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# Overlaying a hypergraph with a graph with bounded maximum degree

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## Abstract

Let  $G$  and  $H$  be respectively a graph and a hypergraph defined on a same set of vertices, and let  $F$  be a fixed graph. We say that  $G$  *F-overlays* a hyperedge  $S$  of  $H$  if  $F$  is a spanning subgraph of the subgraph of  $G$  induced by  $S$ , and that  $G$  *F-overlays*  $H$  if it  $F$ -overlays every hyperedge of  $H$ . Motivated by structural biology, we study the computational complexity of two problems. The first problem,  $(\Delta \leq k)$   $F$ -OVERLAY, consists in deciding whether there is a graph with maximum degree at most  $k$  that  $F$ -overlays a given hypergraph  $H$ . It is a particular case of the second problem MAX  $(\Delta \leq k)$   $F$ -OVERLAY, which takes a hypergraph  $H$  and an integer  $s$  as input, and consists in deciding whether there is a graph with maximum degree at most  $k$  that  $F$ -overlays at least  $s$  hyperedges of  $H$ .

We give a complete polynomial/ $\mathcal{NP}$ -complete dichotomy for the MAX  $(\Delta \leq k)$ - $F$ -OVERLAY problems depending on the pairs  $(F, k)$ , and establish the complexity of  $(\Delta \leq k)$   $F$ -OVERLAY for many pairs  $(F, k)$ .

*Keywords:* graphs, hypergraphs, complexity, algorithm, computational structural biology

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## 1. Introduction

A major problem in structural biology is the characterization of low resolution structures of macro-molecular assemblies [? ?]. To attack this very difficult question, one has to determine the plausible contacts between the subunits (e.g. proteins) of an assembly, given the lists of subunits involved in some complexes. We assume that the composition, in terms of individual subunits, of selected complexes is known. Indeed, a given assembly can be

chemically split into complexes by manipulating chemical conditions. This problem can be conveniently modelled by graphs and hypergraphs. We consider the hypergraph  $H$  whose vertices represent the subunits and whose hyperedges are the complexes. We are then looking for a graph  $G$  with the same vertex set as  $H$  whose edges represent the contacts between subunits, and satisfying (i) some local properties for every complex (*i.e.* hyperedge), and (ii) some other global properties.

For example, consider the following properties: (i) the subset of vertices of every hyperedge must form a connected subgraph of  $G$  and (ii) the number of edges of  $G$  is at most the number of vertices. The first property comes from the fact that the complexes obtained (e.g. by native mass spectrometry) are connected components of a given assembly. The second property is deduced from the constant number of subunits that are in contact to any subunit of an assembly. It means that the total number of edges is linear in the number of vertices (in our example, we assume that the number of edges is at most the number of vertices). Let us now consider a hypergraph  $H$  whose hyperedges are  $\{v_1, v_2, v_3\}$ ,  $\{v_2, v_4, v_5\}$ ,  $\{v_3, v_4, v_5\}$ , and  $\{v_4, v_5, v_6, v_7\}$ . These hyperedges are depicted in Figure 1 (left) in which a vertex labeled  $i$  represents  $v_i$ . The graph  $G$  described in Figure 1 (right) satisfies the previous properties: the set of vertices  $\{v_4, v_5, v_6, v_7\}$  induces a connected subgraph (same observation for the three other hyperedges) and the number of edges of  $G$  is exactly the number of vertices.

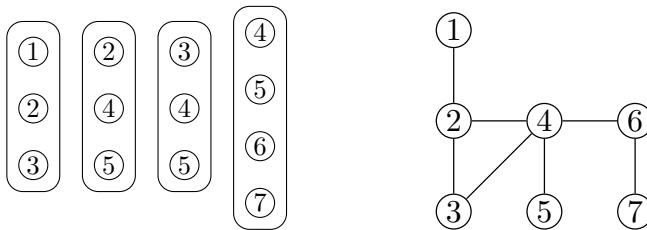


Figure 1: A hypergraph composed of seven vertices and four hyperedges (right). A graph  $G$  satisfying local properties (every hyperedge induces a connected subgraph in  $G$ ) and a global property (the total number of edges is at most the number of vertices).

We first focus on the local properties. They are usually modelled by a (possibly infinite) family  $\mathcal{F}$  of admissible graphs to which each complex must belong: to this end, we define the notion of *enforcement* of a hyperedge and a hypergraph. A graph  $G$   $\mathcal{F}$ -enforces a hyperedge  $S \in E(H)$  if the subgraph  $G[S]$  of  $G$  induced by  $S$  belongs to  $\mathcal{F}$ , and  $G$   $\mathcal{F}$ -enforces  $H$  if  $G$   $\mathcal{F}$ -enforces

all hyperedges of  $H$ . Very often, the considered family  $\mathcal{F}$  is closed on taking edge supergraphs [? ?]: if  $F \in \mathcal{F}$ , then every graph obtained from  $G$  by adding edges is also in  $\mathcal{F}$ . Such a family is completely defined by its set  $\mathcal{M} = \mathcal{M}(\mathcal{F})$  of minimal graphs that are the elements of  $\mathcal{F}$  which are not edge supergraphs of any other. In this case, a graph  $G$   $\mathcal{F}$ -enforcing  $S$  is such that there is an element of  $\mathcal{M}$  which is a spanning subgraph of  $G[S]$ . This leads to the following notion of *overlayment* when considering minimal graph families.

**Definition 1.** A graph  $G$   $\mathcal{F}$ -overlays a hyperedge  $S$  if there exists  $F \in \mathcal{F}$  such that  $F$  is a spanning subgraph of  $G[S]$ , and  $G$   $\mathcal{F}$ -overlays  $H$  if  $G$   $\mathcal{F}$ -overlays every hyperedge of  $H$ .

As said previously, the graph sought will also have to satisfy some global constraints. Since in a macro-molecular assembly the number of contacts is small, the first natural idea is to look for a graph  $G$  with the minimum number of edges. This leads to the MIN- $\mathcal{F}$ -OVERLAY problem: given a hypergraph  $H$  and an integer  $m$ , decide if there exists a graph  $G$   $\mathcal{F}$ -overlying  $H$  such that  $|E(G)| \leq m$ .

A typical example of a family  $\mathcal{F}$  is the set of all connected graphs, in which case  $\mathcal{M}(\mathcal{F})$  is the set of all trees. Agarwal et al. [?] focused on MIN- $\mathcal{M}(\mathcal{F})$ -OVERLAY for this particular family in the aforementioned context of structural biology. However, this problem was previously studied by several communities in other domains, as pointed out by Chen *et al.*[?]. Indeed, it is also known as SUBSET INTERCONNECTION DESIGN, MINIMUM TOPIC-CONNECTED OVERLAY or INTERCONNECTION GRAPH PROBLEM, and was considered (among others) in the design of vacuum systems [? ?], scalable overlay networks [? ?], and reconfigurable interconnection networks [? ?]. Some variants have also been considered in the contexts of inferring a most likely social network [?], determining winners of combinatorial auctions [?], as well as drawing hypergraphs [? ?].

Cohen et al. [?] presented a dichotomy regarding the polynomial vs.  $\mathcal{NP}$ -hard status of the problem MIN- $\mathcal{F}$ -OVERLAY with respect to the considered family  $\mathcal{F}$ . Roughly speaking, they showed that the easy cases one can think of (e.g. when edgeless graphs of the right sizes are in  $\mathcal{F}$ , or if  $\mathcal{F}$  contains only cliques) are the only families giving rise to a polynomial-time solvable problem: all others are  $\mathcal{NP}$ -complete. They also considered the FPT/W[1]-hard dichotomy for several families  $\mathcal{F}$ .

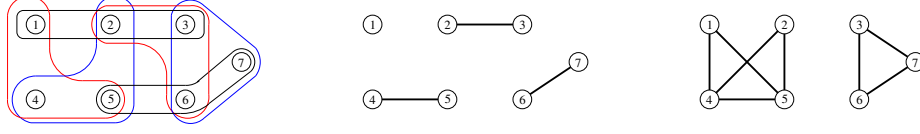


Figure 2: Example of  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY and MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY. In the figure, an instance  $H$  (left), a graph  $G$  with  $\Delta(G) \leq 1$  that  $O_3$ -overlays  $H$  (with  $O_3$  being the graph with three vertices and one edge) (center), and a solution to MAX  $(\Delta \leq 3)$ - $C_3$ -OVERLAY (with  $C_3$  being the cycle on three vertices) (right).

In this paper, we consider the variant in which the additional constraint is that  $G$  must have a bounded maximum degree: this constraint is motivated by the context of structural biology, since a subunit (e.g. a protein) cannot be connected to many other subunits. This yields the following problem for any family  $\mathcal{F}$  of graphs and an integer  $k$ .

$(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY

Input: A hypergraph  $H$ .

Question: Does there exist a graph  $G$   $\mathcal{F}$ -overlying  $H$  such that  $\Delta(G) \leq k$ ?

We denote by  $over_{\mathcal{F}}(H, G)$  the number of hyperedges of  $H$  that are  $\mathcal{F}$ -overlaid by  $G$ . A natural generalization is to find  $over_{\mathcal{F}}(H, k)$ , the maximum number of hyperedges  $\mathcal{F}$ -overlaid by a graph with maximum degree at most  $k$ .

MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY

Input: A hypergraph  $H$  and a positive integer  $s$ .

Question: Does there exist a graph  $G$  such that  $\Delta(G) \leq k$  and  $over_{\mathcal{F}}(H, G) \geq s$ ?

Observe that there is an obvious reduction from  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY to MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY (by setting  $s = |E(H)|$ ).

In this paper, we mainly consider the case when the family  $\mathcal{F}$  contains a unique graph  $F$ . We abbreviate  $(\Delta \leq k)$ - $\{F\}$ -OVERLAY and MAX  $(\Delta \leq k)$ - $\{F\}$ -OVERLAY to  $(\Delta \leq k)$ - $F$ -OVERLAY and MAX  $(\Delta \leq k)$ - $F$ -OVERLAY, respectively. By definition those two problems really make sense only for  $|F|$ -uniform hypergraphs *i.e.* hypergraphs whose hyperedges are of size  $|F|$ . Therefore, we always assume the hypergraph to be  $|F|$ -uniform.

If  $F$  is a graph with maximum degree greater than  $k$ , then solving  $(\Delta \leq k)$ - $F$ -OVERLAY or MAX  $(\Delta \leq k)$ - $F$ -OVERLAY is trivial as the answer is always ‘No’. So we only study the problems when  $\Delta(F) \leq k$ .

115 If  $F$  is an anticomplete graph (i.e. with no edges), then  $\text{MAX } (\Delta \leq k)$ -  
 116  $F$ -OVERLAY is also trivial, because for any hypergraph  $H$ , the anticomplete  
 117 graph on  $V(H)$  vertices  $F$ -overlays  $H$ . Hence the first natural interesting  
 118 cases are the graphs with one edge. For every integer  $p \geq 2$ , we denote by  $O_p$   
 119 the graph with  $p$  vertices and one edge. In Section 2, we prove the following  
 120 dichotomy theorem.

121 **Theorem 1.** *Let  $k \geq 1$  and  $p \geq 2$  be integers. If  $p = 2$  or if  $k = 1$  and*  
 122  *$p = 3$ , then  $\text{MAX } (\Delta \leq k)$ - $O_p$ -OVERLAY and  $(\Delta \leq k)$ - $O_p$ -OVERLAY are*  
 123 *polynomial-time solvable. Otherwise, they are  $\mathcal{NP}$ -complete.*

124 Then, in Section 3, we give a complete polynomial/ $\mathcal{NP}$ -complete di-  
 125 chotomy for the  $\text{MAX } (\Delta \leq k)$ - $F$ -OVERLAY problems.

126 **Theorem 2.**  *$\text{MAX } (\Delta \leq k)$ - $F$ -OVERLAY is polynomial-time solvable if ei-*  
 127 *ther  $\Delta(F) > k$ , or  $F$  is an anticomplete graph, or  $F = O_2$ , or  $k = 1$  and*  
 128  *$F = O_3$ . Otherwise it is  $\mathcal{NP}$ -complete.*

129 In Section 4, we investigate the complexity of  $(\Delta \leq k)$ - $F$ -OVERLAY and  
 130  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY problems. We believe that each such problem is either  
 131 polynomial-time solvable or  $\mathcal{NP}$ -complete. However the dichotomy seems  
 132 to be more complicated than the one for  $\text{MAX } (\Delta \leq k)$ - $F$ -OVERLAY. We  
 133 exhibit several pairs  $(F, k)$  such that  $(\Delta \leq k)$ - $F$ -OVERLAY is polynomial-  
 134 time solvable, while  $\text{MAX } (\Delta \leq k)$ - $F$ -OVERLAY is  $\mathcal{NP}$ -complete. This is  
 135 in particular the case when  $F$  is a complete graph (Proposition 3),  $F$  is  
 136 connected  $k$ -regular (Proposition 4),  $F$  is a path and  $k = 2$  (Theorem 7),  
 137 and when  $F$  is the cycle on 4 vertices and  $k \leq 3$  (Theorem 6).

138 Most notations of this paper are standard. We now recall some of them,  
 139 and we refer the reader to [?] for any undefined terminology. The degree  
 140 of a vertex  $v$  in a graph is denoted by  $d(v)$ . For a positive integer  $p$ , let  
 141  $[p] = \{1, \dots, p\}$ . In some proofs of the paper we might use the modulo  
 142 operation to keep indices in bounds.

143 Given  $S \subseteq V(G)$ , we denote by  $G[S]$  the subgraph induced by  $S$ , that is  
 144 the subgraph with vertex set  $S$  and edge set  $\{uv \in E(G) \mid u, v \in S\}$ . We  
 145 denote by  $\overline{K}_k$  the *anticomplete* graph on  $k$  vertices, that is the graph with  $k$   
 146 vertices and no edges. The *disjoint union* of two graphs  $F$  and  $G$  is denoted  
 147 by  $F + G$ .

148 Let  $H$  be a hypergraph. Two hyperedges are *adjacent* if their intersection  
 149 has size at least 2. A hypergraph is *neat* if any two distinct hyperedges

150 intersect in at most one vertex. In other words, a hypergraph is neat if  
 151 there is no pair of adjacent hyperedges. We denote by  $K(H)$ , the graph  
 152 obtained by replacing each hyperedge by a complete graph. In other words,  
 153  $V(K(H)) = V(H)$  and  $E(K(H)) = \{xy \mid \exists S \in E(H), \{x, y\} \subseteq S\}$ . The  
 154 *edge-weight function induced by  $H$  on  $K(H)$* , denoted by  $w_H$ , is defined by  
 155  $w_H(e) = |\{S \in E(H) \mid e \subseteq S\}|$ . In words,  $w_H(e)$  is the number of hyperedges  
 156 of  $H$  containing  $e$ . A hypergraph  $H$  is *connected* if  $K(H)$  is connected, and  
 157 the *connected components* of a hypergraph  $H$  are the connected components  
 158 of  $K(H)$ . Finally, a graph  $G$   $F$ -overlaying  $H$  with maximum degree at most  
 159  $k$  is called an  $(F, H, k)$ -*graph* (resp.  $(\mathcal{F}, H, k)$ -*graph*) for  $F$  is a graph (resp.  
 160 graph family).

## 161 2. The graphs with one edge

162 In this section, we establish Theorem 1. Let  $p \geq 2$ , and  $H$  be a  $p$ -  
 163 uniform hypergraph. Consider the edge-weighted graph  $(K(H), w_H)$ . For  
 164 every matching  $M$  with at least one edge of this graph, let  $G_M = (V(H), M)$ .  
 165 Every hyperedge  $O_p$ -overlaid by  $G_M$  is a superset of at least one edge of  $M$   
 166 and at most  $\lfloor \frac{p}{2} \rfloor$  edges of  $M$ . We thus have the following:

167 **Observation 1.** *For every matching  $M$  of  $K(H)$ , we have:*

$$\frac{1}{\lfloor \frac{p}{2} \rfloor} w_H(M) \leq \text{over}_{O_p}(H, G_M) \leq w_H(M), \quad (1)$$

168 where  $w_H(M) = \sum_{e \in M} w_H(e)$ .

169 Consider first the case when  $p = 2$ . Let  $H$  be a 2-uniform hypergraph.  
 170 Every hyperedge is an edge, so  $K(H) = H$ . Moreover, a (hyper)edge of  $H$   
 171 is  $O_2$ -overlaid by  $G$  if and only if it is in  $E(G)$ . Hence MAX  $(\Delta \leq k)$ - $O_2$ -  
 172 OVERLAY is equivalent to finding a maximum  $k$ -matching (that is a subgraph  
 173 with maximum degree at most  $k$ ) in  $K(H)$ . This problem is polynomial-time  
 174 solvable, see [? , Chap. 31], hence:

175 **Proposition 1.** MAX  $(\Delta \leq k)$ - $O_2$ -OVERLAY is polynomial-time solvable for  
 176 all positive integer  $k$ .

177 If  $p = 3$ , Inequalities (1) are equivalent to  $\text{over}_{O_3}(H, G_M) = w_H(M)$ .  
 178 Since the edge set of a graph with maximum degree 1 is a matching, MAX  
 179  $(\Delta \leq 1)$ - $O_3$ -OVERLAY is equivalent to finding a maximum-weight matching  
 180 in the edge-weighted graph  $(K(H), w_H)$ . This can be done in polynomial-  
 181 time, see [? , Chap. 14].

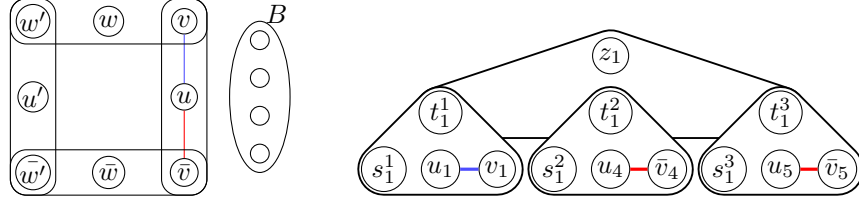


Figure 3: The construction of a variable gadget (left) which means that a blue (resp. red) edge in this gadget corresponds to that the variable is *true* (resp. *false*). And an example of clause gadget of  $C_1 = (x_1 \vee \bar{x}_4 \vee \bar{x}_5)$  (right) with three edges  $u_1v_1, u_4\bar{v}_4, u_5\bar{v}_5$  which corresponds to an assignment  $\phi$  such that  $\phi(x_1) = \phi(\bar{x}_4) = \phi(\bar{x}_5) = \text{true}$ .

182 **Proposition 2.**  $\text{MAX } (\Delta \leq 1)\text{-}O_3\text{-OVERLAY}$  is polynomial-time solvable.

183 We shall now prove that if  $p \geq 4$ , or  $p = 3$  and  $k \geq 2$ , then  $\text{MAX } (\Delta \leq k)\text{-}$   
 184  $O_p\text{-OVERLAY}$  is  $\mathcal{NP}$ -complete. We prove it by a double induction on  $k$  and  
 185  $p$ . Theorems 3 and 4 first prove the base cases of the induction and Lemma 1  
 186 corresponds to the inductive steps.

187 **Theorem 3.**  $(\Delta \leq 1)\text{-}O_4\text{-OVERLAY}$  is  $\mathcal{NP}$ -complete.

188 *Proof.* We reduce 3-SAT to  $(\Delta \leq 1)\text{-}O_4\text{-OVERLAY}$ .

189 We shall need the following gadget hypergraph  $J = J(A, B)$ .  
 190  $V(J) = A \cup B$  with  $|A| = 3$  and  $|B| = 4$  and  $E(J) = \{A \cup \{b\} \mid b \in B\}$ .

191 **Claim 1.** Every  $(O_4, J, 1)$ -graph contains exactly one edge with both endver-  
 192 tices in  $A$ .

193 *Proof of claim:* Let  $G$  be an  $(O_4, J, 1)$ -graph. Each hyperedge of  $J$  contains  
 194 an edge of  $G$ , which necessarily has an endvertex in  $A$ . Since  $\Delta(G) \leq 1$ ,  $G$   
 195 has at most three edges with an endvertex in  $A$ . Since  $J$  has four hyperedges,  
 196 there is an edge of  $G$  contained in two hyperedges of  $J$ . The two endvertices  
 197 of this edge are in  $A$ .  $\triangleleft$

198 Let  $\Phi$  be an instance of 3-SAT with  $m$  clauses  $C_1, \dots, C_m$  on  $n$  variables  
 199  $x_1, \dots, x_n$ . Let  $H = H(\Phi)$  be the 4-uniform hypergraph constructed as  
 200 follows (see Figure 3).

- 201 • For each variable  $x_i$ , we create the variable gadget  $VG_i$  as follows.  
 202 Its vertex set is  $\{u_i, v_i, \bar{v}_i, w_i, \bar{w}_i, w'_i, \bar{w}'_i, u'_i\} \cup B_i$ , with  $|B_i| = 4$ . We add  
 203 the hyperedges of  $J(\{u_i, v_i, \bar{v}_i\}, B_i)$ ,  $J(\{v_i, w_i, w'_i\}, B_i)$ ,  $J(\{\bar{v}_i, \bar{w}_i, \bar{w}'_i\}, B_i)$ ,  
 204 and  $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i)$ .



- For each clause  $C_j = \ell_1 \vee \ell_2 \vee \ell_3$ , we create the clause hypergraph  $CG_j$ . Its vertex set is  $\{z_j\} \cup \bigcup_{k=1}^3 \{s_j^k, t_j^k, u_j^k, v_j^k\}$ . Set  $Z_j = \{t_j^1, t_j^2, t_j^3, z_j\}$  and for  $k \in \{1, 2, 3\}$ , set  $T_j^k = \{s_j^k, t_j^k, u_j^k, v_j^k\}$ . The hyperedge set of  $CG_j$  is  $\{Z_j, T_j^1, T_j^2, T_j^3\}$ . Then for each  $k \in \{1, 2, 3\}$ , we identify  $u_j^k$  and  $v_j^k$  with  $u_i$  and  $v_i$  if  $\ell_k = x_i$  and with  $u_i$  and  $\bar{v}_i$  if  $\ell_k = \bar{x}_i$ .

Let us prove that there is an  $(O_4, H, 1)$ -graph if and only if  $\Phi$  is satisfiable. The general idea of the proof is that the variable gadget enforces only one of edges  $u_i v_i$ ,  $u_i \bar{v}_i$  to belong to  $G$  and that  $u_i v_i \in E(G)$  (resp.  $u_i \bar{v}_i \in E(G)$ ) if and only if the corresponding variable  $x_i$  is *true* (resp. *false*), thus the appearance of such an edge in a clause gadget implies that there is a true literal in the clause.

Assume first that there is an  $(O_4, H, 1)$ -graph  $G$ .

**Claim 2.** *For each  $1 \leq i \leq n$ ,  $G$  contains either the edge  $u_i v_i$  or the edge  $u_i \bar{v}_i$ . Moreover, for each  $x \in \{u_i, v_i, \bar{v}_i\}$  there is some  $x' \in VG_i$  such that  $x$  and  $x'$  are adjacent in  $G$ .*

*Proof of claim:* Since  $VG_i$  contains  $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i)$ , by Claim 1,  $G$  contains an edge  $e''$  with both endvertices in  $\{u'_i, w'_i, \bar{w}'_i\}$ . Assume that  $e''$  is incident to  $w'_i$ . Now  $VG_i$  contains  $J(\{v_i, w_i, w'_i\}, B_i)$ . So, by Claim 1,  $G$  contains an edge  $e'$  with both endvertices in  $\{v_i, w_i, w'_i\}$ . Since  $\Delta(G) \leq 1$ ,  $e'$  is not incident to  $w'_i$ , so  $e' = v_i w_i$ . Now  $VG_i$  contains  $J(\{u_i, v_i, \bar{v}_i\}, B_i)$ . So, by Claim 1,  $G$  contains an edge  $e$  with both endvertices in  $\{u_i, v_i, \bar{v}_i\}$ . This edge is not incident to  $v_i$ , so  $e = u_i \bar{v}_i$ . Similarly, we get that  $G$  contains  $\bar{v}_i \bar{w}_i$  and  $u_i v_i$  if  $e''$  is incident to  $\bar{w}'_i$ .  $\triangleleft$

In view of this claim, one defines the truth assignment  $\phi$  by  $\phi(x_i) = \text{true}$  if  $u_i v_i \in E(G)$  and  $\phi(x_i) = \text{false}$  if  $u_i \bar{v}_i \in E(G)$ . Let us prove that it satisfies  $\Phi$ . Consider a clause  $C_j = \ell_1 \vee \ell_2 \vee \ell_3$  and its corresponding clause gadget  $CG_j$ . If there is none of  $u_j^k v_j^k$  in  $E(G)$  (for all  $k \in \{1, 2, 3\}$ ), that means both  $u_j^k$  and  $v_j^k$  are incident to an edge which is incident to a vertex in the variable gadget it belongs to,  $T_j^k$  must be  $O_4$ -overlaid by the edge  $s_j^k t_j^k$ . So,  $t_j^k$  (for all  $k \in \{1, 2, 3\}$ ) has degree one in  $G$ . However, the hyperedge  $Z_j$  is  $O_4$ -overlaid by  $z_j t_j^k$  or  $t_j^k t_j^{k'}$  for  $k' \neq k$  and  $k, k' \in \{1, 2, 3\}$ , which implies that there is one of vertices  $t_j^k$  which is of degree one in  $G[Z_j]$ . It is a contradiction, thus there is an edge  $u_j^k v_j^k \in E(G)$  (for some  $k \in \{1, 2, 3\}$ ). Hence  $\phi(\ell_k) = \text{true}$ . Hence  $\phi$  satisfies  $\Phi$ .

239 Conversely, assume  $\phi$  is a truth assignment satisfying  $\Phi$ . Let  $G$  be the  
 240 graph with vertex set  $V(H)$  and whose edge set is constructed as follows.  
 241 For every  $1 \leq i \leq n$ , add  $\{u_i v_i, w_i w'_i, u'_i \bar{w}_i, \bar{w}_i \bar{v}_i\}$  to  $E(G)$  if  $\phi(x_i) = \text{true}$   
 242 and  $\{u_i \bar{v}_i, \bar{w}_i \bar{w}'_i, u'_i w'_i, w_i v_i\}$  to  $E(G)$  if  $\phi(x_i) = \text{false}$ . For every clause  $C_j =$   
 243  $\ell_1 \vee \ell_2 \vee \ell_3$ , there exists  $k \in \{1, 2, 3\}$  such that  $\phi(\ell_k) = \text{true}$ . Add the edges  
 244  $z_j t_j^k$ ,  $s_j^{k+1} t_j^{k+1}$  and  $s_j^{k+2} t_j^{k+2}$  (the superscript are modulo 3) to  $E(G)$ . One  
 245 easily checks that  $G$  is an  $(O_4, H, 1)$ -graph.  $\square$

246 **Theorem 4.**  $(\Delta \leq 2)$ - $O_3$ -OVERLAY is  $\mathcal{NP}$ -complete.

247 *Proof.* In order to simplify the proof, we will prove that the problem  $(\Delta \leq 2)$ -  
 248  $O_3$ -OVERLAY WITH HANDICAP is  $\mathcal{NP}$ -complete, where, in addition to a  
 249 hypergraph  $H$ , we are given a subset of vertices  $A \subseteq V(H)$  (called *handicap*  
 250 *vertices*), and we ask for a graph  $G$  that  $O_3$ -overlays  $H$  with the additional  
 251 constraint that for every  $v \in A$ , the degree of  $v$  is at most 1. Observe  
 252 that we can easily reduce this problem to  $(\Delta \leq 2)$ - $O_3$ -OVERLAY: for every  
 253  $v \in A$ , add six vertices  $t_v^1, \dots, t_v^6$ , and hyperedges  $\{v, t_v^1, t_v^p\}$  for every  $p \in$   
 254  $\{2, 3, 4, 5, 6\}$ . Let  $H'$  be the obtained hypergraph. Observe that an  $(O_3, H, 2)$ -  
 255 graph  $G$  in which every  $v \in A$  has degree at most 1 can be extended into an  
 256  $(O_3, H', 2)$ -graph by adding the edge  $vt_v^1$  (and all vertices of  $V(H') \setminus V(H)$ ).  
 257 Conversely,  $(O_3, H', 2)$ -graph  $G'$  must contain the edge  $vt_v^1$ , and thus  $v$  is of  
 258 degree at most 1 in  $G'[V(H)]$ .

259 We reduce 3-SAT to  $(\Delta \leq 2)$ - $O_3$ -OVERLAY WITH HANDICAP. Let  $\Phi$   
 260 be an instance of 3-SAT with  $m$  clauses  $C_1, \dots, C_m$  on  $n$  variables  $x_1, \dots,$   
 261  $x_n$ . We construct a 3-uniform hypergraph  $H$  as follows (see Figure 4 for an  
 262 example):

- 263 • for every variable  $x_i$ , we create a variable gadget  $VG_i$  composed of  
 264 vertices  $a_i^1, \dots, a_i^{p_i}, \alpha_i^1, \dots, \alpha_i^{p_i}, b_i^1, \dots, b_i^{p_i}, \beta_i^1, \dots, \beta_i^{p_i}, z_i^1, \dots, z_i^{p_i},$   
 265 and  $\zeta_i^1, \dots, \zeta_i^{p_i}$  where  $p_i$  is the number of clauses where  $x_i$  appears.  
 266 Then, for every  $j \in [p_i]$ , we create the following hyperedges (additions  
 267 in superscripts are modulo  $p_i$ , and we write  $p_i$  instead of 0):  $h_{i,j}^1 =$   
 268  $\{z_i^j, a_i^j, \alpha_i^j\}$ ,  $h_{i,j}^2 = \{z_i^j, b_i^j, \beta_i^j\}$ ,  $h_{i,j}^3 = \{a_i^j, z_i^j, b_i^j\}$ ,  $h_{i,j}^4 = \{z_i^j, b_i^j, \zeta_i^j\}$ ,  
 269  $h_{i,j}^5 = \{b_i^j, \zeta_i^j, a_i^{j+1}\}$ ,  $h_{i,j}^6 = \{\zeta_i^j, a_i^{j+1}, z_i^{j+1}\}$ .

270 Moreover, all vertices  $z_i^1, \dots, z_i^{p_i}$ , and  $\zeta_i^1, \dots, \zeta_i^{p_i}$  are handicap vertices.

- 271 • for every clause  $C_j$ , we create a clause gadget  $CG_j$  composed of vertices  
 272  $c_j^1, \dots, c_j^5$ , and, for every  $p \in [5]$ , of the hyperedge  $\{c_j^p, c_j^{p+1}, c_j^{p+3}\}$

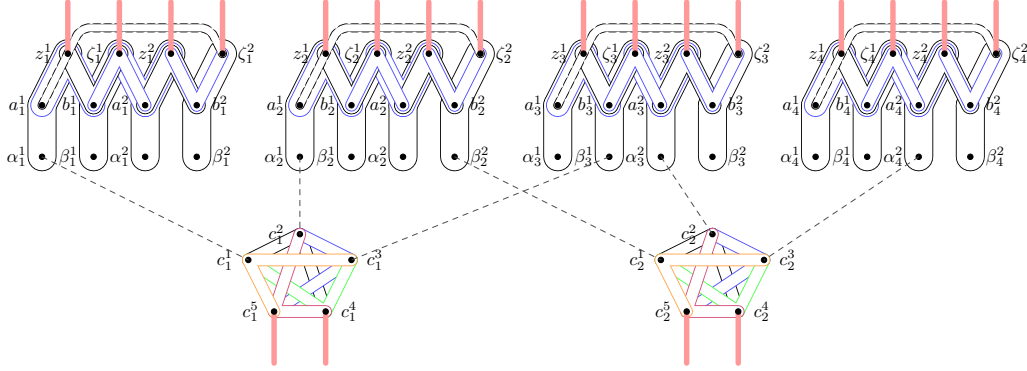


Figure 4: An example of the reduction for  $\Phi = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3 \vee x_4)$ . The dotted lines join two vertices that are identified. The thick red lines indicate the handicap vertices.

(additions are modulo 5, and we write 5 instead of 0). The vertices  $c_j^4$  and  $c_j^5$  are handicap vertices.

- finally, for every clause  $C_j = \ell_1 \vee \ell_2 \vee \ell_3$ , if  $\ell_p = x_i$  ( $\ell_p = \bar{x}_i$ ) for some  $p \in \{1, 2, 3\}$ , then we identify the vertex  $c_j^p$  of  $CG_j$  with the vertex  $\alpha_i^j$  ( $\beta_i^j$ ) of  $VG_i$ , respectively.

The general idea of the proof is the following. There are only two possibilities to overlay a variable gadget  $VG_i$  (either by a set  $T_i$  or a set  $F_i$  of edges). See Claim 3. This two possibilities correspond to the two possible values (*true* or *false*) of the variable  $x_i$ . Then a clause gadget  $CG_j$  imposes that not all literals of the clauses are assigned *false* with this correspondence.

Assume first that there is an  $(O_3, H, 2)$ -graph  $G$  such that all handicap vertices have degree at most 1.

For  $i \in [n]$ , let  $T_i = \bigcup_{j=1}^{p_i} \{z_i^j \alpha_i^j, b_i^j \beta_i^j, b_i^j \zeta_i^j\}$  and  $F_i = \bigcup_{j=1}^{p_i} \{a_i^j \alpha_i^j, z_i^j b_i^j, \zeta_i^j a_i^{j+1}\}$ .

**Claim 3.** *Let  $i \in [n]$ . We have the following:*

1.  $z_i^j \zeta_i^j \notin E(G)$  and  $\zeta_i^j z_i^{j+1} \notin E(G)$ ,  $j \in [m]$ .
2. Either  $T_i \subseteq E(G)$  or  $F_i \subseteq E(G)$ .

*Proof of claim:* 1. Assume for a contradiction that  $z_i^j \zeta_i^j \in E(G)$ . Since  $z_i^j$  and  $\zeta_i^j$  are handicap vertices, the hyperedges  $h_{i,j}^2, h_{i,j}^3, h_{i,j}^5$  must be  $O_3$ -overlaid by the edges  $b_i^j \beta_i^j, a_i^j b_i^j, b_i^j a_i^{j+1}$ , respectively, contradicting the fact that  $b_i^j$  has

293 degree 2 in  $G$ .  
 294 Similarly, if  $\zeta_i^j z_i^{j+1} \in E(G)$ , then the hyperedges  $h_{i,j}^5, h_{i,j+1}^1, h_{i,j+1}^3$  must be  $O_3$ -  
 295 overlaid by the edges  $b_i^j a_i^{j+1}, a_i^{j+1} \alpha_i^j, a_i^{j+1} b_i^{j+1}$ , respectively, a contradiction.

296 2. By 1. the hyperedge  $h_{i,j}^4$  is  $O_3$ -overlaid by  $z_i^j b_i^j$  or  $\zeta_i^j b_i^j$ .

297 In the first case  $z_i^j b_i^j \in E(G)$ , since  $z_i^j$  is a handicap vertex, the hyperedge  
 298  $h_{i,j-1}^6$  must be  $O_3$ -overlaid by  $\zeta_i^{j-1} a_i^j$ , and so hyperedge  $h_{i,j-1}^3$  is  $O_3$ -overlaid  
 299 by  $z_i^{j-1} b_i^{j-1}$ . And by induction, we get that  $\bigcup_{j=1}^m \{\zeta_i^j a_i^{j+1}, z_i^j b_i^j\} \subseteq E(G)$ .  
 300 Since the  $z_i^j$  are handicap vertices, the hyperedge  $h_{i,j}^1$  is  $O_3$ -overlaid by  $a_i^j \alpha_i^j$ .  
 301 Hence  $F_i \subseteq E(G)$ .

302 In the second case  $\zeta_i^j b_i^j \in E(G)$ , a symmetrical argument shows  $T_i \subseteq$   
 303  $E(G)$ .  $\triangleleft$

304 We define the truth assignment  $\phi$  by  $\phi(x_i) = \text{true}$  if  $T_i \subseteq E(G)$  and  
 305  $\phi(x_i) = \text{false}$  if  $F_i \subseteq E(G)$ . This is well-defined by Claim 3 and the fact  
 306 that  $E(G)$  cannot contain both  $T_i$  and  $F_i$  because it has maximum degree 2.

307 Now, for any  $j \in [m]$ , there exists  $p \in \{1, 2, 3\}$  such that  $c_j^p$  has degree 2  
 308 in  $G[CG_j]$  (If not, then there are at most 2 edges. Hence there is a vertex  
 309  $c_j^p$  of degree 0, and thus there is a hyperedge containing  $c_j^p$  which is not  
 310 overlaid). Therefore, it has degree 0 in the variable gadget  $c_j^p$  belongs to. By  
 311 our definition of  $\phi$ , we get that  $\phi(\ell_p) = \text{true}$ . Hence  $\phi$  satisfies  $\Phi$ .

312 Suppose now that  $\phi : \{x_1, \dots, x_n\} \rightarrow \{\text{true}, \text{false}\}$  is a truth assignment  
 313 satisfying  $\Phi$ .

314 Let us first select a set of edges in the variable gadgets. For every  $i \in [n]$ , if  
 315  $\phi(x_i) = \text{true}$ , then select edges  $\{z_i^j, a_i^j\}, \{b_i^j, \beta_i^j\}$  and  $\{\zeta_i^j, b_i^j\}$  for all  $j \in [p_i]$ .  
 316 If  $\phi(x_i) = \text{false}$ , then select edges  $\{z_i^j, b_i^j\}, \{a_i^j, \alpha_i^j\}$  and  $\{\zeta_i^j, a_i^{j+1}\}$  for all  
 317  $j \in [p_i]$  (additions are modulo  $p_i$ ). Let  $E_1$  be the set of selected edges, and  
 318  $G_1 = (V(H), E_1)$ . Observe that  $G_1$   $O_3$ -overlays the hyperedges of all variable  
 319 gadgets: for every  $i \in [n]$  and every  $j \in [p_i]$ ,

- 320 •  $h_{i,j}^1$  contains the edge  $\{z_i^j, a_i^j\}$  if  $\phi(x_i) = \text{true}$ , and the edge  $\{a_i^j, \alpha_i^j\}$   
 321 otherwise.
- 322 •  $h_{i,j}^2$  contains the edge  $\{b_i^j, \beta_i^j\}$  if  $\phi(x_i) = \text{true}$ , and the edge  $\{z_i^j, b_i^j\}$   
 323 otherwise.
- 324 •  $h_{i,j}^3$  contains the edge  $\{z_i^j, a_i^j\}$  if  $\phi(x_i) = \text{true}$ , and the edge  $\{z_i^j, b_i^j\}$   
 325 otherwise.

- 326 •  $h_{i,j}^4$  contains the edge  $\{\zeta_i^j, b_i^j\}$  if  $\phi(x_i) = \text{true}$ , and the edge  $\{z_i^j, b_i^j\}$   
327 otherwise.
- 328 •  $h_{i,j}^5$  contains the edge  $\{\zeta_i^j, b_i^j\}$  if  $\phi(x_i) = \text{true}$ , and the edge  $\{\zeta_i^j, a_i^{j+1}\}$   
329 otherwise.
- 330 •  $h_{i,j}^6$  contains the edge  $\{z_i^{j+1}, a_i^{j+1}\}$  if  $\phi(x_i) = \text{true}$ , and the edge  $\{\zeta_i^j, a_i^{j+1}\}$   
331 otherwise.

332 Moreover the vertices  $z_i^1, \dots, z_i^{p_i}$ , and  $\zeta_i^1, \dots, \zeta_i^{p_i}$  have degree 1 in  $G_1$  and the  
333 vertices  $\alpha_i^j$  have degree 0 in  $G_1$  if and only if  $\phi(x_i) = \text{true}$  and the vertices  
334  $\beta_i^j$  have degree 0 in  $G_1$  if and only if  $\phi(x_i) = \text{false}$ .

335 We now select a set of edges in the clause gadgets for every clause  $C_j =$   
336  $\ell_1 \vee \ell_2 \vee \ell_3$ . Since  $\phi$  satisfies  $\Phi$ , there exists  $p = p_j \in \{1, 2, 3\}$  such that  
337  $\phi(\ell_p) = \text{true}$ . Select the edges  $\{c_j^p, c_j^{p+2}\}$ ,  $\{c_j^p, c_j^{p+3}\}$  and  $\{c_j^{p+1}, c_j^{p+4}\}$ . Let  
338  $E_2$  be the set of selected edges in clause gadgets, and  $G_2 = (V(H), E_2)$ .  
339 Clearly,  $G_2$   $O_3$ -overlays all hyperedges of the clause gadgets and the vertices  
340  $c_j^4$  and  $c_j^5$ ,  $j \in [m]$ , have degree 1 in  $G_2$ . Moreover, for all  $j \in [m]$  the vertex  
341  $c_j^{p_j}$  is identified with the vertex  $\alpha_i^j$  or  $\beta_i^j$ , which has degree 0 in  $G_1$  since  
342  $\phi(\ell_p) = \text{true}$ . Thus  $G := (V(H), E_1 \cup E_2)$  has maximum degree 2, which  
343 proves that it is an  $(O_3, H, 2)$ -graph.  $\square$

344 **Lemma 1.** *If  $(\Delta \leq k)$ - $O_p$ -OVERLAY is  $\mathcal{NP}$ -complete, then  $(\Delta \leq k)$ - $O_{p+1}$ -*  
345 *OVERLAY and  $(\Delta \leq k + 1)$ - $O_p$ -OVERLAY are  $\mathcal{NP}$ -complete.*

346 Let  $k$  and  $p$  be two positive integers with  $p \geq 2$ , let  $H_{k,p}$  be the  $p$ -uniform  
347 hypergraph on  $(k+1)(p-1)$  vertices whose hyperedge set contains all possible  
348  $p$ -subsets of vertices.

349 We first need the following claim in order to prove Lemma 1.

350 **Claim 4.** *Every  $(O_p, H_{k,p}, k)$ -graph is  $k$ -regular. Moreover, there exists an*  
351  *$(O_p, H_{k,p}, k)$ -graph.*

352 *Proof of claim:* Let  $G$  be an  $(O_p, H_{k,p}, k)$ -graph. Assume for a contradic-  
353 tion that it has a vertex  $v$  of degree at most  $k - 1$ . Then, observe that  
354  $|V(G) \setminus N[v]| \geq (k+1)(p-2) + 1$ . Now, by removing successively the closed  
355 neighbourhood of a vertex of the graph (chosen arbitrarily) until the graph  
356 is anticomplete, we can construct, together with  $v$ , an independent set  $S$  of  
357  $G$  of size  $p$ . This is impossible, since  $S$  is a hyperedge of  $H_{k,p}$  which is not  
358  $O_p$ -overlaid by  $G$ .

359 Finally, observe that the disjoint union of  $p - 1$  cliques of size  $k + 1$  is a  
 360  $k$ -regular graph which  $O_p$ -overlays  $H_{k,p}$ .  $\triangleleft$

361 *Proof of Lemma 1.* First, we will reduce from  $(\Delta \leq k)$ - $O_p$ -OVERLAY to  
 362  $(\Delta \leq k)$ - $O_{p+1}$ -OVERLAY. Let  $H$  be a  $p$ -uniform hypergraph. We construct  
 363 a  $(p + 1)$ -uniform hypergraph  $H^*$  from a copy of  $H$  and replace, for every hy-  
 364 peredge  $S$ , a hyperedge  $S' = S \cup \{v_s\}$ . We also add, for every hyperedge  $S$ , a  
 365 set of  $(k + 1)p - 1$  new vertices  $N_S$ , and form, together with  $v_s$ , a hypergraph  
 366  $H_S$  isomorphic to  $H_{k,p+1}$ .

367 Let  $G$  be an  $(O_p, H, k)$ -graph. Construct  $G^*$  from  $G$  by adding, for every  
 368 hyperedge  $S$  of  $H$ , a disjoint union of  $p$  cliques of size  $k + 1$  on the vertices  
 369  $N_S \cup \{v_s\}$ . Clearly,  $G^*$  is an  $(O_{p+1}, H^*, k)$ -graph.

370 Conversely, if  $G^*$  is an  $(O_{p+1}, H^*, k)$ -graph, then, by Claim 4,  $v_s$  must  
 371 have degree  $k$  in  $N_S$  for every hyperedge  $S$  of  $H$ . Thus, the restriction of  $G^*$   
 372 to the vertices of  $H$  is an  $(O_p, H, k)$ -graph.

373 Now, we reduce  $(\Delta \leq k)$ - $O_p$ -OVERLAY to  $(\Delta \leq k + 1)$ - $O_p$ -OVERLAY.

374 Let  $H$  be a  $p$ -uniform hypergraph. Let  $H'$  be the  $p$ -uniform hypergraph  
 375 obtained from  $H$  by doing the following for each vertex  $v \in V(H)$ :

- 376 • add the vertices  $x^v, y_1^v, \dots, y_{p-2}^v$  and the hyperedge  $S_v = \{v, x^v, y_1^v, \dots, y_{p-2}^v\}$ .
- 377 • for any  $i = \{1, \dots, p - 2\}$ , add a copy  $H_i^v$  of  $H_{k+1,p}$  whose vertices are  
 378  $y_i^v$  and  $(p - 1)(k + 2) - 1$  new vertices.

379 Assume that  $G'$  is an  $(O_p, H', k + 1)$ -graph. By Claim 4, the graph induced  
 380 by  $G'$  on  $V(H_i^v)$  is  $(k + 1)$ -regular, for all  $v \in V(H)$ ,  $1 \leq i \leq p - 2$ . Therefore  
 381 each  $y_i^v$  has  $k + 1$ -neighbours outside the hyperedge  $S_v$ . Hence  $G'$  must contain  
 382 the edge  $vx^v$  because it  $O_p$ -overlays  $S_v$ . Thus the subgraph  $G = G'[V(H)]$   
 383 has maximum degree at most  $\Delta(G') - 1 \leq k$  and  $G$   $O_p$ -overlays  $H$ . In other  
 384 words,  $G$  is an  $(O_p, H, k)$ -graph.

385 Conversely, assume  $G$  is an  $(O_p, H, k)$ -graph. Let  $G'$  be the graph with  
 386 vertex set  $V(H')$  and whose edge set is the union of  $E(G)$ ,  $\{vx^v \mid v \in V(H)\}$ ,  
 387 and the edge set of an  $(O_p, H_i^v, k + 1)$ -graph (which exists by Claim 4), for all  
 388  $v \in V(H)$ ,  $1 \leq i \leq p - 2$ . One easily checks that  $G'$  is  $(O_p, H', k + 1)$ -graph.

389 Therefore, there is an  $(O_p, H, k)$ -graph if and only if there is an  
 390  $(O_p, H', k + 1)$ -graph.

391  $\square$

392 Propositions 1 and 2, Theorems 3 and 4, and Lemma 1 imply Theorem 1.

### 393 3. Complexity of Max $(\Delta \leq k)$ - $F$ -Overlay

394 The aim of this section is to establish Theorem 2 that gives the  $\mathcal{P}/\mathcal{NP}$ -  
395 complete dichotomy for the MAX  $(\Delta \leq k)$ - $F$ -OVERLAY problems.

396 As noticed in the introduction, if  $\Delta(F) > k$  or  $F$  is an anticomplete graph  
397 then MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY is trivially polynomial-time solvable.  
398 Moreover, by Propositions 1 and 2, MAX  $(\Delta \leq 1)$ - $O_3$ -OVERLAY as well as  
399 MAX  $(\Delta \leq k)$ - $O_2$ -OVERLAY (for all positive integers  $k$ ) are also polynomial-  
400 time solvable.

401 We shall now prove that if we are not in one of the above cases, then  
402 MAX  $(\Delta \leq k)$ - $F$ -OVERLAY is  $\mathcal{NP}$ -complete. We first establish the  $\mathcal{NP}$ -  
403 completeness when  $F$  has no isolated vertices.

404 **Theorem 5.** *Let  $F$  be a graph on at least three vertices with no isolated*  
405 *vertices. If  $k \geq \Delta(F)$ , then MAX  $(\Delta \leq k)$ - $F$ -OVERLAY is  $\mathcal{NP}$ -complete on*  
406 *neat hypergraphs.*

407 *Proof.* Assume  $k \geq \Delta(F)$ . Let  $n = |F|$ ,  $a_1, \dots, a_n$  be an ordering of the  
408 vertices of  $F$  such that  $\delta(F) = d(a_1) \leq d(a_2) \leq \dots \leq d(a_n) = \Delta(F)$ .

409 Let  $\gamma = \lfloor k/\delta(F) \rfloor - 1$ ,  $\beta = k - \gamma\delta(F)$ . Observe that  $\delta(F) \leq \beta \leq 2\delta(F) - 1$ .

410 We shall give a reduction from INDEPENDENT SET which is a well-known  
411  $\mathcal{NP}$ -complete problem even for cubic graphs (see [? ].) We distinguish two  
412 cases depending on whether  $d(a_2) > \beta$  or not.

413 Let  $\gamma_2 = \lfloor (k - d(a_2))/\delta(F) \rfloor$  and  $\gamma_3 = \lfloor (k - d(a_3))/\delta(F) \rfloor$ . And we define  
414 the value  $\gamma_2$  as follows.

- 415 - If  $d(a_2) > \beta$ , let  $\gamma_1 = \gamma_2$ .
- 416 - If  $d(a_2) \leq \beta$ , let  $\gamma_1 = \lfloor (k - d(a_1))/\delta(F) \rfloor$ .

417 Let  $\Gamma$  be a cubic graph. For each vertex  $v \in V(\Gamma)$ , let  $(e_1(v), e_2(v), e_3(v))$   
418 be an ordering of the edges incident to  $v$ . We shall construct the neat hyper-  
419 graph  $H = H(\Gamma)$  as follows.

- 420 • For each vertex  $v \in \Gamma$ , we create a hyperedge  $S_v = \{a_1^v, \dots, a_n^v\}$ . Then,  
421 for  $1 \leq i \leq 3$ , we add  $\gamma_i$   $a_i^v$ -leaves, that are hyperedges containing  $a_i^v$   
422 and  $n - 1$  new vertices.
- 423 • For each edge  $e = uv \in \Gamma$ , let  $i$  and  $j$  be the indices such that  $e =$   
424  $e_i(u) = e_j(v)$ . We create a new vertex  $z_e$  and hyperedges  $S_u^e$  ( $S_v^e$ )  
425 containing  $z_e$ ,  $a_i^u$  ( $a_j^v$ ), and  $n - 2$  new vertices, respectively. Then,

we add  $\gamma$   $z_e$ -leaves, that are hyperedges containing  $z_e$  and  $n - 1$  new vertices.

From the construction, observe that  $\gamma_i$  will be useful to enforce the degree of vertex  $a_i^v$  for  $v \in V(G)$  which could have degree at most  $d(a_3)$  in the hypergraph.

We shall prove that  $over_F(H, k) = (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$ , where  $\alpha(\Gamma)$  denotes the cardinality of a maximum independent set in  $\Gamma$ .

The following claim shows that there are optimal solutions with specific structure. This leads to the inequality:

$$over_F(H, k) \leq (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$$

**Claim 5.** *There is a graph  $G$  with  $\Delta(G) \leq k$  that  $F$ -overlays  $over_F(H, k)$  hyperedges of  $H$  such that:*

- (a) *each  $x$ -leaf  $L$  is  $F$ -overlaid and  $x$  is incident to  $\delta(F)$  edges in  $G[L]$  (with  $x = a_i^v$  or  $x = z_e$ ).*
- (b) *for each edge  $e = uv \in E(\Gamma)$ , exactly one of the two hyperedges  $S_u^e$  and  $S_v^e$  is  $F$ -overlaid. Moreover if  $S_u^e$  ( $S_v^e$ ) is  $F$ -overlaid, then  $a_i^u$  ( $a_j^v$ ) is incident to at most  $d(a_2)$  edges in  $S_u^e$  ( $S_v^e$ ), respectively.*
- (c) *the set of vertices  $v$  such that  $S_v$  is  $F$ -overlaid is an independent set in  $\Gamma$ .*

*Proof of claim:* Let  $G$  be a graph with  $\Delta(G) \leq k$  that  $F$ -overlays  $over_F(H, k)$  hyperedges of  $H$ . Observe first that since  $H$  is a neat hypergraph, we may assume that, for every hyperedge  $S$  of  $H$ ,  $G[S] = F$  if  $S$  is  $F$ -overlaid and  $G[S]$  has no edge otherwise.

Therefore if an  $x$ -leaf  $L$  is  $F$ -overlaid, then replacing  $G[L]$  by a copy of  $F$  in which  $x$  has degree  $\delta(F)$ , we obtain another graph with maximum degree at most  $k$  that  $F$ -overlays  $over_F(H, k)$  hyperedges of  $H$ . Henceforth we may assume that if an  $x$ -leaf is  $F$ -overlaid then  $x$  is incident to  $\delta(F)$  edges included in it.

Now assume that the  $z_e$ -leaf  $L$ , for  $e \in E(\Gamma)$ , is not  $F$ -overlaid by  $G$ . We have  $d_G(z_e) > k - \delta(F)$  for otherwise adding on  $L$  the edges of a copy of  $F$  in which  $z_e$  has degree  $\delta(F)$ , we obtain a graph  $F$ -overlying more hyperedges than  $G$ . But since there are  $\gamma$   $z_e$ -leaves and  $z_e$  is incident to  $\delta(F)$  edges in each of them, at least one of the hyperedges  $S_u^e$  and  $S_v^e$  is  $F$ -overlaid. Now



replacing the edges of  $G$  in this hyperedge by those of a copy of  $F$  in  $L$ , in which  $z_e$  has degree  $\delta(F)$ , we obtain another graph with maximum degree at most  $k$  that  $F$ -overlays  $over_F(H, k)$  hyperedges of  $H$ . Henceforth we may assume that every  $z_e$ -leaf is  $F$ -overlaid.

A similar reasoning shows that every  $a_i^v$ -leaf is  $F$ -overlaid, for all  $v \in V(\Gamma)$ ,  $i \in \{1, 2, 3\}$ . This proves (a).

Now by (a) each vertex  $z_e$  is incident to  $\gamma\delta(F) = k - \beta > k - 2\delta(F)$  edges in the  $z_e$ -leaves. Thus  $z_e$  is incident to at most  $\beta$  edges in  $S_u^e \cup S_v^e$ . Since  $\beta \leq 2\delta(F) - 1$ , at most one of  $S_u^e$  and  $S_v^e$  is  $F$ -overlaid.

Moreover, if  $S_u^e$  is  $F$ -overlaid and  $\beta < d(a_2)$  (resp.  $d(a_2) \leq \beta$ ), then  $z_e$  must have degree  $d(a_1)$  (resp. at most  $d(a_2)$ ) in  $G[S_u^e]$ .

Assume now that none of  $S_u^e, S_v^e$  is  $F$ -overlaid, and let  $e = e_i(u)$ ,  $a_i^u$  has  $\gamma_i$   $a_i^u$ -leaves, so  $a_i^u$  is incident to  $\gamma_i\delta(F) \leq k - d(a_i)$  edges in these leaves.

Thus, replacing the edges of  $G$  in  $S_u^e$  by those of a copy of  $F$ , in which

- if  $d(a_2) > \beta$ , we have  $d_G(a_i^u) \leq k - d(a_2)$ , then  $z_e$  has degree  $\delta(F)$  and  $a_i^u$  has degree  $d(a_2)$ .
- if  $d(a_2) \leq \beta$ , since  $d_G(a_1^u) \leq k - d(a_1)$ , then for  $i = 1$ ,  $a_i^u$  must have degree  $d(a_1)$ , so  $z_e$  has degree  $d(a_2)$ . And for  $i = \{2, 3\}$ , let  $d(z_e) = d(a_1)$ ,  $d(a_i^u) = d(a_2)$ .

Hence, we obtain another graph with maximum degree at most  $k$  that  $F$ -overlays  $over_F(H, k)$  hyperedges of  $H$ . Henceforth, we may assume that at least one of  $S_u^e$  and  $S_v^e$  is  $F$ -overlaid. This completes the proof of (b).

Consider now a vertex  $v \in V(\Gamma)$  such that  $S_v$  is  $F$ -overlaid. We distinguish the following cases.

- If  $d(a_2) > \beta$ , then by (a), vertex  $a_1^v$  is incident to  $\gamma_1\delta(F)$  edges in the  $a_1^v$ -leaves and at least  $\delta(F) \in S_v$ . Therefore  $S_v^{e_1(v)}$  cannot be  $F$ -overlaid for otherwise, by (b),  $a_1^v$  would have degree at least  $\gamma_1\delta(F) + \delta(F) + d(a_2) > k$ . Similarly,  $S_v^{e_2(v)}$  cannot be  $F$ -overlaid.

Assume now for a contradiction that  $S_v^{e_3(v)}$  is  $F$ -overlaid. By (a) and (b),  $a_3^v$  is incident to  $\gamma_3\delta(F)$  edges in the  $a_3^v$ -leaves and at least  $d(a_2) \in S_v^{e_3(v)}$ . Since  $\gamma_3\delta(F) + d(a_2) + d(a_3) > k$ ,  $a_3^v$  is incident to less than  $d(a_3)$  edges in  $S_v$ . Thus, there is a vertex  $a \in \{a_1^v, a_2^v\}$  which is incident to at least  $d(a_3)$  edges in  $S_v$  and thus at least  $\gamma_1\delta(F) + d(a_3)$  edges in  $G$ . But  $a_3^v$  is adjacent to at least  $\delta(F)$  edges in  $S_v$  so at least  $\gamma_3\delta(F) +$

490  $d(a_2) + \delta(F)$  edges in total. Thus  $\gamma_3\delta(F) + d(a_2) + \delta(F) \leq k$ , so  
 491  $\gamma_3 \leq \gamma_1 - 1$ . Thus  $a$  is adjacent to at least  $(\gamma_3 + 1)\delta(F) + d(a_3) > k$   
 492 edges, a contradiction.

493 - If  $d(a_2) \leq \beta$ , then by (a), vertex  $a_1^v$  is incident to  $\gamma_1\delta(F)$  edges in  
 494 the  $a_1^v$ -leaves and at least  $\delta(F) \in S_v$ . If  $S_v^{e_1(v)}$  is  $F$ -overlaid, then  
 495 (b) yields  $d(a_i^v) = \gamma_i\delta(F) + \delta(F) + \delta(F) > k$ , a contradiction. If  
 496  $S_v^{e_2(v)}$  is  $F$ -overlaid, then  $d(a_2^v) = \delta$  in  $G[S_v]$ . We obtain that  $d(a_2^v) =$   
 497  $\gamma_2\delta + d(a_2) + \delta \leq k$  which implies that  $\gamma_2 \leq \gamma_2 - 1$ , a contradiction.  
 498 If  $S_v^{e_3(v)}$  is  $F$ -overlaid, then we get a contradiction in the same way as  
 499 that in the case  $d(a_2) > \beta$ .

500 To summarize, if  $S_v$  is  $F$ -overlaid, then none of  $S_v^{e_1(v)}$ ,  $S_v^{e_2(v)}$ ,  $S_v^{e_3(v)}$  is  
 501  $F$ -overlaid. Together with (b), this implies (c).  $\triangleleft$

502 Conversely, consider  $W$  a maximum independent set of  $\Gamma$ .

503 Let  $G$  be the graph with vertex  $V(H)$  which is the union of the following  
 504 subgraphs :

- 505 • for each  $x$ -leaf  $L$ , we add a copy of  $F$  on  $L$  in which  $x$  has degree  $\delta(F)$ ;
- 506 • for each vertex  $v \in W$ , we add a copy of  $F$  on  $S_v$  in which  $a_i^v$  has degree  
 507  $d(a_i)$  for all  $1 \leq i \leq n$ .
- 508 • for each edge  $e \in E(\Gamma)$ , we choose an endvertex  $u$  of  $e$  such that  $u \notin W$ ,  
 509 and add a copy of  $F$  in which  $z_e$  has degree  $d(a_1)$  and  $a_i^u$  has degree  
 510  $d(a_2)$  (with  $i$  the index such that  $e_i(u) = e$ ) except the case when  
 511  $d(a_2) \leq \beta$  and  $i = 1$ , we do that but  $d(z_e) = d(a_2)$  and  $d(a_1^u) = d(a_1)$   
 512 in  $G[S_u^{e_1(u)}]$ .

513 It is simple matter to check that  $\Delta(G) \leq k$  and that  $G$   $F$ -overlays  
 514  $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$  hyperedges of  $H$ . Thus  $over_F(H, k) \geq$   
 515  $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$ .

516  $\square$

517 We then establish the following lemma, which allows to derive the  $\mathcal{NP}$ -  
 518 completeness of  $\text{MAX } (\Delta \leq k)\text{-}F\text{-OVERLAY}$  when  $F$  has isolated vertices.

519 **Lemma 2.** *Let  $k$  be a positive integer, let  $F$  be a graph with  $\delta(F) \geq 1$ , and*  
 520 *let  $q$  be a non-negative integer. If  $\text{MAX } (\Delta \leq k)\text{-(}F + \overline{K}_q\text{)-OVERLAY}$  is  $\mathcal{NP}$ -*  
 521 *complete, then  $\text{MAX } (\Delta \leq k)\text{-(}F + \overline{K}_{q+1}\text{)-OVERLAY}$  is also  $\mathcal{NP}$ -complete.*

522 *Proof.* We shall reduce MAX  $(\Delta \leq k)$ -( $F + \overline{K}_q$ )-OVERLAY to MAX  $(\Delta \leq$   
523  $k)$ -( $F + \overline{K}_{q+1}$ )-OVERLAY. Set  $p = |V(F)|$  and  $r = \lfloor k/\delta(F) \rfloor$ . Note that  
524  $r \geq 1$ , for otherwise MAX  $(\Delta \leq k)$ -( $F + \overline{K}_q$ )-OVERLAY would trivially be  
525 polynomial-time solvable.

526 Let  $H'$  be a  $(p+q)$ -uniform hypergraph. Let  $H$  be the  $(p+q+1)$ -uniform  
527 hypergraph obtained from  $H'$  as follows. Its vertex set is partitioned into  
528  $V(H')$ ,  $P_1, \dots, P_r$ ,  $K$ , and  $\{z\}$  with  $|P_i| = p - 1$  for all  $1 \leq i \leq r$  and  
529  $|K| = (m+1)(k+1) + kp$  with  $m = \max\{|E(H')|, q+1\}$ . The hyperedge set  
530 of  $H$  contains  $Q \cup P_i \cup \{z\}$  for all  $(q+1)$ -subsets  $Q$  of  $K$  and all  $1 \leq i \leq r$ ,  
531 and  $S \cup \{z\}$  for all  $S \in E(H')$ .

532 The reduction can be performed in polynomial time, because for each  
533 hyperedge in  $H'$  we construct a polynomial number of hyperedges in  $H$  (recall  
534 that  $q$  is fixed), and each of them is straightforward to obtain. We shall now  
535 prove that  $\text{over}_{F+\overline{K}_{q+1}}(H, k) = \text{over}_{F+\overline{K}_q}(H', k) + r \binom{|K|}{q+1}$ .

536 Let  $G'_0$  be a graph with  $V(G'_0) = V(H')$  and  $\Delta(G'_0) \leq k$  that  $(F + \overline{K}_q)$ -  
537 overlays  $\text{over}_{F+\overline{K}_q}(H', k)$  hyperedges of  $H'$ . For  $1 \leq i \leq r$ , let  $F_i$  be a copy  
538 of  $F$  on  $P_i \cup \{z\}$  in which  $z$  has degree  $\delta(F)$ . Let  $G_0$  be the graph defined  
539 by  $V(G_0) = V(H)$  and  $E(G_0) \cup \bigcup_{i=1}^r E(F_i)$ . Clearly,  $\Delta(G_0) \leq k$  and  $G_0$   $F$ -  
540 overlays all hyperedges of  $H$  except hyperedges  $S \cup \{z\}$  in which  $S \in E(H')$   
541 are not overlaid by  $G'_0$ . So  $\text{over}_{F+\overline{K}_{q+1}}(H, k) \geq \text{over}_{F+\overline{K}_q}(H', k) + r \binom{|K|}{q+1}$ .

542 Conversely, let  $G$  be a graph with  $V(G) = V(H)$  and  $\Delta(G) \leq k$  that  
543 maximizes the number of  $(F + \overline{K}_{q+1})$ -overlaid hyperedges.

544 **Claim 6.** *In  $G$ , vertex  $z$  has at least  $\delta(F)$  neighbours in each  $P_i$ ,  $1 \leq i \leq r$ .*

545 *Proof of claim:* Suppose for a contradiction that  $z$  has less than  $\delta(F)$  neigh-  
546 bours in  $P_i$ . Since  $\Delta(G) \leq k$ , at most  $kp$  vertices of  $K$  are incident to an edge  
547 with an endvertex in  $P_i \cup \{z\}$ . Thus there is a subset  $R$  of  $K$  of  $(m+1)(k+1)$   
548 vertices non-adjacent to all vertices of  $P_i \cup \{z\}$ . Now  $G[R]$  has maximum  
549 degree  $k$  and is thus  $(k+1)$ -colourable. Hence  $G[R]$  has an independent set  
550  $S$  of size  $m+1$ . Observe that for each  $(q+1)$ -subset  $Q$  of  $S$ , the subgraph  
551  $G[Q \cup P_i \cup \{z\}]$  has  $q+2$  vertices of degree less than  $\delta(F)$  (namely the ones of  
552  $Q \cup \{z\}$ ), and so the hyperedge  $Q \cup P_i \cup \{z\}$  is not  $(F + \overline{K}_{q+1})$ -overlaid. Thus  
553 there are at least  $m+1 \geq |E(H')| + 1$  hyperedges not  $(F + \overline{K}_{q+1})$ -overlaid  
554 by  $G$ . This contradicts the maximality of  $G$  as  $G_0$  overlays more hyperedges  
555 of  $H$ .  $\triangleleft$

556 Now  $z$  has degree at least  $r\delta(F)$  in  $\bigcup_{i=1}^r P_i$  and so less than  $\delta(F)$  in each  
557 hyperedge  $S \cup \{z\}$  for  $S \in E(H')$ . Hence if  $G$   $(F + \overline{K}_{q+1})$ -overlays  $S \cup z$ , it

558 must also  $(F + \overline{K}_q)$ -overlay  $S$ . Thus  $over_{F+\overline{K}_{q+1}}(H, k) \leq over_{F+\overline{K}_q}(H', k) +$   
559  $r\left(\frac{|K|}{q+1}\right)$ .  $\square$

560 Now we can prove Theorem 2. As explained in the beginning of the  
561 section, it suffices to prove that  $\text{MAX } (\Delta \leq k)\text{-}F\text{-OVERLAY}$  remains  $\mathcal{NP}$ -  
562 complete when  $\Delta(F) \leq k$ ,  $F \neq \overline{K}_{|F|}$ ,  $|F| \geq 3$  and  $(F, k) \neq (O_3, 1)$ . Assume  
563 that the above conditions are satisfied. Let  $F'$  be the graph induced by the  
564 non-isolated vertices of  $F$ . Then  $F = F' + \overline{K}_q$  with  $q = |F| - |F'|$ . If  $|F'| = 2$ ,  
565 then  $F = O_{|F|}$ , and we have the result by Theorem 1. If  $|F'| \geq 3$ , then the  
566 result follows from Theorem 5, Lemma 2, and an immediate induction.

#### 567 4. Complexity of $(\Delta \leq k)\text{-}F\text{-Overlay}$ and $(\Delta \leq k)\text{-}\mathcal{F}\text{-Overlay}$

568 In this section, we investigate the complexity of  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$   
569 and  $(\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$  problems. We believe that each such problem is  
570 either polynomial-time solvable or  $\mathcal{NP}$ -complete. However the dichotomy is  
571 more complicated than the one we proved in Theorem 2 for  $\text{MAX } (\Delta \leq k)\text{-}F\text{-}$   
572  $\text{OVERLAY}$ . This dichotomy is simple because there are very few pairs  $(F, H)$   
573 for which  $\text{MAX } (\Delta \leq k)\text{-}F\text{-OVERLAY}$  is polynomial-time solvable. We now  
574 exhibit several pairs  $(F, k)$  such that  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$  is polynomial-  
575 time solvable, while  $\text{MAX } (\Delta \leq k)\text{-}F\text{-OVERLAY}$  is  $\mathcal{NP}$ -complete.

##### 576 4.1. Regular graphs

577 **Proposition 3.** *For every complete graph  $K$  and every positive integer  $k$ ,*  
578  *$(\Delta \leq k)\text{-}K\text{-OVERLAY}$  is polynomial-time solvable.*

579 *Proof.* Observe that a  $|V(K)|$ -uniform hypergraph  $H$  is a positive instance  
580 of  $(\Delta \leq k)\text{-}K\text{-OVERLAY}$  if and only if  $K(H)$  is a  $(K, H, k)$ -graph.  $\square$

581 **Proposition 4.** *For every connected  $k$ -regular graph  $F$ ,  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$*   
582 *is polynomial-time solvable.*

583 *Proof.* One easily sees that a  $|V(F)|$ -uniform hypergraph  $H$  admits an  $(F, H, k)$ -  
584 graph if and only if the hyperedges of  $H$  are pairwise non-intersecting.  $\square$

585 Let  $C_4$  denote the cycle on 4 vertices. Proposition 4 implies that  $(\Delta \leq 2)\text{-}$   
586  $C_4\text{-OVERLAY}$  is polynomial-time solvable. We now show that  $(\Delta \leq 3)\text{-}C_4\text{-}$   
587  $\text{OVERLAY}$  is also polynomial-time solvable.

588 **Theorem 6.**  *$(\Delta \leq 3)\text{-}C_4\text{-OVERLAY}$  is polynomial-time solvable.*

589 *Proof.* Let  $H$  be a 4-uniform hypergraph.

590 Let us describe an algorithm to decide whether there is a  $(C_4, H, 3)$ -graph.  
 591 It is sufficient to do it when  $H$  is connected since the disjoint union of the  
 592  $(C_4, K, 3)$ -graphs for connected components  $K$  of  $H$  is a  $(C_4, H, 3)$ -graph.

593 Observe first that if two hyperedges of  $H$  intersect in exactly one vertex  
 594  $u$ , then no such graph exists, since  $u$  must have degree 2 in each of the  
 595 hyperedges if they are  $C_4$ -overlaid, and thus degree 4 in total. Therefore if  
 596 there are two such hyperedges, we return ‘No’. At this point we may assume  
 597 that  $|E(H)| \geq 2$  for otherwise we return ‘Yes’.

598 From now on we may assume that two hyperedges either do not intersect,  
 599 or are adjacent (intersect on at least two vertices).

600 **Claim 7.** *If two hyperedges  $S_1$  and  $S_2$  intersect on three vertices and there*  
 601 *is a  $(C_4, H, 3)$ -graph  $G$ , then  $|V(H)| \leq 6$ .*

602 *Proof of claim:* Assume  $S_1 = \{a_1, b, c, d\}$  and  $S_2 = \{a_2, b, c, d\}$ . Let  $G$  be a  
 603  $(C_4, H, 3)$ -graph. Observe that  $a_1$  and  $a_2$  must have two common neighbours,  
 604 say  $b, c$ , in common in  $\{b, c, d\}$ , for otherwise their unique common vertex  
 605 would have degree 4 in  $G$ . This forces  $d$  to be also adjacent to  $b$  and  $c$ .

606 Consider a hyperedge  $S_3$  intersecting  $S_1 \cup S_2$ . Since it is  $C_4$ -overlaid by  $G$ ,  
 607 at least two edges connect  $S_3 \cap (S_1 \cup S_2)$  to  $S_3 \setminus (S_1 \cup S_2)$ . The endvertices of  
 608 those edges in  $S_1 \cup S_2$  must have degree 2 in  $G[S_1 \cup S_2]$ . Hence, without loss of  
 609 generality, there is a vertex  $e$  not in  $S_1 \cup S_2$  such that either  $S_3 = \{a_1, a_2, b, e\}$   
 610 and  $\{a_1e, a_2e\} \subseteq E(G)$ , or  $S_3 = \{a_1, b, d, e\}$  and  $\{a_1e, de\} \subseteq E(G)$ . Now  
 611 no hyperedge can both intersect  $S_1 \cup S_2 \cup S_3$  and contain a vertex not in  
 612  $S_1 \cup S_2 \cup S_3$ , for such a hyperedge must contain either the vertices  $d, e$  or  
 613  $a_2, e$  which are at distance 3 in  $G[S_1 \cup S_2 \cup S_3]$ . (However there can be more  
 614 hyperedges contained in  $S_1 \cup S_2 \cup S_3$ .) Hence  $|V(H)| \leq 6$ .  $\triangleleft$

615 In view of Claim 7, if there are two hyperedges with three vertices in  
 616 common, either we return ‘No’ if  $|V(H)| > 6$ , or we check all possibilities (or  
 617 follow the proof of the above claim) to return the correct answer otherwise.  
 618 Henceforth, we may assume that any two adjacent hyperedges intersect in  
 619 exactly two vertices.

620 Let  $S_1$  and  $S_2$  be two adjacent hyperedges, say  $S_1 = \{a, b, c, d\}$  and  
 621  $S_2 = \{c, d, e, f\}$ . Note that every  $(C_4, H, 3)$ -graph contains the edges  $ab$ ,  
 622  $cd$  and  $ef$ , and that  $N(c) \cup N(d) = S_1 \cup S_2$  in this  $(C_4, H, 3)$ -graph.

623 **Claim 8.** *If there is another hyperedge than  $S_1$  and  $S_2$  containing  $c$  or  $d$ ,*  
 624 *and there is a  $(C_4, H, 3)$ -graph  $G$ , then  $|V(H)| \leq 8$ .*

625 *Proof of claim:* Without loss of generality, we may assume that  $G$  contains  
626 the cycle  $(a, b, d, f, e, c, a)$  and the edge  $cd$ . Hence the only possible hyper-  
627 edges containing  $c$  or  $d$  and a vertex not in  $S_1 \cup S_2$  are  $S_3 = \{a, c, e, g\}$  for  
628 some  $g \notin S_1 \cup S_2$  and  $S_4 = \{b, d, f, h\}$  for some  $h \notin S_1 \cup S_2$ .

629 If  $H$  contains both  $S_3$  and  $S_4$ , then  $G$  contains the edges  $ag$ ,  $eg$ ,  $bh$   
630 and  $hf$ . If  $G$  contains also  $gh$ , then  $G[S_1 \cup S_2 \cup S_3 \cup S_4]$  is 3-regular, so  
631  $G = G[S_1 \cup S_2 \cup S_3 \cup S_4]$ . If  $G$  does not contain  $gh$ , then the only vertices  
632 of degree 2 in  $G[S_1 \cup S_2 \cup S_3 \cup S_4]$  are  $g$  and  $h$ , and they are at distance at  
633 least 3 in this graph. Thus every hyperedge intersecting  $S_1 \cup S_2 \cup S_3 \cup S_4$  is  
634 contained in this set, so  $|V(H)| = 8$ .

635 Assume now that  $H$  is a superset of only one of  $S_3, S_4$ . Without loss  
636 of generality, we may assume that this is  $S_3$ . Hence  $G$  also contains the  
637 edges  $ag$  and  $eg$ . If  $V(G) \neq S_1 \cup S_2 \cup S_3$ , then there is a hyperedge  $S$  that  
638 intersects  $S_1 \cup S_2 \cup S_3$  and that is not contained in  $S_1 \cup S_2 \cup S_3$ . It does not  
639 contain  $c$  and  $d$ . Hence it must contain one of the vertices  $a$  or  $e$ , because it  
640 intersects each  $S_i$  along an edge of  $G$  or not at all. Without loss of generality,  
641  $a \in S$ . Hence  $S = \{a, b, i, g\}$  for some vertex  $i$  not in  $S_1 \cup S_2 \cup S_3$ , and  $G$   
642 contains the edges  $bi$  and  $ig$ . Now, as previously, either  $i$  and  $f$  are adjacent  
643 and  $G = G[S_1 \cup S_2 \cup S_3 \cup S]$  or they are not adjacent, and every hyperedge  
644 intersecting  $S_1 \cup S_2 \cup S_3 \cup S$  is contained in this set. In both cases,  $|V(H)| = 8$ .

645 ◁

646 We now summarize the algorithm: if  $|V(G)| \leq 8$ , then we solve the  
647 instance by brute force. Otherwise, for every pair of hyperedges  $S_1, S_2$ , if  
648 their intersection is of size 1 or 3, we answer ‘No’. In the remaining cases,  
649 if  $S_1$  and  $S_2$  have non-empty intersection, then, they must intersect on two  
650 vertices  $c$  and  $d$ , and these vertices do not belong to any other hyperedges  
651 but  $S_1$  and  $S_2$ . Hence, if a vertex is in at least 3 hyperedges, then we return  
652 ‘No’. If not, then every hyperedge intersects at most two hyperedges on two  
653 disjoint pairs of vertices. We can then return the  $(C_4, H, 3)$ -graph obtained as  
654 follows. If a hyperedge  $S = \{a, b, c, d\}$  intersects either two other hyperedges  
655  $S_1$  and  $S_2$  is  $\{a, b\}$  and  $\{c, d\}$ , or exactly one hyperedge on  $\{a, b\}$ , then add  
656 the edges  $ab, bc, cd, da$  to  $E(G)$ .

657 Clearly, the above-described algorithm runs in polynomial time. □

#### 658 4.2. Paths

659 Let  $\mathcal{P}$  be the set of all paths. We have the following:

660 **Theorem 7.**  $(\Delta \leq 2)$ - $\mathcal{P}$ -OVERLAY is linear-time solvable.

661 *Proof.* Clearly, if  $H$  is not connected, it suffices to solve the problem on each  
 662 of the components and to return ‘No’ if the answer is negative for at least one  
 663 of the components, and ‘Yes’ otherwise. Henceforth, we shall now assume  
 664 that  $H$  is connected. In such a case, a  $(\mathcal{P}, H, 2)$ -graph is either a path or a  
 665 cycle. However, if  $H$  is  $\mathcal{P}$ -overlaid by a path  $P$ , then it is also  $\mathcal{P}$ -overlaid by  
 666 the cycle obtained from  $P$  by adding an edge between its two endvertices.  
 667 Thus, we focus on the case where  $G$  is a cycle.

668 Let  $\mathcal{S}$  be a family of sets. The *intersection graph* of a family  $\mathcal{S}$  is the  
 669 graph  $IG(\mathcal{S})$  whose vertices are the sets of  $\mathcal{S}$ , and in which two vertices are  
 670 adjacent if the corresponding sets in  $\mathcal{S}$  intersect.

The *intersection graph* of a hypergraph  $H$ , denoted by  $IG(H)$ , is the  
 intersection graph of its hyperedge set. We define two functions  $l_H$  and  $s_H$   
 as follows:

$$l_H(S) = |S| - 1 \text{ for all } S \in E(H) \text{ and } s_H(S, S') = |S \cap S'| - 1 \text{ for all } S, S' \in E(H).$$

671 Let  $\mathbb{C}_\ell$  be the circle of circumference  $\ell$ . We identify the points of  $\mathbb{C}_\ell$  with  
 672 the integer numbers (points) of the segment  $[0, \ell]$ , (with 0 identified with  $\ell$ ).  
 673 A *circular-arc graph* is the intersection graph of a set of arcs on  $\mathbb{C}_\ell$ . A set  
 674  $\mathcal{A}$  of arcs such that  $IG(\mathcal{A}) = G$  is called an *arc representation* of  $G$ . We  
 675 denote by  $A_v$  the arc corresponding to  $v$  in  $\mathcal{A}$ . Let  $G$  be a graph and let  
 676  $l : V(G) \rightarrow \mathbb{N}$  and  $s : E(G) \rightarrow \mathbb{N}$  be two functions. An arc representation  $\mathcal{A}$   
 677 of  $G$  is  *$l$ -respecting* if  $A_v$  has length  $l(v)$  for any  $v \in V(G)$ ,  *$s$ -respecting* if  
 678  $A_v \cap A_u$  has length  $s(u, v)$  for all  $uv \in E(G)$ , and  *$(l, s)$ -respecting* if it is both  
 679  $l$ -respecting and  $s$ -respecting. One can easily adapt the algorithm given by  
 680 Köbler et al. [?] for  $(l, s)$ -respecting interval representations to decide in  
 681 linear time whether a graph admits an  $(l, s)$ -respecting arc representation in  
 682  $\mathbb{C}_n$  for every integer  $n$ .

683 **Claim 9.** *Let  $H$  be a connected hypergraph on  $n$  vertices. There is a cy-*  
 684 *cle  $\mathcal{P}$ -overlying  $H$  if and only if  $IG(H)$  admits an  $(l_H, s_H)$ -respecting arc*  
 685 *representation into  $\mathbb{C}_n$ .*

686 *Proof of claim:* Assume that  $H$  is  $\mathcal{P}$ -overlaid by a cycle  $C = (v_0, v_1, \dots, v_{n-1}, v_0)$ .  
 687 There is a canonical embedding of  $C$  to  $\mathbb{C}_n$  in which every vertex  $v_i$  is mapped  
 688 to  $i$  and every edge  $v_i v_{i+1}$  to the circular arc  $[i, i + 1]$ . For every hyperedge  
 689  $S \in E(H)$ ,  $P[S]$  is a subpath, which is mapped to the circular arc  $A_S$  of  $\mathbb{C}_n$   
 690 that is the union of the circular arcs to which its edges are mapped. Clearly,  
 691  $\mathcal{A} = \{A_S \mid S \in E(H)\}$  is an  $(l_H, s_H)$ -respecting interval representation of  
 692  $IG(H)$ .

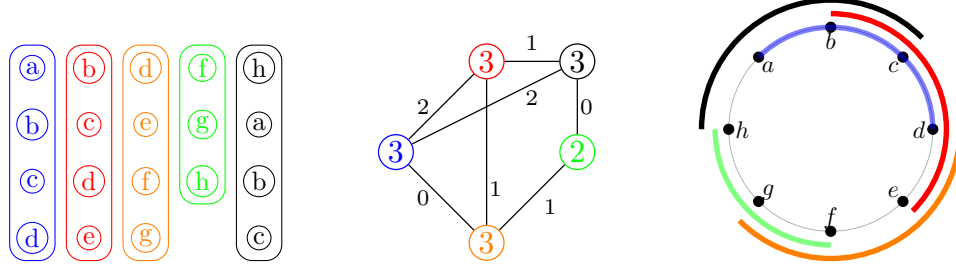


Figure 5: Given a hypergraph  $H$  (left), we determine its graph  $IG(H)$  (middle) with each vertex represents the a hyperedge of the same color in which the value of any vertex  $S \in V(H)$  is  $l_H(S)$  and the weight of an edge between  $S, S'$  is  $s_H(S, S')$ . And a circular arc representation (each arc is associated to a hyperedge of the same color) is shown in the right which is  $(l, s)$ -respecting.

Conversely, assume that  $IG(H)$  admits an  $(l_H, s_H)$ -respecting interval representation  $\mathcal{A} = \{A_S \mid S \in E(H)\}$  into  $\mathbb{C}_n$ . Let  $S_0$  be a hyperedge of minimum size. Free to rotate all intervals, we may assume that  $A_{S_0}$  is  $[1, |S_0|]$ . Now since  $\mathcal{A}$  is  $(l_H, s_H)$ -respecting and  $H$  is connected, we deduce that the extremities of  $A_S$  are integers for all  $S \in E(H)$ . Let  $v_1$  be a vertex of  $H$  that belongs to the hyperedges whose corresponding arcs of  $\mathcal{A}$  contain 1. Then for all  $i = 2$  to  $n = |V(H)|$ , denote by  $v_i$  an arbitrary vertex not in  $\{v_1, \dots, v_{i-1}\}$  that belongs to the hyperedges whose corresponding arcs of  $\mathcal{A}$  contain  $i$ . Such a vertex exists because  $\mathcal{A}$  is  $(l_H, s_H)$ -respecting. Observe that such a construction yields  $S = \{v_i \mid i \in A_S\}$  for all  $S \in E(H)$ . Furthermore, the cycle  $C = (v_1, \dots, v_n, v_1)$   $\mathcal{P}$ -overlays  $H$ . Indeed, for each  $S \in E(H)$ ,  $C[S]$  is the subpath corresponding to  $A_S$ , that is  $V(C[S]) = \{v_i \mid i \in A_S\}$  and  $E(C[S]) = \{v_i v_{i+1} \mid [i, i+1] \subseteq A_S\}$ .  $\triangleleft$

The algorithm to solve  $(\Delta \leq 2)$ - $\mathcal{P}$ -OVERLAY for a connected hypergraph  $H$  in linear time is thus the following:

1. Construct the intersection graph  $IG(H)$  and compute the associated functions  $l_H$  and  $s_H$ .
2. Check whether graph  $IG(H)$  has an  $(l_H, s_H)$ -respecting interval representation. If it is the case, return ‘Yes’. If not return ‘No’.

□

**Remark 1.** We can also detect in polynomial time whether a connected hypergraph  $H$  is  $\mathcal{P}$ -overlaid by a path. Indeed, similarly to Claim 9, one can



show that there is a path  $\mathcal{P}$ -overlying  $H$  if and only if  $IG(H)$  admits an  $(l_H, s_H)$ -respecting interval representation.

#### 4.3. On neat hypergraphs

Let  $F = (V, E)$  be a graph. We denote by  $V_i$  the set of vertices having degree  $i$  in  $F$ . For every increasing sequence of nonnegative integers  $i_1 < i_2 < \dots < i_p$ , an  $(i_1, i_2, \dots, i_p)$ -graph is a graph  $F$  such that  $V_i = \emptyset$  if  $i \notin \{i_1, i_2, \dots, i_p\}$ . In particular, an  $(r)$ -graph is a  $r$ -regular graph.

Given a hypergraph  $H$ , we denote by  $d_H(v)$  the number of hyperedges containing  $v$  for any  $v \in V(H)$ .

**Theorem 8.** *Let  $i_1, i_2, k$  be three non-negative integers with  $i_1 < i_2$ . For every  $(i_1, i_2)$ -graph  $F$ , one can solve  $(\Delta \leq k)$ - $F$ -OVERLAY in polynomial time on neat hypergraphs.*

*Proof.* Let  $H$  be a neat  $|F|$ -uniform hypergraph.

If there is a vertex  $v$  such that  $i_1 \cdot d_H(v) > k$ , then we answer ‘No’. Indeed in an  $(H, F, k)$ -graph  $G$ , the vertex  $v$  has degree at least  $i_1$  in each subgraph of  $G$  induced by a hyperedge to which it belongs, and so degree at least  $i_1 \cdot d_H(v)$  in  $G$  because  $H$  is neat. Hence, we assume that  $i_1 \cdot d_H(v) \leq k$  for every vertex.

Let  $B(H)$  be the incidence graph of  $H$ : its vertex set is  $E(H) \cup V(H)$  and there is an edge between a hyperedge  $S \in E(H)$  and a vertex  $v \in V(H)$  if and only if  $v \in S$ .

**Claim 10.** *There is an  $(H, F, k)$ -graph if and only if  $B(H)$  has a subgraph in which every hyperedge vertex has degree  $|V_{i_2}|$  and every vertex has degree at most  $\lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$ .*

*Proof of claim:* Assume that there is an  $(H, F, k)$ -graph  $G$ . Observe that since  $H$  is a neat hypergraph, free to remove some edges, we may assume that  $G[S]$  is isomorphic to  $F$  for every hyperedge  $S$  of  $H$ . Let  $B'$  be the subgraph of  $B(H)$  in which a vertex  $v$  is adjacent to a hyperedge  $S$  if and only if  $v$  has degree  $i_2$  in  $G[S]$ . By definition every hyperedge has degree  $|V_{i_2}|$  in  $B'$ . Consider now a vertex  $v$ . Its degree in  $G$  is the sum of its degree in the subgraphs induced by the hyperedges  $S$  to which it belongs because  $H$  is neat. Hence  $d_G(v) = i_2 \cdot d_{B'}(v) + i_1(d_H(v) - d_{B'}(v)) = i_1 \cdot d_H(v) + (i_2 - i_1)d_{B'}(v)$ . But  $G$  is an  $(H, F, k)$ -graph, so  $d_G(v) \leq k$ . With the previous equality, this yields  $d_{B'}(v) \leq \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1}$ , and so  $d_{B'}(v) \leq \lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$ .

749 Conversely, assume  $B(H)$  has a subgraph  $B'$  in which every  $S \in E(H)$   
750 has degree  $|V_{i_2}|$  and any  $v \in V(H)$  has degree at most  $\lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$ . We  
751 construct  $G$  as follows: for each hyperedge  $S$ , the induced subgraph  $G[S]$   
752 is a copy of  $F$  such that any vertex  $v$  where  $(S, v) \in E(B')$  is of degree  
753  $i_2$  (there are  $|V_{i_2}|$  such vertices) and the remaining vertices have degree  $i_1$ .  
754 Clearly,  $G$   $F$ -overlays  $H$ . Moreover every vertex  $v \in V(H)$  has degree  $i_2$   
755 in at most  $\lfloor \frac{k-i_1 \cdot d_H(v)}{i_2-i_1} \rfloor$  subgraphs and degree  $i_1$  in the remaining subgraphs  
756 induced by hyperedges to which it belongs. Hence  $v$  has degree at most  $k$  in  
757  $G$ . Therefore  $G$  is an  $(H, F, k)$ -graph.  $\triangleleft$

758 Now the existence of a subgraph of  $B(H)$  as shown in Claim 10 can be  
759 done in polynomial time (see [?] Chap. 10).  $\square$

760 **Theorem 9.** *Let  $i_1, i_2, i_3, k$  be three non-negative integers with  $i_1 < i_2 < i_3$ .  
761 If  $F$  is a  $(i_1, i_2, i_3)$ -graph  $F$  such that all vertices of  $V_{i_1} \cup V_{i_2}$  have the same  
762 number of neighbours in  $V_{i_3}$  and  $k < i_3 + i_1$ , then one can solve  $(\Delta \leq k)$ - $\mathcal{F}$ -  
763 OVERLAY in polynomial time on neat hypergraphs.*

764 *Proof.* Let  $j$  be the number of neighbours in  $V_{i_3}$  that each vertex of  $V_{i_1} \cup V_{i_2}$   
765 has.

766 Let  $H$  be an  $|F|$ -uniform hypergraph. Observe that if a vertex has degree  
767  $i_3$  in the subgraph induced by a hyperedge, then it cannot be in any other  
768 hyperedge, for otherwise it would have degree at least  $i_3 + i_1 > k$ . Therefore,  
769 we first check whether every hyperedge  $S$  contains a set  $I(S)$  of  $|V_{i_3}|$  vertices  
770 that are uniquely in  $S$ . If not, then we return ‘No’. If it is the case, let  $F' =$   
771  $F - V_{i_3}$  and  $H'$  be the hypergraph whose vertex set is  $V(H) \setminus \bigcup_{S \in E(H)} I(S)$   
772 and whose hyperedge set is  $\{S \setminus I(S) \mid S \in E(H)\}$ , and let  $k'(v) = k - j \cdot d_H(v)$   
773 for every  $v \in V(H')$ . Since all vertices of  $V_{i_1} \cup V_{i_2}$  have exactly  $j$  neighbours  
774 in  $V_{i_3}$ , it is then easy to prove the following claim:

775 **Claim 11.** *There is an  $(H, F, k)$ -graph  $G$  if and only if there is a graph  $G'$   
776  $F'$ -overlying  $H'$  such that  $d_G(v) \leq k'(v)$  for all  $v \in V(H')$ .*

777 Deciding whether such a graph  $G'$  exists can be done similarly as the  
778 proof of Theorem 8 replacing  $k$  by  $k'(v)$ .  $\square$

## 779 5. Further research

780 Theorem 2 characterizes the complexity of MAX  $(\Delta \leq k)$ - $\mathcal{F}$ -OVERLAY  
781 when  $\mathcal{F}$  contains a unique graph. It would be nice to extend this character-  
782 ization to families  $\mathcal{F}$  of arbitrary size.

783 **Problem 1.** *Characterize the pairs  $(\mathcal{F}, k)$  for which  $\text{MAX } (\Delta \leq k)\text{-}\mathcal{F}$ -*  
784 *OVERLAY is polynomial-time solvable and those for which it is  $\mathcal{NP}$ -complete.*

785 Theorem 1 and the results obtained in Section 4 give a first view of the  
786 complexity of  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$ . A natural problem is to close the  
787 dichotomy:

788 **Problem 2.** *Characterize the pairs  $(F, k)$  for which  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$*   
789 *is polynomial-time solvable and those for which it is  $\mathcal{NP}$ -complete.*

790 It would be interesting to consider the complexity of this problem when  
791  $F$  is  $k$ -regular but non-connected, and when  $F$  is a cycle. In order to attack  
792 Problem 2, it would be helpful to prove the following conjecture.

793 **Conjecture 1.** *If  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$  is  $\mathcal{NP}$ -complete, then  $(\Delta \leq k+1)\text{-}$*   
794  *$F\text{-OVERLAY}$  is also  $\mathcal{NP}$ -complete.*

795 Furthermore, for each pair  $(\mathcal{F}, k)$  such that  $\text{MAX } (\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$   
796 is  $\mathcal{NP}$ -complete and  $(\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$  is polynomial-time solvable, it  
797 is natural to consider the parameterized complexity of  $\text{MAX } (\Delta \leq k)\text{-}\mathcal{F}$ -  
798  $\text{OVERLAY}$  when parameterized by  $|E(H)| - s$ , because  $(\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$   
799 is the case  $s = 0$ .

800 It would also be interesting to obtain approximation algorithms for  $\text{MAX}$   
801  $(\Delta \leq k)\text{-}\mathcal{F}\text{-OVERLAY}$  when this problem is  $\mathcal{NP}$ -complete.

802 As shown in Subsection 4.3, we proved that  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$  is some-  
803 times easier when restricted to neat hypergraphs. It would then be interesting  
804 to determine the complexity of  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$  on neat hypergraphs.  
805 So far, we only proved some of these problems to be polynomial-time solvable,  
806 but we believe some are  $\mathcal{NP}$ -complete.

807 **Problem 3.** *Characterize the pairs  $(F, k)$  for which  $(\Delta \leq k)\text{-}F\text{-OVERLAY}$*   
808 *on neat hypergraphs is polynomial-time solvable and those for which it is*  
809  *$\mathcal{NP}$ -complete.*