HAL
open science

Schauder Estimates for a Class of Potential Mean Field Games of Controls

Joseph Frédéric Bonnans, Saeed Hadikhanloo, Laurent Pfeiffer

To cite this version:

Joseph Frédéric Bonnans, Saeed Hadikhanloo, Laurent Pfeiffer. Schauder Estimates for a Class of Potential Mean Field Games of Controls. Applied Mathematics and Optimization, 2019, pp.34. hal-02048437

HAL Id: hal-02048437

https://inria.hal.science/hal-02048437v1

Submitted on 25 Feb 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Schauder Estimates for a Class of Potential Mean Field Games of Controls*

J. Frédéric Bonnans† Saeed Hadikhanloo‡ Laurent Pfeiffer§

February 25, 2019

Abstract

An existence result for a class of mean field games of controls is provided. In the considered model, the cost functional to be minimized by each agent involves a price depending at a given time on the controls of all agents and a congestion term. The existence of a classical solution is demonstrated with the Leray-Schauder theorem; the proof relies in particular on a priori bounds for the solution, which are obtained with the help of a potential formulation of the problem.

Keywords: Mean field games of controls, extended mean field games, strongly coupled mean field games, potential formulation, Hölder estimates.

AMS Classification: 91A13, 49N70.

1 Introduction

The goal of this work is to prove the existence and uniqueness of a classical solution to the following system of partial differential equations:

$$\begin{cases} (i) & -\partial_t u - \sigma\Delta u + H(x,t,\nabla u(x,t) + \phi(x,t)^\intercal P(t)) = f(x,t,m(t)) & (x,t)\in Q, \\ (ii) & \partial_t m - \sigma\Delta m + \operatorname{div}(vm) = 0 & (x,t)\in Q, \\ (iii) & P(t) = \Psi\left(t, \int_{\mathbb{T}^d} \phi(x,t)v(x,t)m(x,t)\,\mathrm{d}x\right) & t\in[0,T], \\ (iv) & v(x,t) = -H_p(x,t,\nabla u(x,t) + \phi(x,t)^\intercal P(t)) & (x,t)\in Q, \\ (v) & m(x,0) = m_0(x), \quad u(x,T) = g(x) & x\in\mathbb{T}^d, \end{cases} \tag{MFGC}$$

where

$$u = u(x,t)\in\mathbb{R}, \quad m = m(x,t)\in\mathbb{R}, \quad v = v(x,t)\in\mathbb{R}^d, \quad P = P(t)\in\mathbb{R}^k,$$

*The first author acknowledges support from the FiME Lab (Institut Europlace de Finance). The two first authors acknowledges support from the PGMO project 'Optimal control of conservation equations', itself supported by iCODE (IDEX Paris-Saclay) and the Hadamard Mathematics LabEx.

†Inria and CMAP, Ecole Polytechnique, France. E-mail: frederic.bonnans@inria.fr

‡Inria and CMAP, Ecole Polytechnique, France. E-mail: saeed.hadikhanloo@inria.fr

§Institute of Mathematics, University of Graz, Austria. E-mail: laurent.pfeiffer@uni-graz.at

with $(x,t) \in Q := \mathbb{T}^d \times [0,T]$. The parameters $T > 0$, $\sigma > 0$ are given and

$$H\colon (x,t,p) \in Q \times \mathbb{R}^d \to \mathbb{R}, \qquad \Psi\colon (t,z) \in [0,T] \times \mathbb{R}^k \to \mathbb{R}^k, \qquad \phi\colon (x,t) \in Q \to \mathbb{R}^{k\times d},$$
$$f\colon (x,t,m) \in Q \times \mathcal{D}_1(\mathbb{T}^d) \to \mathbb{R}, \qquad m_0\colon x \in \mathbb{T}^d \to \mathbb{R}_+, \qquad g\colon x \in \mathbb{T}^d \to \mathbb{R}$$

are given data. The set $\mathcal{D}_1(\mathbb{T}^d)$ is defined as

$$\mathcal{D}_1(\mathbb{T}^d) = \Big\{ m \in L^\infty(\mathbb{T}^d) \,|\, m \geq 0, \int_{\mathbb{T}^d} m(x)\, \mathrm{d}x = 1 \Big\}. \tag{1}$$

Here we work with $\mathbb{Z}^d$-periodic data and we set the state set as the d-dimensional torus $\mathbb{T}^d$, that is a quotient set $\mathbb{R}^d/\mathbb{Z}^d$. The Hamiltonian H is assumed to be such that $H(x,t,p) = L^*(x,t,-p)$, for some mapping L, where $L^*(x,t,-p)$ denotes the Fenchel transform with respect to p. The mapping L is assumed to be convex in its third variable.

The function u, as a solution to the Hamilton-Jacobi-Bellman (HJB) in equation (i)(MFGC) is the value function corresponding to the stochastic optimal control problem:

$$\inf_\alpha \mathbb{E}\Big[\int_0^T L(X_t,t,\alpha_t) + \langle \phi(X_t,t)^\intercal P(t), \alpha_t \rangle + f(X_t,t,m(t))\, \mathrm{d}t + g(X_T) \Big], \tag{2}$$

subject to the stochastic dynamics $\mathrm{d}X_t = \alpha_t\, \mathrm{d}t + \sqrt{2\sigma}\, \mathrm{d}B_t$, $X_0 = x \in \mathbb{T}^d$. The feedback law v given by (iv)(MFGC) is then optimal for this stochastic optimal control problem. Equation (ii)(MFGC) is the Fokker-Planck equation which describes the evolution of $m_t = \mathcal{L}(X_t)$ when the optimal feedback law is employed. At last, (iii)(MFGC) makes the quantity $P(t)$ endogenous.

An interpretation of the system (MFGC) is as follows. Consider a stock trading market. A typical trader, with an initial level of stock $X_0 = x$, controls its level of stock $(X_t)_{t\in[0,T]}$ through the purchasing rate α_t with stochastic dynamic $\mathrm{d}X_t = \alpha_t \mathrm{d}t + \sqrt{2\sigma}\mathrm{d}B_t$. The agent aims at minimizing the expected cost (2) where $P(t)$ is the price of the stock at time t. The agent is considered to be infinitesimal and has no impact on $P(t)$, so it assumes the price as given in its optimization problem. On the other hand, in the equilibrium configuration, the price $P(t)$ ($t \in [0,T]$) becomes endogenous and indeed, is a function of the optimal behaviour of the whole population of agents as formulated in (iii)(MFGC). The expression $D(t) := \int_{\mathbb{T}^d} \phi(x,t) v(x,t) m(x,t)\, \mathrm{d}x$ can be considered as a weigted net demand formulation and the relation $P = \Psi(D)$ is the result of supply-demand relation which determines the price of the good at the market. Thus, this system captures an equilibrium configuration. Similar models have been proposed in the electrical engineering literature, see for example [2, 10, 11] and the references therein.

In most mean field game models, the individual players interact through their position only, that is, via the variable m. The problem that we consider belongs to the more general class of problems, called *extended mean field games*, for which the players interact through the joint probability distribution μ of states and controls. Several existence results have been obtained for such models: in [13] for stationary mean field games, in [15] for deterministic mean field games. In [6, Section 5], a class of problems where μ enters in the drift and the integral cost of the agents is considered. We adopt the terminology *mean field games of controls* employed by the authors of the latter reference. Let us mention that our existence proof is different from the one of [6], which includes control bounds. In [3, Section 1], a model where the drift of the players depends on μ is analyzed. In [14], a mean field game model is considered where at all time t, the average control (with respect to all players) is prescribed. We finally mention that extended mean field games

have been studied with a probabilistic approach in [1, 8] and in [7, Section 4.6], and that a class of linear-quadratic extended mean field games has been analyzed in [19].

A difficulty in the study of mean field games of controls is the fact that the control variable, at a given time t, cannot be expressed in an explicit fashion as a function of $m(\cdot, t)$ and $u(\cdot, t)$. Instead, one has to analyze the well-posedness and the stability of a fixed point equation (see for example [6, Lemma 5.2]). In our model, if we combine (iii) and (iv)(MFGC), we obtain the fixed point equation

$$v = -H_p(\nabla u + \Psi(\textstyle\int \phi v m)) \tag{3}$$

for the control variable v. A central idea of the present article is the following: equation (3) is equivalent to the optimality conditions of a convex optimization problem, when L is convex and Ψ is the gradient of a convex function Φ. This observation allows to show the existence and uniqueness of a solution v (to equation (3)) and to investigate its dependence with respect to ∇u and m in a natural way. More precisely, we prove that this dependence is locally Hölder continuous.

The existence of a classical solution of (MFGC) is established with the Leray-Schauder theorem and classical estimates for parabolic equations. A similar approach has been employed in [16] and [17] for the analysis of a mean field game problem proposed by Chan and Sircar in [9]. In this model, each agent exploits an exhaustible resource and fixes its price. The evolution of the capacity of a given producer depends on the price set by the producer, but also on the average price (with respect to all producers).

The application of the Leray-Schauder theorem relies on a priori bounds for fixed points. These bounds are obtained in particular with a potential formulation of the mean field game problem: we prove that all solutions to (MFGC) are also solutions to an optimal control problem of the Fokker-Planck equation. We are not aware of any other publication making use of such a potential formulation for a mean field game of controls, with the exception of [17] for the Chan and Sircar model. Let us mention that besides the derivation of a priori bounds, the potential formulation of the problem can be very helpful for the numerical resolution of the problem and the analysis of learning procedures (which are out of the scope of the present work).

The article is structured as follows. We list in Section 2 the assumptions employed all along. The main result (Theorem 3.1) is stated in Section 3. We provide in Section 4 a first incomplete potential formulation of the problem, incomplete in so far as the term $f(m)$ is not integrated. We also introduce some auxiliary mappings, which allow to express P and v as functions of m and u. We give some regularity properties for these mappings in Section 5. In Section 6 we establish some a priori bounds for solutions to the coupled system. We finally prove our main result in Section 7. In the last section, we give a full potential formulation of the problem, prove the uniqueness of the solution to (MFGC) and prove that $(u, P, f(m))$ is the solution to an optimal control problem of the HJB equation, under an additional monotonicity condition on f.

2 Assumptions on data and parabolic estimates

2.1 Notation and assumptions

Let us introduce the main notation used in the article. Recall that $\mathcal{D}_1(\mathbb{T}^d)$ was defined in (1). For all $m \in \mathcal{D}_1(\mathbb{T}^d)$, for all measurable functions $v\colon \mathbb{T}^d \to \mathbb{R}^d$ such that $|v(\cdot)|^2 m(\cdot)$ is integrable, the

following inequality holds true,

$$\Big|\int_{\mathbb{T}^d} v(x)m(x)\,\mathrm{d}x\Big|^2 \leq \int_{\mathbb{T}^d} |v(x)|^2 m(x)\,\mathrm{d}x. \tag{4}$$

by the Cauchy-Schwarz inequality.

The gradient of the data functions with respect to some variable is denoted with an index, for example, H_p denotes the gradient of H with respect to p. The same notation is used for the Hessian matrix. The gradient of u with respect to x is denoted by ∇u. Let us mention that very often, the variables x and t are omitted, to alleviate the calculations. We also denote by $\int \phi v m$ the integral $\int_{\mathbb{T}^d} \phi v m\,\mathrm{d}x$ when used as a second argument of Ψ. For a given normed space X, the ball of center 0 and radius R is denoted $B(X,R)$.

Along the article, we use the following Hölder spaces: $\mathcal{C}^\alpha(Q)$, $\mathcal{C}^{2+\alpha}(\mathbb{T}^d)$, and $\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)$, defined as usual with $\alpha \in (0,1)$. Sobolev spaces are denoted by $W^{k,p}$, the order of derivation k being possibly non-integral. We fix now a real number p such that

$$p > d+2.$$

We will also make use of the following Banach space:

$$W^{2,1,p}(Q) = L^p(0,T;W^{2,p}(\mathbb{T}^d)) \cap W^{1,p}(Q).$$

Convexity assumptions We collect below the required assumptions on the data. As announced in the introduction, H is related to the convex conjugate of a mapping $L\colon Q\times\mathbb{R}^d \to \mathbb{R}$ as follows:

$$H(x,t,p) = L^*(x,t,-p) = \sup_{v\in\mathbb{R}^n} -\langle p,v\rangle - L(x,t,v). \tag{5}$$

The mapping L is assumed to be strongly convex in its third variable, uniformly in x and t, that is, we assume that L is differentiable with respect to v and that there exists $C>0$ such that

$$\langle L_v(x,t,v_2) - L_v(x,t,v_1), v_2 - v_1\rangle \geq \frac{1}{C}|v_2-v_1|^2, \tag{A1}$$

for all $(x,t)\in Q$ and for all v_1 and $v_2 \in \mathbb{R}^d$. This ensures that H takes finite values and that H is continuously differentiable with respect to p, as can be easily checked. Moreover, the supremum in (5) is reached for a unique v, which is then given by $v = -H_p(x,t,p)$, i.e.

$$H(x,t,p) + L(x,t,v) + \langle p,v\rangle = 0 \Longleftrightarrow v = -H_p(x,t,p), \quad \forall (x,t,p,v)\in Q\times\mathbb{R}^d\times\mathbb{R}^d. \tag{6}$$

We also assume that Ψ has a potential, that is, there exists a mapping $\Phi\colon [0,T]\times\mathbb{R}^k \to \mathbb{R}$, differentiable in its second argument, such that

$$\Psi(t,z) = \Phi_z(t,z), \quad \forall (t,z)\in[0,T]\times\mathbb{R}^k. \tag{7}$$

Regularity assumptions We assume that L_v is differentiable with respect to x and v and that ϕ is differentiable with respect to x. All along the article, we make use of the following assumptions.

Growth assumptions: there exists $C > 0$ such that for all $(x,t) \in Q$, $y \in \mathbb{T}^d$, $v \in \mathbb{R}^d$, $z \in \mathbb{R}^k$, and $m \in \mathcal{D}_1(\mathbb{T}^d)$,

- $L(x,t,v) \leq C|v|^2 + C$ (A2)
- $|L(y,t,v) - L(x,t,v)| \leq C|y-x|(1+|v|^2)$ (A3)
- $|\Psi(t,z)| \leq C|z| + C$ (A4)
- $|f(x,t,m)| \leq C.$ (A5)

Hölder continuity assumptions:

- For all $R > 0$, there exists $\alpha \in (0,1)$ such that
$$\begin{cases} L \in \mathcal{C}^\alpha(B_R), \\ L_v \in \mathcal{C}^\alpha(B_R, \mathbb{R}^d), \\ L_{vx} \in \mathcal{C}^\alpha(B_R, \mathbb{R}^{d\times d}), \\ L_{vv} \in \mathcal{C}^\alpha(B_R, \mathbb{R}^{d\times d}), \end{cases} \qquad \begin{cases} \Psi \in \mathcal{C}^\alpha(B'_R, \mathbb{R}^d), \\ \phi \in \mathcal{C}^\alpha(Q, \mathbb{R}^{k\times d}), \\ D_x\phi \in \mathcal{C}^\alpha(Q, \mathbb{R}^{k\times d\times d}), \end{cases} \tag{A6}$$
where $B_R = Q \times B(\mathbb{R}^d, R)$ and $B'_R = [0,T] \times B(\mathbb{R}^k, R)$.
- There exists $\alpha \in (0,1)$ and $C > 0$ such that
$$|f(x_2,t_2,m_2) - f(x_1,t_1,m_1)| \leq C\big(|x_2 - x_1| + |t_2 - t_1|^\alpha + \|m_2 - m_1\|^\alpha_{L^\infty(\mathbb{T}^d)}\big), \tag{A7}$$
for all (x_1,t_1) and $(x_2,t_2) \in Q$ and for all m_1 and $m_2 \in \mathcal{D}_1(\mathbb{T}^d)$.
- There exists $\alpha \in (0,1)$ such that $m_0 \in \mathcal{C}^{2+\alpha}(\mathbb{T}^d)$ and $g \in \mathcal{C}^{2+\alpha}(\mathbb{T}^d)$. (A8)

We finally assume that
$$m_0(x) \geq 0, \quad \forall x \in \mathbb{T}^d \quad \text{and} \quad \int_{\mathbb{T}^d} m_0(x)\,\mathrm{d}x = 1. \tag{A9}$$

Let us mention here that the variables $C > 0$ and $\alpha \in (0,1)$ used all along the article are generic constants. The value of C may increase from an inequality to the next one and the value of the exponent α may decrease.

Some lower bounds for L and for Φ can be easily deduced from the convexity assumptions. By Assumption (A6), $L(x,t,0)$ and $L_v(x,t,0)$ are bounded. It follows then from the strong convexity assumption (A1) and (A2) that there exists a constant $C > 0$ such that
$$\frac{1}{C}|v|^2 - C \leq L(x,t,v), \qquad \text{for all } (x,t,v) \in Q \times \mathbb{R}^d. \tag{8}$$
Without loss of generality, we can assume that $\Phi(t,0) = 0$, for all $t \in [0,T]$. Since Φ is convex, we have that $\Phi(t,z) \geq \langle \Psi(t,0), z\rangle$, for all $z \in \mathbb{R}^k$. We deduce then from Assumption (A4) that
$$\Phi(t,z) \geq -C|z|, \quad \text{for all } z \in \mathbb{R}^k, \tag{9}$$
where C is independent of t and z.

Some regularity properties for the Hamiltonian can be deduced from the convexity assumption (A1) and the Hölder continuity of L and its derivatives (Assumption (A6)). They are collected in the following lemma.

Lemma 2.1. *The Hamiltonian H is differentiable with respect to p and H_p is differentiable with respect to x and p. Moreover, for all $R > 0$, there exists $\alpha \in (0,1)$ such that*

$$\begin{aligned} &H \in \mathcal{C}^\alpha(B_R), && H_p \in \mathcal{C}^\alpha(B_R, \mathbb{R}^d), \\ &H_{px} \in \mathcal{C}^\alpha(B_R, \mathbb{R}^{d\times d}), && H_{pp} \in \mathcal{C}^\alpha(B_R, \mathbb{R}^{d\times d}). \end{aligned}$$

Proof. For a given $(x,t,p) \in Q \times \mathbb{R}^d$, there exists a unique $v := v(x,t,p)$ maximizing the function $v \in \mathbb{R}^d \mapsto -\langle p, v\rangle - L(x,t,p)$, which is strongly concave by (A1). It is then easy to deduce from (8) and the boundedness of $L(x,t,0)$ that there exists a constant C, independent of (x,t,p), such that

$$|v(x,t,p)| \le C(|p| + 1). \tag{10}$$

For all $(x,t,p) \in Q \times \mathbb{R}^d$,

$$p + L_v(x,t,v(x,t,p)) = 0. \tag{11}$$

Since L_v is continuously differentiable with respect to x and v, we obtain with the inverse mapping theorem that $v(x,t,p)$ is continuously differentiable with respect to x and p. Let $R > 0$ and let (x_1,t_1,p_1) and $(x_2,t_2,p_2) \in Q \times B_R$. Let $v_i = v(x_i,t_i,p_i)$ for $i = 1,2$. By (10), we have $|v_i| \le C$, where C does not depend on x_i, t_i, and p_i (but depends on R). Moreover, we have

$$\begin{aligned} \langle p_2 - p_1, v_2 - v_1\rangle + \langle L_v(x_2,t_2,v_2) - L_v(x_1,t_1,v_2), v_2 - v_1\rangle \\ + \langle L_v(x_1,t_1,v_2) - L_v(x_1,t_1,v_1), v_2 - v_1\rangle = 0. \end{aligned}$$

We deduce from (A1), Young's inequality, and (A6) that there exists $C > 0$ and $\alpha \in (0,1)$, both independent of x_i, t_i, and p_i (but dependent of R) such that

$$\begin{aligned} \frac{1}{C}|v_2 - v_1|^2 &\le |\langle p_2 - p_1, v_2 - v_1\rangle| + |\langle L_v(x_2,t_2,v_2) - L_v(x_1,t_1,v_2), v_2 - v_1\rangle| \\ &\le \frac{1}{2\varepsilon}|p_2 - p_1|^2 + \varepsilon|v_2 - v_1|^2 + \frac{C}{\varepsilon}\big(|x_2 - x_1|^\alpha + |t_2 - t_1|^\alpha\big), \end{aligned}$$

for all $\varepsilon > 0$. Taking $\varepsilon = \frac{1}{2C}$, we deduce that the mapping $(x,t,p) \in B_R \mapsto v(x,t,p)$ is Hölder continuous. Since L is Hölder continuous on bounded sets, we obtain that the Hamiltonian $H(x,t,p) = -\langle p, v(x,t,p)\rangle - L(x,t,v(x,t,p))$ is Hölder continuous on B_R.

One can easily check that $H_p(x,t,p) = -v(x,t,p)$, which proves that H_p is Hölder continuous on B_R. Finally, differentiating relation (11) with respect to x and p, we obtain that

$$\begin{aligned} D_x v(x,t,p) &= -L_{vv}(x,t,v(x,t,p))^{-1} L_{vx}(x,t,v(x,t,p)) \\ D_p v(x,t,p) &= -L_{vv}(x,t,v(x,t,p))^{-1}. \end{aligned}$$

We deduce then with Assumption (A6) that $D_x v(x,t,p)$ and $D_p v(x,t,p)$ (and thus H_{px} and H_{pp}) are Hölder continuous on B_R, as was to be proved. □

An example of coupling term We finish this subsection with an example of a mapping f satisfying the regularity assumptions (A5) and (A7). Let $\varphi \in L^\infty(\mathbb{R}^d)$ be a given Lipschitz continuous mapping, with modulus C_1. Let us set $C_2 = \|\varphi\|_{L^\infty(\mathbb{R}^d)}$. Let $K\colon Q \times [-C_2, C_2] \to \mathbb{R}$ be a measurable mapping satisfying the following assumptions:

1. The mapping $x \in \mathbb{T}^d \mapsto K(x,0,0)$ lies in $L^1(\mathbb{T}^d)$.

2. There exist a mapping $C_3 \in L^1(\mathbb{T}^d)$ and $\alpha \in (0,1)$ such that for a.e. $x \in \mathbb{T}^d$, for all t_1 and $t_2 \in [0,T]$ and for all w_1 and $w_2 \in [-C_2, C_2]$,

$$|K(x,t_2,w_2) - K(x,t_1,w_1)| \leq C_3(x)\big(|t_2-t_1|^\alpha + |w_2-w_1|^\alpha\big).$$

Let us set $\tilde{\varphi}(x) := \varphi(-x)$. We identify $m \in L^\infty(\mathbb{T}^d)$, with its extension by 0 over $\mathbb{R}^d$ so that the convolution product below is well-defined:

$$m * \varphi(x) := \int_{\mathbb{R}^d} m(x-y)\varphi(y)\mathrm{d}y, \quad x \in \mathbb{T}^d. \tag{12}$$

We keep in mind that $m * \varphi$ is a function over $\mathbb{T}^d$. Then

$$\|m * \varphi\|_\infty \leq \|\varphi\|_\infty, \quad \text{for all } m \in \mathcal{D}_1(\mathbb{T}^d). \tag{13}$$

In a similar way we can define

$$f_K(x,t,m) = (K(\cdot,t,m*\varphi(\cdot)) * \tilde{\varphi})(x), \tag{14}$$

and we have that

$$\|f_K(x,t,m)\|_\infty \leq \|K(\cdot,t,m*\varphi)\|_1 \|\tilde{\varphi}\|_\infty \leq \||K(\cdot,t,m*\varphi)\|_\infty \|\varphi\|_\infty \tag{15}$$

The following lemma is inspired from [4, Example 1.1].

Lemma 2.2. *The above mapping f_K satisfies Assumptions (A5) and (A7).*

Proof. Assumption (A5) follows from (15). We next prove (A7). Let (x_1,t_1) and $(x_2,t_2) \in Q$, let m_1 and $m_2 \in \mathcal{D}_1(\mathbb{T}^d)$. Then

$$\begin{aligned} |f_K(x_2,t_2,m_2) - f_K(x_1,t_2,m_2)| &\leq \|K(y,t_2,m_2*\varphi(y))\|_1 \|\varphi(x_2-\cdot) - \varphi(x_1-\cdot)\|_\infty \\ &\leq C_1 C |x_2 - x_1|. \end{aligned}$$

Also,

$$\begin{aligned} |f_K(x_1,t_2,m_2) - f_K(x_1,t_1,m_1)| &= \|K(y,t_2,m_2*\varphi(y)) - K(y,t_1,m_1*\varphi(y))\|_1 \|\varphi\|_\infty \\ &\leq C_2 \|C_3\|_{L^1(\mathbb{T}^d)} \big(|t_2-t_1|^\alpha + \|(m_2-m_1)*\varphi\|^\alpha_{L^\infty(\mathbb{T}^d)}\big). \end{aligned}$$

Finally, we have $\|(m_2-m_1)*\varphi\|_{L^\infty(\mathbb{T}^d)} \leq \|m_2-m_1\|_{L^\infty(\mathbb{T}^d)} \|\varphi\|_{L^\infty(\mathbb{T}^d)}$ and thus, assumption (A7) follows. □

3 Main result and general approach

Theorem 3.1. *There exists $\alpha \in (0,1)$ such that (MFGC) has a classical solution (m,u,P,v), with*

$$\begin{cases} m \in \mathcal{C}^{2+\alpha,1+\alpha/2}(Q), \\ u \in \mathcal{C}^{2+\alpha,1+\alpha/2}(Q), \\ P \in \mathcal{C}^\alpha(0,T;\mathbb{R}^k), \\ v \in \mathcal{C}^\alpha(Q,\mathbb{R}^d), \ D_x v \in \mathcal{C}^\alpha(Q,\mathbb{R}^{d\times d}). \end{cases} \tag{16}$$

The result is obtained with the Leray-Schauder theorem, recalled below.

Theorem 3.2 (Leray-Schauder, [12], Theorem 11.6)**.** *Let X be a Banach space and let $\mathcal{T}\colon X \times [0,1] \to X$ be a continuous and compact mapping. Assume that $\mathcal{T}(x,0) = 0$ for all $x \in X$ and assume there exists $C > 0$ such that $\|x\|_X < C$ for all $(x,\tau) \in X \times [0,1]$ such that $\mathcal{T}(x,\tau) = x$. Then, there exists $x \in X$ such that $\mathcal{T}(x,1) = x$.*

The application of the Leray-Schauder theorem and the construction of $\mathcal{T}$ will be detailed in Section 7. Let us mention that the set of fixed points of $\mathcal{T}(\cdot,\tau)$, for $\tau \in [0,1]$, will coincide with the set of solutions of the following parametrization of (MFGC):

$$\begin{cases} (i) & -\partial_t u - \sigma\Delta u + \tau H(x,t,\nabla u(x,t) + \phi(x,t)^\intercal P(t)) = \tau f(x,t,m(t)) & (x,t) \in Q, \\ (ii) & \partial_t m - \sigma\Delta m + \tau \mathrm{div}(mv) = 0 & (x,t) \in Q, \\ (iii) & P(t) = \Psi\left(t, \int_{\mathbb{T}^d} \phi(x,t)v(x,t)m(x,t)\,\mathrm{d}x\right) & t \in [0,T], \\ (iv) & v(x,t) = -H_p(x,t,\nabla u(x,t) + \phi(x,t)^\intercal P(t)) & (x,t) \in Q, \\ (v) & m(x,0) = m_0(x), \quad u(x,T) = \tau g(x) & x \in \mathbb{T}^d, \end{cases} \tag{MFGC$_\tau$}$$

Of course, (MFGC$_\tau$) corresponds to (MFGC) for $\tau = 1$. Let us introduce the space X and X', used for the formulation of the fixed-point equation:

$$X := \left(W^{2,1,p}(Q)\right)^2, \quad X' := X \times L^\infty(0,T;\mathbb{R}^k) \times L^\infty(Q,\mathbb{R}^d).$$

The HJB equation (i) and the Fokker-Planck equation (ii) are classically understood in the viscosity and weak sense, respectively. However, due to the choice of the solution spaces, we may interpret these equations as equalities in $L^p(Q)$. A first and important step of our analysis is the construction of auxiliary mappings allowing to express v and P as functions of m and u. These mappings cannot be obtained in a straightforward way, since in (iii), P depends on v and in (iv), v depends on P.

Lemma 3.1. *Let $\tau \in [0,1]$, let $(m,v) \in W^{2,1,p}(Q) \times L^\infty(Q,\mathbb{R}^d)$ be a weak solution to the Fokker-Planck equation $\partial_t m - \sigma\Delta m + \tau\mathrm{div}(vm) = 0$, $m(\cdot,0) = m_0(\cdot)$. Then $m \geq 0$ and for all $t \in [0,T]$, $\int_{\mathbb{T}^d} m(x,t)\,\mathrm{d}x = 1$.*

Proof. Multiply (MFGC$_\tau$)(ii) by $\mu(x,t) := \min(0, m(x,t))$. Use $\nabla\mu(x,t) = \mathbf{1}_{\{m(x,t)<0\}}\nabla m(x,t)$, so that integrating (by parts) over $Q_t := \mathbb{T}^d \times (0,t)$, since v is essentially bounded, we get that

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{T}^d} \mu(x,t)^2\,\mathrm{d}x + \sigma \iint_{Q_t} |\nabla\mu(x,s)|^2\,\mathrm{d}x\,\mathrm{d}s &= \tau \iint_{Q_t} \langle v, \nabla\mu\rangle m\,\mathrm{d}x\,\mathrm{d}s \\ &= \tau \iint_{Q_t} \langle v, \nabla\mu\rangle \mu\,\mathrm{d}x\,\mathrm{d}s \leq C\int_{Q_t} |\mu|^2\,\mathrm{d}x\,\mathrm{d}s + \sigma\iint_{Q_t} |\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}s, \end{aligned}$$

so that after cancellation of the contribution of $\nabla\mu$, we obtain, applying Gronwall's lemma to $a(t) := \int_{\mathbb{T}^d} \mu(x,t)^2$, that $a(t) = 0$ for all t which means that m is non-negative. Moreover, for all $t \in [0,T]$,

$$\int_{\mathbb{T}^d} m(x,t)\,\mathrm{d}x = \int_{\mathbb{T}^d} m(x,0)\,\mathrm{d}x + \iint_{Q_t} \sigma\nabla m - \tau\mathrm{div}(vm)\,\mathrm{d}x\,\mathrm{d}s.$$

Integrating by parts the double integral we see that it is equal to 0, and we conclude by noting that $\int_{\mathbb{T}^d} m(x,0)\,\mathrm{d}x = \int_{\mathbb{T}^d} m_0(x)\,\mathrm{d}x = 1$. □

4 Potential formulation

In this section, we first establish a potential formulation of the mean field game problem (MFGC$_\tau$), that is to say, we prove that for $(u_\tau, m_\tau, v_\tau, P_\tau) \in X'$ satisfying (MFGC$_\tau$), (m_τ, v_τ) is a solution to an optimal control problem. We prove then that for all t, $v_\tau(\cdot, t)$ is the unique solution of some optimization problem, which will enable us to construct the announced auxiliary mappings.

Let us introduce the cost functional $B\colon W^{2,1,p}(Q) \times L^\infty(Q, \mathbb{R}^d) \times L^\infty(Q) \to \mathbb{R}$, defined by

$$B(m, v; \tilde{f}) = \iint_Q \big(L(x,t,v(x,t)) + \tilde{f}(x,t)\big) m(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x) m(x,T)\,\mathrm{d}x + \int_0^T \Phi\Big(t, \int_{\mathbb{T}^d} \phi(x,t) v(x,t) m(x,t)\,\mathrm{d}x\Big)\,\mathrm{d}t. \tag{17}$$

We have the following result.

Lemma 4.1. *For all $\tau \in [0,1]$, for all $(u_\tau, m_\tau, v_\tau, P_\tau) \in X'$ satisfying (MFGC$_\tau$), the pair (m_τ, v_τ) is the solution to the following optimization problem:*

$$\min_{\substack{m \in W^{2,1,p}(Q) \\ v \in L^\infty(Q,\mathbb{R}^d)}} B(m, v; \tilde{f}_\tau), \quad \textit{subject to:} \begin{cases} \partial_t m - \sigma \Delta m + \tau \mathrm{div}(vm) = 0, \\ m(x,0) = m_0(x), \end{cases} \tag{18}$$

where $\tilde{f}_\tau(x,t) = f(x,t,m_\tau(t))$.

Remark 4.1. Let us emphasize that the above optimal control problem is only an incomplete potential formulation, since the term $\tilde{f}_\tau$ still depends on m_τ.

Proof of Lemma 4.1. Let us consider the case where $\tau \in (0,1]$. Let $(m,v) \in W^{2,1,p}(Q) \times L^\infty(Q,\mathbb{R}^d)$ be a feasible pair. For all $(x,t) \in Q$, $v_\tau = -H_p(\nabla u_\tau + \phi^\intercal P_\tau)$. Therefore, by (5) and (6), we have that

$$\begin{aligned} L(v) &\geq -H(\nabla u_\tau + \phi^\intercal P_\tau) - \langle \nabla u_\tau + \phi^\intercal P_\tau, v \rangle, \\ L(v_\tau) &= -H(\nabla u_\tau + \phi^\intercal P_\tau) - \langle \nabla u_\tau + \phi^\intercal P_\tau, v_\tau \rangle, \end{aligned}$$

for all $(x,t) \in Q$. Moreover, by Lemma 3.1, $m \geq 0$ and $m_\tau \geq 0$. Therefore,

$$L(v)m - L(v_\tau)m_\tau \geq -H(\nabla u_\tau + \phi^\intercal P_\tau)(m - m_\tau) - \langle \nabla u_\tau + \phi^\intercal P_\tau, vm - v_\tau m_\tau \rangle. \tag{19}$$

Using (i)(MFGC$_\tau$), we obtain

$$L(v)m - L(v_\tau)m_\tau \geq \frac{1}{\tau}(-\partial_t u_\tau - \sigma \Delta u_\tau - \tau \tilde{f}_\tau)(m - m_\tau) - \langle \nabla u_\tau + \phi^\intercal P_\tau, vm - v_\tau m_\tau \rangle.$$

After integration with respect to x, we obtain that for all t,

$$\begin{aligned} &\int_{\mathbb{T}^d} (L(v)m - L(v_\tau)m_\tau) + \tilde{f}_\tau(m - m_\tau)\,\mathrm{d}x \\ &\quad \geq \frac{1}{\tau}\int_{\mathbb{T}^d} (-\partial_t u_\tau - \sigma \Delta u_\tau)(m - m_\tau)\,\mathrm{d}x - \int_{\mathbb{T}^d} \langle \nabla u_\tau, vm - v_\tau m_\tau \rangle\,\mathrm{d}x - \langle P_\tau, \textstyle\int \phi(vm - v_\tau m_\tau) \rangle. \end{aligned}$$

We obtain with the convexity of Φ and (iii)(MFGC$_\tau$) that

$$\begin{aligned}\Phi(\textstyle\int\phi mv) - \Phi(\textstyle\int\phi v_\tau m_\tau) &\geq \langle \Psi(\textstyle\int\phi m_\tau v_\tau), \textstyle\int\phi(vm - v_\tau m_\tau)\rangle \\ &= \langle P_\tau, \textstyle\int\phi(mv - m_\tau v_\tau)\rangle. \end{aligned} \tag{20}$$

Using the previous calculations to bound $B(m, v; \tilde{f}_\tau) - B(m_\tau, v_\tau; \tilde{f}_\tau)$ from below, we observe that the term $\langle P_\tau, \int\phi(m - m_\tau v_\tau)\rangle$ cancels out and obtain that

$$\begin{aligned} B(m, v; \tilde{f}_\tau) - B(m_\tau, v_\tau; \tilde{f}_\tau) \geq & \iint_Q \frac{1}{\tau}(-\partial_t u_\tau - \sigma\Delta u_\tau) - \langle \nabla u_\tau, mv - m_\tau v_\tau\rangle \,\mathrm{d}x\,\mathrm{d}t \\ & + \int_{\mathbb{T}^d} g(x)(m(x,T) - m_\tau(x,T))\,\mathrm{d}x. \end{aligned}$$

Integrating by parts and using (ii)(MFGC$_\tau$), we finally obtain that

$$B(m, v; \tilde{f}_\tau) - B(m_\tau, v_\tau; \tilde{f}_\tau) \geq \frac{1}{\tau}\int_{\mathbb{T}^d} u_\tau(0,x)(m(0,x) - m_\tau(0,x))\,\mathrm{d}x = 0,$$

as was to be proved. We do not detail the proof for the case $\tau = 0$, which is actually simpler. Indeed, for $\tau = 0$, the solution to the Fokker-Planck equation is independent of v and thus $m = m_\tau$ in the above calculations. □

We have proved that the pair (m_τ, v_τ) is the solution to an optimal control problem. Therefore, for all t, $v_\tau(\cdot, t)$ minimizes the Hamiltonian associated with problem (18). Let us introduce some notation, in order to exploit this property. For $m \in \mathcal{D}_1(\mathbb{T}^d)$, we denote by $L^2_m(\mathbb{T}^d, \mathbb{R}^d)$ the Hilbert space of measurable mappings $v\colon \mathbb{T}^d \to \mathbb{R}^d$ such that $\int_{\mathbb{T}^d} |v|^2 m < \infty$, equipped with the scalar product $\int_{\mathbb{T}^d}\langle v_1, v_2\rangle m$. An element of $L^2_m(\mathbb{T}^d)$ is an equivalent class of functions equal m almost everywhere. Note that $L^\infty(\mathbb{T}^d) \subset L^2_m(\mathbb{T}^d)$.

For $t \in [0,T]$, for $m \in \mathcal{D}_1(\mathbb{T}^d)$, and for $w \in L^2_m(\mathbb{T}^d)$, we consider the mapping

$$v \in L^2_m(\mathbb{T}^d, \mathbb{R}^d) \mapsto J(v; t, m, w) := \Phi\big(t, \textstyle\int\phi vm\big) + \displaystyle\int_{\mathbb{T}^d} \big(L(v) + \langle w, v\rangle\big) m\,\mathrm{d}x.$$

Combining inequalities (19) and (20) (with $m = m_\tau$), we directly obtain that for all $t \in [0,T]$, for all $v \in L^2_m(\mathbb{T}^d, \mathbb{R}^d)$ with $m = m_\tau(\cdot, t)$,

$$J\big(v; t, m_\tau(t), \nabla u_\tau(t)\big) \geq J\big(v_\tau(t); t, m_\tau(t), \nabla u_\tau(t)\big).$$

The following lemma will enable us to express $P_\tau(t)$ and $v_\tau(\cdot, t)$ as functions of $m_\tau(\cdot, t)$ and $u_\tau(\cdot, t)$. The key idea is, roughly speaking, to prove the existence and uniqueness of a minimizer to $J(\cdot; t, m, w)$.

Lemma 4.2. *For all $t \in [0,T]$, for all $m \in \mathcal{D}_1(\mathbb{T}^d)$, for all $R > 0$, and for all $w \in L^\infty(\mathbb{T}^d, \mathbb{R}^d)$ such that $\|w\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \leq R$, there exists a unique pair $(v, P) \in L^\infty(\mathbb{T}^d, \mathbb{R}^d) \times \mathbb{R}^k$, such that*

$$\begin{cases} v(x) = & -H_p(x, t, w(x) + \phi(x,t)^\intercal P), \quad \text{for a.e. } x \in \mathbb{T}^d, \\ P = & \Psi(t, \int\phi vm). \end{cases} \tag{21}$$

The pair (v, P) is then denoted $(\mathbf{v}(t, m, w), \mathbf{P}(t, m, w))$. Moreover, we have

$$\|\mathbf{v}(t, m, w)\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \leq C \quad \text{and} \quad |\mathbf{P}(t, m, w)| \leq C, \tag{22}$$

where the constant C is independent of t, m, and w (but depends on R).

Proof. If the pair (v, P) satifies (21), then

$$v = -H_p(w + \phi^\intercal \Psi(\textstyle\int \phi v m)) \quad \text{a.e. on } \mathbb{T}^d. \tag{23}$$

One can easily check that for proving the existence and uniqueness of a pair (v, P) satisfying (21), it is sufficient to prove the existence and uniqueness of $v \in L^\infty(\mathbb{T}^d, \mathbb{R}^d)$ satisfying (23). For future reference, let us mention that by (6), relation (23) is equivalent to

$$\phi^\intercal(x,t)\Psi(t, \textstyle\int \phi v m) + L_v(x,t,v(x)) + w(x) = 0, \quad \text{for a.e. } x \in \mathbb{T}^d. \tag{24}$$

Step 1: existence and uniqueness of a minimizer of $J(\cdot; t, m, w)$ in $L^2_m(\mathbb{T}^d, \mathbb{R}^d)$.
In view of (A1), $v \mapsto \int_{\mathbb{T}^d} L(v) m \, \mathrm{d}x$ is strongly convex over $L^2_m(\mathbb{T}^d, \mathbb{R}^d)$. Since the sum of a l.s.c. convex function and of a l.s.c. strongly convex function is l.s.c. and strongly convex, so is $J(\cdot; t, m, w)$. Since it is continuous, it possesses a unique minimizer $\bar{v}$ in $L^2_m(\mathbb{T}^d, \mathbb{R}^d)$. We obtain

$$C\|\bar{v}\|^2_{L^2_m(\mathbb{T}^d,\mathbb{R}^d)} - C \leq J(\bar{v}; t, m, w)) \leq J(0; t, m, w)) = C, \tag{25}$$

so that $\|\bar{v}\|^2_{L^2_m(\mathbb{T}^d,\mathbb{R}^d)} \leq C$, with C independent of t, m, and w, but depending on R, as all constants C used in the proof.

Step 2: existence of $\mathbf{v}(t, m, w)$ and a priori bound.
One can check that $J(\cdot; t, m, w)$ is differentiable in $L^\infty(\mathbb{T}^d)$. Since $\bar{v}$ is optimal, $DJ(\bar{v}; t, m, w) = 0$ and thus

$$\big[\phi^\intercal(x,t)\Psi(t, \textstyle\int \phi(x')\bar{v}(x')m(x')\,\mathrm{d}x') + L_v(x,t,\bar{v}(x)) + w(x)\big]m(x) = 0, \quad \text{for a.e. } x \in \mathbb{T}^d.$$

Using then the equivalence of (23) and (24), we obtain that

$$m(x) > 0 \Longrightarrow \bar{v}(x) = -H_p\big(x, t, w(x) + \phi^\intercal(x,t)\Psi(t, \textstyle\int \phi(x',t)\bar{v}(x')m(x')\,\mathrm{d}x')\big), \quad \text{for a.e. } x \in \mathbb{T}^d.$$

Consider now the measurable function v defined by

$$v(x) = -H_p\big(x, t, w(x) + \phi^\intercal(x,t)\Psi(t, \textstyle\int \phi(x',t)\bar{v}(x')m(x')\,\mathrm{d}x'\big), \quad \text{for a.e. } x \in \mathbb{T}^d.$$

The two functions v and $\bar{v}$ may not be equal for a.e. x if $m(x) = 0$ on a subset of $\mathbb{T}^d$ of non-zero measure. Still they are equal in $L^2_m(\mathbb{T}^d, \mathbb{R}^d)$, which ensures in particular that $\int \phi(x',t)\bar{v}(x')m(x')\,\mathrm{d}x' = \int \phi(x',t)v(x')m(x')\,\mathrm{d}x'$ and finally that v satisfies (23) and lies in $L^\infty(\mathbb{T}^d, \mathbb{R}^d)$, as a consequence of the continuity of H_p (proved in Lemma 2.1). We also have that $\|\bar{v}\|_{L^2_m(\mathbb{T}^d,\mathbb{R}^d)} = \|v\|_{L^2_m(\mathbb{T}^d,\mathbb{R}^d)} \leq C$, by (25). Using the Cauchy-Schwarz inequality and Assumption (A6), we obtain that $|\int \phi v m| \leq C$. We obtain then with Assumption (A4) that for $P = \Psi(\int \phi v m)$, we have $|P| \leq C$. Using Assumption (A6) and the continuity of H_p, we finally obtain that $\|v\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \leq C$. Thus the bound (22) is satisfied.

Step 3: uniqueness of $\mathbf{v}(t, m, w)$.
Let v_1 and $v_2 \in L^\infty(\mathbb{T}^d, \mathbb{R}^d)$ satisfy (23). Then $DJ(v_1; t, m, w) = DJ(v_2; t, m, w) = 0$, proving that v_1 and v_2 are minimizers of $J(\cdot; t, m, w)$ and thus are equal in $L^2_m(\mathbb{T}^d, \mathbb{R}^d)$. It follows that $\int \phi(x',t)v_1(x')m(x')\,\mathrm{d}x' = \int \phi(x',t)v_2(x')m(x')\,\mathrm{d}x'$ and finally that $v_1 = v_2$, by (23). □

5 Regularity results for the auxiliary mappings

We provide in this section some regularity results for the mappings $\mathbf{v}$ and $\mathbf{P}$. We begin by proving that $\mathbf{P}(\cdot,\cdot,\cdot)$ is locally Hölder continuous. For this purpose, we perform a stability analysis of the optimality condition (24).

Lemma 5.1. *Let t_1 and $t_2 \in [0,T]$, let w_1 and $w_2 \in L^\infty(\mathbb{T}^d,\mathbb{R}^d)$, and let m_1 and $m_2 \in \mathcal{D}_1(\mathbb{T}^d)$. Let $R>0$ be such that $\|w_1\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \le R$ and $\|w_2\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \le R$. Then, there exist a constant $C>0$ and an exponent $\alpha \in (0,1)$, both independent of t_1, t_2, w_1, w_2, m_1, and m_2 but depending on R, such that*

$$|\mathbf{P}(t_2,m_2,w_2)-\mathbf{P}(t_1,m_1,w_1)| \le C\big(|t_2-t_1|^\alpha + \|w_2-w_1\|^\alpha_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} + \|m_2-m_1\|^\alpha_{L^1(\mathbb{T}^d)}\big). \tag{26}$$

Proof. Note that all constants $C>0$ and all exponents $\alpha\in(0,1)$ involved below are independent of t_1, t_2, w_1, w_2, m_1, and m_2. They are also independent of $x\in\mathbb{T}^d$ and $\varepsilon>0$. For $i=1,2$, we set $v_i=\mathbf{v}(t_i,m_i,w_i)$ and $\phi_i=\phi(\cdot,t_i)\in L^\infty(\mathbb{T}^d)$. By (22), we have

$$\|v_i\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)} \le C. \tag{27}$$

By the optimality condition (24), we have that

$$\phi_i^\intercal \Psi\big(t_i,\textstyle\int \phi_i v_i m_i\big) + L_v(t_i,v_i) + w_i = 0, \quad \text{for a.e. } x\in\mathbb{T}^d. \tag{28}$$

Consider the difference of (28) for $i=2$ with (28) for $i=1$. Integrating with respect to x the scalar product of the obtained difference with $v_2m_2-v_1m_1$, we obtain that

$$(a_1)+(a_2)+(a_3)=0,$$

where

$$\begin{aligned}
(a_1) &= \int_{\mathbb{T}^d} \big\langle \phi_2^\intercal\Psi(t_2,\textstyle\int\phi_2v_2m_2) - \phi_1^\intercal\Psi(t_1,\textstyle\int\phi_1v_1m_1), v_2m_2-v_1m_1\big\rangle \,\mathrm{d}x,\\
(a_2) &= \int_{\mathbb{T}^d} \langle L_v(t_2,v_2)-L_v(t_1,v_1), v_2m_2-v_1m_1\rangle \,\mathrm{d}x,\\
(a_3) &= \int_{\mathbb{T}^d} \langle w_2-w_1, v_2m_2-v_1m_1\rangle \,\mathrm{d}x.
\end{aligned}$$

We look for a lower estimate of these three terms. Let us mention that the term $v_2m_2-v_1m_1$, appearing in the three terms, will be estimated only at the end.

Estimation from below of (a_1). We have $(a_1)=(a_{11})+(a_{12})$, where

$$\begin{aligned}
(a_{11}) &= \int_{\mathbb{T}^d} \big\langle \phi_2^\intercal\Psi(t_2,\textstyle\int\phi_2v_2m_2) - \phi_1^\intercal\Psi(t_1,\textstyle\int\phi_1v_2m_2), v_2m_2-v_1m_1\big\rangle \,\mathrm{d}x\\
(a_{12}) &= \int_{\mathbb{T}^d} \big\langle \phi_1^\intercal\Psi(t_1,\textstyle\int\phi_1v_2m_2) - \phi_1^\intercal\Psi(t_1,\textstyle\int\phi_1v_1m_1), v_2m_2-v_1m_1\big\rangle \,\mathrm{d}x.
\end{aligned}$$

By monotonicity of Ψ, we have that

$$(a_{12}) = \Big\langle \Psi(t_1,\textstyle\int\phi_1v_2m_2) - \Psi(t_1,\textstyle\int\phi_1v_1m_1), \displaystyle\int_{\mathbb{T}^d} \phi_1v_2m_2-\phi_1v_1m_1 \,\mathrm{d}x\Big\rangle \ge 0.$$

Let us consider (a_{11}). We set

$$\begin{cases} \Psi_i = \Psi(t_i, \int \phi_i v_2 m_2), & \text{for } i = 1, 2, \\ \xi(x) = \phi_2(x)^\intercal \Psi_2 - \phi_1(x)^\intercal \Psi_1, \end{cases}$$

so that $(a_{11}) = \int_{\mathbb{T}^d} \langle \xi, v_2 m_2 - v_1 m_1 \rangle \, \mathrm{d}x$. Using Assumption (A6), one can check that $|\Psi_i| \leq C$ and that $\left|\Psi_2 - \Psi_1\right| \leq C|t_2 - t_1|^\alpha$. Since $\xi = (\phi_2 - \phi_1)^\intercal \Psi_2 + \phi_1^\intercal (\Psi_2 - \Psi_1)$, we obtain with Assumption (A6) again that

$$\|\xi\|_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} \leq C\big(|\Psi_2 - \Psi_1| + \|\phi_2 - \phi_1\|_{L^\infty(\mathbb{T}^d, \mathbb{R}^{k \times d})}\big) \leq C|t_2 - t_1|^\alpha$$

and further with Young's inequality that

$$|(a_{11})| \leq \frac{C}{\varepsilon}|t_2 - t_1|^\alpha + \frac{\varepsilon}{2}\|v_2 m_2 - v_1 m_1\|^2_{L^1(\mathbb{T}^d, \mathbb{R}^d)}.$$

Estimation from below of (a_2). We have $(a_2) = (a_{21}) + (a_{22}) + (a_{23})$, where

$$\begin{aligned} (a_{21}) &= \int_{\mathbb{T}^d} \langle L_v(t_2, v_2) - L_v(t_1, v_2), v_2 m_2 - v_1 m_1) \rangle \, \mathrm{d}x \\ (a_{22}) &= \int_{\mathbb{T}^d} \langle L_v(t_1, v_2) - L_v(t_1, v_1), v_2 (m_2 - m_1) \rangle \, \mathrm{d}x \\ (a_{23}) &= \int_{\mathbb{T}^d} \langle L_v(t_1, v_2) - L_v(t_1, v_1), (v_2 - v_1) m_1 \rangle \, \mathrm{d}x. \end{aligned}$$

As a consequence of (27), assumption (A6), and Young's inequality, we have that

$$\begin{aligned} |(a_{21})| &\leq \frac{1}{2\varepsilon}\|L_v(t_2, v_2(\cdot) - L_v(t_1, v_2(\cdot))\|^2_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} + \frac{\varepsilon}{2}\|v_2 m_2 - v_1 m_1\|^2_{L^1(\mathbb{T}^d, \mathbb{R}^d)} \\ &\leq \frac{C}{\varepsilon}|t_2 - t_1|^\alpha + \frac{\varepsilon}{2}\|v_2 m_2 - v_1 m_1\|^2_{L^1(\mathbb{T}^d, \mathbb{R}^d)}. \end{aligned}$$

By (27) and assumption (A6), $|L_v(t_1, x, v_i(x))| \leq C$, therefore

$$|(a_{22})| \leq C\|m_2 - m_1\|_{L^1(\mathbb{T}^d, \mathbb{R}^d)}.$$

Finally, we have

$$(a_{23}) \geq \frac{1}{C}\int_{\mathbb{T}^d} |v_2 - v_1|^2 m_1 \, \mathrm{d}x,$$

by assumption (A1) and since $m_1 \geq 0$.

Estimation from below of (a_3). Using (30) and Young's inequality, we obtain that

$$|(a_3)| \leq \frac{1}{2\varepsilon}\|w_2 - w_1\|^2_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} + \frac{\varepsilon}{2}\|v_2 m_2 - v_1 m_1\|^2_{L^1(\mathbb{T}^d, \mathbb{R}^d)}.$$

Conclusion. We have proved that

$$\begin{aligned} \frac{1}{C}\int_{\mathbb{T}^d} |v_2 - v_1|^2 m_1 \, \mathrm{d}x \leq (a_{23}) &= (a_2) - (a_{21}) - (a_{22}) \\ &= -(a_1) - (a_{21}) - (a_{22}) - (a_3) \\ &\leq -(a_{11}) - (a_{21}) - (a_{22}) - (a_3) \\ &\leq \frac{C}{\varepsilon}|t_2 - t_1|^\alpha + \frac{1}{2\varepsilon}\|w_2 - w_1\|^2_{L^\infty(\mathbb{T}^d, \mathbb{R}^d)} + C\|m_2 - m_1\|_{L^1(\mathbb{T}^d, \mathbb{R}^d)} \\ &\quad + \frac{3}{2}\varepsilon\|v_2 m_2 - v_1 m_1\|^2_{L^1(\mathbb{T}^d; \mathbb{R}^d)}. \end{aligned} \tag{29}$$

Let us estimate $\|v_2m_2 - v_1m_1\|_{L^1(\mathbb{T}^d;\mathbb{R}^d)}$. Using the Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}\|v_2m_2 - v_1m_1\|_{L^1(\mathbb{T}^d,\mathbb{R}^d)} &\leq \|v_2(m_2 - m_1)\|_{L^1(\mathbb{T}^d,\mathbb{R}^d)} + \|(v_2 - v_1)m_1\|_{L^1(\mathbb{T}^d,\mathbb{R}^d)} \\ &\leq C\|m_2 - m_1\|_{L^1(\mathbb{T}^d,\mathbb{R}^d)} + \Big(\int_{\mathbb{T}^d} |v_2 - v_1|^2 m_1 \,\mathrm{d}x\Big)^{1/2}. \end{aligned} \tag{30}$$

Injecting this inequality in (29) and taking $\varepsilon = \frac{1}{3C}$, we obtain that

$$\int_{\mathbb{T}^d} |v_2 - v_1|^2 m_1 \leq C\Big(|t_2 - t_1|^\alpha + \|m_2 - m_1\|_{L^1(\mathbb{T}^d)} + \|w_2 - w_1\|^2_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)}\Big). \tag{31}$$

Let us prove (26). We have

$$\begin{aligned}\int_{\mathbb{T}^d} \phi_2 v_2 m_2 \,\mathrm{d}x - \int_{\mathbb{T}^d} \phi_1 v_1 m_1 \,\mathrm{d}x &= \int_{\mathbb{T}^d} (\phi_2 - \phi_1) v_2 m_2 \,\mathrm{d}x \\ &\quad + \int_{\mathbb{T}^d} \phi_1 v_2 (m_2 - m_1) \,\mathrm{d}x + \int_{\mathbb{T}^d} \phi_1 (v_2 - v_1) m_1 \,\mathrm{d}x.\end{aligned}$$

Therefore, using assumption (A6) and (31), we obtain that

$$\begin{aligned}&\Big|\int_{\mathbb{T}^d} \phi_2 v_2 m_2 \,\mathrm{d}x - \int_{\mathbb{T}^d} \phi_1 v_1 m_1 \,\mathrm{d}x\Big| \\ &\quad \leq C\Big(\|\phi_2 - \phi_1\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^{k\times d})} + \|m_2 - m_1\|_{L^1(\mathbb{T}^d)} + \Big(\int_{\mathbb{T}^d} |v_2 - v_1|^2 m_1 \,\mathrm{d}x\Big)^{1/2}\Big) \\ &\quad \leq C\Big(|t_2 - t_1|^\alpha + \|m_2 - m_1\|^{1/2}_{L^1(\mathbb{T}^d)} + \|w_2 - w_1\|_{L^\infty(\mathbb{T}^d,\mathbb{R}^d)}\Big).\end{aligned}$$

Inequality (26) follows, using assumption (A6). □

Given $m \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$ and $w \in L^\infty(Q)$, we consider the Nemytskii operators associated with $\mathbf{v}$ and $\mathbf{P}$, that we still denote by $\mathbf{v}$ and $\mathbf{P}$ without risk of confusion:

$$\begin{aligned}&\mathbf{v}(m,w) \in L^\infty(Q,\mathbb{R}^d), && \mathbf{v}(m,w)(x,t) = \mathbf{v}(t, m(\cdot,t), w(\cdot,t))(x), \\ &\mathbf{P}(m,w) \in L^\infty(0,T;\mathbb{R}^k), && \mathbf{P}(m,w)(t) = \mathbf{P}(t, m(\cdot,t), w(\cdot,t)),\end{aligned}$$

for all $(x,t) \in Q$. We use now Lemma 5.1 to prove regularity properties of the Nemytskii operators $\mathbf{v}$ and $\mathbf{P}$. We recall that $X = \big(W^{2,1,p}(Q)\big)^2$.

Lemma 5.2. *For all $R > 0$, the mapping*

$$(m,w) \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d)) \times B\big(L^\infty(Q,\mathbb{R}^d), R\big) \mapsto \mathbf{P}(m,w) \in L^\infty(0,T;\mathbb{R}^k) \tag{32}$$

and the mapping

$$(u,m) \in B(W^{2,1,p}(Q), R) \times L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d)) \mapsto \mathbf{v}(m,\nabla u) \in L^\infty(Q,\mathbb{R}^d) \cap L^p(0,T;W^{1,p}(\mathbb{T}^d)) \tag{33}$$

are both Hölder continuous, that is, there exist $\alpha \in (0,1)$ and $C > 0$ such that

$$\begin{aligned}\|\mathbf{P}(m_2,w_2) - \mathbf{P}(m_1,w_1)\|_{L^\infty(0,T;\mathbb{R}^k)} &\leq C\big(\|m_2 - m_1\|^\alpha_{L^\infty(Q)} + \|w_2 - w_1\|^\alpha_{L^\infty(Q)}\big) \\ \|\mathbf{v}(m_2,\nabla u_2) - \mathbf{v}(m_1,\nabla u_1)\|_{L^\infty(Q,\mathbb{R}^d)\cap L^p(0,T;W^{1,p}(\mathbb{T}^d))} &\leq C\big(\|u_2 - u_1\|^\alpha_{W^{2,1,p}(Q)} + \|m_2 - m_1\|^\alpha_{L^\infty(Q)}\big),\end{aligned}$$

for all m_1 and $m_2 \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$, for all w_1 and $w_2 \in B(L^\infty(Q,\mathbb{R}^d), R)$, and for all u_1 and u_2 in $B(W^{2,1,p}(Q), R)$.

Proof. The Hölder continuity of the first mapping is a direct consequence of Lemma 5.1. As a consequence, the mapping

$$(u,m) \in B(W^{2,1,p}(Q),R) \times L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d)) \mapsto \nabla u + \phi^\intercal \mathbf{P}(m,\nabla u) \in L^\infty(Q,\mathbb{R}^d)$$

is Hölder continuous. Using then the relations

$$\begin{cases} \mathbf{v}(m,\nabla u) = & -H_p(\nabla u + \phi^\intercal \mathbf{P}(m,\nabla u)), \\ D_x\mathbf{v}(m,\nabla u) = & -H_{px}(\nabla u + \phi^\intercal \mathbf{P}(m,\nabla u)) \\ & \qquad -H_{pp}(\nabla u + \phi^\intercal \mathbf{P}(m,\nabla u))(\nabla^2 u + D\phi^\intercal \mathbf{P}(m,\nabla u)), \end{cases} \tag{34}$$

and the Hölder continuity of H_p, H_{px}, and H_{pp} on bounded sets (Lemma 2.1), we obtain that the second mapping is Hölder continuous. □

Remark 5.1. As a consequence of Lemma 5.2, the images of the mappings given by (32) and (33) are bounded. This fact will be used in the steps 3 and 5 of the proof of Proposition 6.1.

Lemma 5.3. *Let $R > 0$ and $\beta \in (0,1)$. Then, there exists $\alpha \in (0,1)$ and $C > 0$ such that for all $u \in B(W^{2,1,p}(Q),R)$ and for all $m \in B(\mathcal{C}^\beta(Q),R) \cap L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$,*

$$\|\mathbf{P}(m,\nabla u)\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^k)} \leq C.$$

Proof. We recall that by Lemma A.2, $\|\nabla u\|_{\mathcal{C}^\alpha(Q,\mathbb{R}^d)} \leq C\|u\|_{W^{2,1,p}(Q)}$. We obtain then the bound on $\|\mathbf{P}(m,\nabla u)\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^k)}$ with Lemma 5.1. □

Lemma 5.4. *Let $R > 0$ and $\beta \in (0,1)$. There exist $\alpha \in (0,1)$ and $C > 0$ such that for all $u \in B(\mathcal{C}^{2+\beta,1+\beta/2}(Q),R)$ and for all $m \in B(\mathcal{C}^\beta(Q),R) \cap L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$,*

$$\begin{cases} \|\mathbf{v}(m,\nabla u)\|_{\mathcal{C}^\alpha(Q,\mathbb{R}^d)} \leq & C \\ \|D_x\mathbf{v}(m,\nabla u)\|_{\mathcal{C}^\alpha(Q,\mathbb{R}^{d\times d})} \leq & C. \end{cases}$$

Proof. The result follows from relations (34), Lemma 5.3, and from the Hölder continuity of H_p, H_{px}, and H_{pp} on bounded sets. □

6 A priori estimates for fixed points

Proposition 6.1. *There exist a constant $C > 0$ and an exponent $\alpha \in (0,1)$ such that for all $\tau \in [0,1]$, for all $(u_\tau, m_\tau, v_\tau, P_\tau) \in X'$ satisfying (MFGC$_\tau$),*

$$\begin{aligned} m_\tau &\in \mathcal{C}^{2+\alpha,1+\alpha/2}(Q), & \|m_\tau\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} &\leq C, \\ u_\tau &\in \mathcal{C}^{2+\alpha,1+\alpha/2}(Q), & \|u_\tau\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} &\leq C, \\ P_\tau &\in \mathcal{C}^\alpha(0,T;\mathbb{R}^k), & \|P_\tau\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^k)} &\leq C \\ v_\tau &\in \mathcal{C}^\alpha(Q,\mathbb{R}^d), & \|v_\tau\|_{\mathcal{C}^\alpha(Q,\mathbb{R}^d)} &\leq C \\ D_x v_\tau &\in \mathcal{C}^\alpha(Q,\mathbb{R}^{d\times d}), & \|D_x v_\tau\|_{\mathcal{C}^\alpha(Q,\mathbb{R}^{d\times d})} &\leq C. \end{aligned}$$

Proof. The proof is decomposed in 10 steps. Let us fix $\tau \in [0,1]$ and $(u_\tau, m_\tau, v_\tau, P_\tau) \in X'$ satisfying (MFGC$_\tau$). All constants C and all exponents $\alpha \in (0,1)$ involved below are independent of $(u_\tau, m_\tau, v_\tau, P_\tau)$ and τ. Let us recall that $\tilde{f}_\tau \in L^\infty(Q)$ has been defined in Lemma 4.1 by $\tilde{f}_\tau(x,t) = f(x,t,m_\tau(t))$.

Step 1: $\|P_\tau\|_{L^2(0,T;\mathbb{R}^k)} \leq C$.

Let us take $v^0 = 0$ and let m^0 be the solution to the heat equation $\partial_t m^0 - \sigma\Delta m^0 = 0$, $m^0(x,0) = m_0(x)$. By Lemma 4.1, $B(m_\tau, v_\tau; \tilde{f}_\tau) \leq B(m^0, v^0; \tilde{f}_\tau)$. Since $\|\phi\|_{L^\infty(Q,\mathbb{R}^{k\times d})} \leq C$, we have for all $\varepsilon > 0$ and for all $t \in [0,T]$ that

$$\Big|\int_{\mathbb{T}^d} \phi v_\tau m_\tau \,\mathrm{d}x\Big| \leq C\int_{\mathbb{T}^d} |v_\tau| m_\tau \,\mathrm{d}x \leq C\Big(\int_{\mathbb{T}^d} |v_\tau|^2 m_\tau \,\mathrm{d}x\Big)^{1/2} \leq \frac{C}{\varepsilon} + C\varepsilon\int_{\mathbb{T}^d} |v_\tau|^2 m_\tau \,\mathrm{d}x,$$

by the Cauchy-Schwarz inequality and Young's inequality. The constant C is also independent of ε. Using then the lower bounds (8) and (9) and assumptions (A5) and (A8), we obtain that

$$\begin{aligned} C &\geq B(m^0, v^0; \tilde{f}_\tau) \geq B(m_\tau, v_\tau; \tilde{f}_\tau) \\ &\geq \iint_Q \frac{1}{C}|v_\tau|^2 m_\tau \,\mathrm{d}x\,\mathrm{d}t - C\Big|\int_Q \phi v_\tau m_\tau \,\mathrm{d}x\,\mathrm{d}t\Big| - C \\ &\geq \Big(\frac{1}{C} - C\varepsilon\Big)\iint_Q |v_\tau|^2 m_\tau \,\mathrm{d}x\,\mathrm{d}t - C\Big(1 + \frac{1}{\varepsilon}\Big). \end{aligned}$$

Taking $\varepsilon = 1/(2C^2)$, we deduce that $\iint_Q |v_\tau|^2 m_\tau \,\mathrm{d}x\,\mathrm{d}t \leq C$. Using then assumption (A4), the boundedness of ϕ, the Cauchy-Schwarz inequality and the estimate obtained previously, we deduce that

$$\begin{aligned} \|P_\tau\|_{L^2(0,T;\mathbb{R}^k)} &= \int_0^T |\Psi(t, \textstyle\int \phi v_\tau m_\tau)|^2 \,\mathrm{d}t \leq C\int_0^T \Big|\int_{\mathbb{T}^d} \phi v_\tau m_\tau \,\mathrm{d}x\Big|^2 \mathrm{d}t \\ &\leq C\iint_Q |v_\tau|^2 m_\tau \,\mathrm{d}x\,\mathrm{d}t \leq C, \end{aligned} \tag{35}$$

as was to be proved.

Step 2: $\|u_\tau\|_{L^\infty(Q)} \leq C$, $\|\nabla u_\tau\|_{L^\infty(Q,\mathbb{R}^d)} \leq C$.

The argument is classical. We have that u_τ is the unique solution to the HJB equation (i)(MFGC$_\tau$). It is therefore the value function associated with the following stochastic optimal control problem:

$$u_\tau(x,t) = \tau\Big(\inf_{\alpha \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^d)} J_\tau(x,t,\alpha)\Big), \tag{36}$$

where

$$J_\tau(x,t,\alpha) := \mathbb{E}\Big[\int_t^T \big(L(X_s,s,\alpha_s) + \langle \phi(X_s,s)^\intercal P_\tau(s), \alpha_s\rangle + \tilde{f}_\tau(X_s,s)\big)\,\mathrm{d}s + g(X_T)\Big], \tag{37}$$

and $(X_s)_{s\in[t,T]}$ is the solution to the stochastic dynamic $\mathrm{d}X_s = \tau\alpha_s \mathrm{d}s + \sqrt{2\sigma}\mathrm{d}B_s$, $X_t = x$. Here, $L^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$ denotes the set of stochastic processes on (t,T), with values in $\mathbb{R}^d$, adapted to the filtration $\mathbb{F}$ generated by the Brownian motion $(B_s)_{s\in[0,T]}$, and such that $\mathbb{E}\big[\int_t^T |\alpha(s)|^2 \,\mathrm{d}s\big] < \infty$. Then, the boundedness of u_τ from above can be immediately obtained by choosing $\alpha = 0$ in (36)

and using the boundedness of g. We can as well bound u_τ from below since for all $(x,s)\in Q$ and for all $\alpha\in\mathbb{R}^d$, we have

$$\begin{aligned} L(x,s,\alpha)+\langle\phi(x,s)^\intercal P_\tau(s),\alpha\rangle &\geq \frac{1}{C}|\alpha|^2-\|\phi\|_{L^\infty(Q,\mathbb{R}^{k\times d})}|P_\tau(s)||\alpha|-C\\ &\geq \frac{1}{C}|\alpha|^2-C|P_\tau(s)|^2-C, \end{aligned}$$

for some constant C independent of (x,s), α, and $P_\tau(s)$. We already know from the previous step that $\|P_\tau\|_{L^2(0,T;\mathbb{R}^k)}\leq C$. So we can conclude that u_τ is also bounded from below, and thus $\|u_\tau\|_{L^\infty(Q)}\leq C$. We also deduce from the above inequality that for all $\alpha\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$,

$$\mathbb{E}\int_t^T|\alpha_s|^2\,\mathrm{d}s\leq C\big(J_\tau(x,t,\alpha)+1\big). \tag{38}$$

Let us bound ∇u_τ. Choose $\varepsilon\in(0,1)$. For arbitrary (x,t), take an ε-optimal stochastic optimal control $\tilde\alpha$ for (36). We can deduce from the boundedness of the map u_τ and inequality (38) that

$$\mathbb{E}\int_t^T|\tilde\alpha_s|^2\,\mathrm{d}s\leq C\big(J_\tau(x,t,\alpha)+1\big)\leq C(u_\tau(x,t)+\varepsilon+1)\leq C, \tag{39}$$

where C is independent of (τ,x,t) and ε. Let $y\in\mathbb{T}^d$. Set

$$\mathrm{d}X_s=\tau\tilde\alpha_s\mathrm{d}s+\sqrt{2\sigma}\mathrm{d}B_s,\ X_t=x,\quad\text{and}\quad Y_s=X_s-x+y, \tag{40}$$

then obviously $\mathrm{d}Y_s=\tilde\alpha_s\mathrm{d}s+\sqrt{2\sigma}\mathrm{d}B_s,\ Y_t=y$. We have

$$\begin{aligned} u_\tau(x,t)+\varepsilon &\geq \tau\mathbb{E}\Big[\int_t^T\Big(L(X_s,s,\tilde\alpha_s)+\langle P_\tau(s),\phi(X_s,s)^\intercal\tilde\alpha_s\rangle+\tilde f_\tau(X_s,s)\Big)\,\mathrm{d}s+g(X_T)\Big],\\ u_\tau(y,t) &\leq \tau\mathbb{E}\Big[\int_t^T\Big(L(Y_s,s,\tilde\alpha_s)+\langle\phi(Y_s,s)^\intercal P_\tau(s),\tilde\alpha_s\rangle+\tilde f_\tau(Y_s,s)\Big)\,\mathrm{d}s+g(Y_T)\Big]. \end{aligned}$$

Therefore,

$$u_\tau(y,t)-u_\tau(x,t)\leq\varepsilon+|(a)|+|(b)|+|(c)|+|(d)|,$$

where (a), (b), (c), (d) are given by

$$\begin{aligned} (a)&=\tau\mathbb{E}\int_t^T L(Y_s,s,\tilde\alpha_s)-L(X_s,s,\tilde\alpha_s)\,\mathrm{d}s,\\ (b)&=\tau\mathbb{E}\int_t^T\big(\phi(Y_s,s)-\phi(X_s,s)\big)^\intercal P_\tau(s),\tilde\alpha_s\rangle\,\mathrm{d}s,\\ (c)&=\tau\mathbb{E}\big[g(Y_T)-g(X_T)\big],\\ (d)&=\tau\mathbb{E}\int_t^T\big(\tilde f_\tau(Y_s,s)-\tilde f(X_s,s)\big)\,\mathrm{d}s. \end{aligned}$$

First, we have

$$\begin{aligned} |(a)| &\leq \tau\mathbb{E}\int_t^T\big|L(Y_s,s,\tilde\alpha_s)-L(X_s,s,\tilde\alpha_s)\big|\,\mathrm{d}s\\ &\leq C|y-x|\Big(1+\mathbb{E}\Big[\int_t^T|\tilde\alpha_s|^2\,\mathrm{d}s\Big]\Big)\leq C|y-x|, \end{aligned}$$

as a consequence of assumption (A3) and (39). Then, using assumption (A6), (35), and (39), we obtain

$$
\begin{aligned}
|(b)| &\leq \tau \mathbb{E} \int_t^T |\phi(Y_s, s) - \phi(X_s, s)||P_\tau(s)||\tilde{\alpha}(s)| \, \mathrm{d}s \\
&\leq C|y-x| \, \|P_\tau\|_{L^2(0,T;\mathbb{R}^k)} \, \mathbb{E} \int_t^T |\tilde{\alpha}(s)|^2 \, \mathrm{d}s \leq C|y-x|.
\end{aligned}
$$

By assumption (A8), $|(c)| \leq \mathbb{E}\big[|g(Y_T) - g(X_T)|\big] \leq C|y-x|$. Finally, since $\tilde{f}_\tau$ is a Lipschitz function (by assumption (A7)),

$$
|(d)| \leq \tau \mathbb{E} \int_t^T \left|\tilde{f}_\tau(Y_s, s) - \tilde{f}_\tau(X_s, s)\right| \, \mathrm{d}s \leq C|y-x|. \tag{41}
$$

Letting $\varepsilon \to 0$, we obtain that $u_\tau(y,t) - u_\tau(x,t) \leq C|y-x|$. Exchanging x and y, we obtain that u_τ is Lipschitz continuous with modulus C and finally that $\|\nabla u_\tau\|_{L^\infty(Q,\mathbb{R}^d)} \leq C$.

Step 3: $\|P_\tau\|_{L^\infty(0,T;\mathbb{R}^k)} \leq C$.
By Lemma 3.1, $m_\tau \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$. We have that $\|\nabla u_\tau\|_{L^\infty(Q,\mathbb{R}^d)} \leq C$ and $P_\tau = \mathbf{P}(m_\tau, \nabla u_\tau)$. The bound on $\|P_\tau\|_{L^\infty(0,T;\mathbb{R}^k)}$ follows then from Lemma 5.2 and Remark 5.1.

Step 4: $\|u_\tau\|_{W^{2,1,p}(Q)} \leq C$.
By assumption (A6), ϕ is bounded. We have proved that $\|P_\tau\|_{L^\infty(0,T;\mathbb{R}^k)} \leq C$ and by Lemma 2.1, the Hamiltonian H is continuous. Therefore, $\|H(\nabla u_\tau + \phi^\intercal P_\tau)\|_{L^\infty(Q)} \leq C$. By assumption (A5), $\|\tau \tilde{f}_\tau\|_{L^\infty(Q)} \leq C$. It follows that u_τ, as the solution to the HJB equation (i)(MFGC$_\tau$), is the solution to a parabolic equation with bounded coefficients. Thus, by Theorem A.4, $\|u_\tau\|_{W^{2,1,p}(Q)} \leq C$. We also obtain with Lemma A.2 that $\|u_\tau\|_{\mathcal{C}^\alpha(Q)} \leq C$ and $\|\nabla u_\tau\|_{\mathcal{C}^\alpha(Q,\mathbb{R}^d)} \leq C$.

Step 5: $\|v_\tau\|_{L^\infty(Q,\mathbb{R}^d)} \leq C$, $\|D_x v_\tau\|_{L^p(Q,\mathbb{R}^{d\times d})} \leq C$.
We have proved that $v_\tau = \mathbf{v}(m_\tau, \nabla u_\tau)$ and $\|u_\tau\|_{W^{2,1,p}(Q)} \leq C$. The estimate follows directly with Lemma 5.2 and Remark 5.1.

Step 6: $\|m_\tau\|_{\mathcal{C}^\alpha(Q)} \leq C$.
The Fokker-Planck equation can be written in the form of a parabolic equation with coefficients in L^p:

$$
\partial_t m_\tau - \sigma \Delta m_\tau + \tau \langle v_\tau, \nabla m_\tau \rangle + \tau m_\tau \mathrm{div}(v_\tau) = 0,
$$

since $\|D_x v_\tau\|_{L^p(Q,\mathbb{R}^{d\times d})} \leq C$. Combining Theorem A.2 and lemma A.2, we get that $\|m_\tau\|_{\mathcal{C}^\alpha(Q)} \leq C$.

Step 7: $\|P_\tau\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^k)} \leq C$.
We already know that $\|u_\tau\|_{W^{2,1,p}(Q)} \leq C$, that $\|m_\tau\|_{\mathcal{C}^\alpha(Q)} \leq C$, and that $m_\tau \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$. Thus Lemma 5.3 applies and we obtain that $\|P_\tau\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^k)} \leq C$.

Step 8: $\|u_\tau\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$.
We have proved that $\|\nabla u_\tau\|_{\mathcal{C}^\alpha(Q,\mathbb{R}^d)} \leq C$ and $\|P_\tau\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^k)} \leq C$. Moreover, we have assumed that ϕ is Hölder continuous and know that H is Hölder continuous on bounded sets. It follows that

$$
\|H(\nabla u_\tau + \phi^\intercal P_\tau)\|_{\mathcal{C}^\alpha(Q)} \leq C.
$$

It follows from assumption (A7) that $\tau \tilde{f}_\tau$ is Hölder continuous. Since $g \in \mathcal{C}^{2+\alpha}(\mathbb{T}^d)$, we finally obtain that $\|u_\tau\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$, by Theorem A.5.

Step 9: $\|v_\tau\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^d)} \leq C$ and $\|D_x v_\tau\|_{\mathcal{C}^\alpha(0,T;\mathbb{R}^{d\times d})} \leq C$.
We have $\|u_\tau\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$ and $\|m_\tau\|_{\mathcal{C}^\alpha(Q)} \leq C$. Thus Lemma 5.4 applies and the announced estimates hold true.

Step 10: $\|m_\tau\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$.
A direct consequence of Step 9 is that m_τ is the solution to a parabolic equation with Hölder continuous coefficients. Therefore, by Theorem A.5, $\|m_\tau\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$, which concludes the proof of the proposition. □

7 Application of the Leray-Schauder theorem

Proof of Theorem 3.1. Step 1: construction of $\mathcal{T}$.
Let us define the mapping $\mathcal{T}\colon X \times [0,1] \to X$ which is used for the application of the Leray-Schauder theorem. A difficulty is that the auxiliary mappings $\mathbf{P}$ and $\mathbf{v}$ are only defined for $m \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$. Therefore we need a kind of projection operator on this set. Note that $\int_{\mathbb{T}^d} 1 \,\mathrm{d}x = 1$. We consider the mapping

$$\rho : m \in L^\infty(Q) \mapsto \rho(m) \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d)),$$

defined by

$$\rho(m) = \frac{m_+(x,t)}{\max(1, \int m_+(y,t)\,\mathrm{d}y)} + 1 - \frac{\int m_+(y,t)\,\mathrm{d}y}{\max(1, \int m_+(y,t)\,\mathrm{d}y)},$$

where $m_+(x,t) = \max(0, m(x,t))$. The well-posedness of the mapping can be easily checked, as well as the following properties:

- For all $m \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$, $\rho(m) = m$.
- The mapping ρ is Lipschitz continuous, from $L^\infty(Q)$ to $L^\infty(Q)$.
- For all $\alpha \in (0,1)$, there exists a constant $C > 0$ such that if $m \in \mathcal{C}^\alpha(Q)$, then $\rho(m) \in \mathcal{C}^\alpha(Q)$ and $\|\rho(m)\|_{\mathcal{C}^\alpha(Q)} \leq C\|m\|_{\mathcal{C}^\alpha(Q)}$.

For proving the first property, we suggest to consider separately the two cases: $\int m_+(y,t)\,\mathrm{d}y < 1$ and $\int m_+(y,t)\,\mathrm{d}y \geq 1$.

For a given $(u,m,\tau) \in X \times [0,1]$, the pair $(\tilde{u},\tilde{m}) = \mathcal{T}(u,m,\tau)$ is defined as follows: $\tilde{u}$ is the solution to

$$\begin{cases} -\partial_t \tilde{u} - \sigma\Delta\tilde{u} + \tau H(\nabla u + \phi^\intercal \mathbf{P}(\rho(m), \nabla u)) = & \tau f(x,t,\rho(m(t))) \qquad (x,t) \in Q, \\ \tilde{u}(x,T) = & \tau g(x) \qquad\qquad x \in \mathbb{T}^d, \end{cases}$$

and $\tilde{m}$ is the solution to

$$\begin{cases} \partial_t \tilde{m} - \sigma\Delta\tilde{m} + \tau \mathrm{div}(\mathbf{v}(\rho(m), \nabla u)m) = & 0 \qquad (x,t) \in Q, \\ \tilde{m}(x,0) = & m_0(x) \quad x \in \mathbb{T}^d. \end{cases}$$

Let us observe that $\mathcal{T}(u,m,0)$ is not necessarily null, but is independent of (u,m). Thus the Leray-Schauder theorem is still valid, as can be seen with a translation argument that we do not detail.

Step 2: a priori bound.
Let $\tau \in [0,1]$ and let (u_τ, m_τ) be such that $(u_\tau, m_\tau) = \mathcal{T}(u_\tau, m_\tau, \tau)$. Then, by Lemma 3.1, $m_\tau \in L^\infty(0,T;\mathcal{D}_1(\mathbb{T}^d))$. Thus, $m_\tau = \rho(m_\tau)$ and finally, by Lemma 4.2, the quadruplet $(u_\tau, m_\tau, P_\tau, v_\tau)$, with $P_\tau = \mathbf{P}(m_\tau, \nabla u_\tau)$ and $v_\tau = \mathbf{v}(m_\tau, \nabla u_\tau)$, is a solution to (MFGC$_\tau$). We directly conclude with Proposition 6.1 that $\|(u_\tau, m_\tau)\|_X \leq C$, where C is independent of τ.

Step 3: continuity of $\mathcal{T}$.
Using the continuity of ρ, Lemma 5.2, the Hölder continuity of H, and assumption (A7), we obtain that the mappings

$$(u,m) \in X \mapsto H(\nabla u + \phi^\top \mathbf{P}(\rho(m), \nabla u)) - f(\rho(m)) \in L^\infty(Q),$$
$$(u,m) \in X \mapsto \operatorname{div}(\mathbf{v}(\rho(m), \nabla u)m) \in L^p(Q)$$

are continuous. By Theorem A.4, the solution to a parabolic equation of the form (53), with b and c null (in $W^{2,1,p}(Q)$) is a continuous mapping of the right-hand side (in $L^p(Q)$). Thus, $\tilde{u} \in W^{2,1,p}(Q)$ and $\tilde{m} \in W^{2,1,p}(Q)$ depend in a continuous way on $H(\nabla u + \phi^\top \mathbf{P}(\rho(m), \nabla u))$ and $\operatorname{div}(\mathbf{v}(\rho(m), \nabla u)m)$, which finally proves that $\mathcal{T}$ is the composition of continuous mappings and thus is continuous itself.

Step 4: compactness of $\mathcal{T}$.
Let $R > 0$, let $(u,m) \in B(X,R)$. We have $\|\rho(m)\|_{\mathcal{C}^\alpha(Q)} \leq C$, where C is independent of (u,m) (but depends on R). As a consequence of Lemma 5.4 and assumption (A7), and since H is Hölder continuous on bounded sets, we have that

$$\|H(\nabla u + \phi^\intercal \mathbf{P}(\rho(m), \nabla u)) - f(\rho(m))\|_{\mathcal{C}^\alpha(Q)} \leq C,$$
$$\|\operatorname{div}(\mathbf{v}(\rho(m), \nabla u)m)\|_{\mathcal{C}^\alpha(Q)} \leq C,$$

where $C > 0$ and $\alpha \in (0,1)$ are both independent of (u,m) (but depend on R). It follows then with Theorem A.5 that $\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$ and $\|m\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$. The compactness of $\mathcal{T}$ follows, since $\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)$ is compactly embedded in $W^{2,1,p}(Q)$, by the Arzelà–Ascoli theorem.

Step 5: conclusion.
The existence of a fixed point (u,m) to $\mathcal{T}(\cdot,\cdot,1)$ follows. By the maximum principle (Lemma 3.1), we have that $m(t) \in \mathcal{D}_1(\mathbb{T}^d)$ for all t and thus $\rho(m(t)) = m(t)$ for all t. With the same arguments as those used before, we obtain that $(u, m, \mathbf{P}(m, \nabla u), \mathbf{v}(m, \nabla u))$ is a solution to (MFGC$_\tau$) with $\tau = 1$ and finally that (16) holds, by Proposition 6.1. □

8 Uniqueness and duality

In this section we prove the uniqueness of the solution (u,m,P,v) to (MFGC). We also prove that (P,v) is the solution to a dual problem to (18). Both results are obtained under the following additional monotonicity assumption of f: There exists a measurable mapping $F(t,m)\colon [0,T] \times \mathcal{D}_1(\mathbb{T}^d) \to \mathbb{R}$ such that

$$F(t,m_2) - F(t,m_1) \geq \int_{\mathbb{T}^d} f(x,t,m_1)(m_2(x) - m_1(x))\,\mathrm{d}x, \tag{42}$$

for all m_1 and $m_2 \in \mathcal{D}_1(\mathbb{T}^d)$ and for a.e. t. Thus, $F(t,\cdot)$ is a supremum of the exact affine minorants appearing in the above right-hand side, and is therefore a convex function of m.

Remark 8.1. 1. It follows from (42) that f is monotone:

$$\int_{\mathbb{T}^d} (f(x,t,m_2) - f(x,t,m_2))(m_2(x) - m_1(x))\,\mathrm{d}x \geq 0, \tag{43}$$

for all m_1 and $m_2 \in \mathcal{D}_1(\mathbb{T}^d)$ and for a.e. t. Conversely, (42) holds true if (43) is satisfied and if F is a primitive of $f(.,t,.)$ in the sense that

$$F(t,m_2) - F(t,m_1) = \int_0^1 \int_{\mathbb{T}^d} f(x,t,sm_2 + (1-s)m_1)(m_2(x) - m_1(x))\,\mathrm{d}s.$$

We refer to [5, Proposition 1.2] for a further characterization of functions f deriving from a potential.

2. Consider the mapping f_K proposed in Lemma 2.2. Assume that for all $(x,t) \in Q$, $K(x,t,\cdot)$ is non-decreasing and consider the function $\mathcal{K}$, defined by $\mathcal{K}(x,t,w) := \int_0^w K(x,t,w')\,\mathrm{d}w'$, for $(x,t,w) \in Q \times [-C_2, C_2]$. Then inequality (42) holds true with F_K defined by

$$F_K(t,m) = \int_{\mathbb{T}^d} \mathcal{K}(x,t,m * \varphi(x))\,\mathrm{d}x.$$

Indeed, since $\mathcal{K}$ is convex in its third argument, we have

$$\begin{aligned} F_K(t,m_2) - F_K(t,m_1) &= \int_{\mathbb{T}^d} \mathcal{K}(x,t,m_2 * \varphi(x)) - \mathcal{K}(x,t,m_1 * \varphi(x))\,\mathrm{d}x \\ &\geq \int_{\mathbb{T}^d} K(x,t,m_1 * \varphi(x))((m_2 - m_1) * \varphi)(x)\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d} (K(\cdot,t,m_1 * \varphi(\cdot)) * \tilde{\varphi})(x)(m_2(x) - m_1(x))\,\mathrm{d}x \\ &= \int_{\mathbb{T}^d} f_K(x,t,m)(m_2(x) - m_1(x))\,\mathrm{d}x, \end{aligned}$$

as was to be proved.

Without loss of generality, we can assume that $F(t,m_0) = 0$ for a.e. $t \in (0,T)$. It can then be easily deduced from assumption (A5) and (42) that there exists a constant C such that

$$|F(t,m)| \leq C, \quad \forall m \in \mathcal{D}_1(\mathbb{T}^d), \text{ for a.e. } t \in (0,T). \tag{44}$$

Let us consider the potential $B\colon W^{2,1,p}(Q) \times L^\infty(Q;\mathbb{R}^k) \to \mathbb{R}$, defined by

$$\begin{aligned} B(m,v) = \iint_Q L(x,t,v(x,t))m(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T F(t,m(t))\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)m(x,T)\,\mathrm{d}x \\ + \int_0^T \Phi\Big(t, \int_{\mathbb{T}^d} \phi(x,t)v(x,t)m(x,t)\,\mathrm{d}x\Big)\,\mathrm{d}t. \end{aligned} \tag{45}$$

Proposition 8.1. *There exists a unique solution $(u, m, P, v) \in X'$ to (MFGC). Moreover, the pair (m, v) is the unique solution to the following optimal control problem*

$$\min_{\substack{\hat{m}\in W^{2,1,p}(Q)\\ \hat{v}\in L^\infty(Q,\mathbb{R}^d)}} B(\hat{m},\hat{v}), \quad \textit{subject to:} \begin{cases} \partial_t \hat{m} - \sigma\Delta\hat{m} + \operatorname{div}(\hat{v}\hat{m}) = 0, \\ \hat{m}(x,0) = m_0(x). \end{cases} \tag{46}$$

Proof. Let $(u, m, P, v) \in X'$ be a solution to (MFGC). Let us prove that (m, v) is a solution to (46). Let $(\hat{m}, \hat{v})$ be a feasible pair. Denoting $\tilde{f}(x,t) = f(x,t,m(t))$, we have

$$\begin{aligned} B(\hat{m},\hat{v}) - B(m,v) &= \big(B(\hat{m},\hat{v};\tilde{f}) - B(m,v;\tilde{f})\big) \\ &\quad + \Big(\int_0^T F(t,\hat{m}(t)) - F(t,m(t)) - \int_{\mathbb{T}^d} \tilde{f}(x,t)(\hat{m}(x,t) - m(x,t))\,\mathrm{d}x\,\mathrm{d}t\Big). \end{aligned}$$

The two terms in the right-hand side are both nonnegative, as a consequence of Lemma 4.1 and assumption (42), respectively.

To conclude the proof of the proposition, it suffices to prove that (46) has a unique solution. Let us prove first a classical property: There exists a constant $C > 0$ such that for all $(x,t) \in Q$, for all $p \in \mathbb{R}^d$ and for all $v \in \mathbb{R}^d$,

$$H(x,t,p) + L(x,t,v) + \langle p, v\rangle \geq \frac{1}{2C}|v + H_p(x,t,p)|^2. \tag{47}$$

Let us set $\bar{v} = -H_p(x,t,p)$. For a fixed triple (x,t,p), we have $H(x,t,p) = -\langle p,\bar{v}\rangle - L(x,t,\bar{v})$. Moreover, $L_v(x,t,\bar{v}) = -p$ and thus by (A1),

$$L(x,t,v) \geq L(x,t,\bar{v}) - \langle p, v - \bar{v}\rangle + \frac{1}{2C}|v - \bar{v}|^2.$$

Inequality (47) follows.

Let (u_1, m_1, P_1, v_1) and (u_2, m_2, P_2, v_2) be two solutions to (MFGC) in X'. We obtain with inequality (47) that

$$\begin{aligned} L(v_2) &\geq -H(\nabla u_1 + \phi^\intercal P_1) - \langle \nabla u_1 + \phi^\intercal P_1, v_2\rangle + \frac{1}{2C}|v_2 - v_1|^2, \\ L(v_1) &= -H(\nabla u_1 + \phi^\intercal P_1) - \langle \nabla u_1 + \phi^\intercal P_1, v_1\rangle. \end{aligned}$$

Proceeding then exactly like in the proof of Lemma 4.1, we arrive at the following inequality:

$$B(m_2,v_2) - B(m_1,v_1) \geq \frac{1}{2C}\iint_Q |v_2 - v_1|^2 m_2\,\mathrm{d}x\,\mathrm{d}t.$$

We also have that $B(m_1,v_1) - B(m_2,v_2) \geq 0$, thus $\iint_Q |v_2 - v_1|^2 m_2\,\mathrm{d}x\,\mathrm{d}t = 0$. As a consequence, $(v_2 - v_1)m_2 = 0$, since $m_2 \geq 0$. We obtain then that

$$v_2 m_2 - v_1 m_1 = v_1(m_2 - m_1). \tag{48}$$

Let us set $m = m_2 - m_1$. Using relation (48), we obtain that m is the solution to the following parabolic equation:

$$\begin{aligned} \partial_t m - \sigma\Delta m + \operatorname{div}(v_1 m) &= 0 \qquad (x,t) \in Q, \\ m(x,0) &= 0 \qquad x \in \mathbb{T}^d. \end{aligned}$$

Therefore $m = 0$ and $m_2 = m_1$. We already know that $v_2 m_2 = v_1 m_2$, we deduce then that $v_2 m_2 = v_1 m_1$. We obtain further with (iii) that $P_1 = P_2$, then with (i) that $u_1 = u_2$ and finally with (iv) that $v_1 = v_2$, which concludes the proof. □

We finish this section with a duality result. For $\gamma \in L^\infty(\mathbb{T}^d)$, we recall that the convex conjugate of $F(t,\cdot)$ is defined by

$$F^*(t,\gamma) = \sup_{m\in\mathcal{D}_1(\mathbb{T}^d)} \int_{\mathbb{T}^d} \gamma(x)m(x)\,\mathrm{d}x - F(t,m).$$

It directly follows from the above definition that $|F^*(t,\gamma)| \le \|\gamma\|_{L^\infty(\mathbb{T}^d)} + C$, where C is the constant obtained in (44) and thus for $\gamma \in L^\infty(Q)$, $\int_0^T F^*(t,\gamma(\cdot,t))\,\mathrm{d}t$ is well-defined.

Consider the dual criterion $D\colon (u,P,\gamma) \in W^{2,1,p}(Q) \times L^\infty(0,T;\mathbb{R}^k) \times L^\infty(Q) \mapsto D(u,p,\gamma) \in \mathbb{R}\cup\{-\infty\}$, defined by

$$D(u,P,\gamma) = \int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x - \int_0^T \Phi^*(t,P(t))\,\mathrm{d}t - \int_0^T F^*(t,\gamma(t))\,\mathrm{d}t.$$

The function Φ^* is the convex conjugate of Φ with respect to its second argument. Since $\Phi(t,0)=0$, we have that $\Phi^*(t,\cdot)\ge 0$ and thus the first integral is well-defined in $\mathbb{R}\cup\{\infty\}$.

Lemma 8.1. *Let $(\bar u,\bar m,\bar v,\bar P)$ be the solution to (MFGC). Let $\tilde f$ be defined by $\tilde f(x,t) = f(x,t,\bar m(t))$. Then, $(\bar u,\bar P,\tilde f)$ is the solution to the following problem:*

$$\max_{\substack{u\in W^{2,1,p}(Q)\\ P\in L^\infty(0,T;\mathbb{R}^k)\\ \gamma\in L^\infty(Q)}} D(u,P,\gamma), \quad \textit{subject to:} \begin{cases} -\partial_t u - \sigma\Delta u + H(\nabla u + \phi^\intercal P) \le \gamma \\ u(x,T) \le g(x). \end{cases} \tag{49}$$

Moreover, for all solutions (u,P,γ) to the dual problem, $P=\bar P$. If in addition, $\gamma=\tilde f$ and the above inequalities hold as equalities, then $u=\bar u$.

Proof. For all $t\in[0,T]$, we have

$$-\int_0^T \Phi^*(P)\,\mathrm{d}t = \int_0^T \Phi(\textstyle\int \phi\bar v\bar m)\,\mathrm{d}t - \iint_Q \langle \phi^\intercal P, \bar v\bar m\rangle\,\mathrm{d}x\,\mathrm{d}t + (a) \tag{50}$$

with

$$(a) = \int_0^T -\Phi^*(P) - \Phi(\textstyle\int\phi\bar v\bar m) + \langle P, \int\phi\bar v\bar m\rangle\,\mathrm{d}t \le 0.$$

We also have that

$$-\int_0^T F^*(t,\gamma(t))\,\mathrm{d}t + \iint_Q \gamma(x,t)\bar m(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T F(t,\bar m(t))\,\mathrm{d}t + (b), \tag{51}$$

where

$$(b) = \iint_Q \gamma(x,t)\bar m(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T F(t,\bar m(t))\,\mathrm{d}t - \int_0^T F^*(t,\gamma(t))\,\mathrm{d}t \le 0.$$

Integrating by parts (in time), we obtain that

$$\begin{aligned}\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x &= \iint_Q -\partial_t u\bar m - u\partial_t\bar m\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} u(x,T)\bar m(x,T)\,\mathrm{d}x \\ &= \iint_Q (\sigma\Delta u + \gamma - H(\nabla u + \phi^\intercal P))\bar m + \iint_Q (-\sigma\Delta\bar m + \operatorname{div}(\bar v\bar m))u\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\mathbb{T}^d} g(x)\bar m(x,T)\,\mathrm{d}x + (c) + (d),\end{aligned}$$

where

$$(c) = \iint_Q (-\partial_t u - \sigma\Delta u + H(\nabla u + \phi^\intercal P) - \gamma)\bar{m}\,\mathrm{d}x\,\mathrm{d}t \leq 0$$
$$(d) = \int_{\mathbb{T}^d} (u(x,T) - g(x))\bar{m}(x,T)\,\mathrm{d}x \leq 0.$$

Integrating by parts (in space), we further obtain that

$$\begin{aligned}\int_{\mathbb{T}^d} u(x,0)m_0(x)\,\mathrm{d}x &= \iint_Q \big(\gamma - H(\nabla u + \phi^\intercal P) - \langle \nabla u, \bar{v}\rangle\big)\bar{m} + \int_{\mathbb{T}^d} g(x)\bar{m}(x,T)\,\mathrm{d}x + (c) + (d)\\ &= \iint_Q \big(L(\bar{v}) + \gamma\big)\bar{m} + \langle \phi^\intercal \bar{P}, \bar{v}\rangle\bar{m}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)\bar{m}(x,T)\,\mathrm{d}x + (c) + (d) + (e), \qquad (52)\end{aligned}$$

where

$$(e) = \iint_Q \big(-H(\nabla u + \phi^\intercal P) - L(\bar{v}) - \langle \nabla u + \phi^\intercal P, \bar{v}\rangle\big)\bar{m}\,\mathrm{d}x\,\mathrm{d}t \leq 0.$$

Combining (50), (51) and (52) together, we finally obtain that

$$\begin{aligned}D(u,P,\gamma) = &\int_0^T \Phi(\textstyle\int \phi\bar{v}\bar{m})\,\mathrm{d}t + \iint_Q L(\bar{v})\bar{m}\,\mathrm{d}x\,\mathrm{d}t + \int_0^T F(t,m(t))\,\mathrm{d}t + \int_{\mathbb{T}^d} g(x)\bar{m}(x,T)\,\mathrm{d}x\\ &+ (a) + (b) + (c) + (d) + (e)\\ = &\, B(\bar{m},\bar{v}) + (a) + (b) + (c) + (d) + (e).\end{aligned}$$

The five terms (a), (b), (c), (d), (e) are non-negative and equal to zero if $(u,P,\gamma) = (\bar{u},\bar{P},\tilde{f})$, as can be easily verified. This proves the optimality of $(\bar{u},\bar{P},\tilde{f})$. Moreover, since Φ is differentiable, (a) is null if and only if

$$P(t) = \Phi_x(\textstyle\int \phi\bar{v}\bar{m}) = \bar{P}(t), \quad \text{for a.e. } t \in [0,T].$$

Therefore, for all optimal solutions (u,P,γ), $P = \bar{P}$. If moreover $\gamma = \tilde{f}$ and the inequality constraints in (49) hold as equalities, then (since the HJB equation has a unique solution) $u = \bar{u}$, which concludes the proof. □

Conclusion

The existence and uniqueness of a classical solution to a mean field game of controls have been demonstrated. A particularly important aspect of the analysis is the fact that the equations (iii) and (iv)(MFGC), encoding the coupling of the agents through the controls, are equivalent to the optimality system of a convex problem. This observation has eventually enabled us to eliminate the variables v and P from the coupled system.

The analysis done in this article can be extended in different ways. A more complex interaction between the agents could be considered. For example, it would be possible to replace equations (iii) and (iv) by the following ones:

$$\begin{aligned}P(t) = &\quad \Psi(t, \textstyle\int_{\mathbb{T}^d} \varphi(x,t,v(x,t))m(x,t)\,\mathrm{d}x)\\ v(x,t) = &\quad -H_p(x,t,\nabla u(x,t)D_v\varphi(x,t,v(x,t)^\intercal P(t)),\end{aligned}$$

assuming that φ is convex with respect to v and $\Psi \geq 0$. For a fixed $t \in [0,T]$, this system is equivalent to the optimality system associated with the following convex problem:

$$\inf_{v\colon \mathbb{T}^d \to \mathbb{R}^d} \Phi\Big(t, \int_{\mathbb{T}^d} \varphi(x,t,v(x))m(x,t)\,\mathrm{d}x\Big) + \int_{\mathbb{T}^d} \big(L(v(x)) + \langle \nabla u(x,t), v(x)\rangle\big)m(x,t)\,\mathrm{d}x.$$

Another possibility of extension of our analysis would be to add convex constraints on the control variable.

Future research will aim at exploiting the potential structure of the problem, which can be used to solve it numerically and to prove the convergence of learning procedures, as was done in [5].

A A priori bounds for parabolic equations

In this appendix we provide estimates for the following parabolic equation:

$$\begin{aligned} \partial_t u - \sigma\Delta u + \langle b, \nabla u\rangle + cu = &\quad h, && (x,t) \in Q, \\ u(x,0) = &\quad u_0(x), && x \in \mathbb{T}^d, \end{aligned} \tag{53}$$

for different assumptions on b, c, h, and u_0. The technique is based on the following idea. By standard parabolic estimates detailed below, (53) has a unique solution u in $L^2(0,T;H^1(\mathbb{T}^d))$, that we may identify with a periodic function over $\mathbb{R}^d$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be of class C^∞, with value 1 in a neighbourhood of the closure of $\mathbb{T}^d$, and with compact support in $\Omega := B(0,2)$. Set $Q' := \Omega \times (0,T)$. Then $v := u\varphi$ is solution of

$$\begin{aligned} \partial_t v - \sigma\Delta v + \langle b, \nabla v\rangle + cv = &\quad h[u], && (x,t) \in Q', \\ v(x,0) = &\quad v_0(x), && x \in \Omega, \end{aligned} \tag{54}$$

with $v_0 := u_0\varphi$ and

$$h[u] := h\varphi - 2\sigma\langle \nabla\varphi, \nabla u\rangle - \sigma u\Delta\varphi + \langle b, \nabla\phi\rangle u. \tag{55}$$

Observe that the solution v of (54) is equal to 0 in a vicinity of $(\partial\Omega) \times (0,T)$, and hence, satisfies the homogeneous Neumann condition; this allows us to apply some results of [18].

Lemma A.1. *Let $y \in W^{2,1,q}(Q')$, with $q \in (1,\infty)$. Then $y \in L^{q'}(Q')$ and $\nabla y \in L^{q''}(Q')$, where*

$$\begin{cases} \frac{1}{q''} = \frac{1}{q} - \frac{1}{d+2}, & \text{if } q < 2+d, \\ q'' = \infty, & \text{otherwise}, \end{cases} \qquad \begin{cases} \frac{1}{q'} = \frac{1}{q} - \frac{2}{d+2}, & \text{if } q < 1+\frac{d}{2}, \\ q' = \infty, & \text{otherwise}, \end{cases} \tag{56}$$

with continuous inclusion:

$$\|y\|_{L^{q'}(Q')} + \|\nabla y\|_{L^{q''}(Q')} \leq c(q)\|y\|_{W^{2,1,q}(Q')}. \tag{57}$$

Proof. See [18, Lemma 3.3, page 80]. □

Theorem A.1. *Let $q \in (1,\infty)$, $w_0 \in W^{2-2/q,q}(\Omega)$, and $h \in L^q(Q')$. Then the heat equation*

$$\begin{aligned} \partial_t w - \sigma\Delta w = &\quad h, && (x,t) \in Q', \\ w(x,0) = &\quad w_0(x), && x \in \Omega, \end{aligned} \tag{58}$$

with homogeneous Neumann boundary condition on $\partial\Omega \times (0,T)$, has a unique solution in $W^{2,1,q}(Q')$ that satisfies

$$\|w\|_{W^{2,1,q}(Q')} \leq C\big(\|w_0\|_{W^{2-2/q,q}(\Omega)} + \|h\|_{L^q(Q')}\big).$$

Proof. See [18, Theorem IV.9.1, page 341]. □

Theorem A.2. *Let $p > d+2$. For all $R > 0$, there exists $C > 0$ such that for all $u_0 \in W^{2-2/p,p}(\mathbb{T}^d)$, for all $b \in L^p(Q, \mathbb{R}^d)$, for all $c \in L^p(Q)$, for all $h \in L^p(Q)$, satisfying*

$$\|u_0\|_{W^{2-2/p,p}(\mathbb{T}^d)} \leq R, \quad \|b\|_{L^p(Q,\mathbb{R}^d)} \leq R, \quad \|c\|_{L^p(Q)} \leq R, \quad \|h\|_{L^p(Q)} \leq R,$$

equation (53) has a unique solution u in $W^{2,1,p}(Q)$ which moreover satisfies $\|u\|_{W^{2,1,p}(Q)} \leq C$.

Proof. We first check that there is a solution in the standard variational setting with spaces $H := L^2(\mathbb{T}^d)$, $V := H^1(\mathbb{T}^d)$. Let us show that, if $y \in V$, then $\langle b, \nabla y\rangle$ and cy belong to V^*. By the Sobolev inclusion, $V \subset L^{q_1}(\mathbb{T}^d)$, $1/q_1 = 1/2 - 1/d$, with dense inclusion, so that $V^* \subset L^{q_1}(\mathbb{T}^d)^* = L^{q_2}(\mathbb{T}^d)$, with $1/q_2 = 1 - 1/q_1 = 1/2 + 1/d$. Now $\langle b, \nabla y\rangle \in L^r(\mathbb{T}^d)$ with

$$\frac{1}{r} = \frac{1}{2} + \frac{1}{p} < \frac{1}{2} + \frac{1}{d+2} < \frac{1}{q_2},$$

so that $\langle b, \nabla y\rangle$ belongs to V^*. Similarly, $cy \in L^r(\mathbb{T}^d)$ with

$$\frac{1}{r} = \frac{1}{q_1} + \frac{1}{p} < \frac{1}{2} - \frac{1}{d} + \frac{1}{1+d/2} < \frac{1}{q_2},$$

so that cy belongs to V^*. So, (53) has a unique solution in the space

$$W(0,T) := \{v \in L^2(0,T;V);\ \partial_t v \in L^2(0,T;V^*)\}. \tag{59}$$

Then we easily check that $h[u] \in L^{q_0}(Q')$, for some $q_0 \in (1,2)$. Then, by Theorem A.1, $v \in W^{2,1,q_0}(Q)$. We next compute by induction a finite sequence $(q_k)_{k=0,1,\dots,K}$ such that

$$(i)\ v \in W^{2,1,q_k}(Q'),\ \forall k = 0,...,K, \quad (ii)\ q_k \in (1, d+2),\ \forall k = 0,...,K-1, \quad (iii)\ q_K \geq d+2.$$

The first element q_0 has already been fixed and satifies $v \in W^{2,1,q_0}(Q')$. If $q_0 \geq d+2$, we can stop and set $K = 0$. Let $k \in \mathbb{N}$, assume that $q_k \in (1, d+2)$ and that $v \in W^{2,1,q_k}(Q')$. Then v is solution of

$$\begin{aligned} \partial_t v - \sigma\Delta v = &\quad h''[u], \qquad (x,t) \in Q', \\ u(x,0) = &\quad v_0(x), \qquad x \in \Omega, \end{aligned} \tag{60}$$

where

$$h''[u] := h\varphi - 2\sigma\langle\nabla\varphi, \nabla u\rangle - \sigma u\Delta\varphi + u\langle b, \nabla\varphi\rangle - \varphi(\langle b, \nabla u\rangle + cu). \tag{61}$$

We construct now q_{k+1} in such a way that $h''[u] \in L^{q_{k+1}}(Q')$. Since $v \in W^{2,1,q_k}(Q')$, we have that $u \in W^{2,1,q_k}(Q)$ and thus by Lemma A.1, $\langle b, \nabla u\rangle \in L^{r'}(Q')$ with

$$\frac{1}{r'} = \frac{1}{q_k} + \frac{1}{p} - \frac{1}{d+2}. \tag{62}$$

If $q_k < 1 + d/2$, then $cu \in L^{r''}(Q')$ with

$$\frac{1}{r''} = \frac{1}{q_k} + \frac{1}{p} - \frac{2}{d+2}. \tag{63}$$

Note that $r'' > r'$. If $q_k \geq 1 + d/2$, then $u \in L^\infty(Q')$ and thus $cu \in L^p(Q')$. We set now $q_{k+1} = \min(r', p)$. We observe that in both cases, $cu \in L^{q_{k+1}}(Q')$. One can verify that the other terms of $h''[u]$ also lie in $L^{q_{k+1}}(Q')$. Therefore, by Theorem A.1, $v \in W^{2,1,q_{k+1}}(Q')$. If $q_{k+1} \geq d+2$, we stop the construction of the sequence and set $K = k+1$. It remains to prove that the construction of the sequence stops after finitely many iterations. If that was not the case, we would have that $q_{k+1} = r'$, with r' defined in (62), for all $k \in \mathbb{N}$, implying that

$$\frac{1}{q_k} = \frac{1}{q_0} + k\Big(\frac{1}{p} - \frac{1}{d+2}\Big) \underset{k\to\infty}{\longrightarrow} -\infty,$$

which is a contradiction. Now we know that $v \in W^{2,1,q_K}(Q')$, with $q_K \geq d+2$. This implies that $u \in L^\infty(Q')$ and $\nabla u \in L^\infty(Q', \mathbb{R}^d)$ (by Lemma A.1) and thus that $h''[u] \in L^p(Q')$. Finally, $v \in W^{2,1,p}(Q')$ (by Theorem A.1) and $u \in W^{2,1,p}(Q)$, since u and v coincide on Q.

Observing that q_0,...,q_K only depend on p and d, the reader can check that v (and thus u) can be bounded in $W^{2,1,p}(Q')$ by a constant depending on R only. □

Theorem A.3. *For $q \in (1, \infty)$, the trace at time $t = 0$ of elements of $W^{2,1,q}(Q')$ belongs to $W^{2-2/q,q}(\Omega)$.*

Proof. See [18, Lemma 3.4, page 82]. □

Theorem A.4. *Let $p > d+2$. There exists a constant $C > 0$ such that for all $u_0 \in W^{2-2/p,p}(\mathbb{T}^d)$ and for all $h \in L^p(Q)$, the unique solution u to (53) (with $b = 0$ and $c = 0$) satisfies the following estimate:*

$$\|u\|_{W^{2,1,p}(Q)} \leq C\big(\|u_0\|_{W^{2-2/p,p}(\mathbb{T}^d)} + \|h\|_{L^p(Q)}\big).$$

Proof. Consider the mapping

$$u \in W^{2,1,p}(Q) \mapsto (u(\cdot, 0), \partial_t u - \sigma\Delta u - h) \in W^{2-2/p,p}(\Omega), L^p(Q)).$$

By Theorem A.3, it is continuous and by Theorem A.4, it is bijective. As a consequence of the open mapping theorem, its inverse is also continuous. The result follows. □

Lemma A.2. *Let $p > d+2$. There exists $\delta \in (0,1)$ and $C > 0$ such that for all $u \in W^{2,1,p}(Q)$,*

$$\|u\|_{\mathcal{C}^\delta(Q)} + \|\nabla u\|_{\mathcal{C}^\delta(Q,\mathbb{R}^d)} \leq C\|u\|_{W^{2,1,p}(Q)}.$$

Proof. See [18, Lemma II.3.3, page 80 and Corollary, page 342]. □

Theorem A.5. *Let $p > d+2$. For all $\alpha \in (0,1)$, for all $R > 0$, there exists $C > 0$ such that for all $u_0 \in \mathcal{C}^{2+\alpha}(\mathbb{T}^d)$, $b \in \mathcal{C}^{\alpha,\alpha/2}(Q, \mathbb{R}^d)$, $c \in \mathcal{C}^{\alpha,\alpha/2}(Q)$ and $h \in \mathcal{C}^{\alpha,\alpha/2}(Q)$ satisfying*

$$\|u_0\|_{\mathcal{C}^{2+\alpha}(\mathbb{T}^d)} \leq R, \quad \|b\|_{\mathcal{C}^{\alpha,\alpha/2}(Q,\mathbb{R}^d)} \leq R, \quad \|c\|_{\mathcal{C}^{\alpha,\alpha/2}(Q)} \leq R, \quad \text{and} \quad \|h\|_{\mathcal{C}^{\alpha,\alpha/2}(Q)} \leq R,$$

the solution to (53) lies in $\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)$ and satisfies $\|u\|_{\mathcal{C}^{2+\alpha,1+\alpha/2}(Q)} \leq C$.

Proof. By Theorem IV.5.1, page 320 in [18], the result holds for the homogeneous Neumann condition. By lemma A.2, $h[u]$ is Hölder continuous, so that the result holds for v. Since u and v coincide on $\mathbb{T}^d$, the conclusion follows. □

References

[1] B. Acciaio, J. Backhoff-Veraguas, and R. Carmona. Extended mean field control problems: stochastic maximum principle and transport perspective. *ArXiv preprint*, 2018. 3

[2] C. Alasseur, I. Ben Tahar, and A. Matoussi. An extended mean field game for storage in smart grids. *ArXiv preprint*, 2018. 2

[3] C. Bertucci, J.-M. Lasry, and P.-L. Lions. Some remarks on mean field games. *ArXiv preprint*, 2018. 2

[4] P. Cardaliaguet. Long time average of first order mean field games and weak KAM theory. *Dynamic Games and Applications*, 3(4):473–488, 2013. 7

[5] P. Cardaliaguet and S. Hadikhanloo. Learning in mean field games: The fictitious play. *ESAIM: Control, Optimisation and Calculus of Variations*, 23(2):569–591, 2017. 21, 25

[6] P. Cardaliaguet and C.-A. Lehalle. Mean field game of controls and an application to trade crowding. *Math. Financ. Econ.*, 12(3):335–363, 2018. 2, 3

[7] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I*, volume 83 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2018. 3

[8] R. Carmona and D. Lacker. A probabilistic weak formulation of mean field games and applications. *Ann. Appl. Probab.*, 25(3):1189–1231, 2015. 3

[9] P. Chan and R. Sircar. Fracking, renewables, and mean field games. *SIAM Review*, 59(3):588–615, 2017. 3

[10] R. Couillet, S. Perlaza, H. Tembine, and M. Debbah. Electrical vehicles in the smart grid: A mean field game analysis. *IEEE Journal on Selected Areas in Communications*, 30(6):1086–1096, July 2012. 2

[11] A. De Paola, D. Angeli, and G. Strbac. Distributed control of micro-storage devices with mean field games. *IEEE Transactions on Smart Grid*, 7(2):1119–1127, March 2016. 2

[12] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. springer, 2015. 8

[13] D. A. Gomes, S. Patrizi, and V. Voskanyan. On the existence of classical solutions for stationary extended mean field games. *Nonlinear Anal.*, 99:49–79, 2014. 2

[14] D. A. Gomes and J. Saúde. A mean field game approach to price formation in electricity markets. *ArXiv preprint*, 2018. 2

[15] D. A. Gomes and V. K. Voskanyan. Extended deterministic mean-field games. *SIAM J. Control Optim.*, 54(2):1030–1055, 2016. 2

[16] P. J. Graber and A. Bensoussan. Existence and uniqueness of solutions for bertrand and cournot mean field games. *Applied Mathematics & Optimization*, pages 1–25, 2015. 3

[17] P. J. Graber and C. Mouzouni. Variational mean field games for market competition. *ArXiv preprint*, 2017. 3

[18] O. A. Ladyzhenskaia, V. A. Solonnikov, and N. N. Ural'tseva. *Linear and quasi-linear equations of parabolic type*, volume 23. American Mathematical Soc., 1988. 25, 26, 27

[19] H. Pham and X. Wei. Bellman equation and viscosity solutions for mean-field stochastic control problem. *ESAIM Control Optim. Calc. Var.*, 24(1):437–461, 2018. 3