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Solutions to inverse moment estimation problems in dimension 2, using best constrained approximation

Juliette Leblond*

Elodie Pozzi†

Abstract

We study an inverse problem that consists in estimating the first (zero-order) moment of some \mathbb{R}^2 -valued distribution \mathbf{m} supported within a closed interval $S \subset \mathbb{R}$, from partial knowledge of the solution to the Poisson-Laplace partial differential equation with source term equal to the divergence of \mathbf{m} on another interval parallel to and located at some distance from S . Such a question coincides with a 2D version of an inverse magnetic “net” moment recovery question that arises in paleomagnetism, for thin rock samples. We formulate and constructively solve a best approximation problem under constraint in L^2 and in Sobolev spaces involving the restriction of the Poisson extension of the divergence of \mathbf{m} . Numerical results obtained from the described algorithms for the net moment approximation are also furnished.

Keywords: Poisson-Laplace partial differential equation, inverse problems, regularization, best constrained approximation and bounded extremal problems, Poisson and Hilbert transforms, Hardy spaces, Maxwell’s equations and issues.

1 Introduction

The present study concerns the practical issue of estimating the “net” moment (or the mean value) of some \mathbb{R}^2 -valued distribution (or integrable, or square-integrable function) supported on an interval of the real line, from the partial knowledge of the divergence of its Poisson extension to the upper half-plane, on some interval located on a parallel line at distance $h > 0$ from the first one.

This is a two dimensional (2D) formulation of a three dimensional (3D) inverse magnetic moment recovery problem in paleomagnetism, for thin rock samples. There, the magnetization is assumed to be some \mathbb{R}^3 -valued distribution (or integrable function) supported on a planar sample (square), of which we aim at estimating the net moment, given measurements of the normal component of the generated magnetic field on another planar measurements set taken to be a square parallel to the sample and located at some distance h to it. Moment and more general magnetization recovery issues are considered in [5, 6, 7]. More about the data acquisition process and the use of scanning SQUID (Superconducting Quantum Interference Device) microscopy devices for measuring a component of the magnetic field produced by weakly magnetized pieces of rocks can be found in the introductory sections of these references.

In both situations, the partial differential equation (PDE) model that drives the behaviour of the magnetic field derives from Maxwell equations in magnetostatics, see [15]. They ensure that the magnetic field \mathbf{B} derives from a scalar magnetic potential U which is solution to

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a Poisson-Laplace elliptic PDE with right-hand side of divergence form, that relates the Laplacian of that potential to the divergence of the magnetization distribution \mathbf{m} :

$$\mathbf{B} = -\mu_0 \text{grad } U, \quad \Delta U = \text{div } \mathbf{m} \text{ in } \mathbb{R}^n,$$

with $n = 2, 3$, where both \mathbf{B} , \mathbf{m} are \mathbb{R}^n -valued quantities, and μ_0 is the permeability of the free space. We stick to situations where \mathbf{m} has a compact support contained in the hyperplane $\{x_n = 0\}$ and measurements of the vertical component b_n of the field \mathbf{B} are available on another compact subset of $\{x_n = h\}$ for some $h > 0$.

The determination of the magnetic moment (mean value of the magnetization) provides useful preliminary information for the full inversion, i.e. for magnetization estimation, in particular in unidirectional situations where its components are proportional one to the others (the unknown quantity being a \mathbb{R} -valued distribution, the direction/orientation vector being fixed).

In 3D situation, these issues were efficiently analyzed with tools from harmonic analysis, specifically Poisson kernel and Riesz transforms [27], in [5, 6, 7], see also references therein, using their links with Hardy spaces of harmonic gradients. The existence of silent sources in 3D, elements of the non-zero kernel of the non injective magnetization-to-field operator (the operator that maps the magnetization to the measured component of the magnetic field) was established together with their characterization in [5]. This makes non unique the solution to the inverse magnetization issue from the corresponding field data. The mean value of the magnetization however is uniquely determined by the field data.

In the present 2D case, we will use similar tools, Poisson kernel, Hilbert transform and harmonic conjugation, links with Hardy spaces of functions of the complex variable [13, 14, 23]; see also [21, 25] for related issues. Considering square-summable magnetizations \mathbf{m} supported on an interval S of the real line, we will establish that the magnetization-to-field operator $\mathbf{m} \mapsto b_2[\mathbf{m}]$ is injective, whence there are no silent sources, and that the mean value $\langle \mathbf{m} \rangle$ of the magnetization is yet uniquely determined by the field data $b_2[\mathbf{m}]$, provided by values of the vertical component on the measurement interval K .

Our purpose is to establish the existence and to build linear estimators for the mean values $\langle m_i \rangle$ on S of the components m_i of \mathbf{m} for $i = 1, 2$, as was done in [6] for the 3D case. Indeed, though an academic version of the related physical issue, the present 2D situation possesses its own mathematical interest and specificity. These linear estimators consist in square summable functions ϕ_i such that their effect (scalar product) against the data on K is as close as possible to the moment components $\langle m_i \rangle$, for all \mathbf{m} bounded by some L^2 norm (the functions being built once and for all). With $\mathbf{e}_1 = (\chi_S, 0)$, $\mathbf{e}_2 = (0, \chi_S)$, observe that: $\langle m_i \rangle = \langle \mathbf{m}, \mathbf{e}_i \rangle_{L^2(S, \mathbb{R}^2)}$, whence, if b_2^* denotes the adjoint operator to b_2 , we have:

$$|\langle b_2[\mathbf{m}], \phi_i \rangle_{L^2(K)} - \langle m_i \rangle| \leq \|\mathbf{m}\|_{L^2(S, \mathbb{R}^2)} \|b_2^*[\phi_i] - \mathbf{e}_i\|_{L^2(S, \mathbb{R}^2)},$$

where ϕ_i is a square integrable function on K . We will see that the above minimization problem is still ill-posed, even if uniqueness is granted. Specifically, there exist a sequence of functions such that their scalar product on K against $b_2[\mathbf{m}]$ converges to $\langle m_i \rangle$ as $n \rightarrow \infty$. Their quadratic norm however diverge, which reflects an unstable behaviour, as is classical in such inverse problems, see [1]. Regularization is thus needed (of Tykhonov type), see [20] in order to set up and to solve a well-posed problem.

In order to construct such a numerical magnetometer, we then face the best constrained approximation issue (bounded extremal problem, BEP) of finding the function ϕ_i on K satisfying some norm constraint there, such that its scalar product against $b_2[\mathbf{m}]$ is as close as possible to $\langle m_i \rangle$.

Such a norm constraint will be considered both in the Lebesgue space $L^2(K)$ and in the Sobolev space $W_0^{1,2}(K) \subset W^{1,2}(K)$. In those Hilbert spaces, the above problems will be shown to be well-posed, the approximation subsets being closed and convex. The results obtained in $L^2(K)$ and in $W_0^{1,2}(K)$ are different. In particular, the solutions in $W_0^{1,2}(K)$ demonstrate less oscillations at the endpoints of K than the ones in $L^2(K)$, and allow

to incorporate vanishing boundary conditions. The results for moment estimation will be compared between them.

Preliminary numerical computations in the 3D setting, with planar squared support and measurement set, were run on appropriate finite elements bases, see [6]; they are actually heavy. In the present 2D setting and on intervals, they are of course much less costly and we perform some of them using expansions on the Fourier bases, in Lebesgue and Sobolev spaces.

The overview of the present work is as follows. In Section 2, we introduce some notation, recall definitions and properties, concerning Lebesgue, Sobolev and Hardy spaces of functions, together with integral Poisson and Hilbert transforms. We then study in Section 3 the properties of the operator $b_2 : \mathbf{m} \rightarrow b_2[\mathbf{m}]$, which maps the magnetization to the second (vertical) component of the produced magnetic field, and of its adjoint. Section 4 is devoted to the bounded extremal problems of which we consider two versions, in $L^2(K)$ and in $W_0^{1,2}(K)$, establishing their well-posedness and characterizing their solutions. Computational algorithms together with results of preliminary numerical simulations are provided in Section 5, with figures in an appendix. We finally discuss in Section 6 some concluding remarks and perspectives.

2 Preliminaries

Let $\mathbb{C}_+ \simeq \mathbb{R}_+^2$ be the upper half-plane $\mathbb{C}_+ = \{x + iy : y > 0\} \subset \mathbb{C}$, $\mathbb{R}_+^2 = \{(x, y) : y > 0\} \subset \mathbb{R}^2$. The partial derivatives with respect to x and y will be denoted by ∂_x and ∂_y respectively.

2.1 Lebesgue and Sobolev spaces

Let $I \subseteq \mathbb{R}$ a non empty open interval. For $1 \leq p < \infty$, we will denote by $L^p(I, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or \mathbb{R}^2 the space of \mathbb{K} -valued functions f such that $|f|^p$ is integrable on I ; we will simply write $L^p(I) = L^p(I, \mathbb{R})$.

We equip the space $L^2(I, \mathbb{R}^2)$ with the following inner product: for $f = (f_1, f_2)$ and $g = (g_1, g_2) \in L^2(I, \mathbb{R}^2)$,

$$\langle f, g \rangle_{L^2(I, \mathbb{R}^2)} = \langle (f_1, f_2), (g_1, g_2) \rangle_{L^2(I, \mathbb{R}^2)} = \langle f_1, g_1 \rangle_{L^2(I)} + \langle f_2, g_2 \rangle_{L^2(I)},$$

while $L^2(I, \mathbb{C})$ is equipped with:

$$\langle f, g \rangle_{L^2(I, \mathbb{C})} = \langle \operatorname{Re} f, \operatorname{Re} g \rangle_{L^2(I)} + \langle \operatorname{Im} f, \operatorname{Im} g \rangle_{L^2(I)} \text{ for } f, g \in L^2(I, \mathbb{C}). \quad (1)$$

For $I \subsetneq \mathbb{R}$, for $\phi \in L^2(I)$, we will write $\widetilde{\phi} = \phi \vee 0 \in L^2(\mathbb{R})$, the function equal to ϕ on I and equal to 0 on $\mathbb{R} \setminus I$. Note that we identify $\widetilde{L^2(I)}$ to $\chi_I L^2(\mathbb{R})$ and to the subspace of $L^2(\mathbb{R})$ of functions supported on I . We also have $\widetilde{\phi}|_I = R_I \widetilde{\phi} = \phi$ for $\phi \in L^2(I)$ and the adjoint of the restriction operator R_I from $L^2(\mathbb{R})$ to $L^2(I)$ is the extension operator $\widetilde{\cdot}$ on $L^2(I)$.

When I is bounded, $\widetilde{L^2(I)} \subset L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Moreover, functions in $\widetilde{L^2(I)}$ are compactly supported on \mathbb{R} .

For $I \subseteq \mathbb{R}$, the Sobolev space $W^{1,2}(I)$ is the space of real-valued functions $f \in L^2(I)$ such that their (distributional) derivative f' belongs to $L^2(I)$; $W^{1,2}(I)$ is equipped with the norm (see [9, Sec. 8.2]):

$$\|f\|_{W^{1,2}(I)} = \|f\|_{L^2(I)} + \|f'\|_{L^2(I)}, \quad f \in W^{1,2}(I).$$

A function in $W^{1,2}(I)$ can be extended to a function in $W^{1,2}(\mathbb{R})$, see the extension operator in [9, Thm 8.6]. The space $C(\bar{I})$ denotes the space of continuous functions on I with the norm

$$\|f\|_{C(\bar{I})} = \sup_{x \in \bar{I}} |f(x)|, \quad f \in C(\bar{I}).$$

Note that $W^{1,2}(I) \subset C(\bar{I})$, the space of continuous real-valued functions on \bar{I} , with compact (hence continuous) injection, see [9, Thm 8.5]. If I is bounded, $C^1(\bar{I}) \subset W^{1,2}(I)$ will denote

the space of functions in $C(\bar{I})$ with (strong) derivative in $C(\bar{I})$. The space is equipped with the norm

$$\|f\|_{C^1(\bar{I})} = \sup_{x \in \bar{I}} |f(x)| + \sup_{x \in \bar{I}} |f'(x)|, \quad f \in C^1(\bar{I}),$$

where the (strong) derivative coincides with the distributional derivative. The space $C^\infty(\mathbb{R})$ will denote the space of functions such that derivatives at all order are continuous. The subspace $W_0^{1,2}(I)$ of $W^{1,2}(I)$ is the collection of functions $f \in W^{1,2}(I)$ such that $f = 0$ at the endpoints of I . A consequence is that $\tilde{f} \in W^{1,2}(\mathbb{R})$ for $f \in W_0^{1,2}(I)$; in other words, $\tilde{\cdot}$ is the extension operator from $W_0^{1,2}(I)$ to $W^{1,2}(\mathbb{R})$ (see [9, Sec 8.3, Remark 16]). We equip $W_0^{1,2}(K)$ with the norm:

$$\|\phi\|_{W_0^{1,2}(K)} = \|\phi'\|_{L^2(K)}, \quad \phi \in W_0^{1,2}(K),$$

which is equivalent to $\|\phi\|_{W^{1,2}(K)}$ by Poincaré's inequality (see [9, Prop 8.13]).

2.2 Poisson kernel, conjugate Poisson kernel and Hilbert transform

For $y > 0$, we denote by P_y the Poisson kernel of the upper half-plane and by Q_y the conjugate Poisson kernel. For $x \in \mathbb{R}$:

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \quad Q_y(x) = \frac{1}{\pi} \frac{x}{x^2 + y^2}.$$

see [13, 14, 27]. Let $x \in \mathbb{R}$, $y > 0$. Then

$$P_y \star u(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} u(t) dt, \quad u \in L^2(\mathbb{R}),$$

and $\mathcal{P}_y : u \mapsto P_y \star u$ is a bounded (real linear) operator from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ ([27, Thm 1.3]).

The map, $x \mapsto P_y(x)$ belongs to $L^1(\mathbb{R})$ (where its norm is equal to 1). Moreover, $(x, y) \mapsto P_y(x)$ belongs to $C^\infty(\mathbb{R}_+^2)$. Therefore, its partial derivatives are also $C^\infty(\mathbb{R}_+^2)$ -smooth functions, whence for $y > 0$, $x \mapsto (\partial_x P_y)(x)$ and $x \mapsto (\partial_y P_y)(x)$ belong to $C^\infty(\mathbb{R})$. The functions P_y and Q_y satisfy the Cauchy-Riemann equations on \mathbb{C}_+ : $\partial_y P_y = -\partial_x Q_y$, $\partial_x P_y = \partial_y Q_y$.

It can be checked by direct computations that $x \mapsto (\partial_x P_y)(x)$ and $x \mapsto (\partial_y P_y)(x)$ belong to $L^1(\mathbb{R})$, and $\|\partial_x P_y\|_{L^1(\mathbb{R})} = \|\partial_y P_y\|_{L^1(\mathbb{R})} = 2/(\pi y)$. Hence, $u \mapsto (\partial_x P_y) \star u$ and $u \mapsto (\partial_y P_y) \star u$ are bounded operators from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$. Further, for $u \in L^2(\mathbb{R})$, $(x, y) \mapsto \mathcal{P}_y[u](x) = (P_y \star u)(x)$ is also harmonic in \mathbb{R}_+^2 , hence a $C^\infty(\mathbb{R}_+^2)$ -smooth function. Thus, its partial derivatives $x \mapsto \partial_x (P_y \star u)(x)$ and $x \mapsto \partial_y (P_y \star u)(x)$ belong to $C^\infty(\mathbb{R})$, for $y > 0$.

Using the Fourier transform as in [7, Sec. 2.2], we see that convolution by the Poisson kernel and differentiation commute: for $y > 0$ and $u \in L^2(\mathbb{R})$, we have that $\partial_x (P_y \star u) = (\partial_x P_y) \star u$. In particular, the operator \mathcal{P}_y maps continuously $L^2(\mathbb{R})$ onto $W^{1,2}(\mathbb{R})$. By Cauchy-Riemann equations, Q_y satisfies the properties of P_y described above. Indeed, $u \mapsto \partial_x (Q_y \star u)$ and $u \mapsto \partial_y (Q_y \star u)$ are bounded on $L^2(\mathbb{R})$ and for $y > 0$, $x \mapsto \partial_x (Q_y \star u)$ and $x \mapsto \partial_y (Q_y \star u)$ are $C^\infty(\mathbb{R})$.

In order to simplify the notations, we may not use any parentheses in expressions like $\partial_x P_y \star u$, $\partial_y P_y \star u$, $\partial_x Q_y \star u$ and $\partial_y Q_y \star u$ for $u \in L^2(\mathbb{R})$ and $y > 0$.

Let $I, J \subsetneq \mathbb{R}$ be non empty open bounded interval. Then, for $y > 0$, $R_I \mathcal{P}_y$ is a Hilbert-Schmidt operator on $L^2(\widetilde{J, \mathbb{C}}) \subset L^2(\mathbb{R}, \mathbb{C})$ (see [18, p. 264, Ex. 2.19]). Indeed, for $u \in L^2(J)$, we have

$$R_I \mathcal{P}_y[u](x) = R_I (P_y \star u)(x) = \frac{1}{\pi} \int_J \frac{h}{(x-t)^2 + y^2} u(t) dt, \quad x \in I,$$

and $R_I \mathcal{P}_y$ is an integral operator on $L^2(\widetilde{J, \mathbb{C}})$ with kernel $k(x, t) = P_y(x - t)$, $k \in L^2(I \times J)$, since $P_y(x)$ is a smooth function for $y > 0$. Thus, it is compact. The same conclusion holds for the operator $u \mapsto R_I (\partial_x P_y \star u)$ (and $u \mapsto R_I (\partial_y P_y \star u)$) on $L^2(\widetilde{J, \mathbb{C}})$.

For $u \in L^2(\mathbb{R})$, the functions $\mathcal{P}_y[u] = P_y \star u$ and $Q_y \star u$ have non-tangential limits as y tends to zero and

$$\lim_{y \rightarrow 0} P_y \star u(x) = u(x) \quad \text{and} \quad \lim_{y \rightarrow 0} Q_y \star u(x) = \mathcal{H}u(x), \quad (2)$$

where \mathcal{H} is the Hilbert transform from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ defined by

$$\mathcal{H}u(x) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|x-t|>\varepsilon} \frac{u(t)}{x-t} dt.$$

The Hilbert transform is bounded and isometric on $L^2(\mathbb{R})$ and satisfies $\mathcal{H}^2 = -I$, whence its adjoint $\mathcal{H}^* = -\mathcal{H}$. The Hilbert transform commutes with the Poisson operator (which can be proved using the Fourier transform):

$$\mathcal{H}(P_y \star u) = \mathcal{H}(\mathcal{P}_y[u]) = P_y \star \mathcal{H}u = \mathcal{P}_y[\mathcal{H}u]. \quad (3)$$

For $y > 0$, $x \in \mathbb{R}$ and $u \in L^2(\mathbb{R})$, one can prove using [23, Thm 11.6] that:

$$Q_y \star u = P_y \star \mathcal{H}u = \mathcal{P}_y[\mathcal{H}u]. \quad (4)$$

Hence, from (3) and (4), we get that $Q_y = \mathcal{H}P_y$ on $L^2(\mathbb{R})$. We will also use in the sequel the following equality, for $g \in L^2(\mathbb{R})$:

$$\partial_y \mathcal{P}_y[g] = -\mathcal{H}(\partial_x \mathcal{P}_y[g]). \quad (5)$$

It can be obtained using the Fourier transform as in [7, 27]. Actually, for $g \in L^2(\mathbb{R})$, $\partial_y \mathcal{P}_y[g] = -[\partial_x Q_y \star g]$, and $\partial_x \mathcal{P}_y[g] = \partial_y Q_y \star g$. And as a consequence of (5), and the isometric character of \mathcal{H} , we get $\|\partial_y \mathcal{P}_y[g]\|_{L^2(\mathbb{R})} = \|\partial_x \mathcal{P}_y[g]\|_{L^2(\mathbb{R})}$.

We also have the following uniqueness result, from [7, Lem. 2.19]:

Lemma 1 *Let $h > 0$ and $g \in L^2(\mathbb{R})$ such that $\partial_x \mathcal{P}_h[g] = 0$ on a non-empty open subset of \mathbb{R} . Then $g \equiv 0$ on \mathbb{R} .*

As a corollary to Lemma 1, we get that if $g \in L^2(\mathbb{R})$ is such that $\mathcal{P}_h[g] = P_h \star g = 0$ on a non-empty open subset of \mathbb{R} for some $h > 0$, then $g \equiv 0$ on \mathbb{R} .

2.3 Hardy space H^2

For $u \in L^2(\mathbb{R})$, the complex-valued function $P_y \star u + iQ_y \star u$ belongs to $H^2(\mathbb{C}_+) = H^2(\mathbb{C}_+, \mathbb{C})$, the space of analytic functions F on \mathbb{C}_+ such that

$$\|F\|_{H^2(\mathbb{C}_+)} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |F(x + iy)|^2 dx \right)^{1/2} < \infty, \quad (6)$$

see [13, 14]. A function $F \in H^2(\mathbb{C}_+)$ admits a non-tangential limit $F^* \in L^2(\mathbb{R}, \mathbb{C})$ as y tends to 0, almost everywhere on \mathbb{R} . The function F^* belongs to the space $H^2(\mathbb{R}) = H^2(\mathbb{R}, \mathbb{C})$ of functions $f \in L^2(\mathbb{R})$ such that

$$\int_{-\infty}^{\infty} \frac{f(t)}{t - \bar{z}} dt = 0, \quad z \in \mathbb{C}_+.$$

The space $H^2(\mathbb{R})$ coincides with the space of boundary values on \mathbb{R} (also ‘‘traces’’, if smoothness allows) of $H^2(\mathbb{C}_+)$ functions. Indeed, the map $f \in H^2(\mathbb{R}) \mapsto P_y \star f$ is an isometric isomorphism from $H^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$.

It is not difficult to see that $(I + i\mathcal{H})L^2(\mathbb{R}) = H^2(\mathbb{R})$. As a consequence, the map $u \mapsto$

$P_y \star (I + i\mathcal{H})u$ is an isomorphism from $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. A function $f \in H^2(\mathbb{C}_+)$ satisfies the boundary uniqueness Theorem [24, Cor. 6.4.2]: if $F \in H^2(\mathbb{R})$ is such that $F = 0$ on $I \subset \mathbb{R}$ with $|I| > 0$ then $F \equiv 0$ on \mathbb{R} .

Note that if $I \subset \mathbb{R}$, the space $(I + i\mathcal{H})(\widetilde{L^2(I)})$ coincides with the space of functions $F^* \in H^2(\mathbb{R})$ such that $\text{supp Re } F^* \subset I$.

Observe that for $f \in L^2(\mathbb{R}, \mathbb{C})$,

$$(I + i\mathcal{H})f = 2(I + i\mathcal{H})[\text{Re } P_+(f)] = 2P_+(f),$$

where $P_+(f)$ is the orthogonal projection from $L^2(\mathbb{R}, \mathbb{C})$ onto $H^2(\mathbb{R})$, see [21].

Finally, an amazing property of the Hilbert transform is given by the following Lemma (of which a proof can be found in [1, Lem. 2.12]):

Lemma 2 *If $I, J \subset \mathbb{R}$ with $|I|, |J| > 0$ such that $I \cap J = \emptyset$, then for any $g \in L^2(\mathbb{R})$ such that $\mathcal{H}(\chi_{Jg}) = 0$ on I , we have $g = 0$ on J .*

Lemma 2 can be seen as a consequence of the boundary uniqueness Theorem in $H^2(\mathbb{R})$. Indeed, if $\mathcal{H}(\chi_{Jg}) = 0$ on I then the function in $H^2(\mathbb{R})$ given by $F = \chi_{Jg} + i\mathcal{H}(\chi_{Jg})$ is equal to 0 on I (as $J \cap I = \emptyset$). By the boundary uniqueness Theorem, it follows that $F = 0$ on \mathbb{R} and $g = 0$ on J .

When $I \cap J \neq \emptyset$ and $I \not\subseteq J$, Lemma 2 still holds true in some cases:

- $J \subsetneq I$: indeed, the function $F = \chi_{Jg} + i\mathcal{H}(\chi_{Jg})$ is equal to zero on $I \setminus J$ and the boundary uniqueness Theorem implies that $g = 0$ on J , whenever $|I \setminus J| > 0$;
- $I \cap J \neq \emptyset$ and $I \not\subseteq J$: the same function F is equal to zero on $I \setminus (I \cap J)$ and $g = 0$ on J by the boundary uniqueness Theorem, provided that $|I \setminus (I \cap J)| > 0$.

However, if $I \subseteq J$ are bounded intervals, then there exists a function $g \in L^2(\mathbb{R})$, $g \neq 0$, such that $\mathcal{H}(\chi_{Jg}) \neq 0$. With $I = (a, b)$, the (non-zero) function defined on \mathbb{R} by $g(x) = \chi_I(x)/\sqrt{(x-a)(b-x)}$ is such that $\mathcal{H}(\chi_{Jg}) = \mathcal{H}(g) = 0$ on I . Indeed, the function $x \mapsto 1/\sqrt{(x-a)(b-x)}$ is shown in [19, Lem. 2.1] to be an eigenfunction of the operator $\mathcal{H}(\chi_I \cdot)|_I$ associated to the eigenvalue 0.

- If $I = J$, the function g furnishes a solution to the issue.
- If $I \subsetneq J$, one can write $\mathcal{H}(\chi_{Jg}) = \mathcal{H}(\chi_{Ig}) + \mathcal{H}(\chi_{J \setminus I}g)$. The above function g vanishes outside I hence on $J \setminus I$. Thus, $\mathcal{H}(\chi_{J \setminus I}g) = 0$ on \mathbb{R} . Further, $\mathcal{H}(\chi_{Ig}) = 0$ on I , whence $\mathcal{H}(\chi_{Jg}) = 0$ on I , but $g \neq 0$ on J .

3 Main operators

Let S and $K \subset \mathbb{R}$ be two nonempty open bounded intervals. Fix $h > 0$. Let $\mathbf{m} \in L^2(S, \mathbb{R}^2)$. Taking the convolution of the PDE $\Delta U = \text{div } \mathbf{m}$ by the fundamental solution to Laplace equation in dimension $n = 2$ and applying Green formula, we obtain, for $X \in \mathbb{R}^2 \setminus \text{supp } \mathbf{m}$:

$$\begin{aligned} U[\mathbf{m}](X) &= -\frac{1}{2\pi} \iint_{\mathbb{R}^2} (\text{div}_Y \mathbf{m})(Y) \log |X - Y| dY = \frac{1}{2\pi} \iint_{\mathbb{R}^2} \mathbf{m}(Y) \cdot \text{grad}_Y \log |X - Y| dY \\ &= \frac{1}{2\pi} \iint_{\mathbb{R}^2} \frac{\mathbf{m}(Y) \cdot (X - Y)}{|X - Y|^2} dY. \end{aligned}$$

Since the support of \mathbf{m} is a subset of $S \times \{0\}$, we get, at $X = (x, y) \in \mathbb{R}^2 \setminus S \times \{0\}$, with $Y = (t, 0)$:

$$U[\mathbf{m}](x, y) = \frac{1}{2\pi} \int_S \frac{m_1(t)(x-t) + m_2(t)y}{(x-t)^2 + y^2} dt = \frac{1}{2} (Q_y \star \widetilde{m}_1 + P_y \star \widetilde{m}_2)(x, y).$$

Let then $b_2[\mathbf{m}] = -(\partial_y U[\mathbf{m}])|_{K \times \{h\}}$ (from now, we will ignore on the multiplicative constant $\mu_0/2$):

$$\begin{aligned} b_2 : \quad L^2(S, \mathbb{R}^2) &\longrightarrow L^2(K) \\ \mathbf{m} = (m_1, m_2) &\longmapsto -\partial_y (Q_y \star \widetilde{m}_1 + P_y \star \widetilde{m}_2)|_{K \times \{h\}}, \end{aligned}$$

with the use of Cauchy-Riemann equations, which also allow to get

$$\begin{aligned} b_2[\mathbf{m}] &= -(\partial_x \mathcal{P}_y[\widetilde{m}_1] + \partial_y \mathcal{P}_y[\widetilde{m}_2])|_{K \times \{h\}} = -[\partial_x (P_y \star \widetilde{m}_1 - Q_y \star \widetilde{m}_2)]|_{K \times \{h\}} \\ &= -\partial_x \mathcal{P}_y[\widetilde{m}_1 - \mathcal{H}\widetilde{m}_2]|_{K \times \{h\}}, \end{aligned}$$

where the second equality follows from (4). Using the boundedness of the convolution by P_y from $L^2(\mathbb{R})$ to $W^{1,2}(\mathbb{R})$, the boundedness of the Hilbert transform and the equality $\|\widetilde{m}_i\|_{L^2(\mathbb{R})} = \|m_i\|_{L^2(S)}$, for $i \in \{1, 2\}$ (see Section 2), one can prove that the operator b_2 is bounded $L^2(S, \mathbb{R}^2) \rightarrow L^2(K)$.

Proposition 1 *The adjoint operator b_2^* to b_2 is given by*

$$\begin{aligned} b_2^* : \quad L^2(K) &\longrightarrow L^2(S, \mathbb{R}^2) \\ \phi &\longmapsto (\partial_x (P_y \star \widetilde{\phi})|_{S \times \{h\}}, \partial_x (Q_y \star \widetilde{\phi})|_{S \times \{h\}}) \end{aligned}$$

and is such that

$$b_2^*[\phi] = (((\partial_x P_y) \star \widetilde{\phi})|_{S \times \{h\}}, ((\partial_x Q_y) \star \widetilde{\phi})|_{S \times \{h\}}) = \partial_x (P_y \star (I, \mathcal{H})\widetilde{\phi})|_{S \times \{h\}}.$$

Proof: Let $y > 0$ fixed. For $\mathbf{m} \in L^2(S, \mathbb{R}^2)$ and $\phi \in L^2(K)$, we have that

$$\begin{aligned} \langle b_2[\mathbf{m}], \phi \rangle_{L^2(K)} &= -\langle \partial_x (P_y \star \widetilde{m}_1), \phi \rangle_{L^2(K)} + \langle \partial_x (Q_y \star \widetilde{m}_2), \phi \rangle_{L^2(K)} \\ &= -\int_{\mathbb{R}} \int_{\mathbb{R}} \partial_x (P_y(x-t)) \widetilde{m}_1(t) \widetilde{\phi}(x) dt dx + \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_x (Q_y(x-t)) \widetilde{m}_2(t) \widetilde{\phi}(x) dt dx \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \partial_t (P_y(x-t)) \widetilde{\phi}(x) dx \right) \widetilde{m}_1(t) dt + \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \partial_t (Q_y(x-t)) \widetilde{\phi}(x) dx \right) \widetilde{m}_2(t) dt \\ &= \int_S \partial_t \left(\int_{\mathbb{R}} P_y(t-x) \widetilde{\phi}(x) dx \right) m_1(t) dt + \int_S \partial_t \left(\int_{\mathbb{R}} Q_y(t-x) \widetilde{\phi}(x) dx \right) m_2(t) dt \\ &= \langle m_1, \partial_t (P_y \star \widetilde{\phi}) \rangle_{L^2(S)} + \langle m_2, \partial_t (Q_y \star \widetilde{\phi}) \rangle_{L^2(S)} \\ &= \langle (m_1, m_2), (\partial_t (P_y \star \widetilde{\phi}), \partial_t (Q_y \star \widetilde{\phi})) \rangle_{L^2(S, \mathbb{R}^2)}, \end{aligned}$$

with ■

These operators have null kernel, as follows from the next results.

Proposition 2 *The operator b_2 is injective.*

Proof: Let $\mathbf{m} \in \text{Ker } b_2$. Then, $\partial_x P_y \star (\widetilde{m}_1 - \mathcal{H}\widetilde{m}_2) = 0$ on $K \times \{h\}$, which implies that $\widetilde{m}_1 - \mathcal{H}\widetilde{m}_2 = 0$ on \mathbb{R} in view of Lemma 1. Hence, $\widetilde{m}_1 = \mathcal{H}\widetilde{m}_2$ which implies that $\mathcal{H}\widetilde{m}_2 = 0$ outside S whence so does $\widetilde{m}_2 + i\mathcal{H}\widetilde{m}_2$. Because it coincides with the boundary value of a function in $H^2(\mathbb{C}_+)$, namely $P_y \star (\widetilde{m}_2 + i\mathcal{H}\widetilde{m}_2)$, this implies that $m_2 = 0$ (see [13, 14]). Therefore, $m_1 = 0$ and $\text{Ker } b_2 = \{0\}$. ■

In particular, contrarily to the situation in dimension 3 (and in higher dimensions) see [5], there are no non-vanishing ‘‘silent’’ sources (the corresponding so-called ‘‘null-space’’ is reduced to $\{0\}$).

Proposition 3 *The operator b_2^* is injective.*

Proof: Let $f \in \text{Ker } b_2^*$. Then, we have that $\partial_x (P_y \star (I, \mathcal{H})\widetilde{f}) = (0, 0)$ on $S \times \{h\}$ which implies that $\partial_x (P_h \star \widetilde{f}) = \partial_x (P_h \star \mathcal{H}\widetilde{f}) = 0$ on S . By Lemma 1, it follows that $\widetilde{f} = 0$ on \mathbb{R} and $f = 0$ on S . ■

Moreover:

Corollary 1 *The operators b_2 and b_2^* have a dense range.*

Proof: Since b_2^* is continuous (because so is b_2), the following orthogonal decomposition holds true:

$$L^2(S, \mathbb{R}^2) = \text{Ker } b_2 \oplus \overline{\text{Ran } b_2^*}.$$

Hence, using Proposition 3, b_2^* has dense range in $L^2(S, \mathbb{R}^2)$. Similarly,

$$L^2(K) = \text{Ker } b_2^* \oplus \overline{\text{Ran } b_2}.$$

Hence, using Proposition 2, b_2 has dense range in $L^2(K)$. ■

Remark 1 *Using the following equality for $f \in L^2(\mathbb{R}, \mathbb{C})$:*

$$(I + i\mathcal{H})f = (I + i\mathcal{H})(\text{Re } f - \mathcal{H} \text{Im } f). \quad (7)$$

with $\mathcal{H}u = \mathcal{H}[-i^2u] = -i\mathcal{H}[iu]$ whence $\mathcal{H}[iu] = i\mathcal{H}u$, we could consider the analytic versions of b_2 and b_2^* denoted by B_2 and B_2^* respectively (see [21]):

$$\begin{aligned} B_2 : L^2(S, \mathbb{C}) &\longrightarrow L^2(K) \\ m = m_1 + im_2 &\longmapsto -\text{Re } \partial_x(P_y \star (I + i\mathcal{H})\tilde{m})|_{K \times \{h\}}, \\ B_2^* : L^2(K) &\longrightarrow L^2(S, \mathbb{C}) \\ \phi &\longmapsto \partial_x(P_y \star (I + i\mathcal{H})\tilde{\phi})|_{S \times \{h\}}. \end{aligned}$$

Let J be the isomorphism from $L^2(S, \mathbb{C})$ onto $L^2(S, \mathbb{R}^2)$ defined by $J(\phi_1 + i\phi_2) = (\phi_1, \phi_2)$ which is unitary when $L^2(S, \mathbb{C})$ is equipped with the inner product (1). We have that $B_2 = b_2 \circ J$ and $b_2^*[\phi] = J \circ B_2^*[\phi]$, $\phi \in L^2(K)$. As a consequence, B_2 and B_2^* are injective and $\text{Ran } B_2 = L^2(K)$, $\text{Ran } B_2^* = L^2(S, \mathbb{C})$.

Proposition 4 *The ranges of b_2 (respectively B_2) and b_2^* (respectively B_2^*) are not closed: $\text{Ran } b_2 \subsetneq L^2(K)$, $\text{Ran } b_2^* \subsetneq L^2(S, \mathbb{R}^2)$.*

Proof: If we assume that the range of B_2^* is closed, then Corollary 1 and Remark 1 are to the effect that B_2^* is surjective. Hence, for $I \subsetneq S$, with $|I| > 0$, there exists $\phi \in L^2(K)$ such that

$$\partial_x(P_h \star (I + i\mathcal{H})\tilde{\phi})|_S = \chi_I, \quad (8)$$

It follows that $\partial_x(P_h \star (I + i\mathcal{H})\tilde{\phi}) = 0$ on $S \setminus I$ whence $(I + i\mathcal{H})\tilde{\phi} = 0$ on \mathbb{R} by Lemma 1. As in the proof of Proposition 2, this leads to $\phi = 0$ which contradicts Equation (8). We conclude that B_2^* is not surjective. As $b_2^* = J \circ B_2^*$, we get that b_2^* is not surjective. Now, if the range of B_2 is closed, then for any $g \in L^2(K)$, there would be $\phi \in L^2(S, \mathbb{C})$ such that

$$g = -\text{Re } \partial_x(P_h \star (I + i\mathcal{H})\tilde{\phi})|_{K \times \{h\}}. \quad (9)$$

Let $I \subsetneq K$ with $|I| > 0$ and $g = \chi_I$. Then, there exists $\phi \in L^2(S, \mathbb{C})$ such that

$$\text{Re } \partial_x(P_h \star (I + i\mathcal{H})\tilde{\phi})|_K = \partial_x \left(P_h \star \text{Re } (I + i\mathcal{H})\tilde{\phi} \right)|_K = 0 \quad \text{on } K \setminus I.$$

By Lemma 1, we get that $\text{Re } (I + i\mathcal{H})\tilde{\phi} = 0$ on \mathbb{R} , and $\text{Re } \tilde{\phi} - \mathcal{H} \text{Im } \tilde{\phi} = 0$ on \mathbb{R} . This means that $\mathcal{H} \text{Im } \tilde{\phi} = \text{Re } \tilde{\phi}$ on \mathbb{R} whence $\mathcal{H} \text{Im } \tilde{\phi} = 0$ on $\mathbb{R} \setminus S$, which proves that $\text{Im } \tilde{\phi} = 0$ on \mathbb{R} by Lemma 2, and thus $\text{Re } \tilde{\phi} = 0$ on \mathbb{R} . We conclude that $\phi = 0$ on \mathbb{R} which contradicts Equation (9). It follows that the range of B_2 is not closed and thus $b_2 = J^{-1} \circ B_2$ is not surjective. ■

Remark 2 *The ranges of b_2 (and B_2) and b_2^* (and B_2^*) are actually made of C^∞ -smooth functions on K and S , respectively, see Section 2.2.*

Proposition 5 *The operators b_2 , B_2 , b_2^* and B_2^* are compact.*

Proof: Because $u \mapsto (\partial_x P_y \star u)|_S$ and $u \mapsto (\partial_y P_y \star u)|_S$ are compact operators on $L^2(\widehat{K}, \mathbb{C})$ for $y > 0$ (see Section 2.2), so is $B_2^* = (\partial_x \mathcal{P}_h (I + iH))|_S$, and thus also B_2 . As $b_2^* = J \circ B_2^*$ and $b_2 = B_2 \circ J^{-1}$, we get that b_2 and b_2^* are compact. \blacksquare

Remark 3 *We will also make use of the operator a_2 defined by*

$$\begin{aligned} a_2 : L^2(S, \mathbb{R}^2) &\longrightarrow L^2(K) \\ \mathbf{m} = (m_1, m_2) &\longmapsto [P_y \star (\widetilde{m}_1 - \mathcal{H}\widetilde{m}_2)]|_{K \times \{h\}}, \end{aligned}$$

which is such that $b_2[\mathbf{m}] = -\partial_x(a_2[\mathbf{m}])$.

We deduce from the proof of Proposition 1 that $a_2^[\phi] = (P_y \star \widetilde{\phi}, P_y \star \mathcal{H}\widetilde{\phi})|_{S \times \{h\}}$, for $\phi \in L^2(K)$, and from the one of Proposition 5 that a_2 and a_2^* are Hilbert-Schmidt operators. Further, whenever $\phi \in W_0^{1,2}(K)$, then $b_2^*[\phi] = -a_2^*[\partial_x \phi]$.*

4 Bounded extremal problems for moments estimates

We are interested in solving the following bounded extremal problem (BEP), with $\mathbf{e} \in L^2(S, \mathbb{R}^2)$ and $M \geq 0$: find $\phi_o \in L^2(K)$, $\|\phi_o\|_{L^2(K)} \leq M$ that satisfies

$$\|b_2^*[\phi_o] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} = \min_{\substack{\phi \in L^2(K) \\ \|\phi\|_{L^2(K)} \leq M}} \|b_2^*[\phi] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} \quad (\text{BEP}_{m_o})$$

Our motivation is as follows. Let $\mathbf{e}_1 = (\chi_S, 0)$, $\mathbf{e}_2 = (0, \chi_S)$. A solution ϕ_o to (BEP $_{m_o}$) furnishes a linear estimator in $L^2(K)$ for net moment estimates: indeed, for $i \in \{1, 2\}$,

$$|\langle b_2[\mathbf{m}], \phi_i \rangle_{L^2(K)} - \langle m_i \rangle| = |\langle \mathbf{m}, b_2^*[\phi_i] - \mathbf{e}_i \rangle_{L^2(S, \mathbb{R}^2)}| \leq \|\mathbf{m}\|_{L^2(S, \mathbb{R}^2)} \|b_2^*[\phi_i] - \mathbf{e}_i\|_{L^2(S, \mathbb{R}^2)},$$

where $\phi_i = \phi_o[\mathbf{e}_i]$ is the solution of (BEP $_{m_o}$) with $\mathbf{e} = \mathbf{e}_i$ and $\langle m_i \rangle = \int_S m_i(t) dt$.

Below, we consider BEP both in the Lebesgue space $L^2(K)$ and in the Sobolev space $W_0^{1,2}(K)$ in order to control the derivative and the oscillations of the solution.

4.1 BEP in $L^2(K)$

4.1.1 Well-posedness

Proposition 6 *Let $\mathbf{e} \in L^2(S, \mathbb{R}^2)$ and $M \geq 0$. Problem (BEP $_{m_o}$) admits a unique solution $\phi_o \in L^2(K)$. Moreover, if $\mathbf{e} \notin \text{Ran } b_2^*$, then ϕ_o saturates the constraint: $\|\phi_o\|_{L^2(K)} = M$.*

Note that $\|\phi_o\|_{L^2(K)} = M$ also if $\mathbf{e} = b_2^*[\phi]$ for some $\phi \in L^2(K)$ such that $\|\phi\|_{L^2(K)} \geq M$.

Proof: Since $\text{Ran } b_2^*$ is dense in $L^2(S, \mathbb{R}^2)$, the assumptions of [11, Lem. 2.1] (see also [12] and [21, Lem. 1]) are satisfied, to the effect that if $M > 0$, Problem (BEP $_{m_o}$) admits a solution, which is unique and saturates the constraint if $\mathbf{e} \notin \text{Ran } b_2^*$ (or if $\mathbf{e} = b_2^*[\phi]$ for $\phi \in L^2(K)$ such that $\|\phi\|_{L^2(K)} \geq M$). Observe further that because b_2^* is injective (from Proposition 3), uniqueness still holds when $\mathbf{e} \in \text{Ran } b_2^*$. \blacksquare

Existence and uniqueness could also be established by projection onto the closed convex set [9, Thm 5.2] $b_2^*[B_M]$, for $B_M = \{\phi \in L^2(K), \|\phi\|_{L^2(K)} \leq M\}$. Indeed, B_M is weakly compact as a ball in $L^2(K)$ and the weak closedness property of $b_2^*[B_M]$ follows from the continuity of b_2^* , see [6, Prop. 1]. By convexity of B_M and continuity of b_2^* , $b_2^*[\phi_n]$ strongly converges to $b_2^*[\phi]$ in $L^2(S, \mathbb{R}^2)$ whenever ϕ_n weakly converges to ϕ in $L^2(K)$. Note that the compactness of b_2^* is enough to get that $b_2^*[B_M]$ is closed. That the constraint is satisfied if $\mathbf{e} \notin b_2^*[B_M]$ could be established by differentiation of the squared criterion.

Remark 4 The density property of $b_2^* [L^2(K)]$ in $L^2(S, \mathbb{R}^2)$ implies that without the norm constraint, Problem (BEP_{m_o}) above is ill-posed. Indeed, if $\mathbf{e} \notin b_2^* [L^2(K)]$ the density gives the existence of a sequence (ϕ_n) of functions in $L^2(K)$ such that $\|b_2^*[\phi_n] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} \rightarrow 0$. We claim that $\|\phi_n\|_{L^2(K)} \rightarrow \infty$ as $n \rightarrow \infty$. This can be seen by assuming $\|\phi_n\|_{L^2(K)}$ to remain bounded and extracting a weakly convergent subsequence, say (ϕ_n) again, to some $\phi \in b_2^* [B_M]$ (the later being weakly closed as we saw in the proof of Proposition 6). By convexity of $b_2^* [B_M]$, it follows that (ϕ_n) converges strongly to ϕ in $L^2(S, \mathbb{R}^2)$ and this implies that $\mathbf{e} = b_2^*[\phi] \in \text{Ran } b_2^*$ which leads to a contradiction.

4.1.2 A constructive solution

Assume that $\mathbf{e} \notin \text{Ran } b_2^*$. From Proposition 6, the solution of (BEP_{m_o}) is a minimum on $L^2(K)$ of the functional $\phi \mapsto \|b_2^*[\phi] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)}^2$ which satisfies the constraint $\|\phi\|_{L^2(K)}^2 = M^2$. Differentiating the criterion and the constraint w.r.t. ϕ , we get that such a critical point ϕ_o is given by the equation:

$$\langle b_2 b_2^*[\phi_o], \psi \rangle_{L^2(K)} - \langle b_2[\mathbf{e}], \psi \rangle_{L^2(K)} = -\lambda \langle \phi_o, \psi \rangle_{L^2(K)},$$

with $\lambda \in \mathbb{R}$ and for all $\psi \in L^2(K)$. It follows that:

$$\langle b_2 b_2^*[\phi_o] + \lambda \phi_o, \psi \rangle_{L^2(K)} = \langle b_2(\mathbf{e}), \psi \rangle_{L^2(K)}, \quad \text{for all } \psi \in L^2(K), \quad (10)$$

whence:

$$b_2 b_2^*[\phi_o] + \lambda \phi_o = b_2[\mathbf{e}]. \quad (\text{CPE}_{m_o})$$

We have $\lambda > 0$. Indeed, taking $\psi = \phi_o$ in Equation (10), we get

$$-\lambda \langle \phi_o, \phi_o \rangle_{L^2(K)} = -\lambda \|\phi_o\|_{L^2(K)}^2 = \langle b_2^*(\phi_o) - \mathbf{e}, b_2^*(\phi_o) \rangle_{L^2(S, \mathbb{R}^2)}.$$

By [9, Thm 5.2], for any $\mathbf{g} \in L^2(S, \mathbb{R}^2)$,

$$\text{Re} \langle b_2^*(\phi_o) - \mathbf{e}, b_2^*(\phi_o) - \mathbf{g} \rangle_{L^2(S, \mathbb{R}^2)} \leq 0.$$

Taking $\mathbf{g} = 0$, we obtain that $\langle b_2^*(\phi_o) - \mathbf{e}, b_2^*(\phi_o) \rangle_{L^2(S, \mathbb{R}^2)} \leq 0$. Thus, $-\lambda \leq 0$. As $\lambda = 0$ if and only if $\mathbf{e} \in \text{Ran } b_2^*$, we conclude that $-\lambda < 0$.

Assume that $\mathbf{e} \notin \text{Ran } b_2^*$. Then there exists a unique $\lambda > 0$ such that (CPE_{m_o}) holds true and $\|\phi_o\|_{L^2(K)} = M$. In this case,

$$\lambda \|\phi_o\|_{L^2(K)}^2 = \lambda M^2 = -\langle b_2^*(\phi_o) - \mathbf{e}, b_2^*(\phi_o) \rangle_{L^2(S, \mathbb{R}^2)}.$$

From now on, when there is no ambiguity, we will write $f' = \partial_x f$ for the derivative of real-valued differentiable functions f defined on \mathbb{R} or on intervals of \mathbb{R} .

For $f \in L^2(K)$, at $y = h > 0$, we have that:

$$\begin{aligned} b_2 b_2^*[f] &= \left[-P_h \star (\chi_S (P_h \star \tilde{f})') + Q_h \star (\chi_S (Q_h' \star \tilde{f})) \right]'_{|K} \\ &= \left[-P_h' \star (\chi_S (P_h' \star \tilde{f})) + Q_h' \star (\chi_S (Q_h' \star \tilde{f})) \right]_{|K}. \end{aligned}$$

The critical point equation (CPE_{m_o}) can be written as:

$$\left[-P_h' \star (\chi_S (P_h' \star \tilde{\phi}_o)) + Q_h' \star (\chi_S (Q_h' \star \tilde{\phi}_o)) \right]_{|K} + \lambda \phi_o = -P_h' \star (\tilde{e}_1 - \mathcal{H}\tilde{e}_2)_{|K},$$

where $(e_1, e_2) = \mathbf{e}$.

Observe that, if the function $\eta \in L^2(K)$ represents the error on the measurements of $b_2[\mathbf{m}]$, then

$$\begin{aligned} |\langle b_2[\mathbf{m}] + \eta, \phi_i \rangle_{L^2(K)} - \langle m_i \rangle| &= |\langle \mathbf{m}, b_2^*[\phi_i] - \mathbf{e}_i \rangle_{L^2(S, \mathbb{R}^2)} + \langle \eta, \phi_i \rangle_{L^2(K)}| \\ &\leq \|\mathbf{m}\|_{L^2(S, \mathbb{R}^2)} \|b_2^*[\phi_i] - \mathbf{e}_i\|_{L^2(S, \mathbb{R}^2)} + \|\eta\|_{L^2(K)} \|\phi_i\|_{L^2(K)} \\ &\leq \|\mathbf{m}\|_{L^2(S, \mathbb{R}^2)} \|b_2^*[\phi_i] - \mathbf{e}_i\|_{L^2(S, \mathbb{R}^2)} + M \|\eta\|_{L^2(K)}. \end{aligned}$$

For a choice of $M > 0$, the solution of (BEP_{mo}) guarantees that $\|b_2^*[\phi_i] - \mathbf{e}_i\|_{L^2(S, \mathbb{R}^2)}$ is minimal and the term $\|\eta\|_{L^2(K)}$ is controlled by $M = \|\phi_i\|_{L^2(K)}$. The challenge of the numerical implementations is to choose a right $M > 0$ such that ϕ_i satisfies the two conditions mentioned above. We will observe in numerical implementations that the solution ϕ_i can oscillate at the endpoints. However, since $\text{Ran } b_2 \subset W^{1,2}(K)$, then (CPE_{mo}) is to the effect that $\phi_i \in W^{1,2}(K) \subset C(\bar{K})$.

4.2 BEP in $W_0^{1,2}(K)$

An alternative is to formulate the BEP in a space of functions such that these oscillations are controlled. It thus seems natural to search for a solution in the Sobolev space $W_0^{1,2}(K)$. We denote by $b_{2|_W}^*$ the restriction of b_2^* to $W_0^{1,2}(K)$. For $\phi \in W_0^{1,2}(K)$,

$$b_{2|_W}^*[\phi] = b_{2|_W}^*[\phi] = ((P_h \star \tilde{\phi})'_{|_S}, (Q_h \star \tilde{\phi})'_{|_S}) = ((P'_h \star \tilde{\phi})_{|_S}, (Q'_h \star \tilde{\phi})_{|_S}) = ((P_h \star \tilde{\phi}')_{|_S}, (Q_h \star \tilde{\phi}')_{|_S}),$$

where $(\tilde{\phi})' = \tilde{\phi}' = \phi' \vee 0$. Notice that $\tilde{\phi} \in W^{1,2}(\mathbb{R})$ only when $\phi \in W_0^{1,2}(K)$ (see [9]). Let $\mathbf{e} \in L^2(S, \mathbb{R}^2)$ and $M \geq 0$. The problem can thus be stated as the one of finding $\phi_o \in W_0^{1,2}(K)$, $\|\phi_o\|_{W_0^{1,2}(K)} \leq M$ such that:

$$\|b_{2|_W}^*[\phi_o] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} = \min_{\substack{\phi \in W_0^{1,2}(K) \\ \|\phi\|_{W_0^{1,2}(K)} \leq M}} \|b_{2|_W}^*[\phi] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)}. \quad (\text{BEP}_{mo,W})$$

4.2.1 Well-posedness

Proposition 7 *Let $\mathbf{e} \in L^2(S, \mathbb{R}^2)$ and $M \geq 0$. Problem $(\text{BEP}_{mo,W})$ admits a unique solution $\phi \in W_0^{1,2}(K)$. If $\mathbf{e} \notin b_{2|_W}^*[W_0^{1,2}(K)]$, then ϕ_o saturates the constraint: $\|\phi_o\|_{W_0^{1,2}(K)} = M$.*

Proof: We apply again [11, Lemma 2.1] in order to get existence and uniqueness of a solution $\phi_o \in W_0^{1,2}(K)$ to $(\text{BEP}_{mo,W})$ saturating the constraint if $\mathbf{e} \notin b_{2|_W}^*[W_0^{1,2}(K)]$. Yet, because b_2^* is injective (from Proposition 3), uniqueness still holds when $\mathbf{e} \in b_{2|_W}^*[W_0^{1,2}(K)]$. Note that in this case, the constraint is saturated also if $\mathbf{e} = b_2^*[\phi]$ for $\phi \in L^2(K)$ such that $\|\phi\|_{W_0^{1,2}(K)} \geq M$. \blacksquare

Remark 5 *Let $\psi \in L^2(K)$ with zero mean on K . There exists a unique $\phi \in W_0^{1,2}(K)$ such that $\phi' = \psi$ (and $\|\phi\|_{W_0^{1,2}(K)} = \|\psi\|_{L^2(K)}$). Then, $(\text{BEP}_{mo,W})$ is equivalent to the following problem: let $\mathbf{e} \in L^2(S, \mathbb{R}^2)$ and $M \geq 0$, find $\psi_o \in L^2(K)$ such that $\int_K \psi_o(x) dx = 0$, $\|\psi_o\|_{L^2(K)} \leq M$ that satisfies*

$$\|a_2^*[\tilde{\psi}_o] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} = \min_{\substack{\psi \in L^2(K) \\ \int_K \psi(x) dx = 0, \|\psi\|_{L^2(K)} \leq M}} \|a_2^*[\tilde{\psi}] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)}. \quad (11)$$

Proposition 7 is to the effect that a unique solution ϕ_o to $(\text{BEP}_{mo,W})$ does exist, whence also a unique solution ψ_o to (11), which is such that $\phi'_o = \psi_o$.

We also have the following density result.

Lemma 3 $b_2^*[W_0^{1,2}(K)]$ is dense in $L^2(S, \mathbb{R}^2)$.

Proof: First, observe that $W_0^{1,2}(K)$ is dense in $L^2(K)$ since the set $C_c^\infty(K)$ of $C^\infty(K)$ functions with compact support $\subset K$ is dense in $L^2(K)$ (see [9, Cor. 4.23]). As $W^{1,2}(K) \subset L^2(K)$, we use the continuity of b_2^* to get:

$$b_2^* \left[W_0^{1,2}(K) \right] \subset b_2^*(L^2(K)) = b_2^* \left[\overline{W_0^{1,2}(K)}^{L^2(K)} \right] \subset \overline{b_2^* \left[W_0^{1,2}(K) \right]}^{L^2(S, \mathbb{R}^2)},$$

Recall that $b_2^*[L^2(K)]$ is dense in $L^2(S, \mathbb{R}^2)$ (for the $L^2(S, \mathbb{R}^2)$ norm), from Corollary 1. We therefore conclude to the density of $b_2^*[W_0^{1,2}(K)]$ in $L^2(S, \mathbb{R}^2)$. \blacksquare

Yet, Lemma 3 implies that without the norm constraint, Problem (BEP_{mo,W}) is ill-posed. The minimum is in this case an infimum and is equal to 0 while minimizing sequences diverge.

4.2.2 A constructive solution

Observe that $b_2 = (b_{2|_W}^*)^*$, since $b_{2|_W}^* = a_2^*(\phi')$, where ϕ' denotes the derivation from $W_0^{1,2}(K)$ onto the space of $L^2(K)$ -functions with zero mean. Thus, for $f \in W_0^{1,2}(K)$, we get

$$b_2 b_{2|_W}^*[f] = \left[-P'_h \star (\chi_S P_h \star \tilde{f}') + Q'_h \star (\chi_S Q_h \star \tilde{f}') \right]_{|_K}.$$

Assume that $\mathbf{e} \notin b_{2|_W}^*[W_0^{1,2}(K)]$. From Proposition 7, we obtain a critical point equation somewhat different from (CPE_{mo}) for the solution ϕ_o in this case (see also Section 4.1.2):

$$\left\langle b_2 b_{2|_W}^*[\phi_o], \psi \right\rangle_{L^2(K)} - \langle b_2[\mathbf{e}], \psi \rangle_{L^2(K)} = \gamma \langle \phi'_o, \psi' \rangle_{L^2(K)} = -\gamma \langle \phi''_o, \psi \rangle_{L^2(K)},$$

with $\gamma < 0$ and $\forall \psi \in W_0^{1,2}(K)$. The solution $\phi_o \in W_0^{1,2}(K)$ of (BEP_{mo,W}) is thus given by

$$b_2 b_{2|_W}^*[\phi_o] + \gamma \phi''_o = b_2[\mathbf{e}], \quad (\text{CPE}_{mo,W})$$

with $\gamma < 0$, such that $\|\phi_o\|_{W^{1,2}(K)} = \|\phi'_o\|_{L^2(K)} = M$ if $\mathbf{e} \notin \text{Ran } b_{2|_W}^*$. The equation CPE_{mo,W} is considered in the weak sense in $L^2(K)$. Thus,

$$\left[-P'_h \star (\chi_S (P_h \star \tilde{\phi}'_o)) + Q'_h \star (\chi_S (Q_h \star \tilde{\phi}'_o)) \right]_{|_K} + \gamma \phi''_o = -P'_h \star (\tilde{\epsilon}_1 - \mathcal{H}\tilde{\epsilon}_2)_{|_K}.$$

4.2.3 An equivalent problem in H^2

Let $\mathbb{C}_+ = \{x + iy \in \mathbb{C} : x \in \mathbb{R}, y > 0\}$ and $H^2(\mathbb{C}_+)$ denotes the Hardy space of the upper half-plane defined in Section 2.3. As a function $F \in H^2(\mathbb{C}_+)$ can be identified to its boundary function $F^* \in H^2(\mathbb{R})$ so that $F|_{S \times \{0\}}$ can be defined as $F^*_{|_S}$.

Given a function $g \in L^2(S, \mathbb{C})$, for $\varepsilon > 0$, there is a function $F \in H^2(\mathbb{C}_+)$ such that $\|g - F^*_{|_S}\|_{L^2(S, \mathbb{C})} < \varepsilon$. Indeed, if $f \in (H^2(\mathbb{R})_{|_S})^\perp$, then for $g \in H^2(\mathbb{R})$,

$$\langle f, g_{|_S} \rangle_{L^2(S, \mathbb{C})} = \langle \tilde{f}, g \rangle_{L^2(\mathbb{R}, \mathbb{C})} = 0.$$

Taking $g(t) = \frac{1}{t-z}$ with $z \in \mathbb{C}$ such that $\text{Im } z > 0$, it follows that

$$\int_{-\infty}^{+\infty} \frac{\tilde{f}(t)}{t-z} dt = 0, \quad z \in \mathbb{C}_+,$$

and $\tilde{f} \in H^2_-(\mathbb{R})$. By the boundary uniqueness Theorem, $f = 0$ on S so $H^2(\mathbb{R})_{|_S}$ is dense in $L^2(S, \mathbb{C})$.

For $M \geq 0$, let

$$\mathcal{B}_M = \{F \in H^2(\mathbb{C}_+) : \text{supp Re } F^* \subset K \text{ and } \|\text{Re } F^*\|_{L^2(K)} \leq M\} \subset H^2(\mathbb{C}_+).$$

Proposition 8 *Let $g \in L^2(S, \mathbb{C})$. The problem $(\text{BEP}_{mo, W})$ is equivalent to the following bounded extremal problem (with $L^2(S, \mathbb{C})$ norm):*

$$\min_{F \in \mathcal{B}_M} \|F(\cdot + ih)|_S - g\|_{L^2(S, \mathbb{C})}, \quad (\text{BEP}_{C_+} \text{ bis})$$

Proof: Let $\mathbf{e} \in L^2(S, \mathbb{R}^2)$ be such that $J(g) = \mathbf{e}$. By Proposition 7, there exists a function $\phi_o \in W_0^{1,2}(K)$ such that

$$\|b_{2|W}^*[\phi_o] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} = \min_{\substack{\phi \in W_0^{1,2}(K) \\ \|\phi\|_{W_0^{1,2}(K)} \leq M}} \|b_{2|W}^*[\phi] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)}.$$

Using the isomorphism J , we get for any $\phi \in W_0^{1,2}(K)$,

$$\|b_{2|W}^*[\phi] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} = \|J(B_{2|W}^*[\phi]) - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)} = \|B_{2|W}^*[\phi] - J^{-1}\mathbf{e}\|_{L^2(S, \mathbb{C})},$$

where $B_{2|W}^*[\phi] = (P_h \star (I + i\mathcal{H})\tilde{\psi})|_S$ and ψ is the (unique) function of $L^2(K)$ such that $\phi' = \psi$. Let $F_o \in \mathcal{B}_M$ defined by $P_y \star (I + i\mathcal{H})\tilde{\psi}_o = F_o$ with $\phi'_o = \psi_o$. It follows that $F_o(\cdot + ih) = B_{2|W}^*[\phi_o] = (P_h \star (I + i\mathcal{H})\tilde{\psi}_o)$ and $(\text{Re } F_o^*)|_K = \psi_o$. Then,

$$\begin{aligned} \|F_o(\cdot + ih)|_S - g\|_{L^2(S, \mathbb{C})} &= \|B_{2|W}^*[\phi_o] - J^{-1}\mathbf{e}\|_{L^2(S, \mathbb{C})} \\ &= \min_{\substack{\phi \in W_0^{1,2}(K) \\ \|\phi\|_{W_0^{1,2}(K)} \leq M}} \|B_{2|W}^*[\phi] - J^{-1}\mathbf{e}\|_{L^2(S, \mathbb{R}^2)} \\ &= \min_{\substack{\phi \in W_0^{1,2}(K) \\ \|\phi\|_{W_0^{1,2}(K)} \leq M}} \|(P_h \star (I + i\mathcal{H})\tilde{\phi}')|_S - J^{-1}\mathbf{e}\|_{L^2(S, \mathbb{R}^2)} \\ &= \min_{\substack{\psi \in L^2(K) \\ \|\psi\|_{L^2(K)} \leq M}} \|(P_h \star (I + i\mathcal{H})\tilde{\psi})|_S - J^{-1}\mathbf{e}\|_{L^2(S, \mathbb{R}^2)} \\ &= \min_{F \in \mathcal{B}_M} \|F(\cdot + ih)|_S - g\|_{L^2(S, \mathbb{C})}, \end{aligned}$$

where we use that $P_y \star (I + i\mathcal{H})\tilde{\psi} \in \mathcal{B}_M$ for any $\psi \in L^2(K)$. ■

5 Algorithms and numerical computations

5.1 Construction of the algorithms

5.1.1 Bases functions

Let $S = (-s, s)$, $K = (-q, q)$, $s, q \in \mathbb{R}_+$. For $n \in \mathbb{Z}$, let $g_n(x) = \exp(in\pi x/q)$, for $x \in K$. They are eigenfunctions of the Laplacian on K :

$$g_n'' = -\left[\frac{n\pi}{q}\right]^2 g_n = -\mu_n g_n, \quad \text{where } \mu_n = \left[\frac{n\pi}{q}\right]^2 > 0.$$

Up to multiplication by the constant factor $\frac{1}{(2q)^{1/2}}$ for normalization, the family of functions $(g_n)_n$, $n \in \mathbb{Z}$, is the Hilbertian Fourier basis of $L^2(K, \mathbb{C})$ (see [9, Sec. 8.6]). We use it to expand and compute the solutions to the above bounded extremal problems, (BEP_{mo}) in $L^2(K)$ and $(\text{BEP}_{mo, W})$ in $W_0^{1,2}(K)$. Indeed, functions ϕ in $L^2(K)$ can be expanded in Fourier series on the basis $(g_n)_n$, $n \in \mathbb{Z}$:

$$\phi = \sum_{n \in \mathbb{Z}} c_n g_n,$$

with $c_n \in l^2(\mathbb{Z})$, $c_{-n} = \bar{c}_n$, $c_0 \in \mathbb{R}$ (since ϕ is real-valued). Moreover, ϕ belongs to $W^{1,2}(K)$ if, and only if, the coefficients of its expansion on $(g_n)_n$, $n \in \mathbb{Z}$, are such that $nc_n \in l^2(\mathbb{Z})$ and $c_n = -c_{-n} \in i\mathbb{R}$.

Another family of appropriate functions in $L^2(K)$ is made of piecewise constant functions on small intervals (say K_l , $l = 1, \dots, L$) covering $K = (-q, q)$ (so called P_0 finite elements), see also [6]. We use it for the computations of the solutions to the associated forward problem (Section 5.1.3).

5.1.2 Operators, matrices in $L^2(K)$ and $W_0^{1,2}(K)$

For all $\phi \in L^2(K)$, we use the expression

$$b_2^*[\phi] = \left((\partial_x P_y) \star \tilde{\phi}, (\partial_x Q_y) \star \tilde{\phi} \right)_{|_{S \times \{h\}}} = \left(P'_y \star \tilde{\phi}, Q'_y \star \tilde{\phi} \right)_{|_{S \times \{h\}}}.$$

Solutions to (BEP $_{mo}$) in $L^2(K)$ (actually in $W^{1,2}(K) \subset L^2(K)$) are provided by the critical point equation (CPE $_{mo}$).

For moments estimation, we consider the specific functions $\mathbf{e} = \mathbf{e}_1, \mathbf{e}_2$, with $\mathbf{e}_1 = (\chi_S, 0)$ for $\langle m_1 \rangle$, $\mathbf{e}_2 = (0, \chi_S)$ for $\langle m_2 \rangle$. From Lemma 1 and the relation (4), it holds that $\mathbf{e}_i \notin \text{Ran } b_2^*$. Hence, the constraint in the BEP is saturated by the approximant, following Propositions 6 and 7.

Hence, using the Fourier basis (g_n) in order to express solutions to (CPE $_{mo}$) as $\phi_o = \sum_{n \in \mathbb{Z}} c_n g_n$, we get that, $\forall k \in \mathbb{Z}$:

$$\sum_{n \in \mathbb{Z}} c_n \langle b_2^*[g_n], b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)} + \lambda c_k = \langle \mathbf{e}, b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)}, \quad (12)$$

for $\lambda > 0$ such that $\|\phi_o\|_{L^2(K)}^2 = \sum_{n \in \mathbb{Z}} |c_n|^2 = M^2$.

For $t_1, t_2 \in K$, $x \in S$, let:

$$I(t_1, t_2, x) = P'_h(x - t_1) P'_h(x - t_2) + Q'_h(x - t_1) Q'_h(x - t_2),$$

and

$$k(t_1, t_2) = \int_{-s}^s I(t_1, t_2, x) dx.$$

Following the expressions of the operator b_2^* in Section 3, we obtain:

Lemma 4

$$\langle b_2^*[g_n], b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)} = \frac{1}{\pi^2} \int_{-q}^q \int_{-q}^q k(t_1, t_2) e^{i \frac{\pi}{q}(nt_1 + kt_2)} dt_1 dt_2. \quad (13)$$

These are the Fourier coefficients of $\chi_K(t_1) \chi_K(t_2) k(t_1, t_2)$.

Proof: Observe that

$$\begin{aligned} \langle b_2^*[\phi], b_2^*[\psi] \rangle_{L^2(S, \mathbb{R}^2)} &= \left\langle P'_h \star \tilde{\phi}, P'_h \star \tilde{\psi} \right\rangle_{L^2(S)} + \left\langle Q'_h \star \tilde{\phi}, Q'_h \star \tilde{\psi} \right\rangle_{L^2(S)} \\ &= \int_{-s}^s P'_h \star \tilde{\phi}(x) P'_h \star \tilde{\psi}(x) dx + \int_{-s}^s Q'_h \star \tilde{\phi}(x) Q'_h \star \tilde{\psi}(x) dx \\ &= \int_{-s}^s \int_{-q}^q \int_{-q}^q (P'_h(x - t_1) P'_h(x - t_2) + Q'_h(x - t_1) Q'_h(x - t_2)) \phi(t_1) \psi(t_2) dt_1 dt_2 dx. \end{aligned}$$

We have, for $\phi, \psi \in L^2(K)$,

$$\langle b_2^*[\phi], b_2^*[\psi] \rangle_{L^2(S, \mathbb{R}^2)} = \frac{1}{\pi^2} \int_{-s}^s \int_{-q}^q \int_{-q}^q I(t_1, t_2, x) \phi(t_1) \psi(t_2) dt_1 dt_2 dx.$$

We can interchange the integrals, which leads to

$$\langle b_2^*[\phi], b_2^*[\psi] \rangle_{L^2(S, \mathbb{R}^2)} = \frac{1}{\pi^2} \int_{-q}^q \int_{-q}^q k(t_1, t_2) \phi(t_1) \psi(t_2) dt_1 dt_2.$$

Taking $\phi = g_n$ and $\psi = g_k$ leads to the expression (13). ■

This allows to compute the left-hand side of (CPE_{mo}) and of (12).

Similarly, solutions to $(\text{BEP}_{mo,W})$ in $W_0^{1,2}(K)$, are furnished by $(\text{CPE}_{mo,W})$. Notice that $b_{2|W}^*[\phi] = b_2^*[\phi]$ if $\phi \in W_0^{1,2}(K)$. Using (g_n) in order to express the solutions to $(\text{CPE}_{mo,W})$ as $\phi_o = \sum_{n \in \mathbb{Z}} d_n g_n$, we get that, $\forall k \in \mathbb{Z}$:

$$\sum_{n \in \mathbb{Z}} d_n \langle b_2^*[g_n], b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)} - \gamma \mu_k d_k = \langle \mathbf{e}, b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)}. \quad (14)$$

for $\gamma < 0$ such that $\|\phi'_o\|_{L^2(K)}^2 = \sum_{n \in \mathbb{Z}} \mu_n |d_n|^2 = M^2$, which has to be compared with (12). Note that the above functions $I(t_1, t_2, x)$ and $k(t_1, t_2)$ admit explicit expressions which can be used to more precisely compute the quantities $\langle b_2^*[\phi], b_2^*[\psi] \rangle_{L^2(S, \mathbb{R}^2)}$ in Lemma 4.

5.1.3 Forward problems, $b_2[\mathbf{e}_i]$, $b_2[\mathbf{m}]$

Concerning the computation of the right-hand sides $\langle \mathbf{e}, b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)}$ of the critical point equations, following the definition of the operator b_2 in Section 3,

$$b_2[\mathbf{m}] = -(P'_h \star \widetilde{m}_1 - Q'_h \star \widetilde{m}_2)|_K, \quad \forall \mathbf{m} = (m_1, m_2) \in L^2(S, \mathbb{R}^2),$$

we first compute, from the above definition of \mathbf{e}_i for $i = 1, 2$:

$$\begin{aligned} \langle \mathbf{e}_1, b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)} &= \langle b_2[\mathbf{e}_1], g_k \rangle_{L^2(K)} \\ &= \int_{-s}^s \int_{-q}^q P'_h(x-t) e^{ik\pi \frac{t}{q}} dt dx = \int_{-q}^q (P_h(s-t) - P_h(s+t)) e^{ik\pi \frac{t}{q}} dt, \end{aligned}$$

and

$$\begin{aligned} \langle \mathbf{e}_2, b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)} &= \langle b_2[\mathbf{e}_2], g_k \rangle_{L^2(K)} \\ &= \int_{-s}^s \int_{-q}^q Q'_h(x-t) e^{ik\pi \frac{t}{q}} dt dx = \int_{-q}^q (Q_h(s-t) - Q_h(s+t)) e^{ik\pi \frac{t}{q}} dt. \end{aligned}$$

These expressions are computed by discretization on a family of piecewise constant functions on $K = \cup_l K_l$, summing up the contributions on K_l , although they are linked to the Fourier coefficients of $\chi_K P_h$ and $\chi_K Q_h$.

This is all we need in order to solve the linear systems (12) in c_n and (14) in d_n , and to compute the solutions ϕ_i related to \mathbf{e}_i and M .

In order to generate synthetic data, the components m_i of magnetizations $\mathbf{m} \in L^2(S, \mathbb{R}^2)$ are also modelled as finite linear combinations of a family of piecewise constant functions on $S = \cup_l S_l$, $l = 1, \dots, L$.

The quantity $b_2[\mathbf{m}]$ is computed from its Fourier series, while its Fourier coefficients (equal to $\langle b_2[\mathbf{m}], g_k \rangle_{L^2(S)}$) are computed as those of $b_2[\mathbf{e}_i]$ above, summing up the contributions of the intervals S_l , $l = 1, \dots, L$.

We finally compute (with a quadrature method):

$$\langle b_2[\mathbf{m}], \phi_i \rangle_{L^2(K)} = \sum_{n \in \mathbb{Z}} c_n \langle b_2[\mathbf{m}], g_n \rangle_{L^2(K)},$$

which is expected to furnish the approximation $\langle m_i^\varepsilon \rangle = \langle b_2[\mathbf{m}], \phi_i \rangle_{L^2(K)}$ to $\langle m_i \rangle$. For error estimation, we will compare $\langle m_i^\varepsilon \rangle$ to the actual moments $\langle m_i \rangle$ ($i = 1, 2$).

5.2 Numerical illustrations

In this section, we discuss preliminary numerical results concerning the computation of $\langle m_i^e \rangle = \langle b_2[\mathbf{m}], \phi_i \rangle_{L^2(K)}$ in order to estimate the mean value $\langle m_i \rangle$ of each component of \mathbf{m} . Recall that the solutions to (CPE $_{m_0}$) in $L^2(K)$, $\phi_i = \phi_o[\mathbf{e}_i]$, $i = 1, 2$, are given by:

$$\left[-P'_h \star (\chi_S (P'_h \star \tilde{\phi}_i)) + Q'_h \star (\chi_S (Q'_h \star \tilde{\phi}_i)) \right]_{|K} + \lambda \phi_i = \begin{cases} -(P'_h \star \chi_S)_{|K} & i = 1, \\ (P'_h \star \mathcal{H} \chi_S)_{|K} & i = 2, \end{cases}$$

for $\lambda > 0$; we have $\|\phi_i\|_{L^2(K)} = M(\lambda)$.

The solutions $\phi_i = \phi_o[\mathbf{e}_i]$, $i = 1, 2$, to (CPE $_{m_0, W}$) in $W_0^{1,2}(K)$ are given by:

$$\left[-P'_h \star (\chi_S (P_h \star \tilde{\phi}'_i)) + Q'_h \star (\chi_S (Q_h \star \tilde{\phi}'_i)) \right]_{|K} - \lambda \phi''_i = \begin{cases} -(P'_h \star \chi_S)_{|K} & i = 1, \\ (P'_h \star \mathcal{H} \chi_S)_{|K} & i = 2, \end{cases}$$

with $\lambda = -\gamma > 0$, and yet $\|\phi_i\|_{W_0^{1,2}(K)} = M(\lambda)$, in the weak sense in $L^2(K)$.

The quantities (coefficients) $\langle b_2^*[g_n], b_2^*[g_k] \rangle_{L^2(S, \mathbb{R}^2)}$ in (12) and (14) are computed with *fft* (fast Fourier transform), following Lemma 4, while for numerical purposes, the series will be truncated at some order $N \in \mathbb{N}_*$.

Various situations can be examined, with respect to the distance h between the magnetization support \bar{S} contained in $\mathbb{R} \times \{0\}$ and the measurement set $K \subset \mathbb{R} \times \{h\}$, and to their length $|S|$, $|K|$. In order to match with the physics of the model, we assume that $S = (-s, s)$ and $K = (-q, q)$ are centered at the origin, and that $|K| > |S| > h > 0$, with (approximately) $|K| \simeq 1.5 |S|$, $|S| \simeq 20h$. We will take $h = 0.1$, $s = 1$ and $q = 1.5$. The order of truncation will be $N = 250$.

We will compare the estimates $\langle m_i^e \rangle$ of $\langle m_i \rangle$ for functions ϕ_i solutions to the BEP in L^2 or in $W_0^{1,2}$, associated to particular values of λ , for some magnetizations \mathbf{m} . The computations are performed with Matlab, R2017a.

Tables below furnish the estimated moments $\langle m_i^e \rangle$ for $i \in \{1, 2\}$, the corresponding values of the relative errors ε_i and norm constraints M , $\|\phi_i\|_{L^2(K)}$ or $\|\phi_i\|_{W_0^{1,2}(K)}$. Figures are provided in the Appendix.

Below, for the computation of the solutions ϕ_i to the corresponding BEP, we choose $\lambda = \lambda_i$ such that $1 \leq -\log_{10}(\lambda_i) \leq 9$ is an integer and such that the relative error for moments estimation given by

$$\varepsilon_i = \frac{|\langle m_i \rangle - \langle m_i^e \rangle|}{|\langle m_i \rangle|},$$

is small enough among this range for the corresponding data \mathbf{m} , while still providing an acceptable value for the quantity $M(\lambda_i)$, say between 10 and 20, see Table 1. Observe that this trade off is made possible by the fact that the data are available for the present simulations. Elsewhere, in general, $\lambda = \lambda_i$ has to be chosen in terms of the behaviour of the error $\|b_2^*[\phi_i] - \mathbf{e}_i\|_{L^2(S, \mathbb{R}^2)}$ (and of the associated constraint $M(\lambda_i)$), independently of the unknown magnetization \mathbf{m} .

In all the numerical experiments, the net moment will be kept fixed to: $\langle m_1 \rangle = -0.1$ and $\langle m_2 \rangle = 0.1$, whence:

$$\varepsilon_i = 10 |\langle m_i \rangle - \langle m_i^e \rangle| = |(-1)^{i-1} + 10 \langle m_i^e \rangle|.$$

5.2.1 Solutions ϕ_i to BEP

The solutions $\phi_i \in L^2(K)$ for $i = 1, 2$ are plotted in Figure 5, 6 ($\lambda_i = 10^{-3}$ and 10^{-5}). The solutions $\phi_i \in W_0^{1,2}(K)$ can be seen in Figures 7, 8 ($\lambda_i = 10^{-8}$ and 10^{-9}). Their norms are given in Table 1.

ϕ_i	λ_i	$\ \phi_1\ $	$\ \phi_2\ $
in $L^2(K)$	10^{-3}	4.8	4.4
	10^{-5}	14.4	8.2
in $W_0^{1,2}(K)$	10^{-8}	19.9	10.4
	10^{-9}	645.5	221.7

Table 1: Values of λ_i , $M(\lambda_i) = \|\phi_i\|_{L^2(K)}$ or $\|\phi_i\|_{W_0^{1,2}(K)}$, $i = 1, 2$.

The solutions in $L^2(K)$ show quite many oscillations close to the boundary of S , less in $W_0^{1,2}(K)$, as expected. Interestingly, these functions behave as affine or constant in a large interval contained in the interior of S . We also see that as λ_i decreases, the corresponding constraint $M(\lambda_i)$ grows, which also increases the oscillating phenomenon, mainly in $L^2(K)$.

5.2.2 Constant magnetizations

Take $m_1 = -0.05$ on $S = [-1, 1]$ and $m_2 = 0.05$ on S (whence $\mathbf{m} = 0.05(\mathbf{e}_1 - \mathbf{e}_2)$ and $m_i = 0.05(-1)^i \chi_S$). See Figure 1 and Table 2 for plots of m_i and of $b_2[\mathbf{m}]$, computed as explained in Section 5.1.3.

ϕ_i	λ_i	$\langle m_1^e \rangle$	$\langle m_2^e \rangle$	ε_1	ε_2
in $L^2(K)$	10^{-5}	-0.1044	0.09581	$4.4 \cdot 10^{-4}$	$4.2 \cdot 10^{-3}$
in $W_0^{1,2}(K)$	10^{-8}	-0.0996	0.0994	$3.8 \cdot 10^{-3}$	$6.4 \cdot 10^{-3}$

Table 2: \mathbf{m} identically constant.

5.2.3 Magnetizations with large support

Take $m_1 = -0.1$ on $[-1, 0]$, 0 elsewhere in S , and $m_2 = 0.1$ on $[0, 1]$, see Figure 2, together with Table 3.

ϕ_i	λ_i	$\langle m_1^e \rangle$	$\langle m_2^e \rangle$	ε_1	ε_2
in $L^2(K)$	10^{-5}	-0.0999	0.0994	$6.4 \cdot 10^{-4}$	$5.5 \cdot 10^{-3}$
in $W_0^{1,2}(K)$	10^{-8}	-0.1000	0.0995	$4.4 \cdot 10^{-4}$	$4.6 \cdot 10^{-3}$

Table 3: \mathbf{m} with large support.

5.2.4 Other magnetization

Here we take m_1 and m_2 as below. See Figure 3 and Table 4.

$$m_1 = \begin{cases} -0.05 & \text{on } [-0.2, 0], \\ -0.1 & \text{on } [0, 0.2], \\ -0.2 & \text{on } [0.2, 0.4], \\ -0.1 & \text{on } [0.4, 0.6], \\ -0.05 & \text{on } [0.6, 0.8], \\ 0 & \text{elsewhere in } S, \end{cases} \quad m_2 = \begin{cases} 0.05 & \text{on } [-0.8, -0.6], \\ 0.1 & \text{on } [-0.6, -0.4], \\ 0.2 & \text{on } [-0.4, -0.2], \\ 0.1 & \text{on } [-0.2, 0], \\ 0.05 & \text{on } [0, 0.2], \\ 0 & \text{elsewhere in } S. \end{cases}$$

ϕ_i	λ_i	$\langle m_1^\varepsilon \rangle$	$\langle m_2^\varepsilon \rangle$	ε_1	ε_2
in $L^2(K)$	10^{-5}	-0.0981	0.09855	$1.9 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$
in $W_0^{1,2}(K)$	10^{-8}	-0.0977	0.0989	$2.3 \cdot 10^{-2}$	$1.1 \cdot 10^{-2}$

Table 4: Other \mathbf{m} .

5.2.5 Magnetizations with small support

Here we take m_1 and m_2 as described below. See Figure 4 and Table 5.

$$m_1 = \begin{cases} 10 & \text{on } [-0.5, -0.49], \\ -10 & \text{on } [0, 0.0.1], \\ -10 & \text{on } [0.2, 0.21], \\ 0 & \text{elsewhere in } S, \end{cases} \quad m_2 = \begin{cases} 10 & \text{on } [-0.9, -0.89], \\ -10 & \text{on } [-0.3, -0.29], \\ 10 & \text{on } [0.2, 0.21], \\ 0 & \text{elsewhere in } S. \end{cases}$$

ϕ_i	λ_i	$\langle m_1^\varepsilon \rangle$	$\langle m_2^\varepsilon \rangle$	ε_1	ε_2
in $L^2(K)$	10^{-5}	-0.104	0.0958	$4.4 \cdot 10^{-2}$	$4.2 \cdot 10^{-2}$
in $W_0^{1,2}(K)$	10^{-8}	-0.1015	0.0969	$1.5 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$

Table 5: \mathbf{m} with small support.

5.2.6 Comments, discussion

Overall, as presented in the above tables, we obtain quite accurate results for net moment estimation. The net moments of magnetizations with large support are more precisely estimated than those of magnetizations with small support: the value ε_i of the error decreases whenever the size of the support increases, for a same value of the parameter λ_i . This phenomenon is mostly true in $L^2(K)$ for the present examples. These examples may not always provide smaller estimation errors ε_i in $W_0^{1,2}(K)$ than in $L^2(K)$, since the behaviour of ε_i depend on the specific data (smoothness properties of \mathbf{m} , size of its support, ...) and parameters. These properties, together with the influence of the noise in the computations and in the data, will be studied in a forthcoming work.

6 Perspectives, conclusions

In order to complete the results of Section 3, some properties of b_2 remain to be studied. Even if b_2 is injective, it is not coercive (strongly injective), in the sense that $\|b_2[\mathbf{m}]\|_{L^2(K)}$ can be small even when $\|\mathbf{m}\|_{L^2(S, \mathbb{R}^2)}$ is not small: there may exist such ‘‘almost silent’’ source terms \mathbf{m} . However, lower bounds for $\|b_2\|$ can be established if we restrict to truncated Fourier expansions, showing that these cannot be ‘‘almost silent’’. Also, one can estimate the constants involved in the upper bounds of $\|b_2\|$ and $\|b_2^*\|$. The spectral study of the operator $b_2 b_2^*$ remains to be fulfilled, in view of the numerical analysis of the BEP.

The BEP studied in Section 4 can be stated with slightly more general constraints. In particular, let $f \in L^2(K)$. Consider the following BEP in $L^2(K)$:

$$\min_{\substack{\phi \in L^2(K) \\ \|\phi - f\|_{L^2(K)} \leq M}} \|b_2^*[\phi] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)}. \quad (\text{BEP}_f)$$

One can prove existence and uniqueness of the function $\phi_o \in L^2(K)$ solution of (BEP_f) using the same arguments as in the proof of Proposition 6. Moreover, if $\mathbf{e} \notin \text{Ran } b_2^*$ (or

if $\mathbf{e} = b_2^*[\phi]$ for some $\phi \in L^2(K)$ such that $\|\phi - f\|_{L^2(K)} \geq M$, then the solution ϕ_o still saturates the constraint, i.e. $\|\phi_o - f\|_{L^2(K)} = M$, and is given by the following implicit equation, with $\gamma < 0$, see (BEP $_{m_o}$):

$$b_2 b_2^*[\phi_o] - \gamma \phi_o = b_2[\mathbf{e}] + \gamma f.$$

Let now $f \in W_0^{1,2}(K)$. The BEP can be stated as follows:

$$\min_{\substack{\phi \in W_0^{1,2}(K) \\ \|\phi - f\|_{W_0^{1,2}(K)} \leq M}} \|b_2^*[\phi] - \mathbf{e}\|_{L^2(S, \mathbb{R}^2)}. \quad (\text{BEP}_{f,W})$$

Again, as for (BEP $_{m_o,W}$), a solution $\phi_o \in W_0^{1,2}(K)$ exists, is unique, and saturates the constraint if $\mathbf{e} \notin b_{2|W}^*[W_0^{1,2}(K)]$ (or if $\mathbf{e} = b_{2|W}^*[\phi]$ for $\phi \in L^2(K)$ such that $\|\phi - f\|_{W_0^{1,2}(K)} \geq M$): $\|\phi_o - f\|_{W_0^{1,2}(K)} = M$. It satisfies the following equation, with $\gamma < 0$:

$$b_2 b_2^*[\phi_o] + \gamma \phi_o'' = b_2[\mathbf{e}] + \gamma f.$$

The above problem will be considered for $\phi_o \in W^{1,2}(K)$ with norm constraint therein, where it is also well-posed and gives promising numerical results. Moreover, $W^{1,2}(K)$ functions are continuous on K in the present 1-dimensional setting, which is a suitable property of the solution, in view of computing its inner product with the available measurements in a pointwise way. Note however that the continuity of the solution is already granted by the critical point equation (CPE $_{m_o}$) above and the smoothness properties of functions in $\text{Ran } b_2$. Some BEP of mixed type, with constraints in other norms, like uniform, L^1 , BMO, could also be formulated, as well as related extremal problems consist in looking for the best bounded extension of a given data $b_d \simeq b_2[\mathbf{m}]$ from K to \mathbb{R} .

Following Section 5, further numerical analysis and improvements of the computational schemes will be considered. In particular, we expect that refined implementations of the computations in the Fourier domain will provide more efficient and accurate recovery schemes, in the present setting as well as in the 3D one, see [6]. Both a quantitative and a qualitative study of the relations $\lambda \mapsto M(\lambda)$ and $\lambda \mapsto \epsilon_i(\lambda)$ (for the different error terms) remain to be done. This will be the topic of a further work, together with a study concerning the influence of the parameters s, q, h , and of the characteristics of m_i , on the behaviour of the solutions.

Both explicit and asymptotic expressions, in terms of the size q of the measurement set K , relating the first moments of \mathbf{m} to those of $b_2[\mathbf{m}]$, could also be derived, as in [8] for the 3D situation. Besides, the quantities $b_2^*[x^k]$ for monomials x^k could be exactly computed.

Other functions \mathbf{e} may be considered as well, both for the present net moment estimation problem or for higher order or local moments recovery. One can also take the components e_i to be appropriate band-limited basis functions, of Slepian type [22].

Particularly interesting magnetizations are the unidirectional ones, of which the moment recovery using the above linear estimator process still requires a specific study. Also, more general situations where \mathbf{m} belongs to $L^1(S, \mathbb{R}^2)$ or is a measure remain to be considered, see [5], together with magnetizations supported on a two-dimensional set.

Finally, the full inversion problem of recovering \mathbf{m} in $L^2(S, \mathbb{R}^2)$ itself from $b_2[\mathbf{m}]$ will be considered in a further work. It can also be stated as a BEP of which the solution will involve the operator $b_2 b_2^*$. Again more general magnetization distributions will be studied.

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Appendix: numerical illustrations



Figure 1: m_1, m_2 constant, associated $b_2[\mathbf{m}]$.

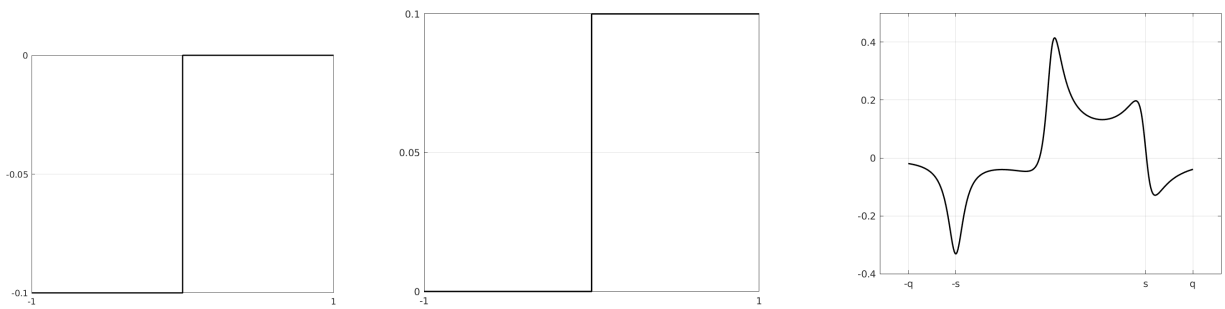


Figure 2: m_1, m_2 with large support, associated $b_2[\mathbf{m}]$.

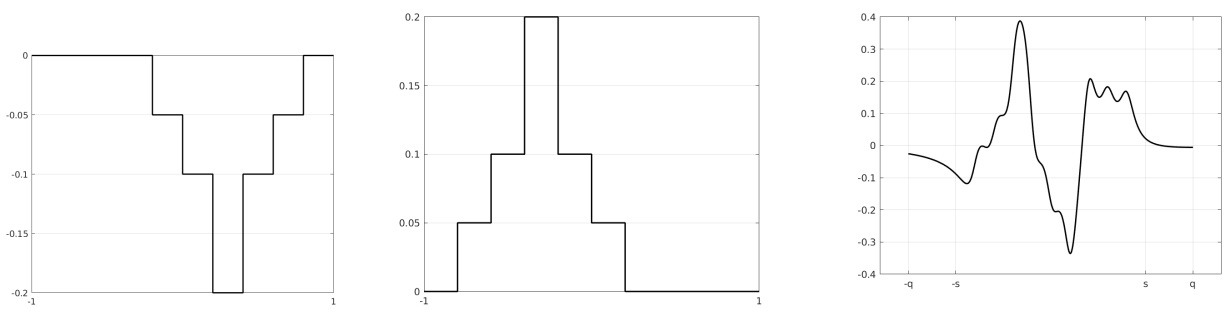


Figure 3: Others m_1, m_2 , associated $b_2[\mathbf{m}]$.

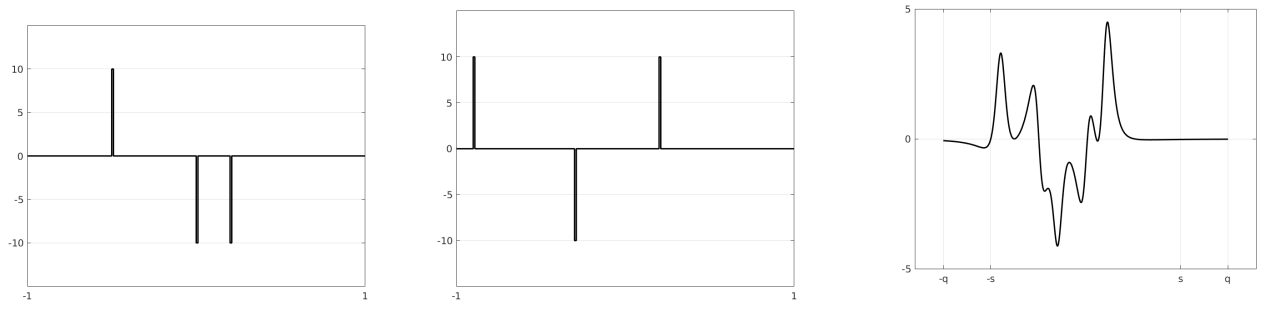


Figure 4: m_1, m_2 with small support, associated $b_2[\mathbf{m}]$.

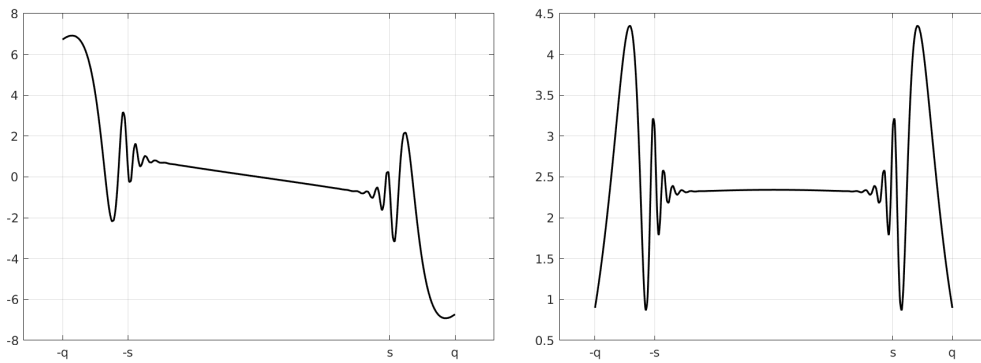


Figure 5: Solutions $\phi_i \in L^2(K)$ for $i = 1, 2$, $\lambda_i = 10^{-3}$.

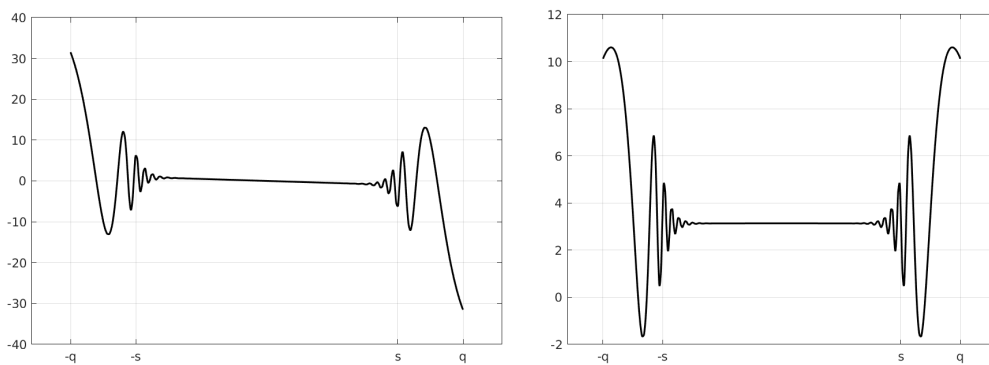


Figure 6: Solutions $\phi_i \in L^2(K)$ for $i = 1, 2$, $\lambda_i = 10^{-5}$.

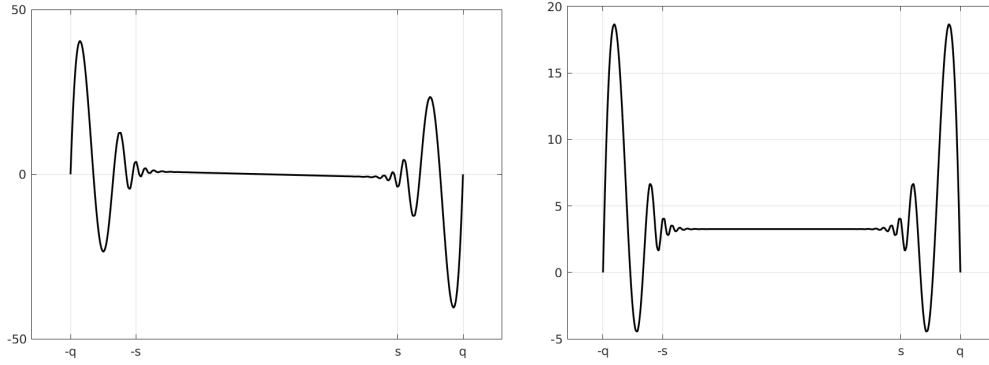


Figure 7: Solutions $\phi_i \in W_0^{1,2}(K)$, $i = 1, 2$, $\lambda_i = 10^{-8}$.

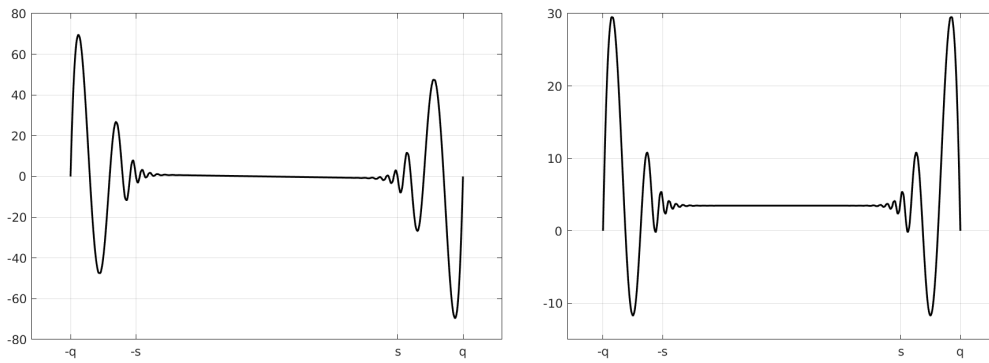


Figure 8: Solutions $\phi_i \in W_0^{1,2}(K)$, $i = 1, 2$, $\lambda_i = 10^{-9}$.