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► **To cite this version:**

Justine Labat, Victor Péron, Sébastien Tordeux. Asymptotic models for the multiple electromagnetic wave scattering problem by small obstacles. Journées Ondes Sud-Ouest (JOSO), Mar 2019, Le Barp, France. hal-02071888

**HAL Id: hal-02071888**

**<https://inria.hal.science/hal-02071888>**

Submitted on 19 Mar 2019

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# Asymptotic models for the multiple electromagnetic wave scattering problem by small obstacles

Justine Labat, Victor Péron, Sébastien Tordeux

PhD student in Applied Mathematics

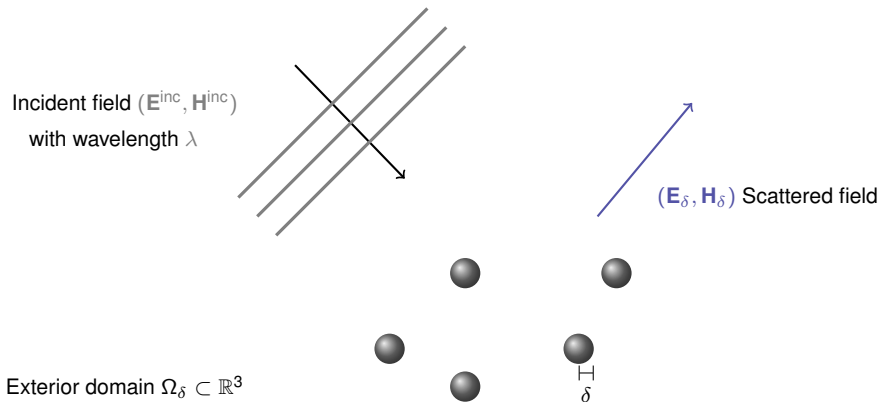
EPC Magique 3D — UPPA-E2S, INRIA Bordeaux Sud-Ouest, LMAP UMR CNRS 5142

Journées Ondes Sud-Ouest

Le Barp, March 13, 2019

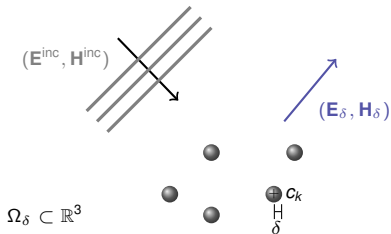


## 3D Scattering problem by small obstacles



Asymptotic assumption :  $\delta \ll \lambda$

# Model problem



- Time-harmonic domain
- Homogeneous & isotropic medium  $\Omega_\delta$
- Perfect conductors of characteristic length  $\delta$

## Time-harmonic Maxwell equations

$$\begin{cases} \mathbf{curl} \mathbf{E}_\delta - i\kappa \mathbf{H}_\delta = 0 & \text{in } \Omega_\delta \\ \mathbf{curl} \mathbf{H}_\delta + i\kappa \mathbf{E}_\delta = 0 & \text{in } \Omega_\delta \end{cases}$$

$$\text{with } \kappa^2 = \omega^2 \mu \left( \epsilon + \frac{i\sigma}{\omega} \right), \Im(\kappa) \geq 0$$

## Boundary condition

$$\mathbf{n} \times \mathbf{E}_\delta = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \quad \text{on } \partial\Omega_\delta$$

## Silver-Müller radiation condition

$$r(\mathbf{H}_\delta \times \hat{\mathbf{x}} - \mathbf{E}_\delta) \xrightarrow[r \rightarrow \infty]{} 0 \quad \text{unif. in } \hat{\mathbf{x}} = \frac{\mathbf{x}}{r}$$

## Mathematical well-posedness

For any  $\mathbf{E}^{\text{inc}} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_\delta)$  there exist a unique solution  $(\mathbf{E}_\delta, \mathbf{H}_\delta) \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega_\delta)^2$  to the exterior Maxwell problem.

# Applications

- Medical imaging
- Civil engineering
- Nuclear industry
- Atmospheric science
- Astronomy
- Climatology
- ...

## Inverse problem ?

- Locate heterogeneities, cracks, ...
- Characterize them

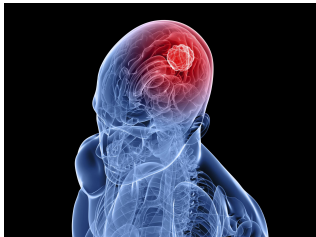


Figure: Small tumor detection<sup>1</sup>



Figure: Non-destructive testing<sup>2</sup>

<sup>1</sup> <https://medcitynews.com/wp-content/uploads/2017/01/GettyImages-94456546-600x450.jpg>

<sup>2</sup> <https://tri-intl.com/wp-content/uploads/2017/07/NonDestructiveTesting.jpg>

## Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

# Workflow

## Asymptotic models

- ✗ Restricted to small obstacles
- ✓ Low computational cost
- ✓ Meshless method

Multiple scattering



## Foldy-Lax model

- ✓ Interactions taken into account
- ✓ Low computational cost
- ✓ Meshless method

Superposition  
principle



## Born approximation

- ✗ No interaction between the obstacles
- ✓ Low computational cost
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### (Non-exhaustive) alternatives

- Boundary integral equations
  - ▶ Boundary element methods
  - ▶ Spectral-based methods
- Enrichment of approximation spaces
  - ▶ Trefftz-based methods
  - ▶ Extended FEM
  - ▶ Partition of Unity method



# Workflow

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- ✗ Restricted to small obstacles
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# Some references

- Historic references
  - ▶ Rayleigh (1884), Foldy (1945), Lax (1951)
- **Small defect** theory
  - ▶ Il'in (1992), Maz'ya-Nazarov-Plamenevskii (2000)
  
- Acoustic obstacle
  - ▶ Ammari-Kang (2003), Ramm (2005), Claeys (2008)
- Time-dependent domain
  - ▶ Mattesi (2014), Korikov (2015), Marmorat (2015)
- **Electromagnetic** obstacle
  - ▶ Vogelius-Volkov (2000), Ammari-Vogelius-Volkov (2001), Korikov-Plamenevskii (2017)
  
- **Foldy** theory
  - ▶ Martin (2004), Cassier-Hazard (2013), Bendali-Cocquet-Tordeux (2014), Challa-Hu-Sini (2014)
- High-order **spectral** algorithms
  - ▶ Xu (1995), Ganesh-Hawkins (2009), Barucq-Chabassier-Pham-Tordeux (2017)
- Inverse problem
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# Outline

## 1. Asymptotic models

- Single-scattering
- Application: Born approximation
- Multiple-scattering: Foldy-Lax model
- Numerical results

## 2. Spectral method: Spherical case

- Discretization
- Numerical convergence
- Comparison with asymptotic models
- Comparison with finite element solutions

## 3. Conclusions and perspectives

# Outline

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- Single-scattering
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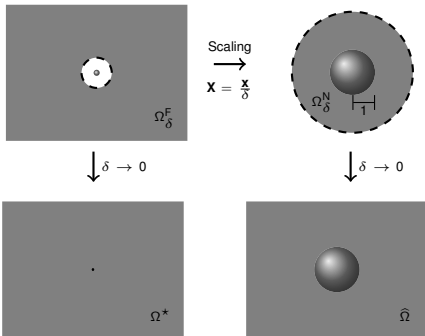
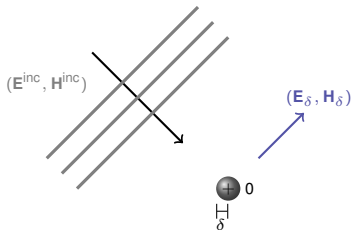
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## 3. Conclusions and perspectives

# Approximation of solution to single scattering

## Method of **matched asymptotic expansions**

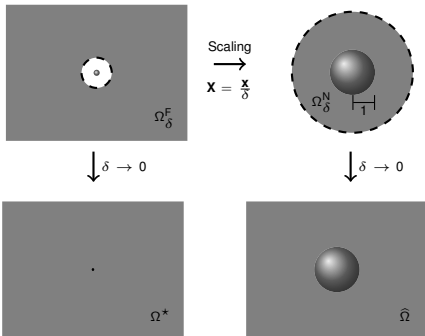
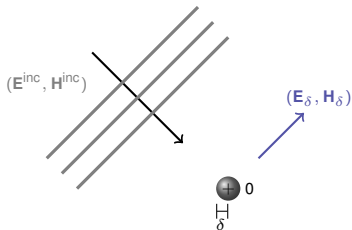
- Domain decomposition
- Local approximations
- Matching procedure



# Approximation of solution to single scattering

## Method of **matched asymptotic expansions**

- Domain decomposition
- **Local approximations**
- Matching procedure



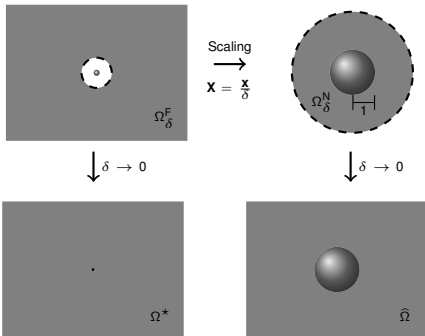
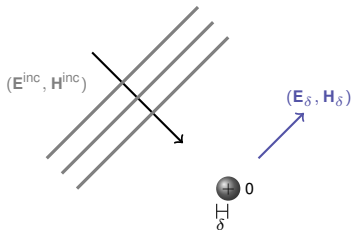
## Asymptotic expansions

- **Far field** expansion in  $\Omega^* = \mathbb{R}^3 \setminus \{0\}$
- **Near field** expansion in  $\hat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0, 1)}$

# Approximation of solution to single scattering

## Method of **matched asymptotic expansions**

- Domain decomposition
- Local approximations
- **Matching procedure**



## Asymptotic expansions

- **Far field** expansion in  $\Omega^* = \mathbb{R}^3 \setminus \{0\}$
- **Near field** expansion in  $\hat{\Omega} = \mathbb{R}^3 \setminus \overline{\mathcal{B}(0, 1)}$

related by an asymptotic process



# Asymptotic expansions

Near field expansion

$$\mathbf{E}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{E}}_0(\mathbf{X}) + \delta\widehat{\mathbf{E}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{E}}_2(\mathbf{X}) + \dots$$

$$\mathbf{H}_\delta(\delta\mathbf{X}) \approx \widehat{\mathbf{H}}_0(\mathbf{X}) + \delta\widehat{\mathbf{H}}_1(\mathbf{X}) + \delta^2\widehat{\mathbf{H}}_2(\mathbf{X}) + \dots$$

$\mathbf{X} = \frac{\mathbf{x}}{\delta}$ : fast variable

Far field expansion

$$\mathbf{E}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{E}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{E}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{E}}_5(\mathbf{x}) + \dots$$

$$\mathbf{H}_\delta(\mathbf{x}) \approx \delta^3\widetilde{\mathbf{H}}_3(\mathbf{x}) + \delta^4\widetilde{\mathbf{H}}_4(\mathbf{x}) + \delta^5\widetilde{\mathbf{H}}_5(\mathbf{x}) + \dots$$

- For an obstacle of arbitrary shape

Numerical solution of elementary problems **independent of  $\delta$**

- ▶ Near-field: **quasi-static** problems
- ▶ Far-field: **time-harmonic** problems + equivalent multipole distributions

- For a **spherical** obstacle

- ▶ Analytical solutions of elementary problems
- ▶ Equivalent **multipole distributions**

# Near-field approximation: Spherical case

Close to the obstacle : **quasi-static** approximation

Superposition of electric fields generated by **multipole distributions**:

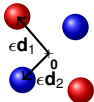
$$\widehat{\mathbf{E}}_p = \sum_{\substack{n=1 \\ |n-p| \text{ odd}}}^{p+1} \boldsymbol{\varepsilon}_{n,\text{elec}}^{\text{stat}}[\mathbf{u}_{1,n}^{p,E}, \dots, \mathbf{u}_{n,n}^{p,E}] + \sum_{\substack{n=1 \\ |n-p| \text{ even}}}^{p+1} \boldsymbol{\varepsilon}_{n,\text{mag}}^{\text{stat}}[\mathbf{u}_{1,n}^{p,H}, \dots, \mathbf{u}_{n,n}^{p,H}] \quad p = 0, 1, 2, \dots$$

Dipole ( $n = 1$ )

Quadrupole ( $n = 2$ )

$\boldsymbol{\varepsilon}_n^{\text{stat}}[\mathbf{d}_1, \dots, \mathbf{d}_n]$  electric fields

Charges



Currents



- generated by **charges** OR **currents**
- electric/magnetic  **$2^n$ -point charges**
  - ▶ electric charge:  $\boldsymbol{\varepsilon} = -\nabla\mathcal{V}$
  - ▶ magnetic current:  $\boldsymbol{\varepsilon} = -\text{curl } \mathcal{A}$
- of **moments** ( $\mathbf{d}_1, \dots, \mathbf{d}_n$ )
- defined by an **asymptotic** process

# Near-field approximation: Spherical case

Close to the obstacle : **quasi-static** approximation

Superposition of electric fields generated by **multipole distributions**:

$$\hat{\mathbf{E}}_p = \sum_{\substack{n=1 \\ |n-p| \text{ odd}}}^{p+1} \boldsymbol{\varepsilon}_{n,\text{elec}}^{\text{stat}} [\mathbf{u}_{1,n}^{p,E}, \dots, \mathbf{u}_{n,n}^{p,E}] + \sum_{\substack{n=1 \\ |n-p| \text{ even}}}^{p+1} \boldsymbol{\varepsilon}_{n,\text{mag}}^{\text{stat}} [\mathbf{u}_{1,n}^{p,H}, \dots, \mathbf{u}_{n,n}^{p,H}] \quad p = 0, 1, 2, \dots$$

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Near field	0	●	●				
	1	●	●	●	●		
	2	●	●	●	●	●	●

● Electric field

● Magnetic field

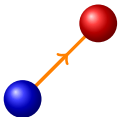
# Far-field approximation

Far from the obstacle: **time-harmonic** approximation

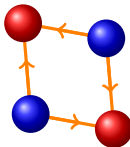
Superposition of electric fields generated by **multipole distributions**:

$$\tilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \boldsymbol{\varepsilon}_{n,\text{elec}}[\mathbf{v}_{1,n}^{\rho,E}, \dots, \mathbf{v}_{n,n}^{\rho,E}] + \boldsymbol{\varepsilon}_{n,\text{mag}}[\mathbf{v}_{1,n}^{\rho,H}, \dots, \mathbf{v}_{n,n}^{\rho,H}] \quad \rho = 0, 1, 2, \dots$$

Dipole ( $n = 1$ )



Quadrupole ( $n = 2$ )



$\boldsymbol{\varepsilon}_n[\mathbf{d}_1, \dots, \mathbf{d}_n]$  electric fields

- generated by charges **AND** currents
  - ▶ related by **charge conservation principle**
- electric/magnetic  $2^n$ -point charges
  - ▶ electric multipole:  $\boldsymbol{\varepsilon} = -\nabla \mathcal{V}_E + i\kappa \mathcal{A}_E$
  - ▶ magnetic multipole:  $\boldsymbol{\varepsilon} = -\text{curl } \mathcal{A}_H$
- of moments ( $\mathbf{d}_1, \dots, \mathbf{d}_n$ )
- defined by an asymptotic process

$$\text{div } \mathcal{J} - i\omega \rho = 0$$

# Far-field approximation

Far from the obstacle: **time-harmonic** approximation

$$\tilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \boldsymbol{\varepsilon}_{n,\text{elec}}[\mathbf{v}_{1,n}^{p,E}, \dots, \mathbf{v}_{n,n}^{p,E}] + \boldsymbol{\varepsilon}_{n,\text{mag}}[\mathbf{v}_{1,n}^{p,H}, \dots, \mathbf{v}_{n,n}^{p,H}]$$

$$\tilde{\mathbf{H}}_{p+3} = \sum_{n=1}^{p+1} \boldsymbol{\mathcal{H}}_{n,\text{mag}}[\mathbf{v}_{1,n}^{p,H}, \dots, \mathbf{v}_{n,n}^{p,H}] + \boldsymbol{\mathcal{H}}_{n,\text{elec}}[\mathbf{v}_{1,n}^{p,E}, \dots, \mathbf{v}_{n,n}^{p,E}]$$

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Far field	3	●	●	●	●		
	4	●	●	●	●		
	5	●	●	●	●	●	●

● Electric field

● Magnetic field

# Far-field approximation

Far from the obstacle: **time-harmonic** approximation

$$\tilde{\mathbf{E}}_{p+3} = \sum_{n=1}^{p+1} \boldsymbol{\varepsilon}_{n,\text{elec}}[\mathbf{v}_{1,n}^{p,E}, \dots, \mathbf{v}_{n,n}^{p,E}] + \boldsymbol{\varepsilon}_{n,\text{mag}}[\mathbf{v}_{1,n}^{p,H}, \dots, \mathbf{v}_{n,n}^{p,H}]$$

$$\tilde{\mathbf{H}}_{p+3} = \sum_{n=1}^{p+1} \boldsymbol{\mathcal{H}}_{n,\text{mag}}[\mathbf{v}_{1,n}^{p,H}, \dots, \mathbf{v}_{n,n}^{p,H}] + \boldsymbol{\mathcal{H}}_{n,\text{elec}}[\mathbf{v}_{1,n}^{p,E}, \dots, \mathbf{v}_{n,n}^{p,E}]$$

- Spherical** case

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Far field	3	•	•	•	•		
	4						
	5	•	•	•	•	•	•

- Collected dipolar** approximation

	Order	Dipole		Quadrupole		Octupole	
		Electric	Magnetic	Electric	Magnetic	Electric	Magnetic
Far field	3	•	•	•	•		
	4						
	5	•	•	•	•		

# Born approximation

- The electromagnetic fields are approximated by the **superposition principle**

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \quad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

- Each obstacle is modeled as a **dipolar source** around  $c_k$

$$\mathbf{E}_{\delta,k}(\mathbf{x}) = \boldsymbol{\varepsilon}_{1,\text{elec}}[\mathbf{d}_{\delta,k}^{\text{E}}](\mathbf{x} - c_k) + \boldsymbol{\varepsilon}_{1,\text{mag}}[\mathbf{d}_{\delta,k}^{\text{H}}](\mathbf{x} - c_k)$$

For **perfectly conducting spheres**:

- ▶ Approximation of order 3

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \mathbf{E}^{\text{inc}}(c_k) \quad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \mathbf{H}^{\text{inc}}(c_k)$$

- ▶ Collected dipolar approximation

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \left(1 + \frac{3(\kappa\delta)^2}{10}\right) \mathbf{E}^{\text{inc}}(c_k) \quad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \left(1 - \frac{3(\kappa\delta)^2}{5}\right) \mathbf{H}^{\text{inc}}(c_k)$$

# Foldy-Lax model

- The electromagnetic fields are approximated by the superposition principle

$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \quad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

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For **perfectly conducting** spheres:

- Approximation of order 3

$$\mathbf{d}_{\delta,k}^{\text{E}} = 4\pi\delta^3 \left( \mathbf{E}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{E}_{\delta,\ell}(c_k) \right) \quad \mathbf{d}_{\delta,k}^{\text{H}} = -2\pi\delta^3 \left( \mathbf{H}^{\text{inc}}(c_k) + \sum_{\substack{\ell=1 \\ \ell \neq k}}^{N_{\text{obs}}} \mathbf{H}_{\delta,\ell}(c_k) \right)$$

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# Foldy-Lax model

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$$\mathbf{E}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{E}_{\delta,k}(\mathbf{x}) \quad \mathbf{H}_\delta(\mathbf{x}) \approx \sum_{k=1}^{N_{\text{obs}}} \mathbf{H}_{\delta,k}(\mathbf{x})$$

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- Vectorial formulation** (3-order approximation)

Find  $\mathbf{d} = ((\mathbf{d}_1^{\text{E}}), \dots, (\mathbf{d}_{N_{\text{obs}}}^{\text{E}}), (\mathbf{d}_1^{\text{H}}), \dots, (\mathbf{d}_{N_{\text{obs}}}^{\text{H}}))^\top \in \mathbb{C}^{6N_{\text{obs}}}$  such that

$$\mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

$$\text{with } \mathbf{f} = \begin{pmatrix} 4\pi \mathbf{E}^{\text{inc}}(c_1) \\ \vdots \\ 4\pi \mathbf{E}^{\text{inc}}(c_{N_{\text{obs}}}) \\ -2\pi \mathbf{H}^{\text{inc}}(c_1) \\ \vdots \\ -2\pi \mathbf{H}^{\text{inc}}(c_{N_{\text{obs}}}) \end{pmatrix}$$

and  $\mathbb{A}$  the “interaction” matrix

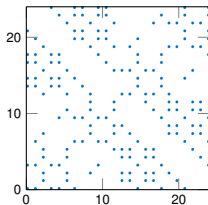


Figure: Skeleton of  $\mathbb{A}$  for  $N_{\text{obs}} = 4$

# Foldy-Lax approximations

- Approximation of order 3

$$\mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

- Collected dipolar approximation

$$\mathbb{D}^{-1} \mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\mathbb{D} = \text{diag}(\underbrace{\alpha, \dots, \alpha}_{3N_{\text{obs}}}, \underbrace{\beta, \dots, \beta}_{3N_{\text{obs}}}) \quad \alpha = 1 + \frac{3(\kappa\delta)^2}{10} \quad \beta = 1 - \frac{6(\kappa\delta)^2}{10}$$

- Modified dipolar approximation

$$\tilde{\mathbb{D}}^{-1} \mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\tilde{\mathbb{D}} = \text{diag}(\underbrace{\tilde{\alpha}, \dots, \tilde{\alpha}}_{3N_{\text{obs}}}, \underbrace{\tilde{\beta}, \dots, \tilde{\beta}}_{3N_{\text{obs}}}) \quad \tilde{\alpha} = \frac{3i}{2(\kappa\delta)^3} \frac{j_1(\kappa\delta)}{h_1^{(1)}(\kappa\delta)} \quad \tilde{\beta} = -\frac{3i}{(\kappa\delta)^3} \frac{j_1(\kappa\delta) + \kappa\delta j_1'(\kappa\delta)}{h_1^{(1)}(\kappa\delta) + \kappa\delta h_1^{(1)'}(\kappa\delta)}$$

# Foldy-Lax approximations

- Approximation of order 3

$$\mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

- Collected dipolar approximation

$$\mathbb{D}^{-1} \mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

where

$$\mathbb{D} = \text{diag}(\underbrace{\alpha, \dots, \alpha}_{3N_{\text{obs}}}, \underbrace{\beta, \dots, \beta}_{3N_{\text{obs}}}) \quad \alpha = 1 + \frac{3(\kappa\delta)^2}{10} \quad \beta = 1 - \frac{6(\kappa\delta)^2}{10}$$

- **Modified** dipolar approximation

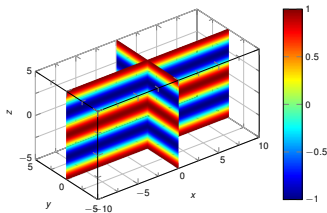
$$\tilde{\mathbb{D}}^{-1} \mathbf{d} - \delta^3 \mathbb{A} \mathbf{d} = \delta^3 \mathbf{f}$$

where

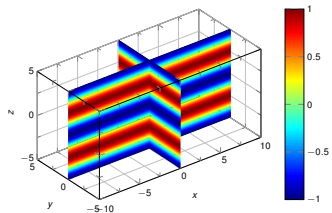
$$\tilde{\mathbb{D}} = \text{diag}(\underbrace{\tilde{\alpha}, \dots, \tilde{\alpha}}_{3N_{\text{obs}}}, \underbrace{\tilde{\beta}, \dots, \tilde{\beta}}_{3N_{\text{obs}}}) \quad \tilde{\alpha} = \frac{3i}{2(\kappa\delta)^3} \frac{j_1(\kappa\delta)}{h_1^{(1)}(\kappa\delta)} \quad \tilde{\beta} = -\frac{3i}{(\kappa\delta)^3} \frac{j_1(\kappa\delta) + \kappa\delta j_1'(\kappa\delta)}{h_1^{(1)}(\kappa\delta) + \kappa\delta h_1^{(1)'}(\kappa\delta)}$$

## Numerical validation: Spherical case

- Incident field: Electromagnetic plane wave of wavelength  $\lambda = 5$  along  $z$ -axis polarized by  $\mathbf{e}_x$

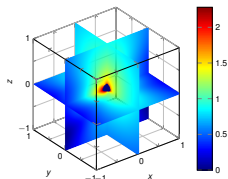


(a) Electric field ( $x$ -component)

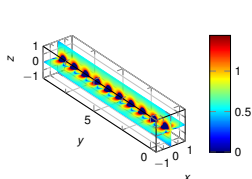


(b) Magnetic field ( $y$ -component)

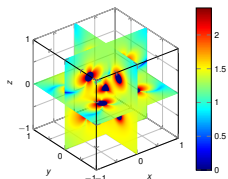
- Reference solution: "Spectral" solution truncated at the order 10



(c) One obstacle



(d) Aligned obstacles



(e) Real 3D-case

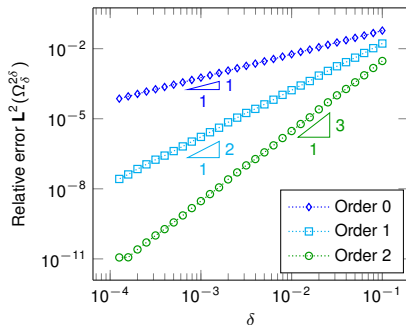
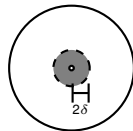
# Single-scattering: Validation of asymptotic expansions

- Near-field approximations

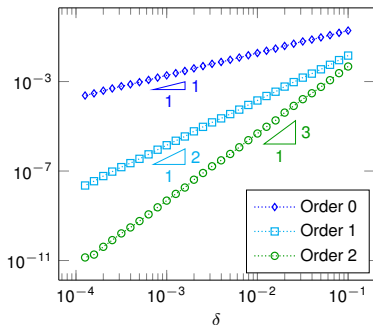
Order 0 :  $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Order 1 :  $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta \widehat{\mathbf{E}}_1(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$

Order 2 :  $\|\mathbf{E}_\delta - \widehat{\mathbf{E}}_0(\frac{\cdot}{\delta}) - \delta \widehat{\mathbf{E}}_1(\frac{\cdot}{\delta}) - \delta^2 \widehat{\mathbf{E}}_2(\frac{\cdot}{\delta})\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})} / \|\mathbf{E}_\delta\|_{\mathbf{L}^2(\Omega_\delta^{2\delta})}$



(f) Electric field



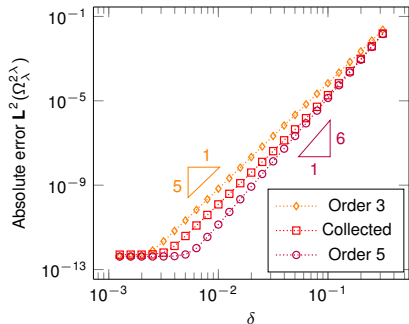
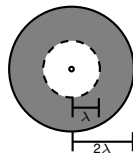
(g) Magnetic Field

# Single-scattering: Validation of asymptotic expansions

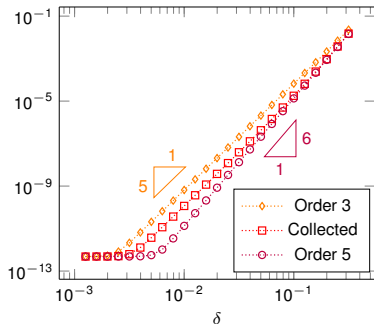
- Far-field approximations

Order 3 :  $\|\mathbf{E}_\delta - \delta^3 \tilde{\mathbf{E}}_3\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$

Order 5 :  $\|\mathbf{E}_\delta - \delta^3 \tilde{\mathbf{E}}_3 - \delta^5 \tilde{\mathbf{E}}_5\|_{\mathbf{L}^2(\Omega_\lambda^{2\lambda})}$



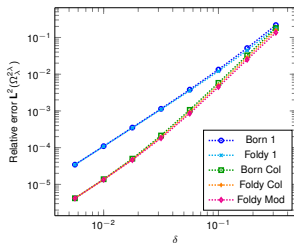
(l) Electric field



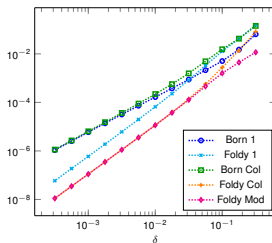
(m) Magnetic field

# Multiple-scattering: Validation of Foldy-Lax model

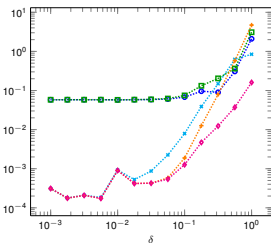
- Order 3: Born and Foldy
- Collected dipole improvement: Collected Born and Collected Foldy
- Spectral improvement: Modified Foldy



(n) Real 3D case with  $N_{\text{obs}} = 13$  (fixed distance)



(o) Aligned obstacles with  $N_{\text{obs}} = 5$  ( $\sqrt{\delta}$ -dependence)



(p) Aligned obstacles with  $N_{\text{obs}} = 5$  ( $\delta$ -dependence)

# Outline

## 1. Asymptotic models

- Single-scattering
- Application: Born approximation
- Multiple-scattering: Foldy-Lax model
- Numerical results

## 2. Spectral method: Spherical case

- Discretization
- Numerical convergence
- Comparison with asymptotic models
- Comparison with finite element solutions

## 3. Conclusions and perspectives



# Spectral method

- The electromagnetic fields are represented by the **Stratton-Chu formula** for  $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_k(\mathbf{y}) ds_{\mathbf{y}}$$

where  $\Phi(\mathbf{x}, \mathbf{y}) = \frac{\exp(i\kappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$

# Spectral method

- The electromagnetic fields are represented by the **Stratton-Chu formula** for  $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_k(\mathbf{y}) ds_{\mathbf{y}}$$

- Each  $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$  satisfies the **boundary integral equation**

$$\sum_{j=1}^{N_{\text{obs}}} \mathcal{M}_\Gamma^{kj} \mathbf{p}_j = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \quad \text{on } \Gamma_\delta^k$$

where  $\mathcal{M}_\Gamma^{kj} : \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^j}, \Gamma_\delta^j) \rightarrow \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$  is the extension of

$$\mathcal{M}_\Gamma^{kj} \boldsymbol{\lambda}(\mathbf{x}_\Gamma) = \mathbf{n}(\mathbf{x}_\Gamma) \times \lim_{\mathbf{x} \rightarrow \mathbf{x}_\Gamma} \mathbf{curl} \int_{\Gamma_\delta^j} \Phi(\mathbf{x}, \mathbf{y}) \boldsymbol{\lambda}(\mathbf{y}) ds_{\mathbf{y}} \quad \boldsymbol{\lambda} \in \mathcal{C}^\infty(\Gamma_\delta^j) \quad \mathbf{x}_\Gamma \in \Gamma_\delta^k$$

# Spectral method

- The electromagnetic fields are represented by the **Stratton-Chu formula** for  $\mathbf{x} \in \Omega_\delta$

$$\mathbf{E}_\delta(\mathbf{x}) = \sum_{k=1}^{N_{\text{obs}}} \mathbf{curl} \int_{\Gamma_\delta^k} \Phi(\mathbf{x}, \mathbf{y}) \mathbf{p}_k(\mathbf{y}) ds_{\mathbf{y}}$$

- Each  $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$  satisfies the **boundary integral equation**

$$\sum_{j=1}^{N_{\text{obs}}} \mathcal{M}_{\Gamma_\delta^k}^{kj} \mathbf{p}_j = -\mathbf{n} \times \mathbf{E}^{\text{inc}} \quad \text{on } \Gamma_\delta^k$$

- Galerkin discretization** of the BIE on **local spectral basis** with  $N_{\text{mod}}$  modes

$$\mathbf{p}_j(\mathbf{x}) = \sum_{n=1}^{N_{\text{mod}}} \sum_{m=-n}^n \mathbf{p}_{n,m}^{j,\perp} \nabla_{S^2} Y_{n,m}(\hat{\mathbf{x}}_j) + \mathbf{p}_{n,m}^{j,\times} \mathbf{curl}_{S^2} Y_{n,m}(\hat{\mathbf{x}}_j) \quad \mathbf{x} \in \Gamma_\delta^j$$

with  $\hat{\mathbf{x}}_j = \frac{\mathbf{x} - \mathbf{c}_j}{|\mathbf{x} - \mathbf{c}_j|}$  and  $\nabla_{S^2} Y_{n,m}$ ,  $\mathbf{curl}_{S^2} Y_{n,m}$ : complex-valued vector spherical harmonics

## Vectorial formulation

- **Variational** formulation: Find  $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$  such that

$$\sum_{j=1}^{N_{\text{obs}}} \langle \mathcal{M}_{\Gamma_\delta^k}^{kj} \mathbf{p}_j, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\text{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \quad \forall \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

- **Vectorial** formulation: Find  $\mathbf{p} = ((\rho_{n,m}^{1,\perp}), \dots, (\rho_{n,m}^{N_{\text{obs}},\perp}), (\rho_{n,m}^{1,\times}), \dots, (\rho_{n,m}^{N_{\text{obs}},\times}))^\top \in \mathbb{C}^{\mathbf{N}}$  s.t.

$$\mathbf{M} \mathbf{p} = \mathbf{f}$$

with  $\mathbf{N} = 2N_{\text{mod}}(N_{\text{mod}} + 2)N_{\text{obs}}$  and

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{\perp\perp} & \mathbf{M}_{\perp\times} \\ \mathbf{M}_{\times\perp} & \mathbf{M}_{\times\times} \end{pmatrix} \quad \text{with} \quad \mathbf{M}_{\alpha\beta} = \begin{pmatrix} \mathbf{M}_{\alpha\beta}^{11} & \mathbf{M}_{\alpha\beta}^{12} & \dots & \mathbf{M}_{\alpha\beta}^{1N_{\text{obs}}} \\ \mathbf{M}_{\alpha\beta}^{21} & \mathbf{M}_{\alpha\beta}^{22} & \dots & \mathbf{M}_{\alpha\beta}^{2N_{\text{obs}}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}_{\alpha\beta}^{N_{\text{obs}}1} & \mathbf{M}_{\alpha\beta}^{N_{\text{obs}}2} & \dots & \mathbf{M}_{\alpha\beta}^{N_{\text{obs}}N_{\text{obs}}} \end{pmatrix}$$

$$= \mathbf{M}_{\alpha\beta}(\delta, \mathbf{d}_{jk})$$

# Vectorial formulation

- Variational formulation: Find  $\mathbf{p}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{div}_{\Gamma_\delta^k}, \Gamma_\delta^k)$  such that

$$\sum_{j=1}^{N_{\text{obs}}} \langle \mathcal{M}_{\Gamma}^{kj} \mathbf{p}_j, \mathbf{v}_k \rangle_{\Gamma_\delta^k} = -\langle \mathbf{n} \times \mathbf{E}^{\text{inc}}, \mathbf{v}_k \rangle_{\Gamma_\delta^k} \quad \forall \mathbf{v}_k \in \mathbf{H}_t^{-\frac{1}{2}}(\text{curl}_{\Gamma_\delta^k}, \Gamma_\delta^k)$$

- Vectorial formulation: Find  $\mathbf{p} = ((\rho_{n,m}^{1,\perp}), \dots, (\rho_{n,m}^{N_{\text{obs}},\perp}), (\rho_{n,m}^{1,\times}), \dots, (\rho_{n,m}^{N_{\text{obs}},\times}))^\top \in \mathbb{C}^{\mathbf{N}}$  s.t.

$$\mathbb{M} \mathbf{p} = \mathbf{f}$$



Computation time  
Ill-conditioned system  
Memory consumption

- Numerical integration (Gauss-Lobatto)
- Large number of unknowns
- Dense matrix



Mex interface (C++)  
Preconditioning (system & matrix)  
Smart storage and assembling

- Code speed-up
- Linear algebra tools
- Sub-blocks  $\mathbb{M}_{\alpha\beta}^{kj}$  depend on  $\mathbf{d}_{jk}$

# Preconditioning

Linear system: **Change of variable** + **Permutation** + **Normalization**

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta) \quad \text{with } \mathbb{M} \mathbf{p}_\delta = \mathbf{f}$$



$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}_\delta) \quad \text{with } \tilde{\mathbb{M}} \tilde{\mathbf{p}}_\delta = \tilde{\mathbf{f}}$$



Dense matrix  $\tilde{\mathbb{M}}$



Error of approximation  
Analytic preconditionner (e.g. dipole)  
Algebraic preconditionner (small criterion)

# Preconditioning

Linear system: Change of variable + Permutation + Normalization

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta) \quad \text{with } \mathbb{M} \mathbf{p}_\delta = \mathbf{f}$$



$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}_\delta) \quad \text{with } \tilde{\mathbb{M}} \tilde{\mathbf{p}}_\delta = \tilde{\mathbf{f}}$$



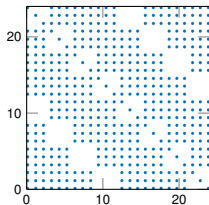
Dense matrix  $\tilde{\mathbb{M}}$



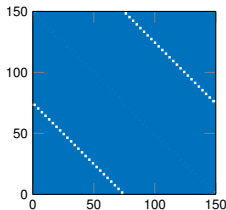
Error of approximation

Analytic preconditionner (e.g. dipole)

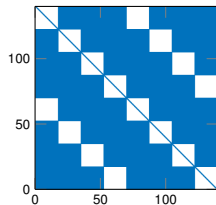
Algebraic preconditionner (small criterion)



(a)  $N_{\text{obs}} = 4$  and  $N_{\text{mod}} = 1$



(b)  $N_{\text{obs}} = 25$  and  $N_{\text{mod}} = 1$



(c)  $N_{\text{obs}} = 4$  and  $N_{\text{mod}} = 5$

Figure: Skeleton of  $\tilde{\mathbb{M}}$

# Preconditioning

Linear system: Change of variable + Permutation + Normalization

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta) \quad \text{with } \mathbb{M} \mathbf{p}_\delta = \mathbf{f}$$



$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}_\delta) \quad \text{with } \tilde{\mathbb{M}} \tilde{\mathbf{p}}_\delta = \tilde{\mathbf{f}}$$



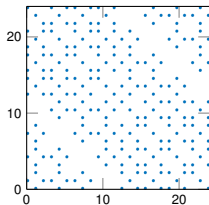
Dense matrix  $\tilde{\mathbb{M}}$



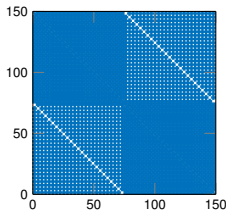
Error of approximation

Analytic preconditionner (e.g. dipole)

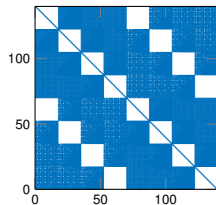
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Figure: Skeleton of  $\tilde{\mathbb{M}}$



# Preconditioning

Linear system: Change of variable + Permutation + Normalization

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta) \quad \text{with } \mathbb{M} \mathbf{p}_\delta = \mathbf{f}$$



$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}_\delta) \quad \text{with } \tilde{\mathbb{M}} \tilde{\mathbf{p}}_\delta = \tilde{\mathbf{f}}$$



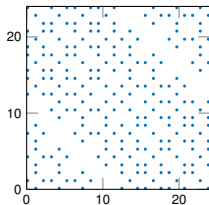
Dense matrix  $\tilde{\mathbb{M}}$



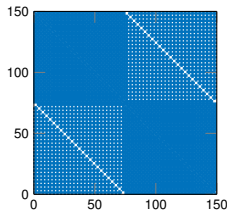
Error of approximation

Analytic preconditionner (e.g. dipole)

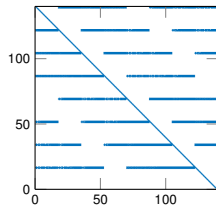
Algebraic preconditionner (small criterion)



(a)  $N_{\text{obs}} = 4$  and  $N_{\text{mod}} = 1$



(b)  $N_{\text{obs}} = 25$  and  $N_{\text{mod}} = 1$



(c)  $N_{\text{obs}} = 4$  and  $N_{\text{mod}} = 5$

Figure: Skeleton of  $\text{Precond}(\tilde{\mathbb{M}})$

# Preconditioning

Linear system: Change of variable + Permutation + Normalization

$$\mathbf{E}_\delta = \phi_\delta(\mathbf{p}_\delta) \quad \text{with } \mathbb{M} \mathbf{p}_\delta = \mathbf{f}$$



$$\mathbf{E}_\delta = \phi(\tilde{\mathbf{p}}_\delta) \quad \text{with } \tilde{\mathbb{M}} \tilde{\mathbf{p}}_\delta = \tilde{\mathbf{f}}$$



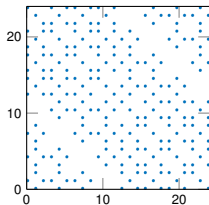
Dense matrix  $\tilde{\mathbb{M}}$



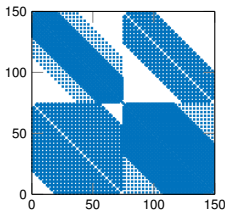
Error of approximation

Analytic preconditionner (e.g. dipole)

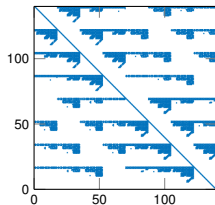
Algebraic preconditionner (small criterion)



(a)  $N_{\text{obs}} = 4$  and  $N_{\text{mod}} = 1$



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(c)  $N_{\text{obs}} = 4$  and  $N_{\text{mod}} = 5$

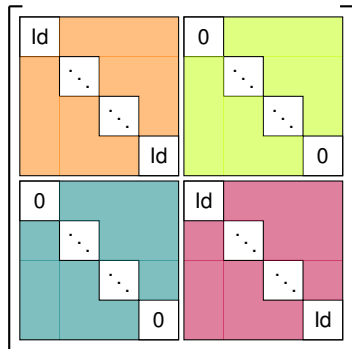
Figure: Skeleton of  $\text{Precond}(\tilde{\mathbb{M}})$

# Smart storage and assembling

- Example: 4 aligned obstacles



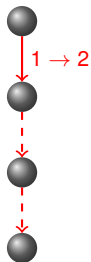
$$\tilde{\mathbf{M}} =$$



The sub-blocks  $M_{\alpha\beta}^{kj}$  depend only on  $\delta$  and  $\mathbf{d}_{jk}$

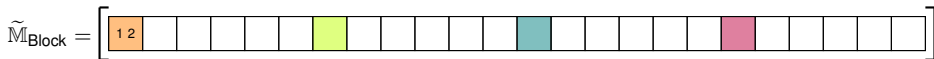
# Smart storage and assembling

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} = \begin{bmatrix} \text{Id} & \begin{smallmatrix} 1 & -2 \\ & \ddots \\ & & 1 \end{smallmatrix} & & \\ & & \begin{smallmatrix} 1 & -2 \\ & \ddots \\ & & 1 \end{smallmatrix} & \\ & & & \text{Id} \\ & & & & 0 \\ 0 & \begin{smallmatrix} 1 & -2 \\ & \ddots \\ & & 1 \end{smallmatrix} & & \\ & & \begin{smallmatrix} 1 & -2 \\ & \ddots \\ & & 1 \end{smallmatrix} & \\ & & & & 0 \\ & & & & & \text{Id} \end{bmatrix}$$

- Storage of sub-blocks

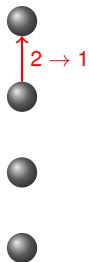






# Smart storage and assembling

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} =$$

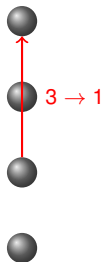
Id	1 - 2	1 - 3	1 - 4	0			
2 - 1	⋮				⋮		
		⋮				⋮	
			Id				0
0				Id			
	⋮				⋮		
		⋮				⋮	
			0				Id

- Storage of sub-blocks



# Smart storage and assembling

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} = \begin{bmatrix} \text{Id} & \begin{matrix} 1 & - & 2 & 1 & - & 3 & 1 & - & 4 \end{matrix} & 0 & \begin{matrix} \text{green} & \text{yellow} & \text{green} & \text{yellow} & \text{green} & \text{yellow} & \text{green} & \text{yellow} \end{matrix} \\ \begin{matrix} 2 & - & 1 \end{matrix} & \begin{matrix} \text{orange} & \text{orange} & \text{orange} & \text{orange} \end{matrix} & \begin{matrix} \text{green} & \text{yellow} & \text{green} & \text{yellow} \end{matrix} & \begin{matrix} \text{green} & \text{yellow} & \text{green} & \text{yellow} \end{matrix} \\ \begin{matrix} 3 & - & 1 \end{matrix} & \begin{matrix} \text{orange} & \text{orange} & \text{orange} & \text{orange} \end{matrix} & \begin{matrix} \text{green} & \text{yellow} & \text{green} & \text{yellow} \end{matrix} & \begin{matrix} \text{green} & \text{yellow} & \text{green} & \text{yellow} \end{matrix} \\ & \text{Id} & 0 & \begin{matrix} \text{green} & \text{yellow} & \text{green} & \text{yellow} \end{matrix} \\ 0 & \begin{matrix} \text{teal} & \text{teal} & \text{teal} & \text{blue} \end{matrix} & \text{Id} & \begin{matrix} \text{pink} & \text{red} & \text{purple} & \text{pink} \end{matrix} \\ \begin{matrix} \text{light blue} & \text{light blue} & \text{light blue} & \text{light blue} \end{matrix} & \begin{matrix} \text{teal} & \text{teal} & \text{teal} & \text{blue} \end{matrix} & \begin{matrix} \text{pink} & \text{red} & \text{purple} & \text{pink} \end{matrix} & \begin{matrix} \text{pink} & \text{red} & \text{purple} & \text{pink} \end{matrix} \\ \begin{matrix} \text{light blue} & \text{light blue} & \text{light blue} & \text{light blue} \end{matrix} & \begin{matrix} \text{teal} & \text{teal} & \text{teal} & \text{blue} \end{matrix} & \begin{matrix} \text{pink} & \text{red} & \text{purple} & \text{pink} \end{matrix} & \begin{matrix} \text{pink} & \text{red} & \text{purple} & \text{pink} \end{matrix} \\ & 0 & \text{Id} & \begin{matrix} \text{pink} & \text{red} & \text{purple} & \text{pink} \end{matrix} \end{bmatrix}$$

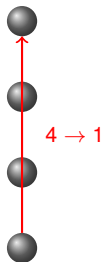
- Storage of sub-blocks

$$\tilde{\mathbf{M}}_{\text{Block}} = \begin{bmatrix} \begin{matrix} 1 & 2 & 1 & 3 & 1 & 4 & 2 & 1 & 3 & 1 \end{matrix} & \begin{matrix} \text{green} & \text{yellow} & \text{green} & \text{yellow} & \text{green} & \text{yellow} & \text{green} & \text{yellow} \end{matrix} & \begin{matrix} \text{teal} & \text{teal} & \text{teal} & \text{blue} & \text{light blue} & \text{light blue} & \text{light blue} & \text{light blue} \end{matrix} & \begin{matrix} \text{pink} & \text{red} & \text{purple} & \text{pink} & \text{pink} & \text{red} & \text{purple} & \text{pink} \end{matrix} \end{bmatrix}$$



# Smart storage and assembling

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} =$$

Id	1 - 2	1 - 3	1 - 4	0			
2 - 1	.	.	.		.	.	.
3 - 1	.	.	.		.	.	.
4 - 1	.	.	.	Id			0
0				Id			
	.	.	.		.	.	.
	.	.	.		.	.	.
	.	.	.	0			Id

- Storage of sub-blocks



# Smart storage and assembling

- Example: 4 aligned obstacles



$$\tilde{\mathbf{M}} = \begin{bmatrix} \text{Id} & \begin{matrix} 1-2 & 1-3 & 1-4 \end{matrix} & 0 & \begin{matrix} \text{green} & \text{yellow} & \text{dark green} \end{matrix} \\ \begin{matrix} 2-1 \\ 3-1 \\ 4-1 \end{matrix} & \begin{matrix} \text{orange} & \text{orange} \\ \text{pink} & \text{pink} \\ \text{pink} & \text{pink} \end{matrix} & \begin{matrix} \text{green} \\ \text{green} \\ \text{green} \end{matrix} & \begin{matrix} \text{yellow} \\ \text{yellow} \\ \text{yellow} \end{matrix} \\ 0 & \begin{matrix} \text{teal} & \text{teal} & \text{blue} \end{matrix} & \begin{matrix} \text{olive} & \text{olive} & \text{green} & 0 \end{matrix} \\ \begin{matrix} 0 \\ \text{light blue} \\ \text{light blue} \\ \text{light blue} \end{matrix} & \begin{matrix} \text{teal} & \text{teal} & \text{teal} \\ \text{light blue} & \text{light blue} & \text{light blue} \\ \text{light blue} & \text{light blue} & \text{light blue} \\ \text{light blue} & \text{light blue} & \text{light blue} \end{matrix} & \begin{matrix} \text{pink} & \text{red} & \text{purple} \\ \text{pink} & \text{pink} & \text{pink} \\ \text{pink} & \text{purple} & \text{pink} \\ \text{pink} & \text{purple} & \text{pink} \end{matrix} & \begin{matrix} \text{Id} \\ \text{Id} \\ \text{Id} \\ \text{Id} \end{matrix} \end{bmatrix}$$

- Storage of sub-blocks

$$\tilde{\mathbf{M}}_{\text{Block}} = \begin{bmatrix} \begin{matrix} 1-2 & 1-3 & 1-4 & 2-1 & 3-1 & 4-1 & \text{green} & \text{yellow} & \text{dark green} & \text{green} & \text{olive} & \text{olive} & \text{teal} & \text{teal} & \text{blue} & \text{light blue} & \text{light blue} & \text{light blue} & \text{pink} & \text{red} & \text{purple} & \text{pink} & \text{pink} & \text{pink} & \text{pink} \end{matrix} \end{bmatrix}$$

- Iterative solvers: define the **action** of  $\tilde{\mathbf{M}}_{\text{Block}}$  on  $\mathbf{u}$

$$\mathcal{A}(\tilde{\mathbf{M}}_{\text{Block}}, \mathbf{u}) := \tilde{\mathbf{M}} \mathbf{u}$$

# Smart storage and assembling

- Example: 4 aligned obstacles



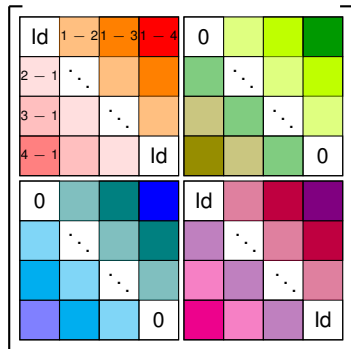
$8(N_{\text{obs}} - 1)$  blocks



instead of  $4(N_{\text{obs}})^2$



$$\tilde{\mathbf{M}} =$$



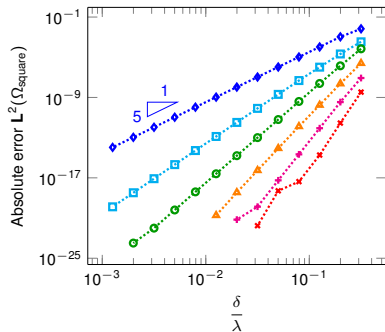
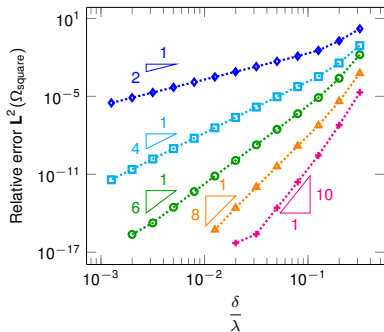
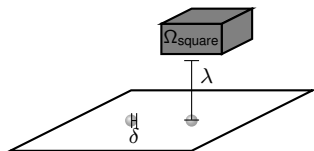
- Storage of sub-blocks



$N_{\text{obs}}$	$4(N_{\text{obs}})^2$	$8(N_{\text{obs}} - 1)$	Ratio (%)	$\tilde{\mathbf{M}}$ (GB)	$\tilde{\mathbf{M}}_{\text{Block}}$ (GB)
100	40 000	792	1.98	$5.76 \cdot 10^{-3}$	$1.15 \cdot 10^{-4}$
1 000	4 000 000	7 992	$1.998 \cdot 10^{-3}$	$5.76 \cdot 10^{-1}$	$1.15 \cdot 10^{-3}$
3 000	36 000 000	23 992	$6.664 \cdot 10^{-4}$	4.01	$3.45 \cdot 10^{-3}$
10 000	400 000 000	79 992	$1.9998 \cdot 10^{-4}$	--	$1.15 \cdot 10^{-2}$

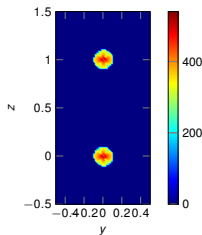
# Numerical validation of spectral solutions

- Incident field: Electromagnetic plane wave along z-axis polarized by  $\mathbf{e}_x$
- Reference solution: Spectral solution with  $N_{\text{mod}} = 10$

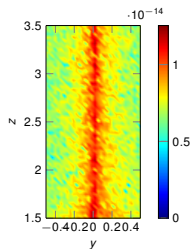


# Comparison with asymptotic models

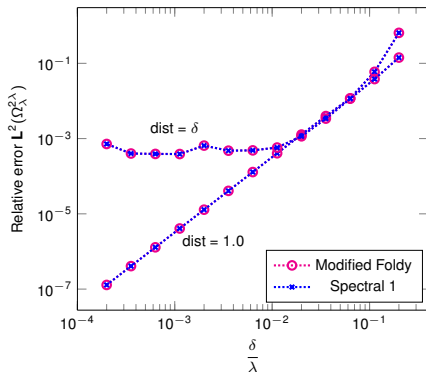
Difference between Spectral 1 and modified Foldy



(c) Close to the obstacles



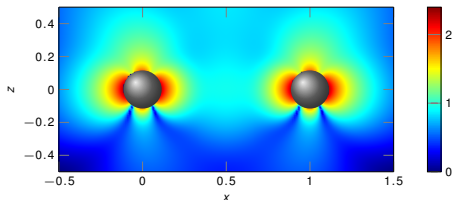
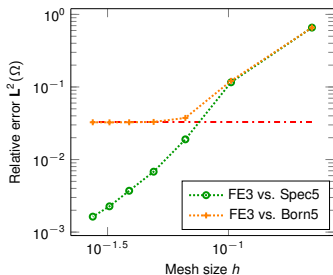
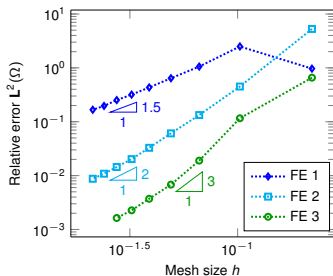
(d) In the far-field domain



	Modified Foldy				Spectral 1			
$N_{\text{obs}}$	100	1000	5000	10000	100	1000	5000	10000
<b>Linear system</b>	0.17	17.71	34.01	43.49	2.66	40.74	352.47	--
<b>Post-processing</b>	5.25	47.84	254.34	459.90	12.17	126.81	> 3600	--

Table: Time comparison in seconds

# Comparison with finite element solutions (Montjoie)



Spectral 3			
$N_{\text{obs}}$	1	2	4
Size of linear system	30	60	120
$L^2$ -Accuracy ( $\times 10^{-6}$ )	4.77	4.48	3.66
<b>Time</b> (s)	15.46	65.55	262.33
<b>Memory</b> (GB)	0.3	2.36	9.3

Finite Element 3			
$N_{\text{obs}}$	1	2	4
Size of linear system	925 422	1 972 800	3 942 936
$L^2$ -Accuracy ( $\times 10^{-3}$ )	0.28	2.27	3.31
<b>Time</b> (min)	26.5	54.25	147.25
<b>Memory</b> (GB)	21.11	36.97	75

Table: Computational costs

# Outline

## 1. Asymptotic models

- Single-scattering
- Application: Born approximation
- Multiple-scattering: Foldy-Lax model
- Numerical results

## 2. Spectral method: Spherical case

- Discretization
- Numerical convergence
- Comparison with asymptotic models
- Comparison with finite element solutions

## 3. Conclusions and perspectives

# Conclusions and Perspectives

## Conclusion

- ✓ Derivation and numerical validation of the matched asymptotic expansions
- ✓ Extension to Born and Foldy-Lax models
- ✓ Derivation and implementation at any order spectral method + Numerical validation

## On-going work

- ~ Comparison of preconditionners and iterative solvers
- ~ Smart assembling for particular configurations (plane, cubic volume)
- ~ Contribution to mathematical justification of the asymptotic expansions

## Perspectives

- ✗ Existence of electromagnetic centers?
- ✗ Extension to obstacles of arbitrary shape
- ✗ Extension to time-dependent domain



Thank you for your attention