb{R}$  then  $\mathbf{d}(s)z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$  for all  $s \in \mathbb{R}$ , i.e.  $\mathbf{d}$  is a group of linear invertible operators on  $L^p(\mathbb{R}^n, \mathbb{R}^m)$ . Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha - \beta/p)s} \|z\|_p, \quad \|z\|_p := \left( \int_{\mathbb{R}^n} |z(x)|^p dx \right)^{1/p}.$$

Hence, the operator  $\mathbf{d}(s) : L^p(\mathbb{R}^n, \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^m)$  is bounded for any  $s \in \mathbb{R}$  and  $\mathbf{d}$  is a dilation group on  $L^p(\mathbb{R}^n, \mathbb{R}^m)$  provided that  $\alpha - \beta/p > 0$ .

Finally, if  $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$  then

$$\text{ess sup}|z(x)| = \text{ess sup}|z(e^{\beta s} x)| < +\infty$$

for any  $\beta, s \in \mathbb{R}$  and  $\|\mathbf{d}(s)z\|_\infty = e^{\alpha s} \|z\|_\infty$  for any  $s \in \mathbb{R}$ .

## 2.2 $\mathbf{d}$ -homogeneous operators

In this section we introduce a notion of  $\mathbf{d}$ -homogeneous (symmetric with respect to a group  $\mathbf{d}$ ) operators in a vector space  $\mathbb{X}$  and present a couple of examples.

**Definition 2** An operator  $F : \mathcal{D}(F) \subset \mathbb{X} \rightarrow \mathbb{X}$  is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if  $\mathbf{d}(s)\mathcal{D}(F) \subset \mathcal{D}(F)$  for any  $s \in \mathbb{R}$  and

$$(F \circ \mathbf{d}(s))u = e^{\mu s} (\mathbf{d}(s) \circ F)u \quad \text{for } s \in \mathbb{R}, \quad u \in \mathcal{D}(F), \quad (2)$$

where  $\mathbf{d}$  is a group of invertible operators on  $\mathbb{X}$ .

A lot of examples of  $\mathbf{d}$ -homogeneous vector-field for  $\mathbb{X} = \mathbb{R}^n$  can be found in control literature (see e.g. [13], [11], [3] and references therein). For example, the vector function

$$(x_1, x_2) \rightarrow \left( x_1 + x_2^2, |x_2| \sin \left( \frac{x_1 - x_2^2}{|x_1| + x_2^2} \right) \right), \quad (x_1, x_2) \in \mathbb{R}^2$$

is  $\mathbf{d}$ -homogeneous of degree 0 with

$$\mathbf{d}(s)(x_1, x_2) \rightarrow (e^{2s}x_1, e^s x_2), \quad s \in \mathbb{R}.$$

All linear and lot of nonlinear models of mathematical physics are  $\mathbf{d}$ -homogeneous under a proper selection of a dilation group (see e.g. [10]).

Notice that, the identity (2) can always be understood in the weak sense. For shortness we omit  $\mathbb{R}^n$  in the notations for  $\int_{\mathbb{R}^n}$ ,  $L^2(\mathbb{R}^n, \mathbb{R}^n)$ ,  $L^1_{loc}(\mathbb{R}^n, \mathbb{R}^n)$  and  $C^\infty_0(\mathbb{R}^n, \mathbb{R}^n)$  in the examples below.

*Example 3 (d-homogeneity of the Laplace operator)*

Let us consider the Laplace operator

$$\Delta : \mathcal{D}(\Delta) \subset L^2 \rightarrow L^2,$$

with the domain

$$\mathcal{D}(\Delta) = \left\{ u \in L^2 : \exists f \in L^1_{loc} \text{ such that } \int u \cdot \Delta \phi = \int f \cdot \phi, \quad \forall \phi \in C^\infty_c \right\}.$$

Let us show that  $\Delta$  is  $\mathbf{d}$ -homogeneous of degree  $2\beta$  provided that the dilation  $\mathbf{d}$  is given by (1).

By Lemma 1,  $\mathbf{d}$  is a group of linear invertible operators on  $C^\infty_c$  and, consequently (see the beginning of Section 2),  $\mathbf{d}(s)$  maps  $C^\infty_c$  onto  $C^\infty_c$ . Notice that if  $\phi \in C^\infty_c$  then, obviously,

$$(\Delta \circ \mathbf{d}(s))\phi(x) = e^{(\alpha+2\beta)s} (\Delta \phi)(e^{\beta s} x) = e^{2\beta s} ((\mathbf{d}(s) \circ \Delta)\phi)(x), \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

In other words, the Laplace operator is  $\mathbf{d}$ -homogeneous as operator  $C^\infty_c \rightarrow C^\infty_c$ . Since  $C^\infty_c$  is dense in  $L^2$  then it is  $\mathbf{d}$ -homogeneous as an operator  $L^2 \rightarrow L^2$ . Let us prove this claim more rigorously.

Let  $u \in \mathcal{D}(\Delta)$  and  $\Delta u = f \in L^1_{loc}$  (in the weak sense). Since  $\mathbf{d}(s)f \in L^1_{loc}$  then using the change-of-variable theorem (see e.g. [6]) in the Lebesgue integral we derive

$$\begin{aligned} e^{2\beta s} \int (\mathbf{d}(s)f) \cdot \phi &= e^{(\alpha+2\beta)s} \int f(e^{\beta s} x) \cdot \phi(x) dx = e^{(\alpha+\beta)s} \int f(x) \cdot \phi(e^{-\beta s} x) dx = \\ &= e^{(2\alpha+\beta)s} \int f \cdot \tilde{\phi} = e^{(2\alpha+\beta)s} \int u \cdot \Delta \tilde{\phi} = e^{(2\alpha+\beta)s} \int u \cdot (\Delta \circ \mathbf{d}(-s))\phi = \\ &= e^{(\alpha-\beta)s} \int u(x) \cdot \Delta \phi(e^{-\beta s} x) dx = e^{\alpha s} \int u(e^{\beta s} x) \cdot \Delta \phi(x) dx = \int (\mathbf{d}(s)u) \cdot \Delta \phi, \end{aligned}$$

where  $\tilde{\phi} = \mathbf{d}(-s)\phi \in C^\infty_c$ . Hence,  $\mathbf{d}(s)u \in \mathcal{D}(\Delta)$  and  $(\Delta \circ \mathbf{d}(s))u = e^{\mu s} \mathbf{d}(s)f = e^{\mu s} (\mathbf{d}(s) \circ \Delta)u$  (in the weak sense) for any  $s \in \mathbb{R}$ ,  $u \in \mathcal{D}(\Delta)$ .

*Example 4* Let us show that the operator

$$F : \mathcal{D}(F) \subset L^2 \rightarrow L^2, \quad F(u) := (u \cdot \nabla)u, \quad u \in \mathcal{D}(F)$$

with the domain

$$\mathcal{D}(F) = \left\{ u \in L^2 : \exists f \in L^1_{loc} \text{ such that } \int u \cdot (u \cdot \nabla \phi) + \int f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty \right\}$$

is  $\mathbf{d}$ -homogeneous of degree  $\alpha + \beta$  provided that the dilation  $\mathbf{d}$  is given by (1).

If  $\phi \in C_0^\infty$  then, obviously,

$$((\nabla \circ \mathbf{d}(s))\phi)(x) = e^{(\alpha+\beta)s}(\nabla\phi)(e^{\beta s}x), \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

Let  $u \in \mathcal{D}(F)$  and  $(u \cdot \nabla)u = f \in L^1_{loc}$  (in the weak sense), i.e.

$$\int u \cdot (u \cdot \nabla \phi) + \int f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty.$$

Since  $\mathbf{d}(s)f \in L^1_{loc}$  then using the change-of-variable theorem (see e.g. [6]) in the Lebesgue integral we derive

$$\begin{aligned} 0 &= \int u(x) \cdot (u(x) \cdot \nabla \phi(x)) + \int f(x) \cdot \phi(x) dx = \\ &e^{\beta s} \int u(e^{\beta s}x) \cdot (u(e^{\beta s}x) \cdot \nabla \phi(e^{\beta s}x)) + e^{\beta s} \int f(e^{\beta s}x) \cdot \phi(e^{\beta s}x) dx = \\ &e^{-3\alpha s} \int (\mathbf{d}(s)u) \cdot ((\mathbf{d}(s)u) \cdot \nabla \tilde{\phi} + e^{(\beta-2\alpha)s} \int (\mathbf{d}(s)f) \cdot \tilde{\phi}, \end{aligned}$$

where  $\tilde{\phi} = \mathbf{d}(s)\phi$ . Since  $\mathbf{d}(s)$  maps  $C_0^\infty$  to  $C_0^\infty$  then

$$\int (\mathbf{d}(s) \circ u) \cdot ((\mathbf{d}(s) \circ u) \cdot \nabla \tilde{\phi} + e^{(\alpha+\beta)s} \int (\mathbf{d}(s)f) \cdot \tilde{\phi} = 0, \quad \forall C_0^\infty$$

Hence,  $\mathbf{d}(s)u \in \mathcal{D}(F)$  and taking into account  $(u \cdot \nabla)u = f$  we conclude

$$(F \circ \mathbf{d}(s))u = e^{(\alpha+\beta)s} \mathbf{d}(s)f = e^{(\alpha+\beta)s} (\mathbf{d}(s) \circ F)u$$

(in the weak sense) for any  $u \in \mathcal{D}(F)$ ,  $s \in \mathbb{R}$ .

### 3 Dilation Symmetry of Evolution Equations

#### 3.1 Linear evolution equation

The dilation symmetry of an operator is inherited by other objects induced (generated) by this operator.



**Lemma 2** *Let a linear closed densely defined operator  $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$  generate a strongly continuous semigroup  $\Phi := \{\Phi(t)\}_{t \geq 0}$  of linear bounded operators on  $\mathbb{B}$  and  $\mathbf{d}$  be a group of linear bounded invertible operators on  $\mathbb{B}$ . If the operator  $A$  is  $\mathbf{d}$ -homogeneous of degree  $\mu$  then*

$$\Phi(t) \circ \mathbf{d}(s) = \mathbf{d}(s) \circ \Phi(e^{\mu s} t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}. \quad (3)$$

*Proof* Since  $\Phi$  is generated by  $A$  then  $\Phi(e^{\mu s} t)u \in \mathcal{D}(A)$  for any  $u \in \mathcal{D}(A)$  (see e.g. [8, page 5]).

Let  $s \in \mathbb{R}$  and  $u \in \mathcal{D}(A)$  be selected arbitrary. Since the operator  $A$  is  $\mathbf{d}$ -homogeneous then  $\mathcal{D}(A)$  is invariant with respect to the transformation  $\mathbf{d}(s)$ , i.e.  $\mathbf{d}(s)z \in \mathcal{D}(A), \forall z \in \mathcal{D}(A)$ , and, consequently,

$$y_1(t) := (\Phi(t) \circ \mathbf{d}(s))u \in \mathcal{D}(A), \quad \text{and} \quad y_2(t) := (\mathbf{d}(s) \circ \Phi(e^{\mu s} t))u \in \mathcal{D}(A), \quad t \geq 0.$$

Being generated by  $A$  the semigroup  $\Phi$  satisfy (see e.g. [8, page 5])

$$\frac{d}{dt} \Phi(t)z = (A \circ \Phi(t))z = (\Phi(t) \circ A)z, \quad \forall t > 0, \quad \forall z \in \mathcal{D}(A).$$

Taking into account that  $A$  is  $\mathbf{d}$ -homogeneous of degree  $\mu$  and  $\mathbf{d}(s)$  is a linear bounded operator on  $\mathbb{B}$ , we derive

$$\frac{d}{dt} y_2(t) = e^{\mu s} (\mathbf{d}(s) \circ A \circ \Phi(e^{\mu s} t))u = (A \circ \mathbf{d}(s) \circ \Phi(e^{\mu s} t))u = Ay_2(t), \quad \forall t > 0.$$

On the other hand, we have

$$\frac{d}{dt} y_1(t) = (A \circ \Phi(t) \circ \mathbf{d}(s))u = Ay_1(t), \quad \forall t > 0.$$

Since  $y_1(0) = y_2(0) = \mathbf{d}(s)u$  then due to uniqueness of the semigroup  $\Phi$  generated by  $A$  (see [8, page 6]) we derive  $y_1(t) = y_2(t)$  for all  $t \geq 0$  and

$$(\Phi(t) \circ \mathbf{d}(s))u = (\mathbf{d}(s) \circ \Phi(e^{\mu s} t))u, \quad \forall t \geq 0, \quad \forall u \in \mathcal{D}(A).$$

Since  $\Phi(t) \circ \mathbf{d}(s)$  and  $\mathbf{d}(s) \circ \Phi(e^{\mu s} t)$  are bounded linear operators and  $\mathcal{D}(A)$  is dense in  $\mathbb{B}$  then the latter identity holds for all  $u \in \mathbb{B}$  and all  $t \geq 0$ .

It is well-known (see e.g. [8, page 100]) that

$$u(t, u_0) = \Phi(t)u_0, \quad t \geq 0$$

is a unique solution of the linear evolution equation

$$\frac{du}{dt} = Au$$

with the initial condition  $u(0) = u_0 \in \mathbb{B}$ . The latter lemma, obviously, proves the symmetry of these solutions:

$$u(t, \mathbf{d}(s)u_0) = \mathbf{d}(s)u(e^{\mu s} t, u_0), \quad s \in \mathbb{R}$$

Below we prove this result for non-linear operators too.

### 3.2 Nonlinear evolution equation

Let us consider the non-linear evolution system

$$\frac{du}{dt} = Au + Gu, \quad t > 0, \quad (4)$$

where a densely defined closed linear operator

$$A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$$

generates a strongly continuous semigroup  $\Phi$  of linear bounded operators on  $\mathbb{B}$ , and

$$G : \mathcal{D}(G) \subset \mathbb{B} \rightarrow \mathbb{B}$$

is a possibly nonlinear operator.

**Definition 3** A continuous function  $u : [0, T) \rightarrow \mathbb{B}$  is said to be

- a *mild* solution of the system (4) if  $Gu \in L^1((0, T), \mathbb{B})$  and

$$u(t) = \Phi(t)u(0) + \int_0^t (\Phi(t - \tau) \circ G)u(\tau) d\tau, \quad t \in (0, T); \quad (5)$$

- a *strong* solution of the evolution equation (4) if  $u \in C([0, T), \mathbb{B})$ ,  $u$  is differentiable almost everywhere on  $(0, T)$ ,  $\frac{du}{dt}, Gu \in L^1((0, T), \mathbb{B})$  and (4) is satisfied almost everywhere on  $(0, T)$ ;
- a *classical* solution of the evolution equation (4) if  $u \in C([0, T), \mathbb{B})$ ,  $\frac{du}{dt} \in C((0, T), \mathbb{B})$ ,  $u(t) \in \mathcal{D}(A) \cap \mathcal{D}(G)$  for all  $t \in (0, T)$  and (4) is satisfied on  $(0, T)$ .

The latter integral is understood in the sense of Bochner ([12, page 132]).

**Theorem 1** Let  $\mathbf{d}$  be a group of linear bounded invertible operators on  $\mathbb{B}$  and let  $A$  and  $G$  be  $\mathbf{d}$ -homogeneous operators of a degree  $\mu \in \mathbb{R}$ .

If  $u : [0, T) \rightarrow \mathbb{B}$  is a mild (strong or classical) solution of the evolution equation (4) and

$$u(t) \stackrel{\text{a.e.}}{\in} \mathcal{D}(G), \quad t \in (0, T),$$

then the function  $u_s : [0, e^{-\mu s}T) \rightarrow \mathbb{B}$  defined as

$$u_s(t) = \mathbf{d}(s)u(e^{\mu s}t), \quad t \in [0, e^{-\mu s}T)$$

is also a mild (strong or classical) solution of the evolution equation (4) and

$$u_s(t) \stackrel{\text{a.e.}}{\in} \mathcal{D}(G), \quad t \in (0, e^{-\mu s}T),$$

for any  $s \in \mathbb{R}$ .

*Proof* If  $u(t) \in \mathcal{D}(G)$  then due to  $\mathbf{d}$ -homogeneity of the operator  $G$  we have  $\mathbf{d}(s)\mathcal{D}(G) \subset \mathcal{D}(G)$  and  $u_s(e^{-\mu s}t) \in \mathcal{D}(G)$ .

1) *The case of mild solutions.*

Since the  $\mathbf{d}$ -homogeneous operator  $A$  generates a strongly continuous semi-group  $\Phi$ , then according Lemma 2 we have

$$\Phi(t) \circ \mathbf{d}(s) = \mathbf{d}(s) \circ \Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}.$$

It is well known [12, page 134] that  $K \int_0^t \xi(s)ds = \int_0^t K\xi(s)ds$  for any bounded linear operator  $K: \mathbb{B} \rightarrow \mathbb{B}$  and any Bochner integrable function  $\xi \in L^1((0, T), \mathbb{B})$ . Hence, using  $G(u) \in L^1((0, T), \mathbb{B})$  we derive

$$\begin{aligned} \mathbf{d}(s) \int_0^t G(u(\tau))d\tau &= \int_0^t \mathbf{d}(s)G(u(\tau))d\tau = e^{-\mu s} \int_0^t G(\mathbf{d}(s)u(\tau))d\tau = \\ &= \int_0^{e^{-\mu s}t} G(\mathbf{d}(s)u(e^{\mu s}\tau))d\tau = \int_0^{e^{-\mu s}t} G(u_s(\tau))d\tau, \end{aligned}$$

i.e.  $G(u_s) \in L^1((0, e^{-\mu s}T), \mathbb{B})$ . Similarly, we derive

$$\begin{aligned} \mathbf{d}(s)u(e^{\mu s}t) &= (\mathbf{d}(s) \circ \Phi(e^{\mu s}t))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)u(\tau) d\tau = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)u(\tau) d\tau = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + e^{\mu s} \int_0^t (\mathbf{d}(s) \circ \Phi(e^{\mu s}(t - \sigma)) \circ G)u(e^{\mu s}\sigma) d\sigma = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + e^{\mu s} \int_0^t (\Phi(t - \sigma) \circ \mathbf{d}(s) \circ G)u(e^{\mu s}\sigma) d\sigma = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + \int_0^t (\Phi(t - \sigma) \circ G)(\mathbf{d}(s)u(e^{\mu s}\sigma)) d\sigma, \end{aligned}$$

where the linearity of the operator  $\Phi(t - \sigma)$  and the  $\mathbf{d}$ -homogeneity of the operator  $G$  are utilized on the last step. Therefore, we have shown that

$$u_s(t) = \Phi(t)u_s(0) + \int_0^t (\Phi(t - \sigma) \circ G)u_s(\sigma) d\sigma,$$

i.e.  $u_s$  is a mild solution of (4).

2) *The case of strong and classical solutions.* Let  $u$  be a strong solution. Since  $u(t)$  is differentiable almost everywhere and  $\mathbf{d}(s)$  is a linear bounded operator then for  $\tau = e^{\mu s}t$  we have

$$\frac{du_s(t)}{dt} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) \frac{du(\tau)}{d\tau} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) (Au(\tau) + G(u(\tau))) \stackrel{a.e.}{=} Au_s(t) + G(u_s(t)).$$

In the case of the classical solution the latter identity holds everywhere. Moreover, if  $u(t) \in \mathcal{D}(A)$  then due to  $\mathbf{d}$ -homogeneity of the operator  $A$  we always have  $u_s(t) \in \mathcal{D}(A)$ . Finally, taking into account that the operator  $\mathbf{d}(s)$  is bounded we conclude  $u_s \in C([0, T], \mathbb{B})$  provided that  $u \in C([0, T], \mathbb{B})$  and  $\frac{d}{dt}u_s \in C([0, T], \mathbb{B})$  provided that  $\frac{d}{dt}u \in C([0, T], \mathbb{B})$ , i.e.  $u_s$  is also a classical solution.

*Remark 1* The proven symmetry of solutions of the evolution equation (4) allows us to guarantee that any local result holds globally. For example, let us consider the initial value problem  $u(0) = u_0$  for (4). If  $\mathcal{D}(G) = \mathbb{B}$  then an existence of a solution for any  $u_0 \in B(r)$  implies existence of solutions for all  $u_0 \in \mathbb{B}$ , where  $B(r)$  is a ball in  $\mathbb{B}$  of the radius  $r > 0$ . Indeed, using the limit property of the dilation  $\mathbf{d}$  we derive that for any  $u_0 \in \mathbb{B}$  there exists  $s \in \mathbb{R}$  such that  $\|\mathbf{d}(s)u_0\| < r$ , i.e.  $\mathbf{d}(s)u_0 \in B(r)$ . By Theorem 1, if  $u(t, \mathbf{d}(-s)u_0)$  is a solution of the initial value problem  $u(0) = \mathbf{d}(-s)u_0 \in B(r)$  then  $u_s(t, u_0) := \mathbf{d}(s)u(e^{\mu s}t, \mathbf{d}(-s)u_0)$  is also a solution of the evolution equation (4). Since  $\mathbf{d}(s)u(0, \mathbf{d}(-s)u_0) = \mathbf{d}(s)\mathbf{d}(-s)u_0 = u_0$  then  $u_s$  is a solution of the initial value problem  $u(0) = u_0 \in \mathbb{B}$  for (4).

Notice that if all solutions with  $u_0 \in B(r)$  exist on  $[0, +\infty)$  then all solutions with  $u_0 \in \mathbb{B}$  also exist on  $[0, +\infty)$ . Below we use this idea to expand regularity of solutions of the Navier-Stokes equations, which was originally proven for small initial conditions.

Some other applications of a dilation symmetry in evolution equations, for example, to problems of mathematical control theory can be found in [10], [9].

### 3.3 Nonlinear implicit evolution equation

Let  $\tilde{\mathbb{X}} = \mathbb{B} \times \mathbb{X}$ , where  $\mathbb{B}$  is Banach space and  $\mathbb{X}$  is a linear (vector) space. Let us consider the nonlinear implicit evolution equation

$$\begin{aligned} \frac{du}{dt} &= Au + G(u, p), & t > 0, \\ \mathbf{0} &= Q(u, p), \end{aligned} \quad (6)$$

where a densely defined closed linear operator

$$A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$$

generates a strongly continuous semigroup  $\Phi$  of linear bounded operators on  $\mathbb{B}$ , and

$$G : \mathcal{D}(G) \subset \tilde{\mathbb{X}} \rightarrow \mathbb{B} \quad \text{and} \quad Q : \mathcal{D}(Q) \subset \tilde{\mathbb{X}} \rightarrow \mathbb{X}$$

are a (possibly) nonlinear operators.

**Definition 4** A pair  $(u, p)$  with  $u : [0, T] \rightarrow \mathbb{B}$  and  $p : [0, T]$  is said to be

- a *mild* solution of the implicit evolution equation (6) if  $u \in C([0, T], \mathbb{B})$ ,  $G(u, p) \in L^1((0, T), \mathbb{B})$  and

$$u(t) = \Phi(t)u(0) + \int_0^t (\Phi(t - \tau) \circ G)(u(\tau), p(\tau)) d\tau, \quad t \in (0, T); \quad (7)$$

$$\mathbf{0} \stackrel{a.e.}{=} Q(u(t), p(t)),$$

- a *strong* solution of the implicit evolution equation (6) if  $u \in C([0, T], \mathbb{B})$ ,  $u$  is differentiable almost everywhere on  $(0, T)$ ,  $\frac{du}{dt}, G(u, p) \in L^1((0, T), \mathbb{B})$  and (6) is satisfied almost everywhere on  $(0, T)$ ;
- a *classical* solution of the evolution equation (6) if  $u \in C([0, T], \mathbb{B})$ ,  $\frac{du}{dt} \in C((0, T), \mathbb{B})$ ,  $u(t) \in \mathcal{D}(A)$  and  $(u(t), p(t)) \in \mathcal{D}(G) \cap \mathcal{D}(Q)$  for  $t \in (0, T)$  and (6) is satisfied on  $(0, T)$ .

A symmetry of solutions of the implicit evolution equation (6) is also preserved provided that the operator, which defines its right-side is  $\mathbf{d}$ -homogeneous.

**Theorem 2** Let a group  $\mathbf{d}$  of invertible operators on  $\tilde{\mathbb{X}}$  be defined as follows

$$\mathbf{d}(s)(u, p) = (\mathbf{d}_1(s)u, \mathbf{d}_2(s)p), \quad s \in \mathbb{R}, u \in \mathbb{B}, p \in \mathbb{X},$$

where  $\mathbf{d}_1$  is a group of linear bounded invertible operators on  $\mathbb{B}$ ,  $\mathbf{d}_2$  is a group of invertible operators on  $\mathbb{X}$  such that

$$\mathbf{d}_2(s)\mathbf{0} = \mathbf{0}, \quad s \in \mathbb{R}.$$

Let us denote  $F := (G, Q)$  and  $\mathcal{D}(F) := \mathcal{D}(G) \cap \mathcal{D}(Q)$ ,

$$F : \mathcal{D}(F) \subset \tilde{\mathbb{X}} \rightarrow \tilde{\mathbb{X}}.$$

Let  $A$  be a  $\mathbf{d}_1$ -homogeneous operator of a degree  $\mu \in \mathbb{R}$  and  $F$  be  $\mathbf{d}$ -homogeneous operator of the same degree  $\mu$ .

If the pair  $(u, p)$  with  $u : [0, T] \rightarrow \mathbb{B}$  and  $p : [0, T] \rightarrow \mathbb{X}$  is a mild (strong or classical) solution of the implicit evolution equation (6) such that

$$(u(t), p(t)) \stackrel{a.e.}{\in} \mathcal{D}(F), \quad t \in (0, T),$$

then the pair  $(u_s, p_s)$  with  $u_s : [0, e^{-\mu s}T] \rightarrow \mathbb{B}$  and  $p : [0, T] \rightarrow \mathbb{X}$  defined as follows

$$(u_s(t), p_s(t)) := \mathbf{d}(s)(u(e^{\mu s}t), p(e^{\mu s}t)) \quad \text{with} \quad t \in [0, e^{-\mu s}T]$$

is also a mild (resp. strong or classical) solution of the evolution equation (4) and

$$(u_s(t), p_s(t)) \stackrel{a.e.}{\in} \mathcal{D}(F), \quad t \in (0, e^{-\mu s} T)$$

for any  $s \in \mathbb{R}$ .

*Proof* Since the operator  $F$  is  $\mathbf{d}$  homogeneous then

$$\mathbf{d}(s)\mathcal{D}(F) \subset \mathcal{D}(F)$$

and

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{\mu s}(\mathbf{d}_1(s) \circ G)(u, p), \quad \forall (u, p) \in \mathcal{D}(F), \forall s \in \mathbb{R},$$

$$Q(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{\mu s}(\mathbf{d}_2(s) \circ Q)(u, p), \quad \forall (u, p) \in \mathcal{D}(F), \forall s \in \mathbb{R}.$$

Hence, if  $(u(t), p(t)) \in \mathcal{D}(F)$  we conclude  $(u_s(e^{-\mu s} t), p_s(e^{-\mu s} t)) \in \mathcal{D}(F)$ .

1) *The case of mild solutions.*

Since the  $\mathbf{d}_1$ -homogeneous operator  $A$  generates a strongly continuous semigroup  $\Phi$ , then according Lemma 2 we have

$$\Phi(t)\mathbf{d}_1(s) = \mathbf{d}_1(s)\Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}.$$

It is well known [12, page 134] that  $K \int_0^t \xi(s)ds = \int_0^t K\xi(s)ds$  for any bounded linear operator  $K : \mathbb{B} \rightarrow \mathbb{B}$  and any Bochner integrable function  $\xi$ . Hence, using  $G(u, p) \in L^1((0, T), \mathbb{B})$  we derive

$$\begin{aligned} \mathbf{d}_1(s) \int_0^t G(u(\tau), p(\tau))d\tau &= \int_0^t \mathbf{d}_1(s)G(u(\tau), p(\tau))d\tau = \\ e^{-\mu s} \int_0^t G(\mathbf{d}_1(s)u(\tau), \mathbf{d}_2(s)p(\tau))d\tau &= \int_0^{e^{-\mu s}t} G(\mathbf{d}_1(s)u(e^{\mu s}\tau), \mathbf{d}_2(s)p(e^{\mu s}\tau))d\tau \\ &= \int_0^{e^{-\mu s}t} G(u_s(\tau), p_s(\tau))d\tau, \end{aligned}$$

i.e.  $G(u_s, p_s) \in L^1((0, e^{-\mu s}T), \mathbb{B})$ . Similarly, we derive

$$\begin{aligned} \mathbf{d}_1(s)u(e^{\mu s}t) &= (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}t))u(0) + \mathbf{d}_1(s) \int_0^{e^{\mu s}t} (\Phi(e^{\mu s}t - \tau) \circ G)(u(\tau), p(\tau)) d\tau \\ &= (\Phi(t) \circ \mathbf{d}_1(s))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)(u(\tau), p(\tau)) d\tau = \\ (\Phi(t) \circ \mathbf{d}_1(s))u(0) + e^{\mu s} \int_0^t (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}(t - \sigma)) \circ G)(u(e^{\mu s}\sigma), p(e^{\mu s}\sigma)) d\sigma &= \\ (\Phi(t) \circ \mathbf{d}_1(s))u(0) + e^{\mu s} \int_0^t (\Phi(t - \sigma) \circ \mathbf{d}_1(s) \circ G)(u(e^{\mu s}\sigma), p(e^{\mu s}\sigma)) d\sigma &= \end{aligned}$$

$$(\Phi(t) \circ \mathbf{d}_1(s))u(0) + \int_0^t (\Phi(t-\sigma) \circ G)(\mathbf{d}_1(s)u(e^{\mu s}\sigma), \mathbf{d}_2(s)p(e^{\mu s}\sigma)) d\sigma,$$

where the linearity of  $\Phi(t-\sigma)$  and a homogeneity of the operator  $G$  is utilized on the last step. Therefore, we have shown that

$$u_s(t) = \Phi(t)u_s(0) + \int_0^t \Phi(t-\sigma)g(u_s(\sigma), p_s(\sigma)) d\sigma.$$

Finally, since  $(u, p)$  is a mild solution on  $[0, T)$  then

$$Q(u(t), p(t))d\sigma \stackrel{a.e.}{=} \mathbf{0}, \quad \forall t \in [0, T)$$

and using the identity  $\mathbf{d}_2(s)\mathbf{0} = \mathbf{0}$  we derive

$$\begin{aligned} Q(u_s(\tau), u_s(\tau)) &= \mathbf{d}_2(s)Q((\mathbf{d}_1(s)u)(\tau), (\mathbf{d}_2(s)p)(\tau)) = \\ e^{\mu s}(\mathbf{d}_2(s) \circ Q)(u(e^{\mu s}\tau), p(e^{\mu s}\tau)) &\stackrel{a.e.}{=} \mathbf{0}, \quad \tau \in [0, e^{-\mu s}T). \end{aligned}$$

2) *The case of strong and classical solutions.* Let  $(u, p)$  be a strong solution. Since  $u(t)$  is differentiable almost everywhere and  $\mathbf{d}(s)$  is a linear bounded operator then for  $\tau = e^{\mu s}t$  we have

$$\frac{du_s(t)}{dt} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) \frac{du(\tau)}{d\tau} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) (Au(\tau) + G(u(\tau), p(\tau))) \stackrel{a.e.}{=} Au_s(t) + G(u_s(t), p_s(t)).$$

In the case of the classical solution the latter identity holds everywhere. Moreover, if  $u(t) \in \mathcal{D}(A)$  then due to  $\mathbf{d}$ -homogeneity of the operator  $A$  we always have  $u_s(t) \in \mathcal{D}(A)$ , i.e.  $u_s$  is a classical solution.

#### 4 Example: Global Existence and Uniqueness of Classical Solutions of Navier-Stokes Equations

The Navier-Stokes equations in  $\mathbb{R}^3$ ,

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla u)u - \nabla p, \\ \mathbf{0} &= \operatorname{div} u \end{aligned}$$

where  $u$  denotes the velocity of a fluid,  $p$  denotes the scalar pressure and  $\nu > 0$  denotes viscosity of the fluid, is the the classical model of the flow of an incompressible viscous fluid.

*Below for shortness we omit  $\mathbb{R}^3$  in the notations of  $\int_{\mathbb{R}^3}$ ,  $L^p$ ,  $C_c^\infty$  and  $C_0^\infty$  spaces if the context is clear.*

The classical idea of analysis is to prove, initially, an existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this scheme has been realized by Jean Leray in 1938 (see, e.g. [5] and the recent review [7]). We follow the same way having in mind  $\mathbf{d}$ -homogeneity (dilation symmetry) of Navier-Stokes equations (see e.g. [1, formula (1.5)]), which will guarantee a global expansion of all local results.

#### 4.1 Homogeneity of Navier-Stokes equation

Navier-Stokes equations can be represented in the form (6). Indeed, notice that for any  $u \in L^2$  we have  $\int (\|u\|_4^2 u) \cdot \nabla \phi = \|u\|_4^2 \int u \cdot \nabla \phi$  and

$$\int u \cdot \nabla \phi = 0 \quad \Leftrightarrow \quad \int (\|u\|_2^4 u) \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty.$$

In the other words, we have

$$\operatorname{div} u = \mathbf{0} \quad \Leftrightarrow \quad \|u\|_2^4 \operatorname{div} u = \mathbf{0}, \quad \forall u \in V,$$

where  $V$  is a set of the so-called *weakly divergence free* velocity fields:

$$V := \left\{ u \in L^2 : \int u \cdot \nabla \phi = 0, \forall \phi \in C_0^\infty \right\}.$$

Due to this the Navier-Stokes equations can be equivalently represented in the form of the implicit evolution equation (6) with the operators

$$A = \Delta, \quad G(u, p) = -\nabla p - (u \cdot \nabla u)u, \quad Q(u, p) = \|u\|_2^4 \operatorname{div} u$$

having the domains  $\mathcal{D}(Q) = V \times L_{loc}^1(\mathbb{R}^n, \mathbb{R})$ ,

$$\mathcal{D}(A) = \left\{ u \in L^2 : \exists f \in L_{loc}^1 \text{ such that } \int u \cdot \Delta \phi = \int f \cdot \phi, \quad \forall \phi \in C_0^\infty \right\},$$

$$\mathcal{D}(G) = \left\{ (u, p) \in L^2 \times L_{loc}^1 : \exists f \in L_{loc}^1, \int u \cdot (u \cdot \nabla \phi) + p \operatorname{div} \phi + f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty \right\}.$$

**Lemma 3** Let  $\mathbf{d}(s) : \tilde{\mathbb{X}} \rightarrow \tilde{\mathbb{X}}$  with  $\tilde{\mathbb{X}} := L^2 \times L_{loc}^1(\mathbb{R}^n, \mathbb{R})$  be defined as

$$\mathbf{d}(s)(u, p) = (\mathbf{d}_1(s)u, \mathbf{d}_2(s)p), \quad s \in \mathbb{R}, u \in L^2, p \in L_{loc}^1(\mathbb{R}^n, \mathbb{R}),$$

where  $\mathbf{d}_1$  is a group of linear bounded invertible operators on  $L^2$ ,

$$(\mathbf{d}_1(s)u)(x) = e^s u(e^s x), \quad u \in L^2, x \in \mathbb{R}^n,$$

and  $\mathbf{d}_2$  is a group of linear invertible operators on  $L_{loc}^1(\mathbb{R}^n, \mathbb{R})$  given by

$$(\mathbf{d}_2(s)p)(x) = e^{2s} p(e^s x) \quad p \in L_{loc}^1, x \in \mathbb{R}^n.$$

Then the operator  $F : \mathcal{D}(F) \subset \tilde{\mathbb{X}} \rightarrow \tilde{\mathbb{X}}$  with  $\mathcal{D}(F) = \mathcal{D}(G) \cap \mathcal{D}(Q)$  and  $F = (G, Q)$  is  $\mathbf{d}$ -homogeneous of degree 1.



*Proof* To complete the proof we must show that

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{2s}(\mathbf{d}_1(s) \circ G)(u, p), \quad \forall (u, p) \in \mathcal{D}(G), \forall s \in \mathbb{R},$$

$$Q(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{2s}(\mathbf{d}_2(s) \circ Q)(u, p), \quad \forall (u, p) \in \mathcal{D}(Q), \forall s \in \mathbb{R}.$$

and the domain  $\mathcal{D}(F)$  is invariant with respect to  $\mathbf{d}(s)$ ,  $\forall s \in \mathbb{R}$ .

1) According to the definition of  $Q$ , the identity  $Q(u, p) = f \in L^1_{loc}$  (in the weak sense) means

$$\int \|u\|_2^4 u \cdot \nabla \phi + f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty.$$

and using a change-of-variable theorem in Lebesgue integral we derive

$$e^s \int \|u\|_2^4 u(e^s x) \cdot (\nabla \phi)(e^s x) + f(e^s x) \cdot \phi(e^s x) dx = 0, \quad \forall \phi \in C_0^\infty.$$

By Lemma 1 we have  $\mathbf{d}_1(s)u \in L^2$  for any  $s \in \mathbb{R}$  and  $\|\mathbf{d}_1(s)u\|_2 = e^{s/2}\|u\|_2$ . Taking into account  $\nabla(\mathbf{d}_1\phi)(x) = e^{2s}(\nabla\phi)(e^s x)$  we obtain

$$\int e^{-4s} \|\mathbf{d}_1(s)u\|_2^4 (\mathbf{d}_1(s)u) \cdot (\nabla \mathbf{d}_1(s)\phi) + e^{-2s}(\mathbf{d}_2(s)f) \cdot (\mathbf{d}_1(s)\phi) = 0, \quad \forall \phi \in C_0^\infty.$$

Since  $\mathbf{d}_1(s)$  maps  $C_0^\infty$  onto  $C_0^\infty$  then taking  $f = \mathbf{0}$  we conclude  $\mathbf{d}_1(s)V \subset V$  for any  $s \in \mathbb{R}$ , and due to  $\mathbf{d}_2(s)f \in L^1_{loc}$  we derive  $(Q \circ \mathbf{d}(s))(u, p) = e^{2s}(\mathbf{d}_2 \circ Q)(u, p)$  (in the weak sense) for all  $s \in \mathbb{R}$  and, at least, on  $\mathcal{D}(Q)$ .

2) The identity  $G(u, p) = f \in L^1$  (in the weak sense) means

$$\int u \cdot (u \cdot \nabla \phi) + p \operatorname{div} \phi + \int f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty.$$

If  $p \in L^1_{loc}$  and  $u \in L^2$  then from Lemma 1 we derive  $\mathbf{d}_1(s)u \in L^2$  and  $\mathbf{d}_2(s)p \in L^1_{loc}$ .

Using the change-of-variable theorem in the Lebesgue integral we derive

$$e^s \int u(e^s x) \cdot (u(e^s x) \cdot (\nabla \phi)(e^s x)) + p(e^s x) (\operatorname{div} \phi)(e^s x) + f(e^s x) \cdot \phi(e^s x) dx = 0.$$

or, equivalently,

$$e^{-3s} \int (\mathbf{d}_1(s)u) \cdot ((\mathbf{d}_1(s)u) \cdot (\nabla \mathbf{d}_1(s)\phi)) + (\mathbf{d}_2(s)p) \operatorname{div}(\mathbf{d}_1(s)\phi) + e^{2s}(\mathbf{d}_1(s)f) \cdot (\mathbf{d}_1(s)\phi) = 0,$$

for all  $\phi \in C_0^\infty$ . Since  $\mathbf{d}_1(s)$  maps  $C_0^\infty$  onto  $C_0^\infty$  then we conclude that  $\mathbf{d}(s)\mathcal{D}(G) \subset \mathcal{D}(G)$  and the identity

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^s \mathbf{d}_1(s)f = e^{2s} \mathbf{d}_1(s)G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p)$$

holds for any  $s \in \mathbb{R}$  and, at least, on  $\mathcal{D}(G)$ .

Therefore,  $F$  is, indeed,  $\mathbf{d}$ -homogeneous of degree 2.

The  $\mathbf{d}$ -homogeneity of the Laplace operator  $\Delta$  is studied in Example 3. Therefore, the Navier-Stokes equation satisfy all conditions required for application of Theorem 2. In other words, if it has mild, strong or classical solutions defined on  $[0, +\infty)$  for small initial values that it has, respectively, mild, strong or classical solutions defined on  $[0, +\infty)$  for large initial values. The following lemma proves this more rigorously.

**Corollary 1** *Let there exist  $\varepsilon > 0$  such that*

$$\forall u_0 \in \{u \in V \cap L^\infty : \|u\|_2 \|u\|_\infty < \varepsilon\}$$

*a mild, strong or classical (in the sense of definition 4) solution  $(u, p)$  of the Navier-Stokes equations with the initial condition  $u(0) = u_0$  exists on  $[0, +\infty)$  and  $u(t) \in V \cap L^\infty$  for all  $t \geq 0$ . Then*

$$\forall u_0 \in V \cap L^\infty$$

*a mild, strong or a classical solution of the Navier-Stokes equations with the initial condition  $u(0) = u_0$  exists on  $[0, +\infty)$  and  $u(t) \in V \cap L^\infty$  for all  $t \geq 0$ .*

*Proof* Let  $\mathbf{d}_1$  be defined as in the proof of Lemma 3 and let an initial value  $u_0 \in V \cap L^\infty$  be such that  $\|u_0\|_2 \|u_0\|_\infty \geq \varepsilon$ . Let us show that in this case the Navier-Stokes equations also have a strong solution on  $[0, +\infty)$ .

In the proof of Lemma 3 we have shown that  $\mathbf{d}_1(s)u_0 \in V$  for any  $s \in \mathbb{R}$  and  $\mathbf{d}_1(s)u_0 \in L^\infty$  for any  $s \in \mathbb{R}$ .

By Lemma 1 we derive  $\|\mathbf{d}_1(s)u_0\|_2 = e^{s/2}\|u_0\|_2$  and  $\|\mathbf{d}_1(s)u_0\|_\infty = e^s\|u_0\|_\infty$ .

$$\|\mathbf{d}_1(s)u_0\|_2 \|\mathbf{d}_1(s)u_0\|_\infty = e^{3s/2} \|u_0\|_2 \|u_0\|_\infty.$$

Let us select

$$-\infty < s_0 < \frac{2}{3} \ln \frac{\varepsilon}{\|u_0\|_2 \|u_0\|_\infty}.$$

In this case, we, obviously, have

$$\|\mathbf{d}_1(s_0)u_0\|_2 \|\mathbf{d}_1(s_0)u_0\|_\infty < \varepsilon \text{ and } \mathbf{d}_1(s_0)u_0 \in V \cap L^\infty.$$

Hence, by assumption of the corollary, a solution  $(u, p)$  with  $u(0) = \mathbf{d}_1(s_0)u_0$  exists on  $[0, +\infty)$ .

By Lemma 3 the Navier-Stokes equations is a homogeneous system and by Theorem 2, the pair  $(\tilde{u}, \tilde{p})$  given by

$$\tilde{u}(t, x) = e^{-s_0 t} u(e^{-2s_0 t}, e^{-s_0 x}), \tilde{p}(t, x) = e^{-2s_0 t} p(e^{-2s_0 t}, e^{-s_0 x}), t \in [0, +\infty), x \in \mathbb{R}^3$$

is also a strong solution of the Navier-Stokes equation. Moreover, since  $\mathbf{d}_1(s)V \subset V$  and  $\mathbf{d}_1(s)L^\infty \subset L^\infty$  then  $\tilde{u}(t) \in V \cap L^\infty$  for all  $t \geq 0$ . Finally, since  $u(0) = \mathbf{d}_1(s_0)u_0$  means that

$$u(0, x) = e^{s_0} u_0(e^{s_0 x}), \quad x \in \mathbb{R}^3$$

then

$$\tilde{u}(0, x) = e^{-s_0} u(0, e^{-s_0} x) = u_0(x),$$

i.e.  $\tilde{u}(0) = u_0$ .

#### 4.2 Global existence of classical solutions

Let us consider also the well-known weak form (see e.g. [5] and the recent review [7]) of Navier-Stokes equations

$$\int u(0) \cdot \phi(0) + \int_0^T \int u \cdot (\phi_t + \Delta \phi) + p \operatorname{div} \phi = \int_0^T \int u \cdot (u \cdot \nabla) \phi, \quad \forall \phi \in C_0^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}^3) \quad (8)$$

with  $u(t) \in V$  for  $t \in (0, T)$ .

**Definition 5** [7, Definition 3.7] A pair  $(u, p)$  is said to be a strong solution of the Navier-Stokes equations on  $[0, T)$  if  $p \in L_{loc}^1((0, T) \times \mathbb{R}^3, \mathbb{R})$ ,  $u(t) \in V$ ,  $t \in (0, T)$

$$u \in C([0, T), L^2) \cap C((0, T), L^\infty),$$

$\|u(t)\|_\infty$  is bounded as  $t \rightarrow 0^+$  and (8) is satisfied.

Let us mention the following properties of the solutions introduced by Definition 5 proven before:

- *Global-in-time existence and boundedness of strong solutions for small initial data* [7, Corollary 3.13 and Lemma 3.10]  
There exists  $\varepsilon > 0$  and there exists  $C > 0$  such that for any  $u_0 \in \{u \in V \cap L^\infty : \|u\|_2 \|u\|_\infty < \varepsilon\}$  a strong solution  $(u, p)$  with the initial data  $u(0) = u_0$  exists on  $[0, +\infty)$  and  $\|u(t)\|_\infty \leq C \|u_0\|_\infty$  for all  $t \geq 0$ .
- *Uniqueness of strong solutions* [7, Theorem 3.9]  
For any  $u_0 \in V \cap L^\infty$  a strong solution with  $u(0) = u_0$  is unique.
- *Smoothness* [7, Corollary 3.3].  
If  $(u, p)$  is a strong solution of the Navier-Stokes equation on  $(0, T)$  then

$$\partial_t^k \nabla^m u, \partial_t^k \nabla^m p \in C((0, T), L^2) \cap C((0, T), L^\infty), \quad \forall m, k \geq 0$$

and, in particular,  $u, p \in C^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}^3)$  constitute a classical solution of the Navier-Stokes equations on  $(0, T) \times \mathbb{R}^3$ .

Therefore, combining the latter results with Corollary 1 we derive

For any initial condition  $u(0) = u_0 \in V \cap L^\infty$  the Navier-Stokes equations have a unique classical solution  $(u, p)$  on  $(0, +\infty) \times \mathbb{R}^3$  and

$$\|u(t)\|_\infty \leq C\|u_0\|.$$

for some constant  $C > 0$  independent of  $u_0$ .

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