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## Overlaying a hypergraph with a graph with bounded maximum degree

<sup>3</sup> Frédéric Havet<sup>a,b</sup>, Dorian Mazauric<sup>a</sup>, Viet-Ha Nguyen<sup>a,b</sup>, Rémi Watrigant<sup>c</sup>

<sup>a</sup> CNRS, I3S, France
<sup>b</sup> Inria, Université Côte d'Azur, France
<sup>c</sup> Université de Lyon, France

#### 7 Abstract

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Let G and H be respectively a graph and a hypergraph defined on a same set of vertices, and let F be a fixed graph. We say that G F-overlays a hyper-9 edge S of H if F is a spanning subgraph of the subgraph of G induced by S, 10 and that G F-overlays H if it F-overlays every hyperedge of H. Motivated 11 by structural biology, we study the computational complexity of two prob-12 lems. The first problem,  $(\Delta \leq k)$  F-OVERLAY, consists in deciding whether 13 there is a graph with maximum degree at most k that F-overlaps a given 14 hypergraph H. It is a particular case of the second problem MAX ( $\Delta \leq k$ ) 15 F-OVERLAY, which takes a hypergraph H and an integer s as input, and 16 consists in deciding whether there is a graph with maximum degree at most k that F-overlays at least s hyperedges of H. 18

We give a complete polynomial/ $\mathcal{NP}$ -complete dichotomy for the MAX  $(\Delta \leq k)$ -F-OVERLAY problems depending on the pairs (F, k), and establish the complexity of  $(\Delta \leq k)$  F-OVERLAY for many pairs (F, k).

Keywords: graphs, hypergraphs, complexity, algorithm, computational
structural biology

#### 21 1. Introduction

A major problem in structural biology is the characterization of low resolution structures of macro-molecular assemblies [? ? ]. To attack this very difficult question, one has to determine the plausible contacts between the subunits (e.g. proteins) of an assembly, given the lists of subunits involved in some complexes. We assume that the composition, in terms of individual subunits, of selected complexes is known. Indeed, a given assembly can be

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chemically split into complexes by manipulating chemical conditions. This problem can be conveniently modelled by graphs and hypergraphs. We consider the hypergraph H whose vertices represent the subunits and whose hyperedges are the complexes. We are then looking for a graph G with the same vertex set as H whose edges represent the contacts between subunits, and satisfying (i) some local properties for every complex (*i.e.* hyperedge), and (ii) some other global properties.

For example, consider the following properties: (i) the subset of vertices 35 of every hyperedge must form a connected subgraph of G and (ii) the number 36 of edges of G is at most the number of vertices. The first property comes from 37 the fact that the complexes obtained (e.g. by native mass spectrometry) are 38 connected components of a given assembly. The second property is deduced 39 from the constant number of subunits that are in contact to any subunit of an 40 assembly. It means that the total number of edges is linear in the number of 41 vertices (in our example, we assume that the number of edges is at most the 42 number of vertices). Let us now consider a hypergraph H whose hyperedges 43 are  $\{v_1, v_2, v_3\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}, \text{ and } \{v_4, v_5, v_6, v_7\}$ . These hyperedges 44 are depicted in Figure 1 (left) in which a vertex labeled i represents  $v_i$ . The 45 graph G described in Figure 1 (right) satisfies the previous properties: the 46 set of vertices  $\{v_4, v_5, v_6, v_7\}$  induces a connected subgraph (same observation 47 for the three other hyperedges) and the number of edges of G is exactly the 48 number of vertices. 49



Figure 1: A hypergraph composed of seven vertices and four hyperedges (right). A graph G satisfying local properties (every hyperdge induces a connected subgraph in G) and a global property (the total number of edges is at most the number of vertices).

We first focus on the local properties. They are usually modelled by a (possibly infinite) family  $\mathcal{F}$  of admissible graphs to which each complex must belong: to this end, we define the notion of *enforcement* of a hyperedge and a hypergraph. A graph  $G \mathcal{F}$ -enforces a hyperedge  $S \in E(H)$  if the subgraph G[S] of G induced by S belongs to  $\mathcal{F}$ , and  $G \mathcal{F}$ -enforces H if  $G \mathcal{F}$ -enforces

all hyperedges of H. Very often, the considered family  $\mathcal{F}$  is closed on taking 55 edge supergraphs [? ?]: if  $F \in \mathcal{F}$ , then every graph obtained from G by 56 adding edges is also in  $\mathcal{F}$ . Such a family is completely defined by its set 57  $\mathcal{M} = \mathcal{M}(\mathcal{F})$  of minimal graphs that are the elements of  $\mathcal{F}$  which are not 58 edge supergraphs of any other. In this case, a graph  $G \mathcal{F}$ -enforcing S is such 59 that there is an element of  $\mathcal{M}$  which is a spanning subgraph of G[S]. This 60 leads to the following notion of *overlayment* when considering minimal graph 61 families. 62

<sup>63</sup> **Definition 1.** A graph  $G \mathcal{F}$ -overlays a hyperedge S if there exists  $F \in \mathcal{F}$ <sup>64</sup> such that F is a spanning subgraph of G[S], and  $G \mathcal{F}$ -overlays H if  $G \mathcal{F}$ -<sup>65</sup> overlays every hyperedge of H.

As said previously, the graph sought will also have to satisfy some global constraints. Since in a macro-molecular assembly the number of contacts is small, the first natural idea is to look for a graph G with the minimum number of edges. This leads to the MIN- $\mathcal{F}$ -OVERLAY problem: given a hypergraph H and an integer m, decide if there exists a graph G  $\mathcal{F}$ -overlaying H such that  $|E(G)| \leq m$ .

A typical example of a family  $\mathcal{F}$  is the set of all connected graphs, in 72 which case  $\mathcal{M}(\mathcal{F})$  is the set of all trees. Agarwal et al. [?] focused on MIN-73  $\mathcal{M}(\mathcal{F})$ -OVERLAY for this particular family in the aforementioned context of 74 structural biology. However, this problem was previously studied by several 75 communities in other domains, as pointed out by Chen *et al.*? ]. Indeed, 76 it is also known as Subset Interconnection Design, Minimum Topic-77 CONNECTED OVERLAY OF INTERCONNECTION GRAPH PROBLEM, and was 78 considered (among others) in the design of vacuum systems [??], scalable 79 overlay networks [??], and reconfigurable interconnection networks [??]. 80 Some variants have also been considered in the contexts of inferring a most 81 likely social network [?], determining winners of combinatorial auctions [? 82 ], as well as drawing hypergraphs [??]. 83

<sup>84</sup> Cohen et al. [?] presented a dichotomy regarding the polynomial vs. <sup>85</sup>  $\mathcal{NP}$ -hard status of the problem MIN- $\mathcal{F}$ -OVERLAY with respect to the con-<sup>86</sup> sidered family  $\mathcal{F}$ . Roughly speaking, they showed that the easy cases one <sup>87</sup> can think of (e.g. when edgeless graphs of the right sizes are in  $\mathcal{F}$ , or if  $\mathcal{F}$ <sup>88</sup> contains only cliques) are the only families giving rise to a polynomial-time <sup>89</sup> solvable problem: all others are  $\mathcal{NP}$ -complete. They also considered the <sup>90</sup> FPT/W[1]-hard dichotomy for several families  $\mathcal{F}$ .



Figure 2: Example of  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY and MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY. In the figure, an instance H (left), a graph G with  $\Delta(G) \leq 1$  that  $O_3$ -overlays H (with  $O_3$  being the graph with three vertices and one edge) (center), and a solution to MAX  $(\Delta \leq 3)$ - $C_3$ -OVERLAY (with  $C_3$  being the cycle on three vertices) (right).

In this paper, we consider the variant in which the additional constraint is that G must have a bounded maximum degree: this constraint is motivated by the context of structural biology, since a subunit (e.g. a protein) cannot be connected to many other subunits. This yields the following problem for any family  $\mathcal{F}$  of graphs and an integer k.

 $(\Delta \leq k) - \mathcal{F} - \text{OVERLAY}$   $\underbrace{\text{Input: A hypergraph } H.}_{\text{Question: Does there exist a graph } G \mathcal{F} - \text{overlaying } H \text{ such that } \Delta(G) \leq k ?$   $\underbrace{\text{We have to be exist a graph } G \mathcal{F} - \text{overlaying } H \text{ such that } \Delta(G) \leq k ?$ 

We denote by  $over_{\mathcal{F}}(H,G)$  the number of hyperedges of H that are  $\mathcal{F}$ overlaid by G. A natural generalization is to find  $over_{\mathcal{F}}(H,k)$ , the maximum number of hyperedges  $\mathcal{F}$ -overlaid by a graph with maximum degree at most k.

Max  $(\Delta \leq k)$ - $\mathcal{F}$ -overlay

<sup>102</sup> Input: A hypergraph H and a positive integer s. Question: Does there exist a graph G such that  $\Delta(G) \leq k$  and  $\overline{over_{\mathcal{F}}(H,G)} \geq s$ ?

<sup>104</sup> Observe that there is an obvious reduction from  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY to <sup>105</sup> MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY (by setting s = |E(H)|).

In this paper, we mainly consider the case when the family  $\mathcal{F}$  contains a unique graph F. We abbreviate  $(\Delta \leq k)$ - $\{F\}$ -OVERLAY and MAX  $(\Delta \leq k)$ - $\{F\}$ -OVERLAY to  $(\Delta \leq k)$ -F-OVERLAY and MAX  $(\Delta \leq k)$ -F-OVERLAY, respectively. By definition those two problems really make sense only for |F|-uniform hypergraphs *i.e.* hypergraphs whose hyperedges are of size |F|. Therefore, we always assume the hypergraph to be |F|-uniform.

II2 If F is a graph with maximum degree greater than k, then solving  $(\Delta \leq k)$ -II3 F-OVERLAY or MAX  $(\Delta \leq k)$ -F-OVERLAY is trivial as the answer is always II4 'No'. So we only study the problems when  $\Delta(F) \leq k$ . If F is an anticomplete graph (i.e. with no edges), then MAX ( $\Delta \leq k$ )- *F*-OVERLAY is also trivial, because for any hypergraph H, the anticomplete graph on V(H) vertices F-overlays H. Hence the first natural interesting cases are the graphs with one edge. For every integer  $p \geq 2$ , we denote by  $O_p$ the graph with p vertices and one edge. In Section 2, we prove the following dichotomy theorem.

**Theorem 1.** Let  $k \ge 1$  and  $p \ge 2$  be integers. If p = 2 or if k = 1 and  $p \ge 2$  be integers. If p = 2 or if k = 1 and  $p \ge 3$ , then MAX ( $\Delta \le k$ )- $O_p$ -OVERLAY and ( $\Delta \le k$ )- $O_p$ -OVERLAY are polynomial-time solvable. Otherwise, they are  $\mathcal{NP}$ -complete.

Then, in Section 3, we give a complete polynomial/ $\mathcal{NP}$ -complete dichotomy for the MAX ( $\Delta \leq k$ )-F-OVERLAY problems.

**Theorem 2.** MAX ( $\Delta \leq k$ )-F-OVERLAY is polynomial-time solvable if either  $\Delta(F) > k$ , or F is an anticomplete graph, or  $F = O_2$ , or k = 1 and  $F = O_3$ . Otherwise it is  $\mathcal{NP}$ -complete.

In Section 4, we investigate the complexity of  $(\Delta \leq k)$ -F-OVERLAY and 129  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY problems. We believe that each such problem is either 130 polynomial-time solvable or  $\mathcal{NP}$ -complete. However the dichotomy seems 131 to be more complicated than the one for MAX ( $\Delta \leq k$ )-F-OVERLAY. We 132 exhibit several pairs (F, k) such that  $(\Delta < k)$ -F-OVERLAY is polynomial-133 time solvable, while MAX ( $\Delta < k$ )-F-OVERLAY is  $\mathcal{NP}$ -complete. This is 134 in particular the case when F is a complete graph (Proposition 3), F is 135 connected k-regular (Proposition 4), F is a path and k = 2 (Theorem 7), 136 and when F is the cycle on 4 vertices and  $k \leq 3$  (Theorem 6). 137

Most notations of this paper are standard. We now recall some of them, and we refer the reader to [?] for any undefined terminology. The degree of a vertex v in a graph is denoted by d(v). For a positive integer p, let  $[p] = \{1, \ldots, p\}$ . In some proofs of the paper we might use the modulo operation to keep indices in bounds.

Given  $S \subseteq V(G)$ , we denote by G[S] the subgraph induced by S, that is the subgraph with vertex set S and edge set  $\{uv \in E(G) \mid u, v \in S\}$ . We denote by  $\overline{K}_k$  the *anticomplete* graph on k vertices, that is the graph with kvertices and no edges. The *disjoint union* of two graphs F and G is denoted by F + G.

Let H be a hypergraph. Two hyperedges are *adjacent* if their intersection has size at least 2. A hypergraph is *neat* if any two distinct hyperedges

intersect in at most one vertex. In other words, a hypergraph is neat if 150 there is no pair of adjacent hyperedges. We denote by K(H), the graph 151 obtained by replacing each hyperedge by a complete graph. In other words, 152 V(K(H)) = V(H) and  $E(K(H)) = \{xy \mid \exists S \in E(H), \{x,y\} \subseteq S\}$ . The 153 edge-weight function induced by H on K(H), denoted by  $w_H$ , is defined by 154  $w_H(e) = |\{S \in E(H) \mid e \subseteq S\}|$ . In words,  $w_H(e)$  is the number of hyperedges 155 of H containing e. A hypergraph H is connected if K(H) is connected, and 156 the *connected components* of a hypergraph H are the connected components 157 of K(H). Finally, a graph G F-overlaying H with maximum degree at most 158 k is called an (F, H, k)-graph (resp.  $(\mathcal{F}, H, k)$ -graph) for F is a graph (resp. 159 graph family). 160

#### <sup>161</sup> 2. The graphs with one edge

In this section, we establish Theorem 1. Let  $p \geq 2$ , and H be a puniform hypergraph. Consider the edge-weighted graph  $(K(H), w_H)$ . For every matching M with at least one edge of this graph, let  $G_M = (V(H), M)$ . Every hyperedge  $O_p$ -overlaid by  $G_M$  is a superset of at least one edge of Mand at most  $\lfloor \frac{p}{2} \rfloor$  edges of M. We thus have the following:

167 **Observation 1.** For every matching M of K(H), we have:

$$\frac{1}{\lfloor \frac{p}{2} \rfloor} w_H(M) \le over_{O_p}(H, G_M) \le w_H(M), \tag{1}$$

168 where  $w_H(M) = \sum_{e \in M} w_H(e)$ .

<sup>169</sup> Consider first the case when p = 2. Let H be a 2-uniform hypergraph. <sup>170</sup> Every hyperedge is an edge, so K(H) = H. Moreover, a (hyper)edge of H<sup>171</sup> is  $O_2$ -overlaid by G if and only if it is in E(G). Hence MAX ( $\Delta \leq k$ )- $O_2$ -<sup>172</sup> OVERLAY is equivalent to finding a maximum k-matching (that is a subgraph <sup>173</sup> with maximum degree at most k) in K(H). This problem is polynomial-time <sup>174</sup> solvable, see [?, Chap. 31], hence:

**Proposition 1.** MAX ( $\Delta \leq k$ )- $O_2$ -OVERLAY is polynomial-time solvable for all positive integer k.

If p = 3, Inequalities (1) are equivalent to  $over_{O_3}(H, G_M) = w_H(M)$ . Since the edge set of a graph with maximum degree 1 is a matching, MAX ( $\Delta \leq 1$ )- $O_3$ -OVERLAY is equivalent to finding a maximum-weight matching in the edge-weighted graph ( $K(H), w_H$ ). This can be done in polynomialtime, see [?, Chap. 14].



Figure 3: The construction of a variable gadget (left) which means that a blue (resp. red) edge in this gadget corresponds to that the variable is *true* (resp. *false*). And an example of clause gadget of  $C_1 = (x_1 \vee \bar{x_4} \vee \bar{x_5})$  (right) with three edges  $u_1v_1, u_4\bar{v}_4, u_5\bar{v}_5$  which corresponds to an assignment  $\phi$  such that  $\phi(x_1) = \phi(\bar{x}_4) = \phi(\bar{x}_5) = true$ .

**Proposition 2.** MAX ( $\Delta \leq 1$ )- $O_3$ -OVERLAY is polynomial-time solvable.

We shall now prove that if  $p \ge 4$ , or p = 3 and  $k \ge 2$ , then MAX ( $\Delta \le k$ )-  $O_p$ -OVERLAY is  $\mathcal{NP}$ -complete. We prove it by a double induction on k and p. Theorems 3 and 4 first prove the base cases of the induction and Lemma 1 corresponds to the inductive steps.

187 Theorem 3.  $(\Delta \leq 1)$ - $O_4$ -Overlay is  $\mathcal{NP}$ -complete.

- 188 Proof. We reduce 3-SAT to  $(\Delta \leq 1)$ -O<sub>4</sub>-OVERLAY.
- We shall need the following gadget hypergraph J = J(A, B).

190  $V(J) = A \cup B$  with |A| = 3 and |B| = 4 and  $E(J) = \{A \cup \{b\} \mid b \in B\}$ .

<sup>191</sup> Claim 1. Every  $(O_4, J, 1)$ -graph contains exactly one edge with both endver-<sup>192</sup> tices in A.

Proof of claim: Let G be an  $(O_4, J, 1)$ -graph. Each hyperedge of J contains an edge of G, which necessarily has an endvertex in A. Since  $\Delta(G) \leq 1$ , G has at most three edges with an endvertex in A. Since J has four hyperedges, there is an edge of G contained in two hyperedges of J. The two endvertices of this edge are in A.

Let  $\Phi$  be an instance of 3-SAT with *m* clauses  $C_1, \ldots, C_m$  on *n* variables  $x_1, \ldots, x_n$ . Let  $H = H(\Phi)$  be the 4-uniform hypergraph constructed as follows (see Figure 3).

- For each variable  $x_i$ , we create the variable gadget  $VG_i$  as follows.
- Its vertex set is  $\{u_i, v_i, \bar{v}_i, w_i, \bar{w}_i, w'_i, \bar{w}'_i, u'_i\} \cup B_i$ , with  $|B_i| = 4$ . We add
- the hyperedges of  $J(\{u_i, v_i, \bar{v}_i\}, B_i), J(\{v_i, w_i, w'_i\}, B_i), J(\{\bar{v}_i, \bar{w}_i, \bar{w}'_i\}, B_i),$
- and  $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i).$

• For each clause  $C_j = \ell_1 \lor \ell_2 \lor \ell_3$ , we create the clause hypergraph  $CG_j$ . Its vertex set is  $\{z_j\} \cup \bigcup_{k=1}^3 \{s_j^k, t_j^k, u_j^k, v_j^k\}$ . Set  $Z_j = \{t_j^1, t_j^2, t_j^3, z_j\}$  and for  $k \in \{1, 2, 3\}$ , set  $T_j^k = \{s_j^k, t_j^k, u_j^k, v_j^k\}$ . The hyperedge set of  $CG_j$ is  $\{Z_j, T_j^1, T_j^2, T_j^3\}$ . Then for each  $k \in \{1, 2, 3\}$ , we identify  $u_j^k$  and  $v_j^k$ with  $u_i$  and  $v_i$  if  $\ell_k = x_i$  and with  $u_i$  and  $\bar{v}_i$  if  $\ell_k = \bar{x}_i$ .

Let us prove that there is an  $(O_4, H, 1)$ -graph if and only if  $\Phi$  is satisfiable. The general idea of the proof is that the variable gadget enforces only one of edges  $u_i v_i$ ,  $u_i \bar{v}_i$  to belongs G and that  $u_i v_i \in E(G)$  (resp.  $u_i \bar{v}_i \in E(G)$ ) if and only if the corresponding variable  $x_i$  is true (resp. false), thus the appearance of such an edge in a clause gadget implies that there is a true literal in the clause.

Assume first that there is an  $(O_4, H, 1)$ -graph G.

**Claim 2.** For each  $1 \leq i \leq n$ , G contains either the edge  $u_i v_i$  or the edge  $u_i \bar{v}_i$ . Moreover, for each  $x \in \{u_i, v_i, \bar{v}_i\}$  there is some  $x' \in VG_i$  such that x and x' are adjacent in G.

*Proof of claim:* Since  $VG_i$  contains  $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i)$ , by Claim 1, G contains 220 an edge e'' with both endvertices in  $\{u'_i, w'_i, \bar{w}'_i\}$ . Assume that e'' is incident 221 to  $w'_i$ . Now  $VG_i$  contains  $J(\{v_i, w_i, w'_i\}, B_i)$ . So, by Claim 1, G contains 222 an edge e' with both endvertices in  $\{v_i, w_i, w'_i\}$ . Since  $\Delta(G) \leq 1$ , e' is not 223 incident to  $w'_i$ , so  $e' = v_i w_i$ . Now  $VG_i$  contains  $J(\{u_i, v_i, \bar{v}_i\}, B_i)$ . So, by 224 Claim 1, G contains an edge e with both endvertices in  $\{u_i, v_i, \bar{v}_i\}$ . This edge 225 is not incident to  $v_i$ , so  $e = u_i \bar{v}_i$ . Similarly, we get that G contains  $\bar{v}_i \bar{w}_i$  and 226  $u_i v_i$  if e'' is incident to  $\bar{w}'_i$ .  $\triangleleft$ 227

In view of this claim, one defines the truth assignment  $\phi$  by  $\phi(x_i) = true$ 228 if  $u_i v_i \in E(G)$  and  $\phi(x_i) = false$  if  $u_i \bar{v}_i \in E(G)$ . Let us prove that it satisfies 229  $\Phi$ . Consider a clause  $C_j = \ell_1 \vee \ell_2 \vee \ell_3$  and its corresponding clause gadget 230  $CG_j$ . If there is none of  $u_j^k v_j^k$  in E(G) (for all  $k \in \{1, 2, 3\}$ ), that means 231 both  $u_j^k$  and  $v_j^k$  are incident to an edge which is incident to a vertex in the 232 variable gadget it they belongs to,  $T_i^k$  must be  $O_4$ -overlaid by the edge  $s_i^k t_i^k$ . 233 So,  $t_i^k$  (for all  $k \in \{1, 2, 3\}$ ) has degree one in G. However, the hyperedge 234  $Z_j$  is  $O_4$ -overlaid by  $z_j t_j^k$  or  $t_j^k t_j^{k'}$  for  $k' \neq k$  and  $k, k' \in \{1, 2, 3\}$ , which 235 implies that there is one of vertices  $t_j^k$  which is of degree one in  $G[Z_j]$ . It is 236 a contradiction, thus there is an edge  $u_i^k v_i^k \in E(G)$  (for some  $k \in \{1, 2, 3\}$ ). 237 Hence  $\phi(\ell_k) = true$ . Hence  $\phi$  satisfies  $\Phi$ . 238

Conversely, assume  $\phi$  is a truth assignment satisfying  $\Phi$ . Let G be the graph with vertex set V(H) and whose edge set is constructed as follows. For every  $1 \leq i \leq n$ , add  $\{u_i v_i, w_i w'_i, u'_i \bar{w}_i, \bar{w}_i \bar{v}_i\}$  to E(G) if  $\phi(x_i) = true$ and  $\{u_i \bar{v}_i, \bar{w}_i \bar{w}'_i, u'_i w'_i, w_i v_i\}$  to E(G) if  $\phi(x_i) = false$ . For every clause  $C_j = \ell_1 \lor \ell_2 \lor \ell_3$ , there exists  $k \in \{1, 2, 3\}$  such that  $\phi(\ell_k) = true$ . Add the edges  $z_{jt}^k_j, s_j^{k+1} t_j^{k+1}$  and  $s_j^{k+2} t_j^{k+2}$  (the superscript are modulo 3) to E(G). One easily checks that G is an  $(O_4, H, 1)$ -graph.  $\Box$ 

#### Theorem 4. $(\Delta \leq 2)$ -O<sub>3</sub>-OVERLAY is $\mathcal{NP}$ -complete.

*Proof.* In order to simplify the proof, we will prove that the problem  $(\Delta \leq 2)$ -247  $O_3$ -OVERLAY WITH HANDICAP is  $\mathcal{NP}$ -complete, where, in addition to a 248 hypergraph H, we are given a subset of vertices  $A \subseteq V(H)$  (called handicap 249 *vertices*), and we ask for a graph G that  $O_3$ -overlays H with the additional 250 constraint that for every  $v \in A$ , the degree of v is at most 1. Observe 251 that we can easily reduce this problem to  $(\Delta \leq 2) - O_3$ -OVERLAY: for every 252  $v \in A$ , add six vertices  $t_v^1, \ldots, t_v^6$ , and hyperedges  $\{v, t_v^1, t_v^p\}$  for every  $p \in v$ 253  $\{2, 3, 4, 5, 6\}$ . Let H' be the obtained hypergraph. Observe that an  $(O_3, H, 2)$ -254 graph G in which every  $v \in A$  has degree at most 1 can be extended into an 255  $(O_3, H', 2)$ -graph by adding the edge  $vt_v^1$  (and all vertices of  $V(H') \setminus V(H)$ ). 256 Conversely,  $(O_3, H', 2)$ -graph G' must contain the edge  $vt_v^1$ , and thus v is of 257 degree at most 1 in G'[V(H)]. 258

We reduce 3-SAT to  $(\Delta \leq 2)$ - $O_3$ -OVERLAY WITH HANDICAP. Let  $\Phi$ be an instance of 3-SAT with *m* clauses  $C_1, \ldots, C_m$  on *n* variables  $x_1, \ldots, x_n$ . We construct a 3-uniform hypergraph *H* as follows (see Figure 4 for an example):

• for every variable  $x_i$ , we create a variable gadget  $VG_i$  composed of vertices  $a_i^1, \ldots, a_i^{p_i}, \alpha_i^1, \ldots, \alpha_i^{p_i}, b_i^1, \ldots, b_i^{p_i}, \beta_i^1, \ldots, \beta_i^{p_i}, z_i^1, \ldots, z_i^{p_i}$ , and  $\zeta_i^1, \ldots, \zeta_i^{p_i}$  where  $p_i$  is the number of clauses where  $x_i$  appears. Then, for every  $j \in [p_i]$ , we create the following hyperedges (additions in superscripts are modulo  $p_i$ , and we write  $p_i$  instead of 0):  $h_{i,j}^1 =$  $\{z_i^j, a_i^j, \alpha_i^j\}, h_{i,j}^2 = \{z_i^j, b_i^j, \beta_i^j\}, h_{i,j}^3 = \{a_i^j, z_i^j, b_i^j\}, h_{i,j}^4 = \{z_i^j, b_i^j, \zeta_i^j\},$  $h_{i,j}^5 = \{b_i^j, \zeta_i^j, a_i^{j+1}\}, h_{i,j}^6 = \{\zeta_i^j, a_i^{j+1}, z_i^{j+1}\}.$ 

- Moreover, all vertices  $z_i^1, \ldots, z_i^{p_i}$ , and  $\zeta_i^1, \ldots, \zeta_i^{p_i}$  are handicap vertices.
- for every clause  $C_j$ , we create a clause gadget  $CG_j$  composed of vertices  $c_j^1, \ldots, c_j^5$ , and, for every  $p \in [5]$ , of the hyperedge  $\{c_j^p, c_j^{p+1}, c_j^{p+3}\}$



Figure 4: An example of the reduction for  $\Phi = (x_1 \lor x_2 \lor x_3) \land (x_2 \lor x_3 \lor x_4)$ . The dotted lines join two vertices that are identified. The thick red lines indicate the handicap vertices.

(additions are modulo 5, and we write 5 instead of 0). The vertices  $c_j^4$ and  $c_j^5$  are handicap vertices.

• finally, for every clause  $C_j = \ell_1 \vee \ell_2 \vee \ell_3$ , if  $\ell_p = x_i$   $(\ell_p = \bar{x}_i)$  for some  $p \in \{1, 2, 3\}$ , then we identify the vertex  $c_j^p$  of  $CG_j$  with the vertex  $\alpha_i^j$ ( $\beta_i^j$ ) of  $VG_i$ , respectively.

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The general idea of the proof is the following. There are only two possibilities to overlay a variable gadget  $VG_i$  (either by a set  $T_i$  or a set  $F_i$  of edges). See Claim 3. This two possibilities correspond to the two possible values (*true* or *false*) of the variable  $x_i$ . Then a clause gagdet  $CG_j$  imposes that not all literals of the clauses are assigned *false* with this correspondence.

Assume first that there is an  $(O_3, H, 2)$ -graph G such that all handicap vertices have degree at most 1.

For  $i \in [n]$ , let  $T_i = \bigcup_{j=1}^{p_i} \{z_i^j a_i^j, b_i^j \beta_i^j, b_i^j \zeta_i^j\}$  and  $F_i = \bigcup_{j=1}^{p_i} \{a_i^j \alpha_i^j, z_i^j b_i^j, \zeta_i^j a_i^{j+1}\}$ . **Claim 3.** Let  $i \in [n]$ . We have the following:

288 1.  $z_i^j \zeta_i^j \notin E(G) \text{ and } \zeta_i^j z_i^{j+1} \notin E(G), \ j \in [m].$ 

289 2. Either  $T_i \subseteq E(G)$  or  $F_i \subseteq E(G)$ .

Proof of claim: 1. Assume for a contradiction that  $z_i^j \zeta_i^j \in E(G)$ . Since  $z_i^j$ and  $\zeta_i^j$  are handicap vertices, the hyperedges  $h_{i,j}^2, h_{i,j}^3, h_{i,j}^5$  must be  $O_3$ -overlaid by the edges  $b_i^j \beta_i^j, a_i^j b_i^j, b_i^j a_i^{j+1}$ , respectively, contradicting the fact that  $b_i^j$  has <sup>293</sup> degree 2 in G.

Similarly, if  $\zeta_i^j z_i^{j+1} \in E(G)$ , then the hyperedges  $h_{i,j}^5, h_{i,j+1}^1, h_{i,j+1}^3$  must be  $O_{3^{-295}}$  overlaid by the edges  $b_i^j a_i^{j+1}, a_i^{j+1} \alpha_i^j, a_i^{j+1} b_i^{j+1}$ , respectively, a contradiction.

296 2. By 1. the hyperedge  $h_{i,j}^4$  is  $O_3$ -overlaid by  $z_i^j b_i^j$  or  $\zeta_i^j b_i^j$ .

In the first case  $z_i^j b_i^j \in E(\tilde{G})$ , since  $z_i^j$  is a handicap vertex, the hyperedge  $h_{i,j-1}^6$  must be  $O_3$ -overlaid by  $\zeta_i^{j-1} a_i^j$ , and so hyperedge  $h_{i,j-1}^3$  is  $O_3$ -overlaid by  $z_i^{j-1} b_i^{j-1}$ . And by induction, we get that  $\bigcup_{j=1}^m \{\zeta_i^j a_i^{j+1}, z_i^j b_i^j\} \subseteq E(G)$ . Since the  $z_i^j$  are handicap vertices, the hyperedge  $h_{i,j}^1$  is  $O_3$ -overlaid by  $a_i^j \alpha_i^j$ . Hence  $F_i \subseteq E(G)$ .

In the second case  $\zeta_i^j b_i^j \in E(G)$ , a symmetrical argument shows  $T_i \subseteq E(G)$ .

We define the truth assignment  $\phi$  by  $\phi(x_i) = true$  if  $T_i \subseteq E(G)$  and 304  $\phi(x_i) = false$  if  $F_i \subseteq E(G)$ . This is well-defined by Claim 3 and the fact 305 that E(G) cannot contain both  $T_i$  and  $F_i$  because it has maximum degree 2. 306 Now, for any  $j \in [m]$ , there exists  $p \in \{1, 2, 3\}$  such that  $c_i^p$  has degree 2 307 in  $G[CG_j]$  (If not, then there are at most 2 edges. Hence there is a vertex 308  $c_j^p$  of degree 0, and thus there is a hyperedge containing  $c_j^p$  which is not 309 overlaid). Therefore, it has degree 0 in the variable gadget  $c_i^p$  belongs to. By 310 our definition of  $\phi$ , we get that  $\phi(\ell_p) = true$ . Hence  $\phi$  satisfies  $\Phi$ . 311

Suppose now that  $\phi : \{x_1, \ldots, x_n\} \to \{true, false\}$  is a truth assignment satisfying  $\Phi$ .

Let us first select a set of edges in the variable gadgets. For every  $i \in [n]$ , if  $\phi(x_i) = true$ , then select edges  $\{z_i^j, a_i^j\}, \{b_i^j, \beta_i^j\}$  and  $\{\zeta_i^j, b_i^j\}$  for all  $j \in [p_i]$ . If  $\phi(x_i) = false$ , then select edges  $\{z_i^j, b_i^j\}, \{a_i^j, \alpha_i^j\}$  and  $\{\zeta_i^j, a_i^{j+1}\}$  for all  $j \in [p_i]$  (additions are modulo  $p_i$ ). Let  $E_1$  be the set of selected edges, and  $G_1 = (V(H), E_1)$ . Observe that  $G_1 O_3$ -overlays the hyperedges of all variable gadgets: for every  $i \in [n]$  and every  $j \in [p_i]$ ,

- $h_{i,j}^1$  contains the edge  $\{z_i^j, a_i^j\}$  if  $\phi(x_i) = true$ , and the edge  $\{a_i^j, \alpha_i^j\}$ otherwise.
- $h_{i,j}^2$  contains the edge  $\{b_i^j, \beta_i^j\}$  if  $\phi(x_i) = true$ , and the edge  $\{z_i^j, b_i^j\}$ otherwise.
- $h_{i,j}^3$  contains the edge  $\{z_i^j, a_i^j\}$  if  $\phi(x_i) = true$ , and the edge  $\{z_i^j, b_i^j\}$ otherwise.

•  $h_{i,j}^4$  contains the edge  $\{\zeta_i^j, b_i^j\}$  if  $\phi(x_i) = true$ , and the edge  $\{z_i^j, b_i^j\}$  otherwise.

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•  $h_{i,j}^5$  contains the edge  $\{\zeta_i^j, b_i^j\}$  if  $\phi(x_i) = true$ , and the edge  $\{\zeta_i^j, a_i^{j+1}\}$ otherwise.

•  $h_{i,j}^6$  contains the edge  $\{z_i^{j+1}, a_i^{j+1}\}$  if  $\phi(x_i) = true$ , and the edge  $\{\zeta_i^j, a_i^{j+1}\}$ otherwise.

Moreover the vertices  $z_i^1, \ldots, z_i^{p_i}$ , and  $\zeta_i^1, \ldots, \zeta_i^{p_i}$  have degree 1 in  $G_1$  and the vertices  $\alpha_i^j$  have degree 0 in  $G_1$  if and only if  $\phi(x_i) = true$  and the vertices  $\beta_i^j$  have degree 0 in  $G_1$  if and only if  $\phi(x_i) = false$ .

We now select a set of edges in the clause gadgets for every clause  $C_i =$ 335  $\ell_1 \vee \ell_2 \vee \ell_3$ . Since  $\phi$  satisfies  $\Phi$ , there exists  $p = p_j \in \{1, 2, 3\}$  such that  $\phi(\ell_p) = true$ . Select the edges  $\{c_j^p, c_j^{p+2}\}, \{c_j^p, c_j^{p+3}\}$  and  $\{c_j^{p+1}, c_j^{p+4}\}$ . Let  $E_2$  be the set of selected edges in clause gadgets, and  $G_2 = (V(H), E_2)$ . 336 337 338 Clearly,  $G_2$   $O_3$ -overlays all hyperedges of the clause gadgets and the vertices 339  $c_j^4$  and  $c_j^5$ ,  $j \in [m]$ , have degree 1 in  $G_2$ . Moreover, for all  $j \in [m]$  the vertex 340  $c_i^{p_j}$  is identified with the vertex  $\alpha_i^j$  or  $\beta_i^j$ , which has degree 0 in  $G_1$  since 341  $\phi(\ell_p) = true$ . Thus  $G := (V(H), E_1 \cup E_2)$  has maximum degree 2, which 342 proves that it is an  $(O_3, H, 2)$ -graph. 343

Lemma 1. If  $(\Delta \leq k)$ - $O_p$ -OVERLAY is  $\mathcal{NP}$ -complete, then  $(\Delta \leq k)$ - $O_{p+1}$ -345 OVERLAY and  $(\Delta \leq k+1)$ - $O_p$ -OVERLAY are  $\mathcal{NP}$ -complete.

Let k and p be two positive integers with  $p \ge 2$ , let  $H_{k,p}$  be the p-uniform hypergraph on (k+1)(p-1) vertices whose hyperedge set contains all possible p-subsets of vertices.

We first need the following claim in order to prove Lemma 1.

**Claim 4.** Every  $(O_p, H_{k,p}, k)$ -graph is k-regular. Moreover, there exists an  $(O_p, H_{k,p}, k)$ -graph.

Proof of claim: Let G be an  $(O_p, H_{k,p}, k)$ -graph. Assume for a contradiction that it has a vertex v of degree at most k - 1. Then, observe that  $|V(G) \setminus N[v]| \ge (k+1)(p-2)+1$ . Now, by removing successively the closed neighbourhood of a vertex of the graph (chosen arbitrarily) until the graph is anticomplete, we can construct, together with v, an independent set S of G of size p. This is impossible, since S is a hyperedge of  $H_{k,p}$  which is not  $O_p$ -overlaid by G. Finally, observe that the disjoint union of p-1 cliques of size k+1 is a *k*-regular graph which  $O_p$ -overlays  $H_{k,p}$ .

Proof of Lemma 1. First, we will reduce from  $(\Delta \leq k) - O_p$ -OVERLAY to  $(\Delta \leq k) - O_{p+1}$ -OVERLAY. Let H be a p-uniform hypergraph. We construct a (p+1)-uniform hypergraph  $H^*$  from a copy of H and replace, for every hyperedge S, a hyperedge  $S' = S \cup \{v_s\}$ . We also add, for every hyperedge S, a set of (k+1)p-1 new vertices  $N_S$ , and form, together with  $v_S$ , a hypergraph  $H_S$  isomorphic to  $H_{k,p+1}$ .

Let G be an  $(O_p, H, k)$ -graph. Construct  $G^*$  from G by adding, for every hyperedge S of H, a disjoint union of p cliques of size k + 1 on the vertices  $N_S \cup \{v_S\}$ . Clearly,  $G^*$  is an  $(O_{p+1}, H^*, k)$ -graph.

<sup>370</sup> Conversely, if  $G^*$  is an  $(O_{p+1}, H^*, k)$ -graph, then, by Claim 4,  $v_S$  must <sup>371</sup> have degree k in  $N_S$  for every hyperedge S of H. Thus, the restriction of  $G^*$ <sup>372</sup> to the vertices of H is an  $(O_p, H, k)$ -graph.

Now, we reduce  $(\Delta \leq k)$ - $O_p$ -OVERLAY to  $(\Delta \leq k+1)$ - $O_p$ -OVERLAY. Let H be a p-uniform hypergraph. Let H' be the p-uniform hypergraph obtained from H by doing the following for each vertex  $v \in V(H)$ :

• add the vertices  $x^v, y_1^v, \ldots, y_{p-2}^v$  and the hyperedge  $S_v = \{v, x^v, y_1^v, \ldots, y_{p-2}^v\}$ .

377 378 • for any  $i = \{1, \ldots, p-2\}$ , add a copy  $H_i^v$  of  $H_{k+1,p}$  whose vertices are  $y_i^v$  and (p-1)(k+2) - 1 new vertices.

Assume that G' is an  $(O_p, H', k+1)$ -graph. By Claim 4, the graph induced by G' on  $V(H_i^v)$  is (k+1)-regular, for all  $v \in V(H)$ ,  $1 \le i \le p-2$ . Therefore each  $y_i^v$  has k+1-neighbours outside the hyperedge  $S_v$ . Hence G' must contain the edge  $vx^v$  because it  $O_p$ -overlays  $S_v$ . Thus the subgraph G = G'[V(H)]has maximum degree at most  $\Delta(G') - 1 \le k$  and  $G O_p$ -overlays H. In other words, G is an  $(O_p, H, k)$ -graph.

Conversely, assume G is an  $(O_p, H, k)$ -graph. Let G' be the graph with vertex set V(H') and whose edge set is the union of E(G),  $\{vx^v \mid v \in V(H)\}$ , and the edge set of an  $(O_p, H_v^i, k+1)$ -graph (which exists by Claim 4), for all  $v \in V(H), 1 \leq i \leq p-2$ . One easily checks that G' is  $(O_p, H', k+1)$ -graph. Therefore, there is an  $(O_p, H, k)$ -graph if and only if there is an  $(O_p, H', k+1)$ -graph.

Propositions 1 and 2, Theorems 3 and 4, and Lemma 1 imply Theorem 1.

### 393 3. Complexity of Max ( $\Delta \leq k$ )-F-Overlay

The aim of this section is to establish Theorem 2 that gives the  $\mathcal{P}/\mathcal{NP}$ complete dichotomy for the MAX ( $\Delta \leq k$ )-F-OVERLAY problems.

As noticed in the introduction, if  $\Delta(F) > k$  or F is an anticomplete graph then MAX ( $\Delta \leq k$ )- $\mathcal{F}$ -OVERLAY is trivially polynomial-time solvable.

Moreover, by Propositions 1 and 2, MAX ( $\Delta \leq 1$ )- $O_3$ -OVERLAY as well as MAX ( $\Delta \leq k$ )- $O_2$ -OVERLAY (for all positive integers k) are also polynomialtime solvable.

We shall now prove that if we are not in one of the above cases, then MAX ( $\Delta \leq k$ )-F-OVERLAY is  $\mathcal{NP}$ -complete. We first establish the  $\mathcal{NP}$ completeness when F has no isolated vertices.

**Theorem 5.** Let F be a graph on at least three vertices with no isolated vertices. If  $k \ge \Delta(F)$ , then MAX ( $\Delta \le k$ )-F-OVERLAY is  $\mathcal{NP}$ -complete on neat hypergraphs.

<sup>407</sup> Proof. Assume  $k \geq \Delta(F)$ . Let  $n = |F|, a_1, \ldots, a_n$  be an ordering of the <sup>408</sup> vertices of F such that  $\delta(F) = d(a_1) \leq d(a_2) \leq \cdots \leq d(a_n) = \Delta(F)$ .

Let  $\gamma = \lfloor k/\delta(F) \rfloor - 1$ ,  $\beta = k - \gamma\delta(F)$ . Observe that  $\delta(F) \leq \beta \leq 2\delta(F) - 1$ . We shall give a reduction from INDEPENDENT SET which is a well-known  $\mathcal{NP}$ -complete problem even for cubic graphs (see [?]). We distinguish two cases depending on whether  $d(a_2) > \beta$  or not.

Let  $\gamma_2 = \lfloor (k - d(a_2))/\delta(F) \rfloor$  and  $\gamma_3 = \lfloor (k - d(a_3))/\delta(F) \rfloor$ . And we define the value  $\gamma_2$  as follows.

415 - If  $d(a_2) > \beta$ , let  $\gamma_1 = \gamma_2$ 

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If 
$$d(a_2) \leq \beta$$
, let  $\gamma_1 = \gamma_2$ .  
If  $d(a_2) \leq \beta$ , let  $\gamma_1 = \lfloor (k - d(a_1)) / \delta(F) \rfloor$ .

Let  $\Gamma$  be a cubic graph. For each vertex  $v \in V(\Gamma)$ , let  $(e_1(v), e_2(v), e_3(v))$ be an ordering of the edges incident to v. We shall construct the neat hypergraph  $H = H(\Gamma)$  as follows.

• For each vertex  $v \in \Gamma$ , we create a hyperedge  $S_v = \{a_1^v, \ldots, a_n^v\}$ . Then, for  $1 \le i \le 3$ , we add  $\gamma_i a_i^v$ -leaves, that are hyperedges containing  $a_i^v$ and n-1 new vertices.

• For each edge  $e = uv \in \Gamma$ , let *i* and *j* be the indices such that  $e = e_i(u) = e_j(v)$ . We create a new vertex  $z_e$  and hyperedges  $S_u^e(S_v^e)$ containing  $z_e$ ,  $a_i^u(a_j^v)$ , and n-2 new vertices, respectively. Then, we add  $\gamma z_e$ -leaves, that are hyperedges containing  $z_e$  and n-1 new vertices.

From the construction, observe that  $\gamma_i$  will be useful to enforce the degree of vertex  $a_i^v$  for  $v \in V(G)$  which could have degree at most  $d(a_3)$  in the hypergraph.

We shall prove that  $over_F(H, k) = (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$ , where  $\alpha(\Gamma)$  denotes the cardinality of a maximum independent set in  $\Gamma$ .

The following claim shows that there are optimal solutions with specific structure. This leads to the inequality:

$$over_F(H,k) \le (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$$

Claim 5. There is a graph G with  $\Delta(G) \leq k$  that F-overlays over<sub>F</sub>(H, k) hyperedges of H such that:

(a) each x-leaf L is F-overlaid and x is incident to  $\delta(F)$  edges in G[L](with  $x = a_i^v$  or  $x = z_e$ ).

(b) for each edge  $e = uv \in E(\Gamma)$ , exactly one of the two hyperedges  $S_u^e$  and  $S_v^e$  is F-overlaid. Moreover if  $S_u^e$  ( $S_v^e$ ) is F-overlaid, then  $a_i^u$  ( $a_j^v$ ) is incident to at most  $d(a_2)$  edges in  $S_u^e$  ( $S_v^e$ ), respectively.

(c) the set of vertices v such that  $S_v$  is F-overlaid is an independent set in  $\Gamma$ .

Proof of claim: Let G be a graph with  $\Delta(G) \leq k$  that F-overlays  $over_F(H, k)$ hyperedges of H. Observe first that since H is a neat hypergraph, we may assume that, for every hyperedge S of H, G[S] = F if S is F-overlaid and G[S] has no edge otherwise.

Therefore if an x-leaf L is F-overlaid, then replacing G[L] by a copy of F in which x has degree  $\delta(F)$ , we obtain another graph with maximum degree at most k that F-overlays  $over_F(H, k)$  hyperedges of H. Henceforth we may assume that if an x-leaf is F-overlaid then x is incident to  $\delta(F)$  edges included in it.

Now assume that the  $z_e$ -leaf L, for  $e \in E(\Gamma)$ , is not F-overlaid by G. We have  $d_G(z_e) > k - \delta(F)$  for otherwise adding on L the edges of a copy of F in which  $z_e$  has degree  $\delta(F)$ , we obtain a graph F-overlaying more hyperedges than G. But since there are  $\gamma z_e$ -leaves and  $z_e$  is incident to  $\delta(F)$  edges in each of them, at least one of the hyperedges  $S_u^e$  and  $S_v^e$  is F-overlaid. Now replacing the edges of G in this hyperedge by those of a copy of F in L, in which  $z_e$  has degree  $\delta(F)$ , we obtain another graph with maximum degree at most k that F-overlays  $over_F(H, k)$  hyperedges of H. Henceforth we may assume that every  $z_e$ -leaf is F-overlaid.

A similar reasoning shows that every  $a_i^v$ -leaf is F-overlaid, for all  $v \in V(\Gamma)$ ,  $i \in \{1, 2, 3\}$ . This proves (a).

Now by (a) each vertex  $z_e$  is incident to  $\gamma \delta(F) = k - \beta > k - 2\delta(F)$  edges in the  $z_e$ -leaves. Thus  $z_e$  is incident to at most  $\beta$  edges in  $S_u^e \cup S_v^e$ . Since  $\beta \leq 2\delta(F) - 1$ , at most one of  $S_u^e$  and  $S_v^e$  is F-overlaid.

Moreover, if  $S_u^e$  is *F*-overlaid and  $\beta < d(a_2)$  (resp.  $d(a_2) \leq \beta$ ), then  $z_e$  must have degree  $d(a_1)$  (resp. at most  $d(a_2)$ ) in  $G[S_u^e]$ .

<sup>467</sup> Assume now that none of  $S_u^e, S_v^e$  is *F*-overlaid, and let  $e = e_i(u), a_i^u$  has  $\gamma_i$ <sup>468</sup>  $a_i^u$ -leaves, so  $a_i^u$  is incident to  $\gamma_i \delta(F) \leq k - d(a_i)$  edges in these leaves.

<sup>469</sup> Thus, replacing the edges of G in  $S_u^e$  by those of a copy of F, in which

470 - if  $d(a_2) > \beta$ , we have  $d_G(a_i^u) \le k - d(a_2)$ , then  $z_e$  has degree  $\delta(F)$  and 471  $a_i^u$  has degree  $d(a_2)$ .

<sup>472</sup> - if  $d(a_2) \leq \beta$ , since  $d_G(a_1^u) \leq k - d(a_1)$ , then for i = 1,  $a_i^u$  must have <sup>473</sup> degree  $d(a_1)$ , so  $z_e$  has degree  $d(a_2)$ . And for  $i = \{2, 3\}$ , let  $d(z_e) = d(a_1)$ ,  $d(a_i^u) = d(a_2)$ .

Hence, we obtain another graph with maximum degree at most k that Foverlays  $over_F(H,k)$  hyperedges of H. Henceforth, we may assume that at least one of  $S_u^e$  and  $S_v^e$  is F-overlaid. This completes the proof of (b).

<sup>478</sup> Consider now a vertex  $v \in V(\Gamma)$  such that  $S_v$  is *F*-overlaid. We distin-<sup>479</sup> guish the following cases.

<sup>480</sup> - If  $d(a_2) > \beta$ , then by (a), vertex  $a_1^v$  is incident to  $\gamma_1 \delta(F)$  edges in the  $a_1^v$ -<sup>481</sup> leaves and at least  $\delta(F) \in S_v$ . Therefore  $S_v^{e_1(v)}$  cannot be *F*-overlaid for <sup>482</sup> otherwise, by (b),  $a_1^v$  would have degree at least  $\gamma_1 \delta(F) + \delta(F) + d(a_2) >$ <sup>483</sup> *k*. Similarly,  $S_v^{e_2(v)}$  cannot be *F*-overlaid.

Assume now for a contradiction that  $S_v^{e_3(v)}$  is *F*-overlaid. By (a) and (b),  $a_3^v$  is incident to  $\gamma_3\delta(F)$  edges in the  $a_3^v$ -leaves and at least  $d(a_2) \in$  $S_v^{e_3(v)}$ . Since  $\gamma_3\delta(F) + d(a_2) + d(a_3) > k$ ,  $a_3^v$  is incident to less than  $d(a_3)$  edges in  $S_v$ . Thus, there is a vertex  $a \in \{a_1^v, a_2^v\}$  which is incident to at least  $d(a_3)$  edges in  $S_v$  and thus at least  $\gamma_1\delta(F) + d(a_3)$  edges in *G*. But  $a_3^v$  is adjacent to at least  $\delta(F)$  edges in  $S_v$  so at least  $\gamma_3\delta(F) +$  490  $d(a_2) + \delta(F)$  edges in total. Thus  $\gamma_3\delta(F) + d(a_2) + \delta(F) \leq k$ , so 491  $\gamma_3 \leq \gamma_1 - 1$ . Thus *a* is adjacent to at least  $(\gamma_3 + 1)\delta(F) + d(a_3) > k$ 492 edges, a contradiction.

<sup>493</sup> - If  $d(a_2) \leq \beta$ , then by (a), vertex  $a_1^v$  is incident to  $\gamma_1\delta(F)$  edges in <sup>494</sup> the  $a_1^v$ -leaves and at least  $\delta(F) \in S_v$ . If  $S_v^{e_1(v)}$  is *F*-overlaid, then <sup>495</sup> (b) yields  $d(a_i^v) = \gamma_i\delta(F) + \delta(F) + \delta(F) > k$ , a contradiction. If <sup>496</sup>  $S_v^{e_2(v)}$  is *F*-overlaid, then  $d(a_2^v) = \delta$  in  $G[S_v]$ . We obtain that  $d(a_2^v) =$ <sup>497</sup>  $\gamma_2\delta + d(a_2) + \delta \leq k$  which implies that  $\gamma_2 \leq \gamma_2 - 1$ , a contradiction. <sup>498</sup> If  $S_v^{e_3(v)}$  is *F*-overlaid, then we get a contradiction in the same way as <sup>499</sup> that in the case  $d(a_2) > \beta$ .

To summarize, if  $S_v$  is *F*-overlaid, then none of  $S_v^{e_1(v)}$ ,  $S_v^{e_2(v)}$ ,  $S_v^{e_3(v)}$  is 501 *F*-overlaid. Together with (b), this implies (c).

<sup>502</sup> Conversely, consider W a maximum independent set of  $\Gamma$ .

Let G be the graph with vertex V(H) which is the union of the following subgraphs :

• for each x-leaf L, we add a copy of F on L in which x has degree  $\delta(F)$ ;

• for each vertex  $v \in W$ , we add a copy of F on  $S_v$  in which  $a_i^v$  has degree  $d(a_i)$  for all  $1 \le i \le n$ .

• for each edge  $e \in E(\Gamma)$ , we choose an endvertex u of e such that  $u \notin W$ , and add a copy of F in which  $z_e$  has degree  $d(a_1)$  and  $a_i^u$  has degree  $d(a_2)$  (with i the index such that  $e_i(u) = e$ ) except the case when  $d(a_2) \leq \beta$  and i = 1, we do that but  $d(z_e) = d(a_2)$  and  $d(a_1^u) = d(a_1)$ in  $G[S_u^{e_1(u)}]$ .

It is simple matter to check that  $\Delta(G) \leq k$  and that G F-overlays  $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$  hyperedges of H. Thus  $over_F(H, k) \geq$  $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$ .

<sup>517</sup> We then establish the following lemma, which allows to derive the  $\mathcal{NP}$ -<sup>518</sup> completeness of MAX ( $\Delta \leq k$ )-*F*-OVERLAY when *F* has isolated vertices.

Lemma 2. Let k be a positive integer, let F be a graph with  $\delta(F) \geq 1$ , and let q be a non-negative integer. If MAX ( $\Delta \leq k$ )- $(F + \overline{K}_q)$ -OVERLAY is  $\mathcal{NP}$ complete, then MAX ( $\Delta \leq k$ )- $(F + \overline{K}_{q+1})$ -OVERLAY is also  $\mathcal{NP}$ -complete. Proof. We shall reduce MAX  $(\Delta \leq k) \cdot (F + \overline{K}_q)$ -OVERLAY to MAX  $(\Delta \leq k) \cdot (F + \overline{K}_{q+1})$ -OVERLAY. Set p = |V(F)| and  $r = \lfloor k/\delta(F) \rfloor$ . Note that  $r \geq 1$ , for otherwise MAX  $(\Delta \leq k) \cdot (F + \overline{K}_q)$ -OVERLAY would trivially be polynomial-time solvable.

Let H' be a (p+q)-uniform hypergraph. Let H be the (p+q+1)-uniform hypergraph obtained from H' as follows. Its vertex set is partitioned into  $V(H'), P_1, \ldots, P_r, K$ , and  $\{z\}$  with  $|P_i| = p - 1$  for all  $1 \le i \le r$  and |K| = (m+1)(k+1) + kp with  $m = \max\{|E(H')|, q+1\}$ . The hyperedge set of H contains  $Q \cup P_i \cup \{z\}$  for all (q+1)-subsets Q of K and all  $1 \le i \le r$ , and  $S \cup \{z\}$  for all  $S \in E(H')$ .

The reduction can be performed in polynomial time, because for each hyperedge in H' we construct a polynomial number of hyperedges in H (recall that q is fixed), and each of them is straightforward to obtain. We shall now prove that  $over_{F+\overline{K}_{q+1}}(H,k) = over_{F+\overline{K}_q}(H',k) + r\binom{|K|}{q+1}$ .

Let  $G'_0$  be a graph with  $V(G'_0) = V(H')$  and  $\Delta(G'_0) \leq k$  that  $(F + \overline{K}_q)$ overlays  $over_{F+\overline{K}_q}(H',k)$  hyperedges of H'. For  $1 \leq i \leq r$ , let  $F_i$  be a copy of F on  $P_i \cup \{z\}$  in which z has degree  $\delta(F)$ . Let  $G_0$  be the graph defined by  $V(G_0) = V(H)$  and  $E(G'_0) \cup \bigcup_{i=1}^r E(F_i)$ . Clearly,  $\Delta(G_0) \leq k$  and  $G_0$  Foverlays all hyperedges of H except hyperedges  $S \cup \{z\}$  in which  $S \in E(H')$ are not overlaid by  $G'_0$ . So  $over_{F+\overline{K}_{q+1}}(H,k) \geq over_{F+\overline{K}_q}(H',k) + r\binom{|K|}{q+1}$ .

<sup>542</sup> Conversely, let G be a graph with V(G) = V(H) and  $\Delta(G) \leq k$  that <sup>543</sup> maximizes the number of  $(F + \overline{K}_{q+1})$ -overlaid hyperedges.

**Claim 6.** In G, vertex z has at least  $\delta(F)$  neighbours in each  $P_i$ ,  $1 \le i \le r$ .

*Proof of claim:* Suppose for a contradiction that z has less than  $\delta(F)$  neigh-545 bours in  $P_i$ . Since  $\Delta(G) \leq k$ , at most kp vertices of K are incident to an edge 546 with an endvertex in  $P_i \cup \{z\}$ . Thus there is a subset R of K of (m+1)(k+1)547 vertices non-adjacent to all vertices of  $P_i \cup \{z\}$ . Now G[R] has maximum 548 degree k and is thus (k+1)-colourable. Hence G[R] has an independent set 549 S of size m+1. Observe that for each (q+1)-subset Q of S, the subgraph 550  $G[Q \cup P_i \cup \{z\}]$  has q+2 vertices of degree less than  $\delta(F)$  (namely the ones of 551  $Q \cup \{z\}$ , and so the hyperedge  $Q \cup P_i \cup \{z\}$  is not  $(F + \overline{K}_{q+1})$ -overlaid. Thus 552 there are at least  $m+1 \ge |E(H')| + 1$  hyperedges not  $(F + K_{q+1})$ -overlaid 553 by G. This contradicts the maximality of G as  $G_0$  overlaps more hyperedges 554 of H. 555  $\triangleleft$ 

Now z has degree at least  $r\delta(F)$  in  $\bigcup_{i=1}^{r} P_i$  and so less than  $\delta(F)$  in each hyperedge  $S \cup \{z\}$  for  $S \in E(H')$ . Hence if  $G(F + \overline{K}_{q+1})$ -overlays  $S \cup z$ , it must also  $(F + \overline{K}_q)$ -overlay S. Thus  $over_{F + \overline{K}_{q+1}}(H, k) \leq over_{F + \overline{K}_q}(H', k) + r\binom{|K|}{q+1}$ .

Now we can prove Theorem 2. As explained in the beginning of the section, it suffices to prove that MAX ( $\Delta \leq k$ )-F-OVERLAY remains  $\mathcal{NP}$ complete when  $\Delta(F) \leq k, F \neq \overline{K}_{|F|}, |F| \geq 3$  and  $(F,k) \neq (O_3,1)$ . Assume that the above conditions are satisfied. Let F' be the graph induced by the non-isolated vertices of F. Then  $F = F' + \overline{K}_q$  with q = |F| - |F'|. If |F'| = 2, then  $F = O_{|F|}$ , and we have the result by Theorem 1. If  $|F'| \geq 3$ , then the result follows from Theorem 5, Lemma 2, and an immediate induction.

#### <sup>567</sup> 4. Complexity of $(\Delta \leq k)$ -F-Overlay and $(\Delta \leq k)$ -F-Overlay

In this section, we investigate the complexity of  $(\Delta \leq k)$ -F-OVERLAY 568 and  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY problems. We believe that each such problem is 569 either polynomial-time solvable or  $\mathcal{NP}$ -complete. However the dichotomy is 570 more complicated than the one we proved in Theorem 2 for MAX ( $\Delta \leq k$ )-F-571 OVERLAY. This dichotomy is simple because there are very few pairs (F, H)572 for which MAX ( $\Delta \leq k$ )-F-OVERLAY is polynomial-time solvable. We now 573 exhibit several pairs (F, k) such that  $(\Delta < k)$ -F-OVERLAY is polynomial-574 time solvable, while MAX ( $\Delta \leq k$ )-F-OVERLAY is  $\mathcal{NP}$ -complete. 575

#### 576 4.1. Regular graphs

**Proposition 3.** For every complete graph K and every positive integer k, <sup>578</sup>  $(\Delta \leq k)$ -K-OVERLAY is polynomial-time solvable.

<sup>579</sup> Proof. Observe that a |V(K)|-uniform hypergraph H is a positive instance <sup>580</sup> of  $(\Delta \leq k)$ -K-OVERLAY if and only if K(H) is a (K, H, k)-graph.  $\Box$ 

Proposition 4. For every connected k-regular graph F,  $(\Delta \leq k)$ -F-OVERLAY is polynomial-time solvable.

Proof. One easily sees that a |V(F)|-uniform hypergraph H admits an (F, H, k)graph if and only if the hyperedges of H are pairwise non-intersecting.

Let  $C_4$  denote the cycle on 4 vertices. Proposition 4 implies that  $(\Delta \leq 2)$ - $C_4$ -OVERLAY is polynomial-time solvable. We now show that  $(\Delta \leq 3)$ - $C_4$ -OVERLAY is also polynomial-time solvable.

**Theorem 6.**  $(\Delta \leq 3)$ -C<sub>4</sub>-OVERLAY is polynomial-time solvable.

<sup>589</sup> *Proof.* Let H be a 4-uniform hypergraph.

Let us describe an algorithm to decide whether there is a  $(C_4, H, 3)$ -graph. It is sufficient to do it when H is connected since the disjoint union of the  $(C_4, K, 3)$ -graphs for connected components K of H is a  $(C_4, H, 3)$ -graph.

<sup>593</sup> Observe first that if two hyperedges of H intersect in exactly one vertex <sup>594</sup> u, then no such graph exists, since u must have degree 2 in each of the <sup>595</sup> hyperedges if they are  $C_4$ -overlaid, and thus degree 4 in total. Therefore if <sup>596</sup> there are two such hyperedges, we return 'No'. At this point we may assume <sup>597</sup> that  $|E(H)| \ge 2$  for otherwise we return 'Yes'.

From now on we may assume that two hyperedges either do not intersect, or are adjacent (intersect on at least two vertices).

**Claim 7.** If two hyperedges  $S_1$  and  $S_2$  intersect on three vertices and there is a  $(C_4, H, 3)$ -graph G, then  $|V(H)| \leq 6$ .

Proof of claim: Assume  $S_1 = \{a_1, b, c, d\}$  and  $S_2 = \{a_2, b, c, d\}$ . Let G be a (C<sub>4</sub>, H, 3)-graph. Observe that  $a_1$  and  $a_2$  must have two common neighbours, say b, c, in common in  $\{b, c, d\}$ , for otherwise their unique common vertex would have degree 4 in G. This forces d to be also adjacent to b and c.

Consider a hyperedge  $S_3$  intersecting  $S_1 \cup S_2$ . Since it is  $C_4$ -overlaid by  $G_3$ . 606 at least two edges connect  $S_3 \cap (S_1 \cup S_2)$  to  $S_3 \setminus (S_1 \cup S_2)$ . The endvertices of 607 those edges in  $S_1 \cup S_2$  must have degree 2 in  $G[S_1 \cup S_2]$ . Hence, without loss of 608 generality, there is a vertex e not in  $S_1 \cup S_2$  such that either  $S_3 = \{a_1, a_2, b, e\}$ 609 and  $\{a_1e, a_1e\} \subseteq E(G)$ , or  $S_3 = \{a_1, b, d, e\}$  and  $\{a_1e, de\} \subseteq E(G)$ . Now 610 no hyperedge can both intersect  $S_1 \cup S_2 \cup S_3$  and contain a vertex not in 611  $S_1 \cup S_2 \cup S_3$ , for such a hyperedge must contain either the vertices d, e or 612  $a_2, e$  which are at distance 3 in  $G[S_1 \cup S_2 \cup S_3]$ . (However there can be more 613 hyperedges contained in  $S_1 \cup S_2 \cup S_3$ .) Hence  $|V(H)| \le 6$ .  $\triangleleft$ 614

In view of Claim 7, if there are two hyperedges with three vertices in common, either we return 'No' if |V(H)| > 6, or we check all possibilities (or follow the proof of the above claim) to return the correct answer otherwise. Henceforth, we may assume that any two adjacent hyperedges intersect in exactly two vertices.

Let  $S_1$  and  $S_2$  be two adjacent hyperedges, say  $S_1 = \{a, b, c, d\}$  and  $S_2 = \{c, d, e, f\}$ . Note that every  $(C_4, H, 3)$ -graph contains the edges ab, cd and ef, and that  $N(c) \cup N(d) = S_1 \cup S_2$  in this  $(C_4, H, 3)$ -graph.

<sup>623</sup> Claim 8. If there is another hyperedge than  $S_1$  and  $S_2$  containing c or d, <sup>624</sup> and there is a  $(C_4, H, 3)$ -graph G, then  $|V(H)| \leq 8$ . Proof of claim: Without loss of generality, we may assume that G contains the cycle (a, b, d, f, e, c, a) and the edge cd. Hence the only possible hyperedges containing c or d and a vertex not in  $S_1 \cup S_2$  are  $S_3 = \{a, c, e, g\}$  for some  $g \notin S_1 \cup S_2$  and  $S_4 = \{b, d, f, h\}$  for some  $h \notin S_1 \cup S_2$ .

If H contains both  $S_3$  and  $S_4$ , then G contains the edges ag, eg, bhand hf. If G contains also gh, then  $G[S_1 \cup S_2 \cup S_3 \cup S_4]$  is 3-regular, so  $G_{31} \quad G = G[S_1 \cup S_2 \cup S_3 \cup S_4]$ . If G does not contain gh, then the only vertices of degree 2 in  $G[S_1 \cup S_2 \cup S_3 \cup S_4]$  are g and h, and they are at distance at least 3 in this graph. Thus every hyperedge intersecting  $S_1 \cup S_2 \cup S_3 \cup S_4$  is contained in this set, so |V(H)| = 8.

Assume now that H is a superset of only one of  $S_3, S_4$ . Without loss 635 of generality, we may assume that this is  $S_3$ . Hence G also contains the 636 edges ag and eg. If  $V(G) \neq S_1 \cup S_2 \cup S_3$ , then there is a hyperedge S that 637 intersects  $S_1 \cup S_2 \cup S_3$  and that is not contained in  $S_1 \cup S_2 \cup S_3$ . It does not 638 contain c and d. Hence it must contain one of the vertices a or e, because it 639 intersects each  $S_i$  along an edge of G or not at all. Without loss of generality, 640  $a \in S$ . Hence  $S = \{a, b, i, g\}$  for some vertex i not in  $S_1 \cup S_2 \cup S_3$ , and G 641 contains the edges bi and ig. Now, as previously, either i and f are adjacent 642 and  $G = G[S_1 \cup S_2 \cup S_3 \cup S]$  or they are not adjacent, and every hyperedge 643 intersecting  $S_1 \cup S_2 \cup S_3 \cup S$  is contained in this set. In both cases, |V(H)| = 8. 644  $\triangleleft$ 645

We now summarize the algorithm: if  $|V(G)| \leq 8$ , then we solve the 646 instance by brute force. Otherwise, for every pair of hyperedges  $S_1$ ,  $S_2$ , if 647 their intersection is of size 1 or 3, we answer 'No'. In the remaining cases, 648 if  $S_1$  and  $S_2$  have non-empty intersection, then, they must intersect on two 649 vertices c and d, and these vertices do not belong to any other hyperedges 650 but  $S_1$  and  $S_2$ . Hence, if a vertex is in at least 3 hyperedges, then we return 651 ' No'. If not, then every hyperedge intersects at most two hyperedges on two 652 disjoint pairs of vertices. We can then return the  $(C_4, H, 3)$ -graph obtained as 653 follows. If a hyperedge  $S = \{a, b, c, d\}$  intersects either two other hyperedges 654  $S_1$  and  $S_2$  is  $\{a, b\}$  and  $\{c, d\}$ , or exactly one hyperedge on  $\{a, b\}$ , then add 655 the edges ab, bc, cd, da to E(G). 656

#### $_{657}$ Clearly, the above-described algorithm runs in polynomial time.

**Theorem 7.**  $(\Delta \leq 2)$ - $\mathcal{P}$ -OVERLAY is linear-time solvable.

<sup>658 4.2.</sup> Paths

Let  $\mathcal{P}$  be the set of all paths. We have the following:

<sup>661</sup> *Proof.* Clearly, if H is not connected, it suffices to solve the problem on each <sup>662</sup> of the components and to return 'No' if the answer is negative for at least one <sup>663</sup> of the components, and 'Yes' otherwise. Henceforth, we shall now assume <sup>664</sup> that H is connected. In such a case, a  $(\mathcal{P}, H, 2)$ -graph is either a path or a <sup>665</sup> cycle. However, if H is  $\mathcal{P}$ -overlaid by a path P, then it is also  $\mathcal{P}$ -overlaid by <sup>666</sup> the cycle obtained from P by adding an edge between its two endvertices. <sup>667</sup> Thus, we focus on the case where G is a cycle.

Let S be a family of sets. The *intersection graph* of a family S is the graph IG(S) whose vertices are the sets of S, and in which two vertices are adjacent if the corresponding sets in S intersect.

The *intersection graph* of a hypergraph H, denoted by IG(H), is the intersection graph of its hyperedge set. We define two functions  $l_H$  and  $s_H$  as follows:

$$l_H(S) = |S| - 1$$
 for all  $S \in E(H)$  and  $s_H(S, S') = |S \cap S'| - 1$  for all  $S, S' \in E(H)$ 

Let  $\mathbb{C}_{\ell}$  be the circle of circumference  $\ell$ . We identify the points of  $\mathbb{C}_{\ell}$  with 671 the integer numbers (points) of the segment  $[0, \ell]$ , (with 0 identified with  $\ell$ ). 672 A circular-arc graph is the intersection graph of a set of arcs on  $\mathbb{C}_{\ell}$ . A set 673  $\mathcal{A}$  of arcs such that  $IG(\mathcal{A}) = G$  is called an *arc representation* of G. We 674 denote by  $A_v$  the arc corresponding to v in  $\mathcal{A}$ . Let G be a graph and let 675  $l: V(G) \to \mathbb{N}$  and  $s: E(G) \to \mathbb{N}$  be two functions. An arc representation  $\mathcal{A}$ 676 of G is *l*-respecting if  $A_v$  has length l(v) for any  $v \in V(G)$ , *s*-respecting if 677  $A_v \cap A_u$  has length s(u, v) for all  $uv \in E(G)$ , and (l, s)-respecting if it is both 678 *l*-respecting and *s*-respecting. One can easily adapt the algorithm given by 679 Köbler et al. [?] for (l, s)-respecting interval representations to decide in 680 linear time whether a graph admits an (l, s)-respecting arc representation in 681  $\mathbb{C}_n$  for every integer n. 682

Claim 9. Let H be a connected hypergraph on n vertices. There is a cycle  $\mathcal{P}$ -overlaying H if and only if IG(H) admits an  $(l_H, s_H)$ -respecting arc representation into  $\mathbb{C}_n$ .

Proof of claim: Assume that H is  $\mathcal{P}$ -overlaid by a cycle  $C = (v_0, v_1, \ldots, v_{n-1}, v_0)$ . There is a canonical embedding of C to  $\mathbb{C}_n$  in which every vertex  $v_i$  is mapped to i and every edge  $v_i v_{i+1}$  to the circular arc [i, i+1]. For every hyperedge  $S \in E(H), P[S]$  is a subpath, which is mapped to the circular arc  $A_S$  of  $\mathbb{C}_n$ that is the union of the circular arcs to which its edges are mapped. Clearly,  $\mathcal{A} = \{A_S \mid S \in E(H)\}$  is an  $(l_H, s_H)$ -respecting interval representation of IG(H).



Figure 5: Given a hypergraph H (left), we determine its graph IG(H) (middle) with each vertex represents the a hyperedge of the same color in which the value of any vertex  $S \in V(H)$  is  $l_H(S)$  and the weight of an edge between S, S' is  $s_H(S, S')$ . And a circular arc representation (each arc is associated to a hyperedge of the same color) is shown in the right which is (l, s)-respecting.

Conversely, assume that IG(H) admits an  $(l_H, s_H)$ -respecting interval 693 representation  $\mathcal{A} = \{A_S \mid S \in E(H)\}$  into  $\mathbb{C}_n$ . Let  $S_0$  be a hyperedge 694 of minimum size. Free to rotate all intervals, we may assume that  $A_{S_0}$  is 695  $[1, |S_0|]$ . Now since  $\mathcal{A}$  is  $(l_H, s_H)$ -respecting and H is connected, we deduce 696 that the extremities of  $A_S$  are integers for all  $S \in E(H)$ . Let  $v_1$  be a vertex 697 of H that belongs to the hyperedges whose corresponding arcs of  $\mathcal{A}$  contain 698 1. Then for all i = 2 to n = |V(H)|, denote by  $v_i$  an arbitrary vertex not in 699  $\{v_1, \ldots, v_{i-1}\}$  that belongs to the hyperedges whose corresponding arcs of  $\mathcal{A}$ 700 contain *i*. Such a vertex exists because  $\mathcal{A}$  is  $(l_H, s_H)$ -respecting. Observe that 701 such a construction yields  $S = \{v_i \mid i \in A_S\}$  for all  $S \in E(H)$ . Furthermore, 702 the cycle  $C = (v_1, \ldots, v_n, v_1) \mathcal{P}$ -overlays H. Indeed, for each  $S \in E(H)$ , 703 C[S] is the subpath corresponding to  $A_S$ , that is  $V(C[S]) = \{v_i \mid i \in A_S\}$ 704 and  $E(C[S]) = \{v_i v_{i+1} \mid [i, i+1] \subseteq A_S\}.$  $\triangleleft$ 705

The algorithm to solve  $(\Delta \leq 2)$ - $\mathcal{P}$ -OVERLAY for a connected hypergraph H in linear time is thus the following:

<sup>708</sup> 1. Construct the intersection graph IG(H) and compute the associated <sup>709</sup> functions  $l_H$  and  $s_H$ .

<sup>710</sup> 2. Check whether graph IG(H) has an  $(l_H, s_H)$ -respecting interval repre-<sup>711</sup> sentation. If it is the case, return 'Yes'. If not return 'No'.

712

<sup>713</sup> **Remark 1.** We can also detect in polynomial time whether a connected <sup>714</sup> hypergraph H is  $\mathcal{P}$ -overlaid by a path. Indeed, similarly to Claim 9, one can

show that there is a path  $\mathcal{P}$ -overlaying H if and only if IG(H) admits an  $(l_H, s_H)$ -respecting interval representation.

#### 717 4.3. On neat hypergraphs

Let F = (V, E) be a graph. We denote by  $V_i$  the set of vertices having degree i in F. For every increasing sequence of nonnegative integers  $i_1 < i_2 < \cdots < i_p$ , an  $(i_1, i_2, \ldots, i_p)$ -graph is a graph F such that  $V_i = \emptyset$  if  $i_2 \notin \{i_1, i_2, \ldots, i_p\}$ . In particular, an (r)-graph is a r-regular graph.

Given a hypergraph H, we denote by  $d_H(v)$  the number of hyperedges containing v for any  $v \in V(H)$ .

Theorem 8. Let  $i_1, i_2, k$  be three non-negative integers with  $i_1 < i_2$ . For every  $(i_1, i_2)$ -graph F, one can solve  $(\Delta \leq k)$ -F-OVERLAY in polynomial time on neat hypergraphs.

<sup>727</sup> *Proof.* Let H be a neat |F|-uniform hypergraph.

If there is a vertex v such that  $i_1 \cdot d_H(v) > k$ , then we answer 'No'. Indeed in an (H, F, k)-graph G, the vertex v has degree at least  $i_1$  in each subgraph of G induced by a hyperedge to which it belongs, and so degree at least  $i_1 \cdot d_H(v)$  in G because H is neat. Hence, we assume that  $i_1 \cdot d_H(v) \le k$  for every vertex.

Let B(H) be the incidence graph of H: its vertex set is  $E(H) \cup V(H)$ and there is an edge between a hyperedge  $S \in E(H)$  and a vertex  $v \in V(H)$ if and only if  $v \in S$ .

Claim 10. There is an (H, F, k)-graph if and only if B(H) has a subgraph in which every hyperedge vertex has degree  $|V_{i_2}|$  and every vertex has degree at most  $\lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$ .

*Proof of claim:* Assume that there is an (H, F, k)-graph G. Observe that 739 since H is a neat hypergraph, free to remove some edges, we may assume 740 that G[S] is isomorphic to F for every hyperedge S of H. Let B' be the 741 subgraph of B(H) in which a vertex v is adjacent to a hyperedge S if and 742 only if v has degree  $i_2$  in G[S]. By definition every hyperedge has degree  $|V_{i_2}|$ 743 in B'. Consider now a vertex v. Its degree in G is the sum of its degree in 744 the subgraphs induced by the hyperedges S to which it belongs because H is 745 neat. Hence  $d_G(v) = i_2 \cdot d_{B'}(v) + i_1(d_H(v) - d_{B'}(v)) = i_1 \cdot d_H(v) + (i_2 - i_1)d_{B'}(v)$ . 746 But G is an (H, F, k)-graph, so  $d_G(v) \leq k$ . With the previous equality, this 747 yields  $d_{B'}(v) \leq \frac{k-i_1 \cdot d_H(v)}{i_2-i_1}$ , and so  $d_{B'}(v) \leq \lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$ . 748

Conversely, assume B(H) has a subgraph B' in which every  $S \in E(H)$ 749 has degree  $|V_{i_2}|$  and any  $v \in V(H)$  has degree at most  $\lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$ . We 750 construct G as follows: for each hyperedge S, the induced subgraph G[S]751 is a copy of F such that any vertex v where  $(S, v) \in E(B')$  is of degree 752  $i_2$  (there are  $|V_{i_2}|$  such vertices) and the remaining vertices have degree  $i_1$ . 753 Clearly, G F-overlays H. Moreover every vertex  $v \in V(H)$  has degree  $i_2$ 754 in at most  $\lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$  subgraphs and degree  $i_1$  in the remaining subgraphs 755 induced by hyperedges to which it belongs. Hence v has degree at most k in 756 G. Therefore G is an (H, F, k)-graph.  $\triangleleft$ 757

Now the existence of a subgraph of B(H) as shown in Claim 10 can be done in polynomial time (see [?] Chap. 10).

**Theorem 9.** Let  $i_1, i_2, i_3, k$  be three non-negative integers with  $i_1 < i_2 < i_3$ . If F is a  $(i_1, i_2, i_3)$ -graph F such that all vertices of  $V_{i_1} \cup V_{i_2}$  have the same number of neighbours in  $V_{i_3}$  and  $k < i_3 + i_1$ , then one can solve  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY in polynomial time on neat hypergraphs.

<sup>764</sup> *Proof.* Let j be the number of neighbours in  $V_{i_3}$  that each vertex of  $V_{i_1} \cup V_{i_2}$ <sup>765</sup> has.

Let H be an |F|-uniform hypergraph. Observe that if a vertex has degree 766  $i_3$  in the subgraph induced by a hyperedge, then it cannot be in any other 767 hyperedge, for otherwise it would have degree at least  $i_3 + i_1 > k$ . Therefore, 768 we first check whether every hyperedge S contains a set I(S) of  $|V_{i_3}|$  vertices 769 that are uniquely in S. If not, then we return 'No'. If it is the case, let F' =770  $F - V_{i_3}$  and H' be the hypergraph whose vertex set is  $V(H) \setminus \bigcup_{S \in E(H)} I(S)$ 771 and whose hyperedge set is  $\{S \setminus I(S) \mid S \in E(H)\}$ , and let  $k'(v) = k - j \cdot d_H(v)$ 772 for every  $v \in V(H')$ . Since all vertices of  $V_{i_1} \cup V_{i_2}$  have exactly j neighbours 773 in  $V_{i_3}$ , it is then easy to prove the following claim: 774

Claim 11. There is an (H, F, k)-graph G if and only if there is a graph G'r<sub>76</sub> F'-overlaying H' such that  $d_G(v) \leq k'(v)$  for all  $v \in V(H')$ .

<sup>777</sup> Deciding whether such a graph G' exists can be done similarly as the <sup>778</sup> proof of Theorem 8 replacing k by k'(v).

#### **5.** Further research

Theorem 2 characterizes the complexity of MAX ( $\Delta \leq k$ )- $\mathcal{F}$ -OVERLAY when  $\mathcal{F}$  contains a unique graph. It would be nice to extend this characterization to families  $\mathcal{F}$  of arbitrary size. **Problem 1.** Characterize the pairs  $(\mathcal{F}, k)$  for which MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is polynomial-time solvable and those for which it is  $\mathcal{NP}$ -complete.

Theorem 1 and the results obtained in Section 4 give a first view of the complexity of  $(\Delta \leq k)$ -F-OVERLAY. A natural problem is to close the dichotomy:

**Problem 2.** Characterize the pairs (F, k) for which  $(\Delta \leq k)$ -F-OVERLAY is polynomial-time solvable and those for which it is  $\mathcal{NP}$ -complete.

It would be interesting to consider the complexity of this problem when F is k-regular but non-connected, and when F is a cycle. In order to attack Problem 2, it would be helpful to prove the following conjecture.

<sup>793</sup> Conjecture 1. If  $(\Delta \leq k)$ -F-OVERLAY is  $\mathcal{NP}$ -complete, then  $(\Delta \leq k+1)$ -<sup>794</sup> F-OVERLAY is also  $\mathcal{NP}$ -complete.

Furthermore, for each pair  $(\mathcal{F}, k)$  such that MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is  $\mathcal{NP}$ -complete and  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is polynomial-time solvable, it is natural to consider the parameterized complexity of MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY when parameterized by |E(H)| - s, because  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is the case s = 0.

It would also be interesting to obtain approximation algorithms for MAX ( $\Delta \leq k$ )- $\mathcal{F}$ -OVERLAY when this problem is  $\mathcal{NP}$ -complete.

As shown in Subsection 4.3, we proved that  $(\Delta \leq k)$ -F-OVERLAY is sometimes easier when restricted to neat hypergraphs. It would then be interesting to determine the complexity of  $(\Delta \leq k)$ -F-OVERLAY on neat hypergraphs. So far, we only proved some of these problems to be polynomial-time solvable, but we believe some are  $\mathcal{NP}$ -complete.

**Problem 3.** Characterize the pairs (F, k) for which  $(\Delta \leq k)$ -F-OVERLAY on neat hypergraphs is polynomial-time solvable and those for which it is  $\mathcal{NP}$ -complete.