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On Expansion of Regularity of Nonlinear Evolution Equations by Means of Dilation Symmetry

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Abstract The paper presents a dilation symmetry based approach to expansion of regularity of nonlinear evolution equations. In particular, it is shown that a symmetry of an operator in a Banach space, which describes a non-linear evolution equation, is inherited by solutions of this equation. In the case of dilation symmetry, the latter implies that global-in-time existence of solutions for small initial data always implies global-in-time existence of solutions for large initial data. As an example, an impossibility of a finite-time blow up of solutions of the Navier-Stokes equation in \mathbb{R}^3 is proven accepting that the existence of global-in-time solutions for small initial data is already proven, for example, in [5] and [7].

Keywords Dilation Symmetry · Nonlinear Evolution Equations

1 Introduction

According to the classical concept of homogeneity introduced by Leonhard Euler in 18th century, homogeneity is a sort of symmetry of an object (e.g. a function or a set) with respect to a group of transformations known today as dilations. For example, a function f in \mathbb{R}^n is homogeneous in the classical (standard) sense if it is symmetric with respect to a uniform dilation of an argument, i.e. there exists $\nu \in \mathbb{R}$ such that

$$f(e^s u) = e^{(\nu+1)s} f(u), \quad u \in \mathbb{R}^n, s \in \mathbb{R}.$$

Homogeneity of a function is inherited by other objects induced by this function. For example, the Euler's Homogeneous Function Theorem implies that

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any derivative of f is homogeneous too. Similarly, if $u(\cdot) : [0, +\infty) \rightarrow \mathbb{R}^n$ is a classical solution of

$$\frac{du}{dt} = f(u), \quad t > 0$$

with the initial condition $u(0) = u_0$ then $u_s(t) := e^s u(e^{\alpha s} t)$ is defined on $[0, +\infty)$, then, due to symmetry, we derive $\frac{du_s}{dt} = e^{(\alpha+1)s} f(u(e^{\alpha s} t)) = f(u_s(t))$, $t > 0$, i.e. u_s is a classical solution of the same differential equation with the initial condition $u_s(0) = e^s u_0$, where $s \in \mathbb{R}$.

Let $\exists \varepsilon > 0$ such that a classical solution of the differential equation exists on $[0, +\infty)$ for any initial value $u(0) = u_0 \in B_\varepsilon := \{u \in \mathbb{R}^n : \|u\| < \varepsilon\}$. Hence, exploiting the symmetry we derive existence of a classical solution for any initial condition $u(0) = u_0 \in \bigcup_{s \in \mathbb{R}} \mathbf{d}(s)B_\varepsilon = \mathbb{R}^n$.

In this paper we show that the same trick can be utilized for non-linear evolution equations in a Banach space \mathbb{B} provided that a dilation group is properly introduced in \mathbb{B} . As an example, we prove global existence and uniqueness of classical solutions of Navier-Stokes equations in \mathbb{R}^3 accepting that the existence of global-in-time solutions for small initial data is already proven, for example, in [5] and [7].

The standard *notation* is utilized through the paper, e.g. \mathbb{R} is the field of real numbers; \mathbb{B} denotes a Banach space; \mathbb{X} is a linear vector space; $L^1_{loc}((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ denotes the space of locally integrable functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$; $L^p(\mathbb{R}^n, \mathbb{R}^m)$, $1 \leq p \leq +\infty$ is a Lebesgue space of function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ with the norm $\|\cdot\|_p$; $C_c^\infty((0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions $(0, T) \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ with compact support and $C_0^\infty([0, T) \times \mathbb{R}^n, \mathbb{R}^m)$ is a space of smooth functions which vanish at infinity, where $0 < T \leq \infty$. The notation $\stackrel{a.e.}{\in}$ and $\stackrel{a.e.}{=}$ is utilized in order to indicate that an inclusion or identity is fulfilled almost everywhere on a domain.

2 Dilation symmetry of nonlinear operators

2.1 Dilation group

Let \mathbb{X} be a (linear) vector space and $\mathbf{d} := \{\mathbf{d}(s)\}_{s \in \mathbb{R}}$ be a family of operators $\mathbf{d}(s) : \mathbb{X} \rightarrow \mathbb{X}$. If

- $\mathbf{d}(0) = I$, where I is an identity operator (i.e. $Iz = z$ for all $z \in \mathbb{X}$);
- $\mathbf{d}(t+s)z = (\mathbf{d}(t) \circ \mathbf{d}(s))z = (\mathbf{d}(s) \circ \mathbf{d}(t))z$ for all $t, s \in \mathbb{R}, z \in \mathbb{X}$

then, by definition, \mathbf{d} is a *group*. Using the group properties for $t = -s$ we derive $(\mathbf{d}(-s) \circ \mathbf{d}(s))z = (\mathbf{d}(s) \circ \mathbf{d}(-s))z = z$, the operator $\mathbf{d}(s)$ is invertible and $[\mathbf{d}(s)]^{-1} = \mathbf{d}(-s)$. Moreover, $\mathbf{d}(s)$ maps \mathbb{X} onto \mathbb{X} for any $s \in \mathbb{R}$. Indeed, suppose the contrary, i.e. $\exists z^* \in \mathbb{X}$ and such $s^* \in \mathbb{R}$ such that $z^* \notin \mathbf{d}(s^*)\mathbb{X}$. Since $u^* := \mathbf{d}(-s^*)z^* \in \mathbb{X}$ then $z^* = \mathbf{d}(s)\mathbf{d}(-s^*)z^* = \mathbf{d}(s)u^* \in \mathbf{d}(s)\mathbb{X}$.

Definition 1 A group \mathbf{d} of operators on a normed vector space \mathbb{X} is said to be a *dilation group* (or simply *dilation*) on \mathbb{X} if $\mathbf{d}(s)\mathbf{0} = \mathbf{0}$ for any $s \in \mathbb{R}$ and the following *limit property* holds

$$\liminf_{s \rightarrow -\infty} \|\mathbf{d}(s)z\| = 0 \quad \text{and} \quad \limsup_{s \rightarrow +\infty} \|\mathbf{d}(s)z\| = \infty \quad \text{for} \quad z \neq \mathbf{0}.$$

The limit property given above specifies a group being a dilation in an abstract space. We refer the reader to [2] for more details about topological characterization of dilations.

Example 1 Let us recall a few well-know dilation groups in \mathbb{R}^n :

1) *Uniform dilation* (L. Euler, 18th century):

$$\mathbf{d}(s) = e^s I, \quad s \in \mathbb{R}$$

where I is the identity matrix $\mathbb{R}^{n \times n}$.

2) *Weighted dilation* [13]:

$$\mathbf{d}(s) = \begin{pmatrix} e^{r_1 s} & 0 & \dots & 0 \\ 0 & e^{r_2 s} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & e^{r_n s} \end{pmatrix},$$

where $r_i > 0$, $i = 1, 2, \dots, n$.

3) *Geometric dilation* (see e.g. [4], [11], [3]) is a flow generated by an Euler vector field ¹.

The uniform dilation $\mathbf{d}(s) = e^s I \in \mathbb{B}$, $s \in \mathbb{R}$ is the simplest example of a dilation in any normed vector space. Let us consider a few other examples.

Example 2 Let \mathbb{B} be a space of bounded uniformly continuous functions $\mathbb{R}^n \rightarrow \mathbb{R}^m$ with the supremum norm. A dilation group \mathbf{d} in \mathbb{B} can be defined as follows

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(x + \beta s),$$

where $s \in \mathbb{R}$ is the group parameter, $z \in \mathbb{X}$, $x \in \mathbb{R}$ and $\alpha > 0$ and $\beta \in \mathbb{R}$ are a constant parameters. Indeed, $\mathbf{d}(s)z \in \mathbb{X}$ if $z \in \mathbb{B}$, $s \in \mathbb{R}$ and for $v = \mathbf{d}(s)z$ we have

$$(\mathbf{d}(t) \circ \mathbf{d}(s)z)(x) = (\mathbf{d}(t)v)(x) = e^{\alpha t} v(x + \beta t) = e^{\alpha t} e^{\alpha s} z(x + \beta s + \beta t) = (\mathbf{d}(s+t)z)(x).$$

The limit property also holds since

$$\|\mathbf{d}(s)z\| = \sup_{x \in \mathbb{R}^n} |e^{\alpha s} z(x + \beta s)| = e^{\alpha s} \sup_{x \in \mathbb{R}^n} |z(x + \beta s)| = e^{\alpha s} \|z\|.$$

¹ A C^1 vector field $\nu : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called Euler if it is complete and $-\nu$ is globally asymptotically stable.

The next lemma introduces the most common dilation in functional spaces.

Lemma 1 *The family \mathbf{d} of the operators $\mathbf{d}(s)$ given by,*

$$(\mathbf{d}(s)z)(x) = e^{\alpha s} z(e^{\beta s} x), \quad (1)$$

where $s \in \tilde{\mathbb{R}}$, z is a function $\mathbb{R}^n \rightarrow \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$ are constant parameters, is

- 1) a group of linear invertible operators on $L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$, $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and on $C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)$;
- 2) a dilation group of bounded linear invertible operators on $L^p(\mathbb{R}^n, \mathbb{R}^m)$,

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha-\beta/p)s} \|z\|_p, \quad z \in L^p(\mathbb{R}^n, \mathbb{R}^m), s \in \mathbb{R}$$

provided that $\beta < p\alpha$;

where $0 < p \leq \infty$.

Proof 1) If $z \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$, i.e. z is smooth and has a compact support, then $\mathbf{d}(s)z \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$, i.e. $\mathbf{d}(s)z$ is also smooth and has a compact support, and

$$(\mathbf{d}(s+t)\phi)(x) = e^{\alpha(t+s)} z(e^{\beta(t+s)} x) = e^{\alpha s} e^{\alpha t} \phi(e^{\beta s} e^{\beta t} x) =$$

$$e^{\alpha s} (\mathbf{d}(t)\phi)(e^{\beta s} x) = (\mathbf{d}(s) \circ \mathbf{d}(t)\phi)(x)$$

for all $s, t \in \mathbb{R}$, $\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$, $x \in \mathbb{R}^n$, i.e. \mathbf{d} is a group of linear invertible operators on $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathbf{d}(s)$ maps $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ onto $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ for any $s \in \mathbb{R}$ (see considerations in the beginning of this section). Considerations for C_0^∞ (smooth functions vanishing at infinity) are identically the same.

If $z \in L^1_{loc}$ then

$$\int_{\mathbb{R}^n} |z \cdot \phi| < +\infty, \quad \forall \phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m).$$

Hence, using the change-of-variable theorem (see e.g. [6]) for the Lebesgue integral we derive

$$\int_{\mathbb{R}^n} |z(x) \cdot \phi(x)| dx = e^{\beta s} \int_{\mathbb{R}^n} |z(e^{s\beta} x) \cdot \phi(e^{\beta s} x)| dx = e^{(\beta-2\alpha)s} \int_{\mathbb{R}^n} |\mathbf{d}(s)z \cdot \tilde{\phi}| dx,$$

where $\tilde{\phi} = \mathbf{d}(s)\phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$. Since $e^{(\beta-2\alpha)s} > 0$ and $\mathbf{d}(s)$ maps $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ onto $C_c^\infty(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n} |\mathbf{d}(s)z \cdot \tilde{\phi}| dx < +\infty, \quad \forall \tilde{\phi} \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m).$$

Similarly, we derive that the group property $\mathbf{d}(s+t)z = (\mathbf{d}(s) \circ \mathbf{d}(t))z$ also holds for any $s, t \in \mathbb{R}$ and $z \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ in the weak sense

$$\int_{\mathbb{R}^n} \mathbf{d}(s+t)z \cdot \phi = \int_{\mathbb{R}^n} (\mathbf{d}(s) \circ \mathbf{d}(t))z \cdot \phi.$$

Hence, $\mathbf{d}(-s) \circ \mathbf{d}(s)z = \mathbf{d}(s) \circ \mathbf{d}(-s)z = z$ for any $z \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ and $\mathbf{d}(s) = [\mathbf{d}(-s)]^{-1}$.

The operator $\mathbf{d}(s)$ is, obviously, linear

$$\mathbf{d}(s)(\mu_1 z_1 + \mu_2 z_2) = \mu_1 \mathbf{d}(s)z_1 + \mu_2 \mathbf{d}(s)z_2, \quad \mu_1, \mu_2 \in \mathbb{R}, z_1, z_2 \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m).$$

2) Similarly, if $z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\int_{\mathbb{R}^n} |z(x)|^p dx < +\infty$$

and the change-of-variable theorem (see e.g. [6]) for the Lebesgue integral we derive

$$\int_{\mathbb{R}^n} |z(x)|^p dx = e^{\beta s} \int_{\mathbb{R}^n} |z(e^{\beta s} x)|^p dx = e^{(\beta - \alpha p)s} \int_{\mathbb{R}^n} |(\mathbf{d}(s)z)(x)|^p dx < +\infty$$

Since $e^{(\beta - \alpha p)s} > 0$ for any $\alpha, \beta, p, s \in \mathbb{R}$ then $\mathbf{d}(s)z \in L^p(\mathbb{R}^n, \mathbb{R}^m)$ for all $s \in \mathbb{R}$, i.e. \mathbf{d} is a group of linear invertible operators on $L^p(\mathbb{R}^n, \mathbb{R}^m)$. Moreover, the latter identities imply that

$$\|\mathbf{d}(s)z\|_p = e^{(\alpha - \beta/p)s} \|z\|_p, \quad \|z\|_p := \left(\int_{\mathbb{R}^n} |z(x)|^p dx \right)^{1/p}.$$

Hence, the operator $\mathbf{d}(s) : L^p(\mathbb{R}^n, \mathbb{R}^m) \rightarrow L^p(\mathbb{R}^n, \mathbb{R}^m)$ is bounded for any $s \in \mathbb{R}$ and \mathbf{d} is a dilation group on $L^p(\mathbb{R}^n, \mathbb{R}^m)$ provided that $\alpha - \beta/p > 0$.

Finally, if $z \in L^\infty(\mathbb{R}^n, \mathbb{R}^m)$ then

$$\text{ess sup}|z(x)| = \text{ess sup}|z(e^{\beta s} x)| < +\infty$$

for any $\beta, s \in \mathbb{R}$ and $\|\mathbf{d}(s)z\|_\infty = e^{\alpha s} \|z\|_\infty$ for any $s \in \mathbb{R}$.

2.2 \mathbf{d} -homogeneous operators

In this section we introduce a notion of \mathbf{d} -homogeneous (symmetric with respect to a group \mathbf{d}) operators in a vector space \mathbb{X} and present a couple of examples.

Definition 2 An operator $F : \mathcal{D}(F) \subset \mathbb{X} \rightarrow \mathbb{X}$ is said to be \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if $\mathbf{d}(s)\mathcal{D}(F) \subset \mathcal{D}(F)$ for any $s \in \mathbb{R}$ and

$$(F \circ \mathbf{d}(s))u = e^{\mu s} (\mathbf{d}(s) \circ F)u \quad \text{for } s \in \mathbb{R}, \quad u \in \mathcal{D}(F), \quad (2)$$

where \mathbf{d} is a group of invertible operators on \mathbb{X} .

A lot of examples of \mathbf{d} -homogeneous vector-field for $\mathbb{X} = \mathbb{R}^n$ can be found in control literature (see e.g. [13], [11], [3] and references therein). For example, the vector function

$$(x_1, x_2) \rightarrow \left(x_1 + x_2^2, |x_2| \sin \left(\frac{x_1 - x_2^2}{|x_1| + x_2^2} \right) \right), \quad (x_1, x_2) \in \mathbb{R}^2$$

is \mathbf{d} -homogeneous of degree 0 with

$$\mathbf{d}(s)(x_1, x_2) \rightarrow (e^{2s}x_1, e^s x_2), \quad s \in \mathbb{R}.$$

All linear and lot of nonlinear models of mathematical physics are \mathbf{d} -homogeneous under a proper selection of a dilation group (see e.g. [10]).

Notice that, the identity (2) can always be understood in the weak sense. For shortness we omit \mathbb{R}^n in the notations for $\int_{\mathbb{R}^n}$, $L^2(\mathbb{R}^n, \mathbb{R}^n)$, $L^1_{loc}(\mathbb{R}^n, \mathbb{R}^n)$ and $C^\infty_0(\mathbb{R}^n, \mathbb{R}^n)$ in the examples below.

Example 3 (d-homogeneity of the Laplace operator)

Let us consider the Laplace operator

$$\Delta : \mathcal{D}(\Delta) \subset L^2 \rightarrow L^2,$$

with the domain

$$\mathcal{D}(\Delta) = \left\{ u \in L^2 : \exists f \in L^1_{loc} \text{ such that } \int u \cdot \Delta \phi = \int f \cdot \phi, \quad \forall \phi \in C^\infty_c \right\}.$$

Let us show that Δ is \mathbf{d} -homogeneous of degree 2β provided that the dilation \mathbf{d} is given by (1).

By Lemma 1, \mathbf{d} is a group of linear invertible operators on C^∞_c and, consequently (see the beginning of Section 2), $\mathbf{d}(s)$ maps C^∞_c onto C^∞_c . Notice that if $\phi \in C^\infty_c$ then, obviously,

$$(\Delta \circ \mathbf{d}(s))\phi(x) = e^{(\alpha+2\beta)s} (\Delta \phi)(e^{\beta s} x) = e^{2\beta s} ((\mathbf{d}(s) \circ \Delta)\phi)(x), \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

In other words, the Laplace operator is \mathbf{d} -homogeneous as operator $C^\infty_c \rightarrow C^\infty_c$. Since C^∞_c is dense in L^2 then it is \mathbf{d} -homogeneous as an operator $L^2 \rightarrow L^2$. Let us prove this claim more rigorously.

Let $u \in \mathcal{D}(\Delta)$ and $\Delta u = f \in L^1_{loc}$ (in the weak sense). Since $\mathbf{d}(s)f \in L^1_{loc}$ then using the change-of-variable theorem (see e.g. [6]) in the Lebesgue integral we derive

$$\begin{aligned} e^{2\beta s} \int (\mathbf{d}(s)f) \cdot \phi &= e^{(\alpha+2\beta)s} \int f(e^{\beta s} x) \cdot \phi(x) dx = e^{(\alpha+\beta)s} \int f(x) \cdot \phi(e^{-\beta s} x) dx = \\ &= e^{(2\alpha+\beta)s} \int f \cdot \tilde{\phi} = e^{(2\alpha+\beta)s} \int u \cdot \Delta \tilde{\phi} = e^{(2\alpha+\beta)s} \int u \cdot (\Delta \circ \mathbf{d}(-s))\phi = \\ &= e^{(\alpha-\beta)s} \int u(x) \cdot \Delta \phi(e^{-\beta s} x) dx = e^{\alpha s} \int u(e^{\beta s} x) \cdot \Delta \phi(x) dx = \int (\mathbf{d}(s)u) \cdot \Delta \phi, \end{aligned}$$

where $\tilde{\phi} = \mathbf{d}(-s)\phi \in C^\infty_c$. Hence, $\mathbf{d}(s)u \in \mathcal{D}(\Delta)$ and $(\Delta \circ \mathbf{d}(s))u = e^{\mu s} \mathbf{d}(s)f = e^{\mu s} (\mathbf{d}(s) \circ \Delta)u$ (in the weak sense) for any $s \in \mathbb{R}$, $u \in \mathcal{D}(\Delta)$.

Example 4 Let us show that the operator

$$F : \mathcal{D}(F) \subset L^2 \rightarrow L^2, \quad F(u) := (u \cdot \nabla)u, \quad u \in \mathcal{D}(F)$$

with the domain

$$\mathcal{D}(F) = \left\{ u \in L^2 : \exists f \in L^1_{loc} \text{ such that } \int u \cdot (u \cdot \nabla \phi) + \int f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty \right\}$$

is \mathbf{d} -homogeneous of degree $\alpha + \beta$ provided that the dilation \mathbf{d} is given by (1).

If $\phi \in C_0^\infty$ then, obviously,

$$((\nabla \circ \mathbf{d}(s))\phi)(x) = e^{(\alpha+\beta)s}(\nabla\phi)(e^{\beta s}x), \quad s \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

Let $u \in \mathcal{D}(F)$ and $(u \cdot \nabla)u = f \in L^1_{loc}$ (in the weak sense), i.e.

$$\int u \cdot (u \cdot \nabla \phi) + \int f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty.$$

Since $\mathbf{d}(s)f \in L^1_{loc}$ then using the change-of-variable theorem (see e.g. [6]) in the Lebesgue integral we derive

$$\begin{aligned} 0 &= \int u(x) \cdot (u(x) \cdot \nabla \phi(x)) + \int f(x) \cdot \phi(x) dx = \\ &e^{\beta s} \int u(e^{\beta s}x) \cdot (u(e^{\beta s}x) \cdot \nabla \phi(e^{\beta s}x)) + e^{\beta s} \int f(e^{\beta s}x) \cdot \phi(e^{\beta s}x) dx = \\ &e^{-3\alpha s} \int (\mathbf{d}(s)u) \cdot ((\mathbf{d}(s)u) \cdot \nabla \tilde{\phi} + e^{(\beta-2\alpha)s} \int (\mathbf{d}(s)f) \cdot \tilde{\phi}, \end{aligned}$$

where $\tilde{\phi} = \mathbf{d}(s)\phi$. Since $\mathbf{d}(s)$ maps C_0^∞ to C_0^∞ then

$$\int (\mathbf{d}(s) \circ u) \cdot ((\mathbf{d}(s) \circ u) \cdot \nabla \tilde{\phi} + e^{(\alpha+\beta)s} \int (\mathbf{d}(s)f) \cdot \tilde{\phi} = 0, \quad \forall C_0^\infty$$

Hence, $\mathbf{d}(s)u \in \mathcal{D}(F)$ and taking into account $(u \cdot \nabla)u = f$ we conclude

$$(F \circ \mathbf{d}(s))u = e^{(\alpha+\beta)s} \mathbf{d}(s)f = e^{(\alpha+\beta)s} (\mathbf{d}(s) \circ F)u$$

(in the weak sense) for any $u \in \mathcal{D}(F)$, $s \in \mathbb{R}$.

3 Dilation Symmetry of Evolution Equations

3.1 Linear evolution equation

The dilation symmetry of an operator is inherited by other objects induced (generated) by this operator.

Lemma 2 *Let a linear closed densely defined operator $A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$ generate a strongly continuous semigroup $\Phi := \{\Phi(t)\}_{t \geq 0}$ of linear bounded operators on \mathbb{B} and \mathbf{d} be a group of linear bounded invertible operators on \mathbb{B} . If the operator A is \mathbf{d} -homogeneous of degree μ then*

$$\Phi(t) \circ \mathbf{d}(s) = \mathbf{d}(s) \circ \Phi(e^{\mu s} t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}. \quad (3)$$

Proof Since Φ is generated by A then $\Phi(e^{\mu s} t)u \in \mathcal{D}(A)$ for any $u \in \mathcal{D}(A)$ (see e.g. [8, page 5]).

Let $s \in \mathbb{R}$ and $u \in \mathcal{D}(A)$ be selected arbitrary. Since the operator A is \mathbf{d} -homogeneous then $\mathcal{D}(A)$ is invariant with respect to the transformation $\mathbf{d}(s)$, i.e. $\mathbf{d}(s)z \in \mathcal{D}(A), \forall z \in \mathcal{D}(A)$, and, consequently,

$$y_1(t) := (\Phi(t) \circ \mathbf{d}(s))u \in \mathcal{D}(A), \quad \text{and} \quad y_2(t) := (\mathbf{d}(s) \circ \Phi(e^{\mu s} t))u \in \mathcal{D}(A), \quad t \geq 0.$$

Being generated by A the semigroup Φ satisfy (see e.g. [8, page 5])

$$\frac{d}{dt} \Phi(t)z = (A \circ \Phi(t))z = (\Phi(t) \circ A)z, \quad \forall t > 0, \quad \forall z \in \mathcal{D}(A).$$

Taking into account that A is \mathbf{d} -homogeneous of degree μ and $\mathbf{d}(s)$ is a linear bounded operator on \mathbb{B} , we derive

$$\frac{d}{dt} y_2(t) = e^{\mu s} (\mathbf{d}(s) \circ A \circ \Phi(e^{\mu s} t))u = (A \circ \mathbf{d}(s) \circ \Phi(e^{\mu s} t))u = Ay_2(t), \quad \forall t > 0.$$

On the other hand, we have

$$\frac{d}{dt} y_1(t) = (A \circ \Phi(t) \circ \mathbf{d}(s))u = Ay_1(t), \quad \forall t > 0.$$

Since $y_1(0) = y_2(0) = \mathbf{d}(s)u$ then due to uniqueness of the semigroup Φ generated by A (see [8, page 6]) we derive $y_1(t) = y_2(t)$ for all $t \geq 0$ and

$$(\Phi(t) \circ \mathbf{d}(s))u = (\mathbf{d}(s) \circ \Phi(e^{\mu s} t))u, \quad \forall t \geq 0, \quad \forall u \in \mathcal{D}(A).$$

Since $\Phi(t) \circ \mathbf{d}(s)$ and $\mathbf{d}(s) \circ \Phi(e^{\mu s} t)$ are bounded linear operators and $\mathcal{D}(A)$ is dense in \mathbb{B} then the latter identity holds for all $u \in \mathbb{B}$ and all $t \geq 0$.

It is well-known (see e.g. [8, page 100]) that

$$u(t, u_0) = \Phi(t)u_0, \quad t \geq 0$$

is a unique solution of the linear evolution equation

$$\frac{du}{dt} = Au$$

with the initial condition $u(0) = u_0 \in \mathbb{B}$. The latter lemma, obviously, proves the symmetry of these solutions:

$$u(t, \mathbf{d}(s)u_0) = \mathbf{d}(s)u(e^{\mu s} t, u_0), \quad s \in \mathbb{R}$$

Below we prove this result for non-linear operators too.

3.2 Nonlinear evolution equation

Let us consider the non-linear evolution system

$$\frac{du}{dt} = Au + Gu, \quad t > 0, \quad (4)$$

where a densely defined closed linear operator

$$A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$$

generates a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} , and

$$G : \mathcal{D}(G) \subset \mathbb{B} \rightarrow \mathbb{B}$$

is a possibly nonlinear operator.

Definition 3 A continuous function $u : [0, T) \rightarrow \mathbb{B}$ is said to be

- a *mild* solution of the system (4) if $Gu \in L^1((0, T), \mathbb{B})$ and

$$u(t) = \Phi(t)u(0) + \int_0^t (\Phi(t - \tau) \circ G)u(\tau) d\tau, \quad t \in (0, T); \quad (5)$$

- a *strong* solution of the evolution equation (4) if $u \in C([0, T), \mathbb{B})$, u is differentiable almost everywhere on $(0, T)$, $\frac{du}{dt} \in L^1((0, T), \mathbb{B})$ and (4) is satisfied almost everywhere on $(0, T)$;
- a *classical* solution of the evolution equation (4) if $u \in C([0, T), \mathbb{B})$, $\frac{du}{dt} \in C((0, T), \mathbb{B})$, $u(t) \in \mathcal{D}(A) \cap \mathcal{D}(G)$ for all $t \in (0, T)$ and (4) is satisfied on $(0, T)$.

The latter integral is understood in the sense of Bochner ([12, page 132]).

Theorem 1 Let \mathbf{d} be a group of linear bounded invertible operators on \mathbb{B} and let A and G be \mathbf{d} -homogeneous operators of a degree $\mu \in \mathbb{R}$.

If $u : [0, T) \rightarrow \mathbb{B}$ is a mild (strong or classical) solution of the evolution equation (4) and

$$u(t) \stackrel{\text{a.e.}}{\in} \mathcal{D}(G), \quad t \in (0, T),$$

then the function $u_s : [0, e^{-\mu s}T) \rightarrow \mathbb{B}$ defined as

$$u_s(t) = \mathbf{d}(s)u(e^{\mu s}t), \quad t \in [0, e^{-\mu s}T)$$

is also a mild (strong or classical) solution of the evolution equation (4) and

$$u_s(t) \stackrel{\text{a.e.}}{\in} \mathcal{D}(G), \quad t \in (0, e^{-\mu s}T),$$

for any $s \in \mathbb{R}$.

Proof If $u(t) \in \mathcal{D}(G)$ then due to \mathbf{d} -homogeneity of the operator G we have $\mathbf{d}(s)\mathcal{D}(G) \subset \mathcal{D}(G)$ and $u_s(e^{-\mu s}t) \in \mathcal{D}(G)$.

1) *The case of mild solutions.*

Since the \mathbf{d} -homogeneous operator A generates a strongly continuous semi-group Φ , then according Lemma 2 we have

$$\Phi(t) \circ \mathbf{d}(s) = \mathbf{d}(s) \circ \Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}.$$

It is well known [12, page 134] that $K \int_0^t \xi(s)ds = \int_0^t K\xi(s)ds$ for any bounded linear operator $K: \mathbb{B} \rightarrow \mathbb{B}$ and any Bochner integrable function $\xi \in L^1((0, T), \mathbb{B})$. Hence, using $G(u) \in L^1((0, T), \mathbb{B})$ we derive

$$\begin{aligned} \mathbf{d}(s) \int_0^t G(u(\tau))d\tau &= \int_0^t \mathbf{d}(s)G(u(\tau))d\tau = e^{-\mu s} \int_0^t G(\mathbf{d}(s)u(\tau))d\tau = \\ &= \int_0^{e^{-\mu s}t} G(\mathbf{d}(s)u(e^{\mu s}\tau))d\tau = \int_0^{e^{-\mu s}t} G(u_s(\tau))d\tau, \end{aligned}$$

i.e. $G(u_s) \in L^1((0, e^{-\mu s}T), \mathbb{B})$. Similarly, we derive

$$\begin{aligned} \mathbf{d}(s)u(e^{\mu s}t) &= (\mathbf{d}(s) \circ \Phi(e^{\mu s}t))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)u(\tau) d\tau = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)u(\tau) d\tau = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + e^{\mu s} \int_0^t (\mathbf{d}(s) \circ \Phi(e^{\mu s}(t - \sigma)) \circ G)u(e^{\mu s}\sigma) d\sigma = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + e^{\mu s} \int_0^t (\Phi(t - \sigma) \circ \mathbf{d}(s) \circ G)u(e^{\mu s}\sigma) d\sigma = \\ &= (\Phi(t) \circ \mathbf{d}(s))u(0) + \int_0^t (\Phi(t - \sigma) \circ G)(\mathbf{d}(s)u(e^{\mu s}\sigma)) d\sigma, \end{aligned}$$

where the linearity of the operator $\Phi(t - \sigma)$ and the \mathbf{d} -homogeneity of the operator G are utilized on the last step. Therefore, we have shown that

$$u_s(t) = \Phi(t)u_s(0) + \int_0^t (\Phi(t - \sigma) \circ G)u_s(\sigma) d\sigma,$$

i.e. u_s is a mild solution of (4).

2) *The case of strong and classical solutions.* Let u be a strong solution. Since $u(t)$ is differentiable almost everywhere and $\mathbf{d}(s)$ is a linear bounded operator then for $\tau = e^{\mu s}t$ we have

$$\frac{du_s(t)}{dt} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) \frac{du(\tau)}{d\tau} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) (Au(\tau) + G(u(\tau))) \stackrel{a.e.}{=} Au_s(t) + G(u_s(t)).$$

In the case of the classical solution the latter identity holds everywhere. Moreover, if $u(t) \in \mathcal{D}(A)$ then due to \mathbf{d} -homogeneity of the operator A we always have $u_s(t) \in \mathcal{D}(A)$. Finally, taking into account that the operator $\mathbf{d}(s)$ is bounded we conclude $u_s \in C([0, T], \mathbb{B})$ provided that $u \in C([0, T], \mathbb{B})$ and $\frac{d}{dt}u_s \in C([0, T], \mathbb{B})$ provided that $\frac{d}{dt}u \in C([0, T], \mathbb{B})$, i.e. u_s is also a classical solution.

Remark 1 The proven symmetry of solutions of the evolution equation (4) allows us to guarantee that any local result holds globally. For example, let us consider the initial value problem $u(0) = u_0$ for (4). If $\mathcal{D}(G) = \mathbb{B}$ then an existence of a solution for any $u_0 \in B(r)$ implies existence of solutions for all $u_0 \in \mathbb{B}$, where $B(r)$ is a ball in \mathbb{B} of the radius $r > 0$. Indeed, using the limit property of the dilation \mathbf{d} we derive that for any $u_0 \in \mathbb{B}$ there exists $s \in \mathbb{R}$ such that $\|\mathbf{d}(s)u_0\| < r$, i.e. $\mathbf{d}(s)u_0 \in B(r)$. By Theorem 1, if $u(t, \mathbf{d}(-s)u_0)$ is a solution of the initial value problem $u(0) = \mathbf{d}(-s)u_0 \in B(r)$ then $u_s(t, u_0) := \mathbf{d}(s)u(e^{\mu s}t, \mathbf{d}(-s)u_0)$ is also a solution of the evolution equation (4). Since $\mathbf{d}(s)u(0, \mathbf{d}(-s)u_0) = \mathbf{d}(s)\mathbf{d}(-s)u_0 = u_0$ then u_s is a solution of the initial value problem $u(0) = u_0 \in \mathbb{B}$ for (4).

Notice that if all solutions with $u_0 \in B(r)$ exist on $[0, +\infty)$ then all solutions with $u_0 \in \mathbb{B}$ also exist on $[0, +\infty)$. Below we use this idea to expand regularity of solutions of the Navier-Stokes equations, which was originally proven for small initial conditions.

Some other applications of a dilation symmetry in evolution equations, for example, to problems of mathematical control theory can be found in [10], [9].

3.3 Nonlinear implicit evolution equation

Let $\tilde{\mathbb{X}} = \mathbb{B} \times \mathbb{X}$, where \mathbb{B} is Banach space and \mathbb{X} is a linear (vector) space. Let us consider the nonlinear implicit evolution equation

$$\begin{aligned} \frac{du}{dt} &= Au + G(u, p), & t > 0, \\ \mathbf{0} &= Q(u, p), \end{aligned} \quad (6)$$

where a densely defined closed linear operator

$$A : \mathcal{D}(A) \subset \mathbb{B} \rightarrow \mathbb{B}$$

generates a strongly continuous semigroup Φ of linear bounded operators on \mathbb{B} , and

$$G : \mathcal{D}(G) \subset \tilde{\mathbb{X}} \rightarrow \mathbb{B} \quad \text{and} \quad Q : \mathcal{D}(Q) \subset \tilde{\mathbb{X}} \rightarrow \mathbb{X}$$

are a (possibly) nonlinear operators.

Definition 4 A pair (u, p) with $u : [0, T] \rightarrow \mathbb{B}$ and $p : [0, T]$ is said to be

- a *mild* solution of the implicit evolution equation (6) if $u \in C([0, T], \mathbb{B})$, $G(u, p) \in L^1((0, T), \mathbb{B})$ and

$$u(t) = \Phi(t)u(0) + \int_0^t (\Phi(t - \tau) \circ G)(u(\tau), p(\tau)) d\tau, \quad t \in (0, T); \quad (7)$$

$$\mathbf{0} \stackrel{a.e.}{=} Q(u(t), p(t)),$$

- a *strong* solution of the implicit evolution equation (6) if $u \in C([0, T], \mathbb{B})$, u is differentiable almost everywhere on $(0, T)$, $\frac{du}{dt}, G(u, p) \in L^1((0, T), \mathbb{B})$ and (6) is satisfied almost everywhere on $(0, T)$;
- a *classical* solution of the evolution equation (6) if $u \in C([0, T], \mathbb{B})$, $\frac{du}{dt} \in C((0, T), \mathbb{B})$, $u(t) \in \mathcal{D}(A)$ and $(u(t), p(t)) \in \mathcal{D}(G) \cap \mathcal{D}(Q)$ for $t \in (0, T)$ and (6) is satisfied on $(0, T)$.

A symmetry of solutions of the implicit evolution equation (6) is also preserved provided that the operator, which defines its right-side is \mathbf{d} -homogeneous.

Theorem 2 Let a group \mathbf{d} of invertible operators on $\tilde{\mathbb{X}}$ be defined as follows

$$\mathbf{d}(s)(u, p) = (\mathbf{d}_1(s)u, \mathbf{d}_2(s)p), \quad s \in \mathbb{R}, u \in \mathbb{B}, p \in \mathbb{X},$$

where \mathbf{d}_1 is a group of linear bounded invertible operators on \mathbb{B} , \mathbf{d}_2 is a group of invertible operators on \mathbb{X} such that

$$\mathbf{d}_2(s)\mathbf{0} = \mathbf{0}, \quad s \in \mathbb{R}.$$

Let us denote $F := (G, Q)$ and $\mathcal{D}(F) := \mathcal{D}(G) \cap \mathcal{D}(Q)$,

$$F : \mathcal{D}(F) \subset \tilde{\mathbb{X}} \rightarrow \tilde{\mathbb{X}}.$$

Let A be a \mathbf{d}_1 -homogeneous operator of a degree $\mu \in \mathbb{R}$ and F be \mathbf{d} -homogeneous operator of the same degree μ .

If the pair (u, p) with $u : [0, T] \rightarrow \mathbb{B}$ and $p : [0, T] \rightarrow \mathbb{X}$ is a mild (strong or classical) solution of the implicit evolution equation (6) such that

$$(u(t), p(t)) \stackrel{a.e.}{\in} \mathcal{D}(F), \quad t \in (0, T),$$

then the pair (u_s, p_s) with $u_s : [0, e^{-\mu s}T] \rightarrow \mathbb{B}$ and $p : [0, T] \rightarrow \mathbb{X}$ defined as follows

$$(u_s(t), p_s(t)) := \mathbf{d}(s)(u(e^{\mu s}t), p(e^{\mu s}t)) \quad \text{with} \quad t \in [0, e^{-\mu s}T]$$

is also a mild (resp. strong or classical) solution of the evolution equation (4) and

$$(u_s(t), p_s(t)) \stackrel{a.e.}{\in} \mathcal{D}(F), \quad t \in (0, e^{-\mu s} T)$$

for any $s \in \mathbb{R}$.

Proof Since the operator F is \mathbf{d} homogeneous then

$$\mathbf{d}(s)\mathcal{D}(F) \subset \mathcal{D}(F)$$

and

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{\mu s}(\mathbf{d}_1(s) \circ G)(u, p), \quad \forall (u, p) \in \mathcal{D}(F), \forall s \in \mathbb{R},$$

$$Q(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{\mu s}(\mathbf{d}_2(s) \circ Q)(u, p), \quad \forall (u, p) \in \mathcal{D}(F), \forall s \in \mathbb{R}.$$

Hence, if $(u(t), p(t)) \in \mathcal{D}(F)$ we conclude $(u_s(e^{-\mu s} t), p_s(e^{-\mu s} t)) \in \mathcal{D}(F)$.

1) *The case of mild solutions.*

Since the \mathbf{d}_1 -homogeneous operator A generates a strongly continuous semigroup Φ , then according Lemma 2 we have

$$\Phi(t)\mathbf{d}_1(s) = \mathbf{d}_1(s)\Phi(e^{\mu s}t), \quad \forall t \geq 0, \quad \forall s \in \mathbb{R}.$$

It is well known [12, page 134] that $K \int_0^t \xi(s)ds = \int_0^t K\xi(s)ds$ for any bounded linear operator $K : \mathbb{B} \rightarrow \mathbb{B}$ and any Bochner integrable function ξ . Hence, using $G(u, p) \in L^1((0, T), \mathbb{B})$ we derive

$$\begin{aligned} \mathbf{d}_1(s) \int_0^t G(u(\tau), p(\tau))d\tau &= \int_0^t \mathbf{d}_1(s)G(u(\tau), p(\tau))d\tau = \\ e^{-\mu s} \int_0^t G(\mathbf{d}_1(s)u(\tau), \mathbf{d}_2(s)p(\tau))d\tau &= \int_0^{e^{-\mu s}t} G(\mathbf{d}_1(s)u(e^{\mu s}\tau), \mathbf{d}_2(s)p(e^{\mu s}\tau))d\tau \\ &= \int_0^{e^{-\mu s}t} G(u_s(\tau), p_s(\tau))d\tau, \end{aligned}$$

i.e. $G(u_s, p_s) \in L^1((0, e^{-\mu s}T), \mathbb{B})$. Similarly, we derive

$$\begin{aligned} \mathbf{d}_1(s)u(e^{\mu s}t) &= (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}t))u(0) + \mathbf{d}_1(s) \int_0^{e^{\mu s}t} (\Phi(e^{\mu s}t - \tau) \circ G)(u(\tau), p(\tau)) d\tau \\ &= (\Phi(t) \circ \mathbf{d}_1(s))u(0) + \int_0^{e^{\mu s}t} (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}t - \tau) \circ G)(u(\tau), p(\tau)) d\tau = \\ (\Phi(t) \circ \mathbf{d}_1(s))u(0) + e^{\mu s} \int_0^t (\mathbf{d}_1(s) \circ \Phi(e^{\mu s}(t - \sigma)) \circ G)(u(e^{\mu s}\sigma), p(e^{\mu s}\sigma)) d\sigma = \\ (\Phi(t) \circ \mathbf{d}_1(s))u(0) + e^{\mu s} \int_0^t (\Phi(t - \sigma) \circ \mathbf{d}_1(s) \circ G)(u(e^{\mu s}\sigma), p(e^{\mu s}\sigma)) d\sigma = \end{aligned}$$

$$(\Phi(t) \circ \mathbf{d}_1(s))u(0) + \int_0^t (\Phi(t-\sigma) \circ G)(\mathbf{d}_1(s)u(e^{\mu s}\sigma), \mathbf{d}_2(s)p(e^{\mu s}\sigma)) d\sigma,$$

where the linearity of $\Phi(t-\sigma)$ and a homogeneity of the operator G is utilized on the last step. Therefore, we have shown that

$$u_s(t) = \Phi(t)u_s(0) + \int_0^t \Phi(t-\sigma)g(u_s(\sigma), p_s(\sigma)) d\sigma.$$

Finally, since (u, p) is a mild solution on $[0, T)$ then

$$Q(u(t), p(t))d\sigma \stackrel{a.e.}{=} \mathbf{0}, \quad \forall t \in [0, T)$$

and using the identity $\mathbf{d}_2(s)\mathbf{0} = \mathbf{0}$ we derive

$$\begin{aligned} Q(u_s(\tau), u_s(\tau)) &= \mathbf{d}_2(s)Q((\mathbf{d}_1(s)u)(\tau), (\mathbf{d}_2(s)p)(\tau)) = \\ e^{\mu s}(\mathbf{d}_2(s) \circ Q)(u(e^{\mu s}\tau), p(e^{\mu s}\tau)) &\stackrel{a.e.}{=} \mathbf{0}, \quad \tau \in [0, e^{-\mu s}T). \end{aligned}$$

2) *The case of strong and classical solutions.* Let (u, p) be a strong solution. Since $u(t)$ is differentiable almost everywhere and $\mathbf{d}(s)$ is a linear bounded operator then for $\tau = e^{\mu s}t$ we have

$$\frac{du_s(t)}{dt} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) \frac{du(\tau)}{d\tau} \stackrel{a.e.}{=} e^{\mu s} \mathbf{d}(s) (Au(\tau) + G(u(\tau), p(\tau))) \stackrel{a.e.}{=} Au_s(t) + G(u_s(t), p_s(t)).$$

In the case of the classical solution the latter identity holds everywhere. Moreover, if $u(t) \in \mathcal{D}(A)$ then due to \mathbf{d} -homogeneity of the operator A we always have $u_s(t) \in \mathcal{D}(A)$, i.e. u_s is a classical solution.

4 Example: Global Existence and Uniqueness of Classical Solutions of Navier-Stokes Equations

The Navier-Stokes equations in \mathbb{R}^3 ,

$$\begin{aligned} \partial_t u &= \nu \Delta u - (u \cdot \nabla u)u - \nabla p, \\ \mathbf{0} &= \operatorname{div} u \end{aligned}$$

where u denotes the velocity of a fluid, p denotes the scalar pressure and $\nu > 0$ denotes viscosity of the fluid, is the the classical model of the flow of an incompressible viscous fluid.

Below for shortness we omit \mathbb{R}^3 in the notations of $\int_{\mathbb{R}^3}$, L^p , C_c^∞ and C_0^∞ spaces if the context is clear.

The classical idea of analysis is to prove, initially, an existence and regularity of weak solutions and next to show that any weak solution is smooth. For Navier-Stokes equations this scheme has been realized by Jean Leray in 1938 (see, e.g. [5] and the recent review [7]). We follow the same way having in mind \mathbf{d} -homogeneity (dilation symmetry) of Navier-Stokes equations (see e.g. [1, formula (1.5)]), which will guarantee a global expansion of all local results.

4.1 Homogeneity of Navier-Stokes equation

Navier-Stokes equations can be represented in the form (6). Indeed, notice that for any $u \in L^2$ we have $\int (\|u\|_4^2 u) \cdot \nabla \phi = \|u\|_4^2 \int u \cdot \nabla \phi$ and

$$\int u \cdot \nabla \phi = 0 \quad \Leftrightarrow \quad \int (\|u\|_2^4 u) \cdot \nabla \phi = 0, \quad \forall \phi \in C_0^\infty.$$

In the other words, we have

$$\operatorname{div} u = \mathbf{0} \quad \Leftrightarrow \quad \|u\|_2^4 \operatorname{div} u = \mathbf{0}, \quad \forall u \in V,$$

where V is a set of the so-called *weakly divergence free* velocity fields:

$$V := \left\{ u \in L^2 : \int u \cdot \nabla \phi = 0, \forall \phi \in C_0^\infty \right\}.$$

Due to this the Navier-Stokes equations can be equivalently represented in the form of the implicit evolution equation (6) with the operators

$$A = \Delta, \quad G(u, p) = -\nabla p - (u \cdot \nabla u)u, \quad Q(u, p) = \|u\|_2^4 \operatorname{div} u$$

having the domains $\mathcal{D}(Q) = V \times L_{loc}^1(\mathbb{R}^n, \mathbb{R})$,

$$\mathcal{D}(A) = \left\{ u \in L^2 : \exists f \in L_{loc}^1 \text{ such that } \int u \cdot \Delta \phi = \int f \cdot \phi, \quad \forall \phi \in C_0^\infty \right\},$$

$$\mathcal{D}(G) = \left\{ (u, p) \in L^2 \times L_{loc}^1 : \exists f \in L_{loc}^1, \int u \cdot (u \cdot \nabla \phi) + p \operatorname{div} \phi + f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty \right\}.$$

Lemma 3 Let $\mathbf{d}(s) : \tilde{\mathbb{X}} \rightarrow \tilde{\mathbb{X}}$ with $\tilde{\mathbb{X}} := L^2 \times L_{loc}^1(\mathbb{R}^n, \mathbb{R})$ be defined as

$$\mathbf{d}(s)(u, p) = (\mathbf{d}_1(s)u, \mathbf{d}_2(s)p), \quad s \in \mathbb{R}, u \in L^2, p \in L_{loc}^1(\mathbb{R}^n, \mathbb{R}),$$

where \mathbf{d}_1 is a group of linear bounded invertible operators on L^2 ,

$$(\mathbf{d}_1(s)u)(x) = e^s u(e^s x), \quad u \in L^2, x \in \mathbb{R}^n,$$

and \mathbf{d}_2 is a group of linear invertible operators on $L_{loc}^1(\mathbb{R}^n, \mathbb{R})$ given by

$$(\mathbf{d}_2(s)p)(x) = e^{2s} p(e^s x) \quad p \in L_{loc}^1, x \in \mathbb{R}^n.$$

Then the operator $F : \mathcal{D}(F) \subset \tilde{\mathbb{X}} \rightarrow \tilde{\mathbb{X}}$ with $\mathcal{D}(F) = \mathcal{D}(G) \cap \mathcal{D}(Q)$ and $F = (G, Q)$ is \mathbf{d} -homogeneous of degree 1.

Proof To complete the proof we must show that

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{2s}(\mathbf{d}_1(s) \circ G)(u, p), \quad \forall (u, p) \in \mathcal{D}(G), \forall s \in \mathbb{R},$$

$$Q(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^{2s}(\mathbf{d}_2(s) \circ Q)(u, p), \quad \forall (u, p) \in \mathcal{D}(Q), \forall s \in \mathbb{R}.$$

and the domain $\mathcal{D}(F)$ is invariant with respect to $\mathbf{d}(s)$, $\forall s \in \mathbb{R}$.

1) According to the definition of Q , the identity $Q(u, p) = f \in L^1_{loc}$ (in the weak sense) means

$$\int \|u\|_2^4 u \cdot \nabla \phi + f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty.$$

and using a change-of-variable theorem in Lebesgue integral we derive

$$e^s \int \|u\|_2^4 u(e^s x) \cdot (\nabla \phi)(e^s x) + f(e^s x) \cdot \phi(e^s x) dx = 0, \quad \forall \phi \in C_0^\infty.$$

By Lemma 1 we have $\mathbf{d}_1(s)u \in L^2$ for any $s \in \mathbb{R}$ and $\|\mathbf{d}_1(s)u\|_2 = e^{s/2}\|u\|_2$. Taking into account $\nabla(\mathbf{d}_1\phi)(x) = e^{2s}(\nabla\phi)(e^s x)$ we obtain

$$\int e^{-4s} \|\mathbf{d}_1(s)u\|_2^4 (\mathbf{d}_1(s)u) \cdot (\nabla \mathbf{d}_1(s)\phi) + e^{-2s}(\mathbf{d}_2(s)f) \cdot (\mathbf{d}_1(s)\phi) = 0, \quad \forall \phi \in C_0^\infty.$$

Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ then taking $f = \mathbf{0}$ we conclude $\mathbf{d}_1(s)V \subset V$ for any $s \in \mathbb{R}$, and due to $\mathbf{d}_2(s)f \in L^1_{loc}$ we derive $(Q \circ \mathbf{d}(s))(u, p) = e^{2s}(\mathbf{d}_2 \circ Q)(u, p)$ (in the weak sense) for all $s \in \mathbb{R}$ and, at least, on $\mathcal{D}(Q)$.

2) The identity $G(u, p) = f \in L^1$ (in the weak sense) means

$$\int u \cdot (u \cdot \nabla \phi) + p \operatorname{div} \phi + \int f \cdot \phi = 0, \quad \forall \phi \in C_0^\infty.$$

If $p \in L^1_{loc}$ and $u \in L^2$ then from Lemma 1 we derive $\mathbf{d}_1(s)u \in L^2$ and $\mathbf{d}_2(s)p \in L^1_{loc}$.

Using the change-of-variable theorem in the Lebesgue integral we derive

$$e^s \int u(e^s x) \cdot (u(e^s x) \cdot (\nabla \phi)(e^s x)) + p(e^s x) (\operatorname{div} \phi)(e^s x) + f(e^s x) \cdot \phi(e^s x) dx = 0.$$

or, equivalently,

$$e^{-3s} \int (\mathbf{d}_1(s)u) \cdot ((\mathbf{d}_1(s)u) \cdot (\nabla \mathbf{d}_1(s)\phi)) + (\mathbf{d}_2(s)p) \operatorname{div}(\mathbf{d}_1(s)\phi) + e^{2s}(\mathbf{d}_1(s)f) \cdot (\mathbf{d}_1(s)\phi) = 0,$$

for all $\phi \in C_0^\infty$. Since $\mathbf{d}_1(s)$ maps C_0^∞ onto C_0^∞ then we conclude that $\mathbf{d}(s)\mathcal{D}(G) \subset \mathcal{D}(G)$ and the identity

$$G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p) = e^s \mathbf{d}_1(s)f = e^{2s} \mathbf{d}_1(s)G(\mathbf{d}_1(s)u, \mathbf{d}_2(s)p)$$

holds for any $s \in \mathbb{R}$ and, at least, on $\mathcal{D}(G)$.

Therefore, F is, indeed, \mathbf{d} -homogeneous of degree 2.

The \mathbf{d} -homogeneity of the Laplace operator Δ is studied in Example 3. Therefore, the Navier-Stokes equation satisfy all conditions required for application of Theorem 2. In other words, if it has mild, strong or classical solutions defined on $[0, +\infty)$ for small initial values that it has, respectively, mild, strong or classical solutions defined on $[0, +\infty)$ for large initial values. The following lemma proves this more rigorously.

Corollary 1 *Let there exist $\varepsilon > 0$ such that*

$$\forall u_0 \in \{u \in V \cap L^\infty : \|u\|_2 \|u\|_\infty < \varepsilon\}$$

a mild, strong or classical (in the sense of definition 4) solution (u, p) of the Navier-Stokes equations with the initial condition $u(0) = u_0$ exists on $[0, +\infty)$ and $u(t) \in V \cap L^\infty$ for all $t \geq 0$. Then

$$\forall u_0 \in V \cap L^\infty$$

a mild, strong or a classical solution of the Navier-Stokes equations with the initial condition $u(0) = u_0$ exists on $[0, +\infty)$ and $u(t) \in V \cap L^\infty$ for all $t \geq 0$.

Proof Let \mathbf{d}_1 be defined as in the proof of Lemma 3 and let an initial value $u_0 \in V \cap L^\infty$ be such that $\|u_0\|_2 \|u_0\|_\infty \geq \varepsilon$. Let us show that in this case the Navier-Stokes equations also have a strong solution on $[0, +\infty)$.

In the proof of Lemma 3 we have shown that $\mathbf{d}_1(s)u_0 \in V$ for any $s \in \mathbb{R}$ and $\mathbf{d}_1(s)u_0 \in L^\infty$ for any $s \in \mathbb{R}$.

By Lemma 1 we derive $\|\mathbf{d}_1(s)u_0\|_2 = e^{s/2}\|u_0\|_2$ and $\|\mathbf{d}_1(s)u_0\|_\infty = e^s\|u_0\|_\infty$.

$$\|\mathbf{d}_1(s)u_0\|_2 \|\mathbf{d}_1(s)u_0\|_\infty = e^{3s/2} \|u_0\|_2 \|u_0\|_\infty.$$

Let us select

$$-\infty < s_0 < \frac{2}{3} \ln \frac{\varepsilon}{\|u_0\|_2 \|u_0\|_\infty}.$$

In this case, we, obviously, have

$$\|\mathbf{d}_1(s_0)u_0\|_2 \|\mathbf{d}_1(s_0)u_0\|_\infty < \varepsilon \text{ and } \mathbf{d}_1(s_0)u_0 \in V \cap L^\infty.$$

Hence, by assumption of the corollary, a solution (u, p) with $u(0) = \mathbf{d}_1(s_0)u_0$ exists on $[0, +\infty)$.

By Lemma 3 the Navier-Stokes equations is a homogeneous system and by Theorem 2, the pair (\tilde{u}, \tilde{p}) given by

$$\tilde{u}(t, x) = e^{-s_0 t} u(e^{-2s_0 t}, e^{-s_0 x}), \tilde{p}(t, x) = e^{-2s_0 t} p(e^{-2s_0 t}, e^{-s_0 x}), t \in [0, +\infty), x \in \mathbb{R}^3$$

is also a strong solution of the Navier-Stokes equation. Moreover, since $\mathbf{d}_1(s)V \subset V$ and $\mathbf{d}_1(s)L^\infty \subset L^\infty$ then $\tilde{u}(t) \in V \cap L^\infty$ for all $t \geq 0$. Finally, since $u(0) = \mathbf{d}_1(s_0)u_0$ means that

$$u(0, x) = e^{s_0} u_0(e^{s_0 x}), \quad x \in \mathbb{R}^3$$

then

$$\tilde{u}(0, x) = e^{-s_0} u(0, e^{-s_0} x) = u_0(x),$$

i.e. $\tilde{u}(0) = u_0$.

4.2 Global existence of classical solutions

Let us consider also the well-known weak form (see e.g. [5] and the recent review [7]) of Navier-Stokes equations

$$\int u(0) \cdot \phi(0) + \int_0^T \int u \cdot (\phi_t + \Delta \phi) + p \operatorname{div} \phi = \int_0^T \int u \cdot (u \cdot \nabla) \phi, \quad \forall \phi \in C_0^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}^3) \quad (8)$$

with $u(t) \in V$ for $t \in (0, T)$.

Definition 5 [7, Definition 3.7] A pair (u, p) is said to be a strong solution of the Navier-Stokes equations on $[0, T)$ if $p \in L_{loc}^1((0, T) \times \mathbb{R}^3, \mathbb{R})$, $u(t) \in V$, $t \in (0, T)$

$$u \in C([0, T), L^2) \cap C((0, T), L^\infty),$$

$\|u(t)\|_\infty$ is bounded as $t \rightarrow 0^+$ and (8) is satisfied.

Let us mention the following properties of the solutions introduced by Definition 5 proven before:

- *Global-in-time existence and boundedness of strong solutions for small initial data* [7, Corollary 3.13 and Lemma 3.10]
There exists $\varepsilon > 0$ and there exists $C > 0$ such that for any $u_0 \in \{u \in V \cap L^\infty : \|u\|_2 \|u\|_\infty < \varepsilon\}$ a strong solution (u, p) with the initial data $u(0) = u_0$ exists on $[0, +\infty)$ and $\|u(t)\|_\infty \leq C \|u_0\|_\infty$ for all $t \geq 0$.
- *Uniqueness of strong solutions* [7, Theorem 3.9]
For any $u_0 \in V \cap L^\infty$ a strong solution with $u(0) = u_0$ is unique.
- *Smoothness* [7, Corollary 3.3].
If (u, p) is a strong solution of the Navier-Stokes equation on $(0, T)$ then

$$\partial_t^k \nabla^m u, \partial_t^k \nabla^m p \in C((0, T), L^2) \cap C((0, T), L^\infty), \quad \forall m, k \geq 0$$

and, in particular, $u, p \in C^\infty((0, T) \times \mathbb{R}^3, \mathbb{R}^3)$ constitute a classical solution of the Navier-Stokes equations on $(0, T) \times \mathbb{R}^3$.

Therefore, combining the latter results with Corollary 1 we derive

For any initial condition $u(0) = u_0 \in V \cap L^\infty$ the Navier-Stokes equations have a unique classical solution (u, p) on $(0, +\infty) \times \mathbb{R}^3$ and

$$\|u(t)\|_\infty \leq C\|u_0\|.$$

for some constant $C > 0$ independent of u_0 .

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