



HAL
open science

Overlaying a hypergraph with a graph with bounded maximum degree

Frédéric Havet, Dorian Mazauric, Viet-Ha Nguyen, Rémi Watrigant

► **To cite this version:**

Frédéric Havet, Dorian Mazauric, Viet-Ha Nguyen, Rémi Watrigant. Overlaying a hypergraph with a graph with bounded maximum degree. *Discrete Applied Mathematics*, In press, 319, pp.394-406. 10.1007/978-3-030-39219-2_32 . hal-02025469v3

HAL Id: hal-02025469

<https://inria.hal.science/hal-02025469v3>

Submitted on 15 Jun 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 Overlaying a hypergraph with a graph with
2 bounded maximum degree

3 Frédéric Havet^{1,1}, Dorian Mazauric¹, Viet-Ha Nguyen^{1,1}, Rémi Watrigant¹

4 ^a*CNRS, I3S, France*

5 ^b*Inria, Université Côte d'Azur, France*

6 ^c*Université de Lyon, France*

7 **Abstract**

8 Let G and H be respectively a graph and a hypergraph defined on a same set
9 of vertices, and let F be a fixed graph. We say that G F -overlays a hyper-
10 edge S of H if F is a spanning subgraph of the subgraph of G induced by S ,
11 and that G F -overlays H if it F -overlays every hyperedge of H . Motivated
12 by structural biology, we study the computational complexity of two prob-
13 lems. The first problem, $(\Delta \leq k)$ F -OVERLAY, consists in deciding whether
14 there is a graph with maximum degree at most k that F -overlays a given
15 hypergraph H . It is a particular case of the second problem MAX $(\Delta \leq k)$
16 F -OVERLAY, which takes a hypergraph H and an integer s as input, and
17 consists in deciding whether there is a graph with maximum degree at most
18 k that F -overlays at least s hyperedges of H .

We give a complete polynomial/ \mathcal{NP} -complete dichotomy for the MAX $(\Delta \leq k)$ - F -OVERLAY problems depending on the pairs (F, k) , and establish the complexity of $(\Delta \leq k)$ F -OVERLAY for many pairs (F, k) .

19 *Keywords:* graphs, hypergraphs, complexity, algorithm, computational
20 structural biology

21 **1. Introduction**

22 A major problem in structural biology is the characterization of low res-
23 olution structures of macro-molecular assemblies [? ?]. To attack this very
24 difficult question, one has to determine the plausible contacts between the
25 subunits (e.g. proteins) of an assembly, given the lists of subunits involved
26 in some complexes. We assume that the composition, in terms of individual
27 subunits, of selected complexes is known. Indeed, a given assembly can be

28 chemically split into complexes by manipulating chemical conditions. This
 29 problem can be conveniently modelled by graphs and hypergraphs. We con-
 30 sider the hypergraph H whose vertices represent the subunits and whose
 31 hyperedges are the complexes. We are then looking for a graph G with the
 32 same vertex set as H whose edges represent the contacts between subunits,
 33 and satisfying (i) some local properties for every complex (*i.e.* hyperedge),
 34 and (ii) some other global properties.

35 For example, consider the following properties: (i) the subset of vertices
 36 of every hyperedge must form a connected subgraph of G and (ii) the number
 37 of edges of G is at most the number of vertices. The first property comes from
 38 the fact that the complexes obtained (e.g. by native mass spectrometry) are
 39 connected components of a given assembly. The second property is deduced
 40 from the constant number of subunits that are in contact to any subunit of an
 41 assembly. It means that the total number of edges is linear in the number of
 42 vertices (in our example, we assume that the number of edges is at most the
 43 number of vertices). Let us now consider a hypergraph H whose hyperedges
 44 are $\{v_1, v_2, v_3\}$, $\{v_2, v_4, v_5\}$, $\{v_3, v_4, v_5\}$, and $\{v_4, v_5, v_6, v_7\}$. These hyperedges
 45 are depicted in Figure ?? (left) in which a vertex labeled i represents v_i . The
 46 graph G described in Figure ?? (right) satisfies the previous properties: the
 47 set of vertices $\{v_4, v_5, v_6, v_7\}$ induces a connected subgraph (same observation
 48 for the three other hyperedges) and the number of edges of G is exactly the
 49 number of vertices.

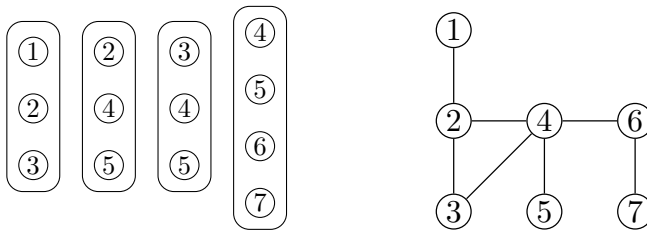


Figure 1: A hypergraph composed of seven vertices and four hyperedges (right). A graph G satisfying local properties (every hyperedge induces a connected subgraph in G) and a global property (the total number of edges is at most the number of vertices).

50 We first focus on the local properties. They are usually modelled by a
 51 (possibly infinite) family \mathcal{F} of admissible graphs to which each complex must
 52 belong: to this end, we define the notion of *enforcement* of a hyperedge and
 53 a hypergraph. A graph G \mathcal{F} -enforces a hyperedge $S \in E(H)$ if the subgraph
 54 $G[S]$ of G induced by S belongs to \mathcal{F} , and G \mathcal{F} -enforces H if G \mathcal{F} -enforces

55 all hyperedges of H . Very often, the considered family \mathcal{F} is closed on taking
 56 edge supergraphs [? ?]: if $F \in \mathcal{F}$, then every graph obtained from G by
 57 adding edges is also in \mathcal{F} . Such a family is completely defined by its set
 58 $\mathcal{M} = \mathcal{M}(\mathcal{F})$ of minimal graphs that are the elements of \mathcal{F} which are not
 59 edge supergraphs of any other. In this case, a graph G \mathcal{F} -enforcing S is such
 60 that there is an element of \mathcal{M} which is a spanning subgraph of $G[S]$. This
 61 leads to the following notion of *overlayment* when considering minimal graph
 62 families.

63 **Definition 1.** A graph G \mathcal{F} -overlays a hyperedge S if there exists $F \in \mathcal{F}$
 64 such that F is a spanning subgraph of $G[S]$, and G \mathcal{F} -overlays H if G \mathcal{F} -
 65 overlays every hyperedge of H .

66 As said previously, the graph sought will also have to satisfy some global
 67 constraints. Since in a macro-molecular assembly the number of contacts is
 68 small, the first natural idea is to look for a graph G with the minimum number
 69 of edges. This leads to the MIN- \mathcal{F} -OVERLAY problem: given a hypergraph
 70 H and an integer m , decide if there exists a graph G \mathcal{F} -overlying H such
 71 that $|E(G)| \leq m$.

72 A typical example of a family \mathcal{F} is the set of all connected graphs, in
 73 which case $\mathcal{M}(\mathcal{F})$ is the set of all trees. Agarwal et al. [?] focused on MIN-
 74 $\mathcal{M}(\mathcal{F})$ -OVERLAY for this particular family in the aforementioned context of
 75 structural biology. However, this problem was previously studied by several
 76 communities in other domains, as pointed out by Chen *et al.*[?]. Indeed,
 77 it is also known as SUBSET INTERCONNECTION DESIGN, MINIMUM TOPIC-
 78 CONNECTED OVERLAY or INTERCONNECTION GRAPH PROBLEM, and was
 79 considered (among others) in the design of vacuum systems [? ?], scalable
 80 overlay networks [? ?], and reconfigurable interconnection networks [? ?].
 81 Some variants have also been considered in the contexts of inferring a most
 82 likely social network [?], determining winners of combinatorial auctions [?
 83], as well as drawing hypergraphs [? ?].

84 Cohen et al. [?] presented a dichotomy regarding the polynomial vs.
 85 \mathcal{NP} -hard status of the problem MIN- \mathcal{F} -OVERLAY with respect to the con-
 86 sidered family \mathcal{F} . Roughly speaking, they showed that the easy cases one
 87 can think of (e.g. when edgeless graphs of the right sizes are in \mathcal{F} , or if \mathcal{F}
 88 contains only cliques) are the only families giving rise to a polynomial-time
 89 solvable problem: all others are \mathcal{NP} -complete. They also considered the
 90 FPT/W[1]-hard dichotomy for several families \mathcal{F} .

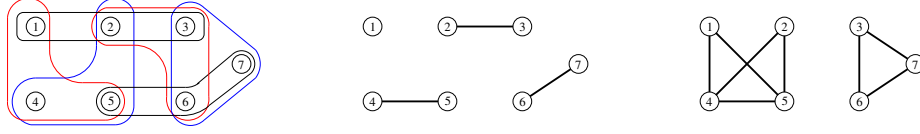


Figure 2: Example of $(\Delta \leq k)$ - \mathcal{F} -OVERLAY and MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY. In the figure, an instance H (left), a graph G with $\Delta(G) \leq 1$ that O_3 -overlays H (with O_3 being the graph with three vertices and one edge) (center), and a solution to MAX $(\Delta \leq 3)$ - C_3 -OVERLAY (with C_3 being the cycle on three vertices) (right).

91 In this paper, we consider the variant in which the additional constraint is
 92 that G must have a bounded maximum degree: this constraint is motivated
 93 by the context of structural biology, since a subunit (e.g. a protein) cannot
 94 be connected to many other subunits. This yields the following problem for
 95 any family \mathcal{F} of graphs and an integer k .

($\Delta \leq k$)- \mathcal{F} -OVERLAY
 96 Input: A hypergraph H .
 Question: Does there exist a graph G \mathcal{F} -overlying H such that $\Delta(G) \leq k$?

97 We denote by $over_{\mathcal{F}}(H, G)$ the number of hyperedges of H that are \mathcal{F} -
 98 overlaid by G . A natural generalization is to find $over_{\mathcal{F}}(H, k)$, the maximum
 99 number of hyperedges \mathcal{F} -overlaid by a graph with maximum degree at most
 100 k .
 101 k .

MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY
 102 Input: A hypergraph H and a positive integer s .
 Question: Does there exist a graph G such that $\Delta(G) \leq k$ and
 103 $over_{\mathcal{F}}(H, G) \geq s$?

104 Observe that there is an obvious reduction from $(\Delta \leq k)$ - \mathcal{F} -OVERLAY to
 105 MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY (by setting $s = |E(H)|$).

106 In this paper, we mainly consider the case when the family \mathcal{F} contains a
 107 unique graph F . We abbreviate $(\Delta \leq k)$ - $\{F\}$ -OVERLAY and MAX $(\Delta \leq k)$ -
 108 $\{F\}$ -OVERLAY to $(\Delta \leq k)$ - F -OVERLAY and MAX $(\Delta \leq k)$ - F -OVERLAY,
 109 respectively. By definition those two problems really make sense only for
 110 $|F|$ -uniform hypergraphs *i.e.* hypergraphs whose hyperedges are of size $|F|$.
 111 Therefore, we always assume the hypergraph to be $|F|$ -uniform.

112 If F is a graph with maximum degree greater than k , then solving $(\Delta \leq k)$ -
 113 F -OVERLAY or MAX $(\Delta \leq k)$ - F -OVERLAY is trivial as the answer is always
 114 ‘No’. So we only study the problems when $\Delta(F) \leq k$.

115 If F is an anticomplete graph (i.e. with no edges), then $\text{MAX } (\Delta \leq k)$ -
 116 F - OVERLAY is also trivial, because for any hypergraph H , the anticomplete
 117 graph on $V(H)$ vertices F -overlays H . Hence the first natural interesting
 118 cases are the graphs with one edge. For every integer $p \geq 2$, we denote by O_p
 119 the graph with p vertices and one edge. In Section ??, we prove the following
 120 dichotomy theorem.

121 **Theorem 1.** *Let $k \geq 1$ and $p \geq 2$ be integers. If $p = 2$ or if $k = 1$ and*
 122 *$p = 3$, then $\text{MAX } (\Delta \leq k)$ - O_p - OVERLAY and $(\Delta \leq k)$ - O_p - OVERLAY are*
 123 *polynomial-time solvable. Otherwise, they are \mathcal{NP} -complete.*

124 Then, in Section ??, we give a complete polynomial/ \mathcal{NP} -complete di-
 125 chotomy for the $\text{MAX } (\Delta \leq k)$ - F - OVERLAY problems.

126 **Theorem 2.** *$\text{MAX } (\Delta \leq k)$ - F - OVERLAY is polynomial-time solvable if ei-*
 127 *ther $\Delta(F) > k$, or F is an anticomplete graph, or $F = O_2$, or $k = 1$ and*
 128 *$F = O_3$. Otherwise it is \mathcal{NP} -complete.*

129 In Section ??, we investigate the complexity of $(\Delta \leq k)$ - F - OVERLAY and
 130 $(\Delta \leq k)$ - \mathcal{F} - OVERLAY problems. We believe that each such problem is either
 131 polynomial-time solvable or \mathcal{NP} -complete. However the dichotomy seems
 132 to be more complicated than the one for $\text{MAX } (\Delta \leq k)$ - F - OVERLAY . We
 133 exhibit several pairs (F, k) such that $(\Delta \leq k)$ - F - OVERLAY is polynomial-
 134 time solvable, while $\text{MAX } (\Delta \leq k)$ - F - OVERLAY is \mathcal{NP} -complete. This is
 135 in particular the case when F is a complete graph (Proposition ??), F is
 136 connected k -regular (Proposition ??), F is a path and $k = 2$ (Theorem ??),
 137 and when F is the cycle on 4 vertices and $k \leq 3$ (Theorem ??).

138 Most notations of this paper are standard. We now recall some of them,
 139 and we refer the reader to [?] for any undefined terminology. The degree
 140 of a vertex v in a graph is denoted by $d(v)$. For a positive integer p , let
 141 $[p] = \{1, \dots, p\}$. In some proofs of the paper we might use the modulo
 142 operation to keep indices in bounds.

143 Given $S \subseteq V(G)$, we denote by $G[S]$ the subgraph induced by S , that is
 144 the subgraph with vertex set S and edge set $\{uv \in E(G) \mid u, v \in S\}$. We
 145 denote by \overline{K}_k the *anticomplete* graph on k vertices, that is the graph with k
 146 vertices and no edges. The *disjoint union* of two graphs F and G is denoted
 147 by $F + G$.

148 Let H be a hypergraph. Two hyperedges are *adjacent* if their intersection
 149 has size at least 2. A hypergraph is *neat* if any two distinct hyperedges

150 intersect in at most one vertex. In other words, a hypergraph is neat if
 151 there is no pair of adjacent hyperedges. We denote by $K(H)$, the graph
 152 obtained by replacing each hyperedge by a complete graph. In other words,
 153 $V(K(H)) = V(H)$ and $E(K(H)) = \{xy \mid \exists S \in E(H), \{x, y\} \subseteq S\}$. The
 154 *edge-weight function induced by H on $K(H)$* , denoted by w_H , is defined by
 155 $w_H(e) = |\{S \in E(H) \mid e \subseteq S\}|$. In words, $w_H(e)$ is the number of hyperedges
 156 of H containing e . A hypergraph H is *connected* if $K(H)$ is connected, and
 157 the *connected components* of a hypergraph H are the connected components
 158 of $K(H)$. Finally, a graph G F -overlying H with maximum degree at most
 159 k is called an (F, H, k) -*graph* (resp. (\mathcal{F}, H, k) -*graph*) for F is a graph (resp.
 160 graph family).

161 2. The graphs with one edge

162 In this section, we establish Theorem ???. Let $p \geq 2$, and H be a p -
 163 uniform hypergraph. Consider the edge-weighted graph $(K(H), w_H)$. For
 164 every matching M with at least one edge of this graph, let $G_M = (V(H), M)$.
 165 Every hyperedge O_p -overlaid by G_M is a superset of at least one edge of M
 166 and at most $\lfloor \frac{p}{2} \rfloor$ edges of M . We thus have the following:

167 **Observation 1.** *For every matching M of $K(H)$, we have:*

$$\frac{1}{\lfloor \frac{p}{2} \rfloor} w_H(M) \leq \text{over}_{O_p}(H, G_M) \leq w_H(M), \quad (1)$$

168 where $w_H(M) = \sum_{e \in M} w_H(e)$.

169 Consider first the case when $p = 2$. Let H be a 2-uniform hypergraph.
 170 Every hyperedge is an edge, so $K(H) = H$. Moreover, a (hyper)edge of H
 171 is O_2 -overlaid by G if and only if it is in $E(G)$. Hence MAX $(\Delta \leq k)$ - O_2 -
 172 OVERLAY is equivalent to finding a maximum k -matching (that is a subgraph
 173 with maximum degree at most k) in $K(H)$. This problem is polynomial-time
 174 solvable, see [? , Chap. 31], hence:

175 **Proposition 1.** MAX $(\Delta \leq k)$ - O_2 -OVERLAY is polynomial-time solvable for
 176 all positive integer k .

177 If $p = 3$, Inequalities (??) are equivalent to $\text{over}_{O_3}(H, G_M) = w_H(M)$.
 178 Since the edge set of a graph with maximum degree 1 is a matching, MAX
 179 $(\Delta \leq 1)$ - O_3 -OVERLAY is equivalent to finding a maximum-weight matching
 180 in the edge-weighted graph $(K(H), w_H)$. This can be done in polynomial-
 181 time, see [? , Chap. 14].

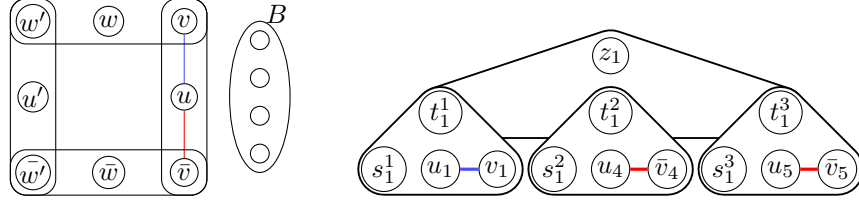


Figure 3: The construction of a variable gadget (left) which means that a blue (resp. red) edge in this gadget corresponds to that the variable is *true* (resp. *false*). And an example of clause gadget of $C_1 = (x_1 \vee \bar{x}_4 \vee \bar{x}_5)$ (right) with three edges $u_1v_1, u_4\bar{v}_4, u_5\bar{v}_5$ which corresponds to an assignment ϕ such that $\phi(x_1) = \phi(\bar{x}_4) = \phi(\bar{x}_5) = \text{true}$.

182 **Proposition 2.** $\text{MAX } (\Delta \leq 1)\text{-}O_3\text{-OVERLAY}$ is polynomial-time solvable.

183 We shall now prove that if $p \geq 4$, or $p = 3$ and $k \geq 2$, then MAX
 184 $(\Delta \leq k)\text{-}O_p\text{-OVERLAY}$ is \mathcal{NP} -complete. We prove it by a double induction
 185 on k and p . Theorems ?? and ?? first prove the base cases of the induction
 186 and Lemma ?? corresponds to the inductive steps.

187 **Theorem 3.** $(\Delta \leq 1)\text{-}O_4\text{-OVERLAY}$ is \mathcal{NP} -complete.

188 *Proof.* We reduce 3-SAT to $(\Delta \leq 1)\text{-}O_4\text{-OVERLAY}$.

189 We shall need the following gadget hypergraph $J = J(A, B)$.
 190 $V(J) = A \cup B$ with $|A| = 3$ and $|B| = 4$ and $E(J) = \{A \cup \{b\} \mid b \in B\}$.

191 **Claim 1.** Every $(O_4, J, 1)$ -graph contains exactly one edge with both endver-
 192 tices in A .

193 *Proof of claim:* Let G be an $(O_4, J, 1)$ -graph. Each hyperedge of J contains
 194 an edge of G , which necessarily has an endvertex in A . Since $\Delta(G) \leq 1$, G
 195 has at most three edges with an endvertex in A . Since J has four hyperedges,
 196 there is an edge of G contained in two hyperedges of J . The two endvertices
 197 of this edge are in A . \triangleleft

198 Let Φ be an instance of 3-SAT with m clauses C_1, \dots, C_m on n variables
 199 x_1, \dots, x_n . Let $H = H(\Phi)$ be the 4-uniform hypergraph constructed as
 200 follows (see Figure ??).

- 201 • For each variable x_i , we create the variable gadget VG_i as follows.
 202 Its vertex set is $\{u_i, v_i, \bar{v}_i, w_i, \bar{w}_i, w'_i, \bar{w}'_i, u'_i\} \cup B_i$, with $|B_i| = 4$. We add
 203 the hyperedges of $J(\{u_i, v_i, \bar{v}_i\}, B_i)$, $J(\{v_i, w_i, w'_i\}, B_i)$, $J(\{\bar{v}_i, \bar{w}_i, \bar{w}'_i\}, B_i)$,
 204 and $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i)$.

- 205 • For each clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$, we create the clause hypergraph CG_j .
 206 Its vertex set is $\{z_j\} \cup \bigcup_{k=1}^3 \{s_j^k, t_j^k, u_j^k, v_j^k\}$. Set $Z_j = \{t_j^1, t_j^2, t_j^3, z_j\}$ and
 207 for $k \in \{1, 2, 3\}$, set $T_j^k = \{s_j^k, t_j^k, u_j^k, v_j^k\}$. The hyperedge set of CG_j
 208 is $\{Z_j, T_j^1, T_j^2, T_j^3\}$. Then for each $k \in \{1, 2, 3\}$, we identify u_j^k and v_j^k
 209 with u_i and v_i if $\ell_k = x_i$ and with u_i and \bar{v}_i if $\ell_k = \bar{x}_i$.

210 Let us prove that there is an $(O_4, H, 1)$ -graph if and only if Φ is satisfiable.
 211 The general idea of the proof is that the variable gadget enforces only one
 212 of edges $u_i v_i$, $u_i \bar{v}_i$ to belong to G and that $u_i v_i \in E(G)$ (resp. $u_i \bar{v}_i \in E(G)$)
 213 if and only if the corresponding variable x_i is *true* (resp. *false*), thus the
 214 appearance of such an edge in a clause gadget implies that there is a true
 215 literal in the clause.

216 Assume first that there is an $(O_4, H, 1)$ -graph G .

217 **Claim 2.** *For each $1 \leq i \leq n$, G contains either the edge $u_i v_i$ or the edge*
 218 *$u_i \bar{v}_i$. Moreover, for each $x \in \{u_i, v_i, \bar{v}_i\}$ there is some $x' \in VG_i$ such that x*
 219 *and x' are adjacent in G .*

220 *Proof of claim:* Since VG_i contains $J(\{u'_i, w'_i, \bar{w}'_i\}, B_i)$, by Claim ??, G con-
 221 tains an edge e'' with both endvertices in $\{u'_i, w'_i, \bar{w}'_i\}$. Assume that e'' is
 222 incident to w'_i . Now VG_i contains $J(\{v_i, w_i, w'_i\}, B_i)$. So, by Claim ??, G
 223 contains an edge e' with both endvertices in $\{v_i, w_i, w'_i\}$. Since $\Delta(G) \leq 1$, e'
 224 is not incident to w'_i , so $e' = v_i w_i$. Now VG_i contains $J(\{u_i, v_i, \bar{v}_i\}, B_i)$. So,
 225 by Claim ??, G contains an edge e with both endvertices in $\{u_i, v_i, \bar{v}_i\}$. This
 226 edge is not incident to v_i , so $e = u_i \bar{v}_i$. Similarly, we get that G contains $\bar{v}_i \bar{w}_i$
 227 and $u_i v_i$ if e'' is incident to \bar{w}'_i . \triangleleft

228 In view of this claim, one defines the truth assignment ϕ by $\phi(x_i) = \text{true}$
 229 if $u_i v_i \in E(G)$ and $\phi(x_i) = \text{false}$ if $u_i \bar{v}_i \in E(G)$. Let us prove that it satisfies
 230 Φ . Consider a clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$ and its corresponding clause gadget
 231 CG_j . If there is none of $u_j^k v_j^k$ in $E(G)$ (for all $k \in \{1, 2, 3\}$), that means
 232 both u_j^k and v_j^k are incident to an edge which is incident to a vertex in the
 233 variable gadget it they belongs to, T_j^k must be O_4 -overlaid by the edge $s_j^k t_j^k$.
 234 So, t_j^k (for all $k \in \{1, 2, 3\}$) has degree one in G . However, the hyperedge
 235 Z_j is O_4 -overlaid by $z_j t_j^k$ or $t_j^k t_j^{k'}$ for $k' \neq k$ and $k, k' \in \{1, 2, 3\}$, which
 236 implies that there is one of vertices t_j^k which is of degree one in $G[Z_j]$. It is
 237 a contradiction, thus there is an edge $u_j^k v_j^k \in E(G)$ (for some $k \in \{1, 2, 3\}$).
 238 Hence $\phi(\ell_k) = \text{true}$. Hence ϕ satisfies Φ .

239 Conversely, assume ϕ is a truth assignment satisfying Φ . Let G be the
 240 graph with vertex set $V(H)$ and whose edge set is constructed as follows.
 241 For every $1 \leq i \leq n$, add $\{u_i v_i, w_i w'_i, u'_i \bar{w}'_i, \bar{w}_i \bar{v}_i\}$ to $E(G)$ if $\phi(x_i) = \text{true}$
 242 and $\{u_i \bar{v}_i, \bar{w}_i \bar{w}'_i, u'_i w'_i, w_i v_i\}$ to $E(G)$ if $\phi(x_i) = \text{false}$. For every clause $C_j =$
 243 $\ell_1 \vee \ell_2 \vee \ell_3$, there exists $k \in \{1, 2, 3\}$ such that $\phi(\ell_k) = \text{true}$. Add the edges
 244 $z_j t_j^k, s_j^{k+1} t_j^{k+1}$ and $s_j^{k+2} t_j^{k+2}$ (the superscript are modulo 3) to $E(G)$. One
 245 easily checks that G is an $(O_4, H, 1)$ -graph. \square

246 **Theorem 4.** $(\Delta \leq 2)$ - O_3 -OVERLAY is \mathcal{NP} -complete.

247 *Proof.* In order to simplify the proof, we will prove that the problem $(\Delta \leq 2)$ -
 248 O_3 -OVERLAY WITH HANDICAP is \mathcal{NP} -complete, where, in addition to a
 249 hypergraph H , we are given a subset of vertices $A \subseteq V(H)$ (called *handicap*
 250 *vertices*), and we ask for a graph G that O_3 -overlays H with the additional
 251 constraint that for every $v \in A$, the degree of v is at most 1. Observe
 252 that we can easily reduce this problem to $(\Delta \leq 2)$ - O_3 -OVERLAY: for every
 253 $v \in A$, add six vertices t_v^1, \dots, t_v^6 , and hyperedges $\{v, t_v^1, t_v^p\}$ for every $p \in$
 254 $\{2, 3, 4, 5, 6\}$. Let H' be the obtained hypergraph. Observe that an $(O_3, H, 2)$ -
 255 graph G in which every $v \in A$ has degree at most 1 can be extended into an
 256 $(O_3, H', 2)$ -graph by adding the edge vt_v^1 (and all vertices of $V(H') \setminus V(H)$).
 257 Conversely, $(O_3, H', 2)$ -graph G' must contain the edge vt_v^1 , and thus v is of
 258 degree at most 1 in $G'[V(H)]$.

259 We reduce 3-SAT to $(\Delta \leq 2)$ - O_3 -OVERLAY WITH HANDICAP. Let Φ
 260 be an instance of 3-SAT with m clauses C_1, \dots, C_m on n variables $x_1, \dots,$
 261 x_n . We construct a 3-uniform hypergraph H as follows (see Figure ?? for an
 262 example):

- 263 • for every variable x_i , we create a variable gadget VG_i composed of
 264 vertices $a_i^1, \dots, a_i^{p_i}, \alpha_i^1, \dots, \alpha_i^{p_i}, b_i^1, \dots, b_i^{p_i}, \beta_i^1, \dots, \beta_i^{p_i}, z_i^1, \dots, z_i^{p_i},$
 265 and $\zeta_i^1, \dots, \zeta_i^{p_i}$ where p_i is the number of clauses where x_i appears.
 266 Then, for every $j \in [p_i]$, we create the following hyperedges (additions
 267 in superscripts are modulo p_i , and we write p_i instead of 0): $h_{i,j}^1 =$
 268 $\{z_i^j, a_i^j, \alpha_i^j\}, h_{i,j}^2 = \{z_i^j, b_i^j, \beta_i^j\}, h_{i,j}^3 = \{a_i^j, z_i^j, b_i^j\}, h_{i,j}^4 = \{z_i^j, b_i^j, \zeta_i^j\},$
 269 $h_{i,j}^5 = \{b_i^j, \zeta_i^j, a_i^{j+1}\}, h_{i,j}^6 = \{\zeta_i^j, a_i^{j+1}, z_i^{j+1}\}.$

270 Moreover, all vertices $z_i^1, \dots, z_i^{p_i}$, and $\zeta_i^1, \dots, \zeta_i^{p_i}$ are handicap vertices.

- 271 • for every clause C_j , we create a clause gadget CG_j composed of vertices
 272 c_j^1, \dots, c_j^5 , and, for every $p \in [5]$, of the hyperedge $\{c_j^p, c_j^{p+1}, c_j^{p+3}\}$

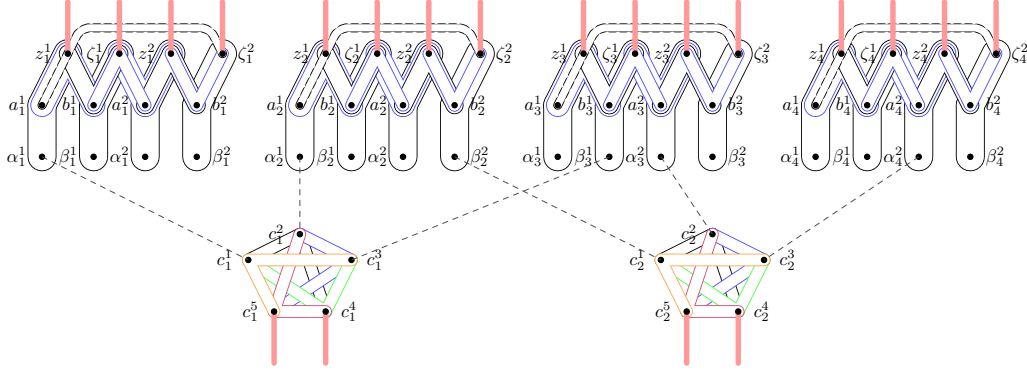


Figure 4: An example of the reduction for $\Phi = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (\bar{x}_2 \vee x_3 \vee x_4)$. The dotted lines join two vertices that are identified. The thick red lines indicate the handicap vertices.

273 (additions are modulo 5, and we write 5 instead of 0). The vertices c_j^4
 274 and c_j^5 are handicap vertices.

- 275 • finally, for every clause $C_j = \ell_1 \vee \ell_2 \vee \ell_3$, if $\ell_p = x_i$ ($\ell_p = \bar{x}_i$) for some
 276 $p \in \{1, 2, 3\}$, then we identify the vertex c_j^p of CG_j with the vertex α_i^j
 277 (β_i^j) of VG_i , respectively.

278

279 The general idea of the proof is the following. There are only two possibil-
 280 ities to overlay a variable gadget VG_i (either by a set T_i or a set F_i of edges).
 281 See Claim ???. This two possibilities correspond to the two possible values
 282 (*true* or *false*) of the variable x_i . Then a clause gadget CG_j imposes that
 283 not all literals of the clauses are assigned *false* with this correspondence.

284 Assume first that there is an $(O_3, H, 2)$ -graph G such that all handicap
 285 vertices have degree at most 1.

286 For $i \in [n]$, let $T_i = \bigcup_{j=1}^{p_i} \{z_i^j \alpha_i^j, b_i^j \beta_i^j, b_i^j \zeta_i^j\}$ and $F_i = \bigcup_{j=1}^{p_i} \{a_i^j \alpha_i^j, z_i^j b_i^j, \zeta_i^j a_i^{j+1}\}$.

287 **Claim 3.** *Let $i \in [n]$. We have the following:*

- 288 1. $z_i^j \zeta_i^j \notin E(G)$ and $\zeta_i^j z_i^{j+1} \notin E(G)$, $j \in [m]$.
- 289 2. Either $T_i \subseteq E(G)$ or $F_i \subseteq E(G)$.

290 *Proof of claim:* 1. Assume for a contradiction that $z_i^j \zeta_i^j \in E(G)$. Since z_i^j
 291 and ζ_i^j are handicap vertices, the hyperedges $h_{i,j}^2, h_{i,j}^3, h_{i,j}^5$ must be O_3 -overlaid
 292 by the edges $b_i^j \beta_i^j, a_i^j b_i^j, b_i^j a_i^{j+1}$, respectively, contradicting the fact that b_i^j has

293 degree 2 in G .
 294 Similarly, if $\zeta_i^j z_i^{j+1} \in E(G)$, then the hyperedges $h_{i,j}^5, h_{i,j+1}^1, h_{i,j+1}^3$ must be O_3 -
 295 overlaid by the edges $b_i^j a_i^{j+1}, a_i^{j+1} \alpha_i^j, a_i^{j+1} b_i^{j+1}$, respectively, a contradiction.

296 2. By 1. the hyperedge $h_{i,j}^4$ is O_3 -overlaid by $z_i^j b_i^j$ or $\zeta_i^j b_i^j$.

297 In the first case $z_i^j b_i^j \in E(G)$, since z_i^j is a handicap vertex, the hyperedge
 298 $h_{i,j-1}^6$ must be O_3 -overlaid by $\zeta_i^{j-1} a_i^j$, and so hyperedge $h_{i,j-1}^3$ is O_3 -overlaid
 299 by $z_i^{j-1} b_i^{j-1}$. And by induction, we get that $\bigcup_{j=1}^m \{\zeta_i^j a_i^{j+1}, z_i^j b_i^j\} \subseteq E(G)$.
 300 Since the z_i^j are handicap vertices, the hyperedge $h_{i,j}^1$ is O_3 -overlaid by $a_i^j \alpha_i^j$.
 301 Hence $F_i \subseteq E(G)$.

302 In the second case $\zeta_i^j b_i^j \in E(G)$, a symmetrical argument shows $T_i \subseteq$
 303 $E(G)$. \triangleleft

304 We define the truth assignment ϕ by $\phi(x_i) = true$ if $T_i \subseteq E(G)$ and
 305 $\phi(x_i) = false$ if $F_i \subseteq E(G)$. This is well-defined by Claim ?? and the fact
 306 that $E(G)$ cannot contain both T_i and F_i because it has maximum degree 2.

307 Now, for any $j \in [m]$, there exists $p \in \{1, 2, 3\}$ such that c_j^p has degree 2
 308 in $G[CG_j]$ (If not, then there are at most 2 edges. Hence there is a vertex
 309 c_j^p of degree 0, and thus there is a hyperedge containing c_j^p which is not
 310 overlaid). Therefore, it has degree 0 in the variable gadget c_j^p belongs to. By
 311 our definition of ϕ , we get that $\phi(\ell_p) = true$. Hence ϕ satisfies Φ .

312 Suppose now that $\phi : \{x_1, \dots, x_n\} \rightarrow \{true, false\}$ is a truth assignment
 313 satisfying Φ .

314 Let us first select a set of edges in the variable gadgets. For every $i \in [n]$, if
 315 $\phi(x_i) = true$, then select edges $\{z_i^j, a_i^j\}, \{b_i^j, \beta_i^j\}$ and $\{\zeta_i^j, b_i^j\}$ for all $j \in [p_i]$.
 316 If $\phi(x_i) = false$, then select edges $\{z_i^j, b_i^j\}, \{a_i^j, \alpha_i^j\}$ and $\{\zeta_i^j, a_i^{j+1}\}$ for all
 317 $j \in [p_i]$ (additions are modulo p_i). Let E_1 be the set of selected edges, and
 318 $G_1 = (V(H), E_1)$. Observe that G_1 O_3 -overlays the hyperedges of all variable
 319 gadgets: for every $i \in [n]$ and every $j \in [p_i]$,

- 320 • $h_{i,j}^1$ contains the edge $\{z_i^j, a_i^j\}$ if $\phi(x_i) = true$, and the edge $\{a_i^j, \alpha_i^j\}$
 321 otherwise.
- 322 • $h_{i,j}^2$ contains the edge $\{b_i^j, \beta_i^j\}$ if $\phi(x_i) = true$, and the edge $\{z_i^j, b_i^j\}$
 323 otherwise.
- 324 • $h_{i,j}^3$ contains the edge $\{z_i^j, a_i^j\}$ if $\phi(x_i) = true$, and the edge $\{z_i^j, b_i^j\}$
 325 otherwise.

- 326 • $h_{i,j}^4$ contains the edge $\{\zeta_i^j, b_i^j\}$ if $\phi(x_i) = true$, and the edge $\{z_i^j, b_i^j\}$
327 otherwise.
- 328 • $h_{i,j}^5$ contains the edge $\{\zeta_i^j, b_i^j\}$ if $\phi(x_i) = true$, and the edge $\{\zeta_i^j, a_i^{j+1}\}$
329 otherwise.
- 330 • $h_{i,j}^6$ contains the edge $\{z_i^{j+1}, a_i^{j+1}\}$ if $\phi(x_i) = true$, and the edge $\{\zeta_i^j, a_i^{j+1}\}$
331 otherwise.

332 Moreover the vertices $z_i^1, \dots, z_i^{p_i}$, and $\zeta_i^1, \dots, \zeta_i^{p_i}$ have degree 1 in G_1 and the
333 vertices α_i^j have degree 0 in G_1 if and only if $\phi(x_i) = true$ and the vertices
334 β_i^j have degree 0 in G_1 if and only if $\phi(x_i) = false$.

335 We now select a set of edges in the clause gadgets for every clause $C_j =$
336 $\ell_1 \vee \ell_2 \vee \ell_3$. Since ϕ satisfies Φ , there exists $p = p_j \in \{1, 2, 3\}$ such that
337 $\phi(\ell_p) = true$. Select the edges $\{c_j^p, c_j^{p+2}\}$, $\{c_j^p, c_j^{p+3}\}$ and $\{c_j^{p+1}, c_j^{p+4}\}$. Let
338 E_2 be the set of selected edges in clause gadgets, and $G_2 = (V(H), E_2)$.
339 Clearly, G_2 O_3 -overlays all hyperedges of the clause gadgets and the vertices
340 c_j^4 and c_j^5 , $j \in [m]$, have degree 1 in G_2 . Moreover, for all $j \in [m]$ the vertex
341 $c_j^{p_j}$ is identified with the vertex α_i^j or β_i^j , which has degree 0 in G_1 since
342 $\phi(\ell_p) = true$. Thus $G := (V(H), E_1 \cup E_2)$ has maximum degree 2, which
343 proves that it is an $(O_3, H, 2)$ -graph. \square

344 **Lemma 1.** *If $(\Delta \leq k)$ - O_p -OVERLAY is \mathcal{NP} -complete, then $(\Delta \leq k)$ - O_{p+1} -
345 OVERLAY and $(\Delta \leq k + 1)$ - O_p -OVERLAY are \mathcal{NP} -complete.*

346 Let k and p be two positive integers with $p \geq 2$, let $H_{k,p}$ be the p -uniform
347 hypergraph on $(k+1)(p-1)$ vertices whose hyperedge set contains all possible
348 p -subsets of vertices.

349 We first need the following claim in order to prove Lemma ??.

350 **Claim 4.** *Every $(O_p, H_{k,p}, k)$ -graph is k -regular. Moreover, there exists an
351 $(O_p, H_{k,p}, k)$ -graph.*

352 *Proof of claim:* Let G be an $(O_p, H_{k,p}, k)$ -graph. Assume for a contradic-
353 tion that it has a vertex v of degree at most $k - 1$. Then, observe that
354 $|V(G) \setminus N[v]| \geq (k+1)(p-2) + 1$. Now, by removing successively the closed
355 neighbourhood of a vertex of the graph (chosen arbitrarily) until the graph
356 is anticomplete, we can construct, together with v , an independent set S of
357 G of size p . This is impossible, since S is a hyperedge of $H_{k,p}$ which is not
358 O_p -overlaid by G .

359 Finally, observe that the disjoint union of $p - 1$ cliques of size $k + 1$ is a
 360 k -regular graph which O_p -overlays $H_{k,p}$. ◁

361 *Proof of Lemma ??.* First, we will reduce from $(\Delta \leq k)$ - O_p -OVERLAY to
 362 $(\Delta \leq k)$ - O_{p+1} -OVERLAY. Let H be a p -uniform hypergraph. We construct
 363 a $(p + 1)$ -uniform hypergraph H^* from a copy of H and replace, for every
 364 hyperedge S , a hyperedge $S' = S \cup \{v_s\}$. We also add, for every hyperedge
 365 S , a set of $(k + 1)p - 1$ new vertices N_S , and form, together with v_S , a
 366 hypergraph H_S isomorphic to $H_{k,p+1}$.

367 Let G be an (O_p, H, k) -graph. Construct G^* from G by adding, for every
 368 hyperedge S of H , a disjoint union of p cliques of size $k + 1$ on the vertices
 369 $N_S \cup \{v_S\}$. Clearly, G^* is an (O_{p+1}, H^*, k) -graph.

370 Conversely, if G^* is an (O_{p+1}, H^*, k) -graph, then, by Claim ??, v_S must
 371 have degree k in N_S for every hyperedge S of H . Thus, the restriction of G^*
 372 to the vertices of H is an (O_p, H, k) -graph.

373 Now, we reduce $(\Delta \leq k)$ - O_p -OVERLAY to $(\Delta \leq k + 1)$ - O_p -OVERLAY.

374 Let H be a p -uniform hypergraph. Let H' be the p -uniform hypergraph
 375 obtained from H by doing the following for each vertex $v \in V(H)$:

- 376 • add the vertices $x^v, y_1^v, \dots, y_{p-2}^v$ and the hyperedge $S_v = \{v, x^v, y_1^v, \dots, y_{p-2}^v\}$.
- 377 • for any $i = \{1, \dots, p - 2\}$, add a copy H_i^v of $H_{k+1,p}$ whose vertices are
 378 y_i^v and $(p - 1)(k + 2) - 1$ new vertices.

379 Assume that G' is an $(O_p, H', k + 1)$ -graph. By Claim ??, the graph
 380 induced by G' on $V(H_i^v)$ is $(k + 1)$ -regular, for all $v \in V(H)$, $1 \leq i \leq p - 2$.
 381 Therefore each y_i^v has $k + 1$ -neighbours outside the hyperedge S_v . Hence
 382 G' must contain the edge vx^v because it O_p -overlays S_v . Thus the subgraph
 383 $G = G'[V(H)]$ has maximum degree at most $\Delta(G') - 1 \leq k$ and G O_p -overlays
 384 H . In other words, G is an (O_p, H, k) -graph.

385 Conversely, assume G is an (O_p, H, k) -graph. Let G' be the graph with
 386 vertex set $V(H')$ and whose edge set is the union of $E(G)$, $\{vx^v \mid v \in V(H)\}$,
 387 and the edge set of an $(O_p, H_i^v, k + 1)$ -graph (which exists by Claim ??), for all
 388 $v \in V(H)$, $1 \leq i \leq p - 2$. One easily checks that G' is $(O_p, H', k + 1)$ -graph.

389 Therefore, there is an (O_p, H, k) -graph if and only if there is an
 390 $(O_p, H', k + 1)$ -graph. ◻

392 Propositions ?? and ??, Theorems ?? and ??, and Lemma ?? imply
 393 Theorem ??.

394 **3. Complexity of Max ($\Delta \leq k$)- F -Overlay**

395 The aim of this section is to establish Theorem ?? that gives the \mathcal{P}/\mathcal{NP} -
396 complete dichotomy for the MAX ($\Delta \leq k$)- F -OVERLAY problems.

397 As noticed in the introduction, if $\Delta(F) > k$ or F is an anticomplete graph
398 then MAX ($\Delta \leq k$)- \mathcal{F} -OVERLAY is trivially polynomial-time solvable.

399 Moreover, by Propositions ?? and ??, MAX ($\Delta \leq 1$)- O_3 -OVERLAY as well as
400 MAX ($\Delta \leq k$)- O_2 -OVERLAY (for all positive integers k) are also polynomial-
401 time solvable.

402 We shall now prove that if we are not in one of the above cases, then
403 MAX ($\Delta \leq k$)- F -OVERLAY is \mathcal{NP} -complete. We first establish the \mathcal{NP} -
404 completeness when F has no isolated vertices.

405 **Theorem 5.** *Let F be a graph on at least three vertices with no isolated*
406 *vertices. If $k \geq \Delta(F)$, then MAX ($\Delta \leq k$)- F -OVERLAY is \mathcal{NP} -complete on*
407 *neat hypergraphs.*

408 *Proof.* Assume $k \geq \Delta(F)$. Let $n = |F|$, a_1, \dots, a_n be an ordering of the
409 vertices of F such that $\delta(F) = d(a_1) \leq d(a_2) \leq \dots \leq d(a_n) = \Delta(F)$.

410 Let $\gamma = \lfloor k/\delta(F) \rfloor - 1$, $\beta = k - \gamma\delta(F)$. Observe that $\delta(F) \leq \beta \leq 2\delta(F) - 1$.

411 We shall give a reduction from INDEPENDENT SET which is a well-known
412 \mathcal{NP} -complete problem even for cubic graphs (see [?].) We distinguish two
413 cases depending on whether $d(a_2) > \beta$ or not.

414 Let $\gamma_2 = \lfloor (k - d(a_2))/\delta(F) \rfloor$ and $\gamma_3 = \lfloor (k - d(a_3))/\delta(F) \rfloor$. And we define
415 the value γ_2 as follows.

- 416 - If $d(a_2) > \beta$, let $\gamma_1 = \gamma_2$.
- 417 - If $d(a_2) \leq \beta$, let $\gamma_1 = \lfloor (k - d(a_1))/\delta(F) \rfloor$.

418 Let Γ be a cubic graph. For each vertex $v \in V(\Gamma)$, let $(e_1(v), e_2(v), e_3(v))$
419 be an ordering of the edges incident to v . We shall construct the neat hyper-
420 graph $H = H(\Gamma)$ as follows.

- 421 • For each vertex $v \in \Gamma$, we create a hyperedge $S_v = \{a_1^v, \dots, a_n^v\}$. Then,
422 for $1 \leq i \leq 3$, we add γ_i a_i^v -leaves, that are hyperedges containing a_i^v
423 and $n - 1$ new vertices.
- 424 • For each edge $e = uv \in \Gamma$, let i and j be the indices such that $e =$
425 $e_i(u) = e_j(v)$. We create a new vertex z_e and hyperedges S_u^e (S_v^e)
426 containing z_e , a_i^u (a_j^v), and $n - 2$ new vertices, respectively. Then,

427 we add γ z_e -leaves, that are hyperedges containing z_e and $n - 1$ new
 428 vertices.

429 From the construction, observe that γ_i will be useful to enforce the degree
 430 of vertex a_i^v for $v \in V(G)$ which could have degree at most $d(a_3)$ in the
 431 hypergraph.

432 We shall prove that $over_F(H, k) = (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$,
 433 where $\alpha(\Gamma)$ denotes the cardinality of a maximum independent set in Γ .

The following claim shows that there are optimal solutions with specific structure. This leads to the inequality:

$$over_F(H, k) \leq (\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$$

434 **Claim 5.** *There is a graph G with $\Delta(G) \leq k$ that F -overlays $over_F(H, k)$
 435 hyperedges of H such that:*

436 (a) *each x -leaf L is F -overlaid and x is incident to $\delta(F)$ edges in $G[L]$
 437 (with $x = a_i^v$ or $x = z_e$).*

438 (b) *for each edge $e = uv \in E(\Gamma)$, exactly one of the two hyperedges S_u^e and
 439 S_v^e is F -overlaid. Moreover if S_u^e (S_v^e) is F -overlaid, then a_i^u (a_j^v) is
 440 incident to at most $d(a_2)$ edges in S_u^e (S_v^e), respectively.*

441 (c) *the set of vertices v such that S_v is F -overlaid is an independent set
 442 in Γ .*

443 *Proof of claim:* Let G be a graph with $\Delta(G) \leq k$ that F -overlays $over_F(H, k)$
 444 hyperedges of H . Observe first that since H is a neat hypergraph, we may
 445 assume that, for every hyperedge S of H , $G[S] = F$ if S is F -overlaid and
 446 $G[S]$ has no edge otherwise.

447 Therefore if an x -leaf L is F -overlaid, then replacing $G[L]$ by a copy of F
 448 in which x has degree $\delta(F)$, we obtain another graph with maximum degree
 449 at most k that F -overlays $over_F(H, k)$ hyperedges of H . Henceforth we may
 450 assume that if an x -leaf is F -overlaid then x is incident to $\delta(F)$ edges included
 451 in it.

452 Now assume that the z_e -leaf L , for $e \in E(\Gamma)$, is not F -overlaid by G . We
 453 have $d_G(z_e) > k - \delta(F)$ for otherwise adding on L the edges of a copy of F
 454 in which z_e has degree $\delta(F)$, we obtain a graph F -overlying more hyperedges
 455 than G . But since there are γ z_e -leaves and z_e is incident to $\delta(F)$ edges in
 456 each of them, at least one of the hyperedges S_u^e and S_v^e is F -overlaid. Now

457 replacing the edges of G in this hyperedge by those of a copy of F in L , in
 458 which z_e has degree $\delta(F)$, we obtain another graph with maximum degree
 459 at most k that F -overlays $over_F(H, k)$ hyperedges of H . Henceforth we may
 460 assume that every z_e -leaf is F -overlaid.

461 A similar reasoning shows that every a_i^v -leaf is F -overlaid, for all $v \in V(\Gamma)$,
 462 $i \in \{1, 2, 3\}$. This proves (a).

463 Now by (a) each vertex z_e is incident to $\gamma\delta(F) = k - \beta > k - 2\delta(F)$ edges
 464 in the z_e -leaves. Thus z_e is incident to at most β edges in $S_u^e \cup S_v^e$. Since
 465 $\beta \leq 2\delta(F) - 1$, at most one of S_u^e and S_v^e is F -overlaid.

466 Moreover, if S_u^e is F -overlaid and $\beta < d(a_2)$ (resp. $d(a_2) \leq \beta$), then z_e must
 467 have degree $d(a_1)$ (resp. at most $d(a_2)$) in $G[S_u^e]$.

468 Assume now that none of S_u^e, S_v^e is F -overlaid, and let $e = e_i(u)$, a_i^u has γ_i
 469 a_i^u -leaves, so a_i^u is incident to $\gamma_i\delta(F) \leq k - d(a_i)$ edges in these leaves.

470 Thus, replacing the edges of G in S_u^e by those of a copy of F , in which

471 - if $d(a_2) > \beta$, we have $d_G(a_i^u) \leq k - d(a_2)$, then z_e has degree $\delta(F)$ and
 472 a_i^u has degree $d(a_2)$.

473 - if $d(a_2) \leq \beta$, since $d_G(a_1^u) \leq k - d(a_1)$, then for $i = 1$, a_i^u must have
 474 degree $d(a_1)$, so z_e has degree $d(a_2)$. And for $i = \{2, 3\}$, let $d(z_e) =$
 475 $d(a_1)$, $d(a_i^u) = d(a_2)$.

476 Hence, we obtain another graph with maximum degree at most k that F -
 477 overlays $over_F(H, k)$ hyperedges of H . Henceforth, we may assume that at
 478 least one of S_u^e and S_v^e is F -overlaid. This completes the proof of (b).

479 Consider now a vertex $v \in V(\Gamma)$ such that S_v is F -overlaid. We distin-
 480 guish the following cases.

481 - If $d(a_2) > \beta$, then by (a), vertex a_1^v is incident to $\gamma_1\delta(F)$ edges in the a_1^v -
 482 leaves and at least $\delta(F) \in S_v$. Therefore $S_v^{e_1(v)}$ cannot be F -overlaid for
 483 otherwise, by (b), a_1^v would have degree at least $\gamma_1\delta(F) + \delta(F) + d(a_2) >$
 484 k . Similarly, $S_v^{e_2(v)}$ cannot be F -overlaid.

485 Assume now for a contradiction that $S_v^{e_3(v)}$ is F -overlaid. By (a) and
 486 (b), a_3^v is incident to $\gamma_3\delta(F)$ edges in the a_3^v -leaves and at least $d(a_2) \in$
 487 $S_v^{e_3(v)}$. Since $\gamma_3\delta(F) + d(a_2) + d(a_3) > k$, a_3^v is incident to less than
 488 $d(a_3)$ edges in S_v . Thus, there is a vertex $a \in \{a_1^v, a_2^v\}$ which is incident
 489 to at least $d(a_3)$ edges in S_v and thus at least $\gamma_1\delta(F) + d(a_3)$ edges in
 490 G . But a_3^v is adjacent to at least $\delta(F)$ edges in S_v so at least $\gamma_3\delta(F) +$

491 $d(a_2) + \delta(F)$ edges in total. Thus $\gamma_3\delta(F) + d(a_2) + \delta(F) \leq k$, so
 492 $\gamma_3 \leq \gamma_1 - 1$. Thus a is adjacent to at least $(\gamma_3 + 1)\delta(F) + d(a_3) > k$
 493 edges, a contradiction.

494 - If $d(a_2) \leq \beta$, then by (a), vertex a_1^v is incident to $\gamma_1\delta(F)$ edges in
 495 the a_1^v -leaves and at least $\delta(F) \in S_v$. If $S_v^{e_1(v)}$ is F -overlaid, then
 496 (b) yields $d(a_i^v) = \gamma_i\delta(F) + \delta(F) + \delta(F) > k$, a contradiction. If
 497 $S_v^{e_2(v)}$ is F -overlaid, then $d(a_2^v) = \delta$ in $G[S_v]$. We obtain that $d(a_2^v) =$
 498 $\gamma_2\delta + d(a_2) + \delta \leq k$ which implies that $\gamma_2 \leq \gamma_2 - 1$, a contradiction.
 499 If $S_v^{e_3(v)}$ is F -overlaid, then we get a contradiction in the same way as
 500 that in the case $d(a_2) > \beta$.

501 To summarize, if S_v is F -overlaid, then none of $S_v^{e_1(v)}$, $S_v^{e_2(v)}$, $S_v^{e_3(v)}$ is
 502 F -overlaid. Together with (b), this implies (c). \triangleleft

503 Conversely, consider W a maximum independent set of Γ .

504 Let G be the graph with vertex $V(H)$ which is the union of the following
 505 subgraphs :

- 506 • for each x -leaf L , we add a copy of F on L in which x has degree $\delta(F)$;
- 507 • for each vertex $v \in W$, we add a copy of F on S_v in which a_i^v has degree
 508 $d(a_i)$ for all $1 \leq i \leq n$.
- 509 • for each edge $e \in E(\Gamma)$, we choose an endvertex u of e such that $u \notin W$,
 510 and add a copy of F in which z_e has degree $d(a_1)$ and a_i^u has degree
 511 $d(a_2)$ (with i the index such that $e_i(u) = e$) except the case when
 512 $d(a_2) \leq \beta$ and $i = 1$, we do that but $d(z_e) = d(a_2)$ and $d(a_1^u) = d(a_1)$
 513 in $G[S_u^{e_1(u)}]$.

514 It is simple matter to check that $\Delta(G) \leq k$ and that G F -overlays
 515 $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$ hyperedges of H . Thus $over_F(H, k) \geq$
 516 $(\gamma_1 + \gamma_2 + \gamma_3)|V(\Gamma)| + (\gamma + 1)|E(\Gamma)| + \alpha(\Gamma)$.

517 \square

518 We then establish the following lemma, which allows to derive the \mathcal{NP} -
 519 completeness of $\text{MAX } (\Delta \leq k)\text{-}F\text{-OVERLAY}$ when F has isolated vertices.

520 **Lemma 2.** *Let k be a positive integer, let F be a graph with $\delta(F) \geq 1$, and*
 521 *let q be a non-negative integer. If $\text{MAX } (\Delta \leq k)\text{-}(F + \overline{K}_q)\text{-OVERLAY}$ is \mathcal{NP} -*
 522 *complete, then $\text{MAX } (\Delta \leq k)\text{-}(F + \overline{K}_{q+1})\text{-OVERLAY}$ is also \mathcal{NP} -complete.*

523 *Proof.* We shall reduce MAX $(\Delta \leq k)$ - $(F + \overline{K}_q)$ -OVERLAY to MAX $(\Delta \leq$
524 $k)$ - $(F + \overline{K}_{q+1})$ -OVERLAY. Set $p = |V(F)|$ and $r = \lfloor k/\delta(F) \rfloor$. Note that
525 $r \geq 1$, for otherwise MAX $(\Delta \leq k)$ - $(F + \overline{K}_q)$ -OVERLAY would trivially be
526 polynomial-time solvable.

527 Let H' be a $(p+q)$ -uniform hypergraph. Let H be the $(p+q+1)$ -uniform
528 hypergraph obtained from H' as follows. Its vertex set is partitioned into
529 $V(H')$, P_1, \dots, P_r , K , and $\{z\}$ with $|P_i| = p - 1$ for all $1 \leq i \leq r$ and
530 $|K| = (m+1)(k+1) + kp$ with $m = \max\{|E(H')|, q+1\}$. The hyperedge set
531 of H contains $Q \cup P_i \cup \{z\}$ for all $(q+1)$ -subsets Q of K and all $1 \leq i \leq r$,
532 and $S \cup \{z\}$ for all $S \in E(H')$.

533 The reduction can be performed in polynomial time, because for each
534 hyperedge in H' we construct a polynomial number of hyperedges in H (recall
535 that q is fixed), and each of them is straightforward to obtain. We shall now
536 prove that $over_{F+\overline{K}_{q+1}}(H, k) = over_{F+\overline{K}_q}(H', k) + r \binom{|K|}{q+1}$.

537 Let G'_0 be a graph with $V(G'_0) = V(H')$ and $\Delta(G'_0) \leq k$ that $(F + \overline{K}_q)$ -
538 overlays $over_{F+\overline{K}_q}(H', k)$ hyperedges of H' . For $1 \leq i \leq r$, let F_i be a copy
539 of F on $P_i \cup \{z\}$ in which z has degree $\delta(F)$. Let G_0 be the graph defined
540 by $V(G_0) = V(H)$ and $E(G_0) \cup \bigcup_{i=1}^r E(F_i)$. Clearly, $\Delta(G_0) \leq k$ and G_0 F -
541 overlays all hyperedges of H except hyperedges $S \cup \{z\}$ in which $S \in E(H')$
542 are not overlaid by G'_0 . So $over_{F+\overline{K}_{q+1}}(H, k) \geq over_{F+\overline{K}_q}(H', k) + r \binom{|K|}{q+1}$.

543 Conversely, let G be a graph with $V(G) = V(H)$ and $\Delta(G) \leq k$ that
544 maximizes the number of $(F + \overline{K}_{q+1})$ -overlaid hyperedges.

545 **Claim 6.** *In G , vertex z has at least $\delta(F)$ neighbours in each P_i , $1 \leq i \leq r$.*

546 *Proof of claim:* Suppose for a contradiction that z has less than $\delta(F)$ neigh-
547 bours in P_i . Since $\Delta(G) \leq k$, at most kp vertices of K are incident to an edge
548 with an endvertex in $P_i \cup \{z\}$. Thus there is a subset R of K of $(m+1)(k+1)$
549 vertices non-adjacent to all vertices of $P_i \cup \{z\}$. Now $G[R]$ has maximum
550 degree k and is thus $(k+1)$ -colourable. Hence $G[R]$ has an independent set
551 S of size $m+1$. Observe that for each $(q+1)$ -subset Q of S , the subgraph
552 $G[Q \cup P_i \cup \{z\}]$ has $q+2$ vertices of degree less than $\delta(F)$ (namely the ones of
553 $Q \cup \{z\}$), and so the hyperedge $Q \cup P_i \cup \{z\}$ is not $(F + \overline{K}_{q+1})$ -overlaid. Thus
554 there are at least $m+1 \geq |E(H')| + 1$ hyperedges not $(F + \overline{K}_{q+1})$ -overlaid
555 by G . This contradicts the maximality of G as G_0 overlays more hyperedges
556 of H . \triangleleft

557 Now z has degree at least $r\delta(F)$ in $\bigcup_{i=1}^r P_i$ and so less than $\delta(F)$ in each
558 hyperedge $S \cup \{z\}$ for $S \in E(H')$. Hence if G $(F + \overline{K}_{q+1})$ -overlays $S \cup z$, it

559 must also $(F + \overline{K}_q)$ -overlay S . Thus $over_{F+\overline{K}_{q+1}}(H, k) \leq over_{F+\overline{K}_q}(H', k) +$
 560 $r\binom{|K|}{q+1}$. \square

561 Now we can prove Theorem ???. As explained in the beginning of the
 562 section, it suffices to prove that MAX $(\Delta \leq k)$ - F -OVERLAY remains \mathcal{NP} -
 563 complete when $\Delta(F) \leq k$, $F \neq \overline{K}_{|F|}$, $|F| \geq 3$ and $(F, k) \neq (O_3, 1)$. Assume
 564 that the above conditions are satisfied. Let F' be the graph induced by the
 565 non-isolated vertices of F . Then $F = F' + \overline{K}_q$ with $q = |F| - |F'|$. If $|F'| = 2$,
 566 then $F = O_{|F|}$, and we have the result by Theorem ??. If $|F'| \geq 3$, then the
 567 result follows from Theorem ??, Lemma ??, and an immediate induction.

568 4. Complexity of $(\Delta \leq k)$ - F -Overlay and $(\Delta \leq k)$ - \mathcal{F} -Overlay

569 In this section, we investigate the complexity of $(\Delta \leq k)$ - F -OVERLAY
 570 and $(\Delta \leq k)$ - \mathcal{F} -OVERLAY problems. We believe that each such prob-
 571 lem is either polynomial-time solvable or \mathcal{NP} -complete. However the di-
 572 chotomy is more complicated than the one we proved in Theorem ?? for
 573 MAX $(\Delta \leq k)$ - F -OVERLAY. This dichotomy is simple because there are
 574 very few pairs (F, H) for which MAX $(\Delta \leq k)$ - F -OVERLAY is polynomial-
 575 time solvable. We now exhibit several pairs (F, k) such that $(\Delta \leq k)$ - F -
 576 OVERLAY is polynomial-time solvable, while MAX $(\Delta \leq k)$ - F -OVERLAY is
 577 \mathcal{NP} -complete.

578 4.1. Regular graphs

579 **Proposition 3.** *For every complete graph K and every positive integer k ,*
 580 *$(\Delta \leq k)$ - K -OVERLAY is polynomial-time solvable.*

581 *Proof.* Observe that a $|V(K)|$ -uniform hypergraph H is a positive instance of
 582 of $(\Delta \leq k)$ - K -OVERLAY if and only if $K(H)$ is a (K, H, k) -graph. \square

583 **Proposition 4.** *For every connected k -regular graph F , $(\Delta \leq k)$ - F -OVERLAY*
 584 *is polynomial-time solvable.*

585 *Proof.* One easily sees that a $|V(F)|$ -uniform hypergraph H admits an (F, H, k) -
 586 graph if and only if the hyperedges of H are pairwise non-intersecting. \square

587 Let C_4 denote the cycle on 4 vertices. Proposition ?? implies that $(\Delta \leq$
 588 $2)$ - C_4 -OVERLAY is polynomial-time solvable. We now show that $(\Delta \leq 3)$ -
 589 C_4 -OVERLAY is also polynomial-time solvable.

590 **Theorem 6.** $(\Delta \leq 3)$ - C_4 -OVERLAY is polynomial-time solvable.

591 *Proof.* Let H be a 4-uniform hypergraph.

592 Let us describe an algorithm to decide whether there is a $(C_4, H, 3)$ -graph.
 593 It is sufficient to do it when H is connected since the disjoint union of the
 594 $(C_4, K, 3)$ -graphs for connected components K of H is a $(C_4, H, 3)$ -graph.

595 Observe first that if two hyperedges of H intersect in exactly one vertex
 596 u , then no such graph exists, since u must have degree 2 in each of the
 597 hyperedges if they are C_4 -overlaid, and thus degree 4 in total. Therefore if
 598 there are two such hyperedges, we return ‘No’. At this point we may assume
 599 that $|E(H)| \geq 2$ for otherwise we return ‘Yes’.

600 From now on we may assume that two hyperedges either do not intersect,
 601 or are adjacent (intersect on at least two vertices).

602 **Claim 7.** *If two hyperedges S_1 and S_2 intersect on three vertices and there*
 603 *is a $(C_4, H, 3)$ -graph G , then $|V(H)| \leq 6$.*

604 *Proof of claim:* Assume $S_1 = \{a_1, b, c, d\}$ and $S_2 = \{a_2, b, c, d\}$. Let G be a
 605 $(C_4, H, 3)$ -graph. Observe that a_1 and a_2 must have two common neighbours,
 606 say b, c , in common in $\{b, c, d\}$, for otherwise their unique common vertex
 607 would have degree 4 in G . This forces d to be also adjacent to b and c .

608 Consider a hyperedge S_3 intersecting $S_1 \cup S_2$. Since it is C_4 -overlaid by G ,
 609 at least two edges connect $S_3 \cap (S_1 \cup S_2)$ to $S_3 \setminus (S_1 \cup S_2)$. The endvertices of
 610 those edges in $S_1 \cup S_2$ must have degree 2 in $G[S_1 \cup S_2]$. Hence, without loss of
 611 generality, there is a vertex e not in $S_1 \cup S_2$ such that either $S_3 = \{a_1, a_2, b, e\}$
 612 and $\{a_1e, a_2e\} \subseteq E(G)$, or $S_3 = \{a_1, b, d, e\}$ and $\{a_1e, de\} \subseteq E(G)$. Now
 613 no hyperedge can both intersect $S_1 \cup S_2 \cup S_3$ and contain a vertex not in
 614 $S_1 \cup S_2 \cup S_3$, for such a hyperedge must contain either the vertices d, e or
 615 a_2, e which are at distance 3 in $G[S_1 \cup S_2 \cup S_3]$. (However there can be more
 616 hyperedges contained in $S_1 \cup S_2 \cup S_3$.) Hence $|V(H)| \leq 6$. \triangleleft

617 In view of Claim ??, if there are two hyperedges with three vertices in
 618 common, either we return ‘No’ if $|V(H)| > 6$, or we check all possibilities (or
 619 follow the proof of the above claim) to return the correct answer otherwise.
 620 Henceforth, we may assume that any two adjacent hyperedges intersect in
 621 exactly two vertices.

622 Let S_1 and S_2 be two adjacent hyperedges, say $S_1 = \{a, b, c, d\}$ and
 623 $S_2 = \{c, d, e, f\}$. Note that every $(C_4, H, 3)$ -graph contains the edges ab ,
 624 cd and ef , and that $N(c) \cup N(d) = S_1 \cup S_2$ in this $(C_4, H, 3)$ -graph.

625 **Claim 8.** *If there is another hyperedge than S_1 and S_2 containing c or d ,*
626 *and there is a $(C_4, H, 3)$ -graph G , then $|V(H)| \leq 8$.*

627 *Proof of claim:* Without loss of generality, we may assume that G contains
628 the cycle (a, b, d, f, e, c, a) and the edge cd . Hence the only possible hyper-
629 edges containing c or d and a vertex not in $S_1 \cup S_2$ are $S_3 = \{a, c, e, g\}$ for
630 some $g \notin S_1 \cup S_2$ and $S_4 = \{b, d, f, h\}$ for some $h \notin S_1 \cup S_2$.

631 If H contains both S_3 and S_4 , then G contains the edges ag, eg, bh
632 and hf . If G contains also gh , then $G[S_1 \cup S_2 \cup S_3 \cup S_4]$ is 3-regular, so
633 $G = G[S_1 \cup S_2 \cup S_3 \cup S_4]$. If G does not contain gh , then the only vertices
634 of degree 2 in $G[S_1 \cup S_2 \cup S_3 \cup S_4]$ are g and h , and they are at distance at
635 least 3 in this graph. Thus every hyperedge intersecting $S_1 \cup S_2 \cup S_3 \cup S_4$ is
636 contained in this set, so $|V(H)| = 8$.

637 Assume now that H is a superset of only one of S_3, S_4 . Without loss
638 of generality, we may assume that this is S_3 . Hence G also contains the
639 edges ag and eg . If $V(G) \neq S_1 \cup S_2 \cup S_3$, then there is a hyperedge S that
640 intersects $S_1 \cup S_2 \cup S_3$ and that is not contained in $S_1 \cup S_2 \cup S_3$. It does not
641 contain c and d . Hence it must contain one of the vertices a or e , because it
642 intersects each S_i along an edge of G or not at all. Without loss of generality,
643 $a \in S$. Hence $S = \{a, b, i, g\}$ for some vertex i not in $S_1 \cup S_2 \cup S_3$, and G
644 contains the edges bi and ig . Now, as previously, either i and f are adjacent
645 and $G = G[S_1 \cup S_2 \cup S_3 \cup S]$ or they are not adjacent, and every hyperedge
646 intersecting $S_1 \cup S_2 \cup S_3 \cup S$ is contained in this set. In both cases, $|V(H)| = 8$.

647 ◁

648 We now summarize the algorithm: if $|V(G)| \leq 8$, then we solve the
649 instance by brute force. Otherwise, for every pair of hyperedges S_1, S_2 , if
650 their intersection is of size 1 or 3, we answer ‘No’. In the remaining cases,
651 if S_1 and S_2 have non-empty intersection, then, they must intersect on two
652 vertices c and d , and these vertices do not belong to any other hyperedges
653 but S_1 and S_2 . Hence, if a vertex is in at least 3 hyperedges, then we return
654 ‘No’. If not, then every hyperedge intersects at most two hyperedges on two
655 disjoint pairs of vertices. We can then return the $(C_4, H, 3)$ -graph obtained as
656 follows. If a hyperedge $S = \{a, b, c, d\}$ intersects either two other hyperedges
657 S_1 and S_2 is $\{a, b\}$ and $\{c, d\}$, or exactly one hyperedge on $\{a, b\}$, then add
658 the edges ab, bc, cd, da to $E(G)$.

659 Clearly, the above-described algorithm runs in polynomial time. ◻

660 4.2. Paths

661 Let \mathcal{P} be the set of all paths. We have the following:

662 **Theorem 7.** $(\Delta \leq 2)$ - \mathcal{P} -OVERLAY is linear-time solvable.

663 *Proof.* Clearly, if H is not connected, it suffices to solve the problem on each
 664 of the components and to return ‘No’ if the answer is negative for at least one
 665 of the components, and ‘Yes’ otherwise. Henceforth, we shall now assume
 666 that H is connected. In such a case, a $(\mathcal{P}, H, 2)$ -graph is either a path or a
 667 cycle. However, if H is \mathcal{P} -overlaid by a path P , then it is also \mathcal{P} -overlaid by
 668 the cycle obtained from P by adding an edge between its two endvertices.
 669 Thus, we focus on the case where G is a cycle.

670 Let \mathcal{S} be a family of sets. The *intersection graph* of a family \mathcal{S} is the
 671 graph $IG(\mathcal{S})$ whose vertices are the sets of \mathcal{S} , and in which two vertices are
 672 adjacent if the corresponding sets in \mathcal{S} intersect.

The *intersection graph* of a hypergraph H , denoted by $IG(H)$, is the
 intersection graph of its hyperedge set. We define two functions l_H and s_H
 as follows:

$$l_H(S) = |S| - 1 \text{ for all } S \in E(H) \text{ and } s_H(S, S') = |S \cap S'| - 1 \text{ for all } S, S' \in E(H).$$

673 Let \mathbb{C}_ℓ be the circle of circumference ℓ . We identify the points of \mathbb{C}_ℓ with
 674 the integer numbers (points) of the segment $[0, \ell]$, (with 0 identified with ℓ).
 675 A *circular-arc graph* is the intersection graph of a set of arcs on \mathbb{C}_ℓ . A set
 676 \mathcal{A} of arcs such that $IG(\mathcal{A}) = G$ is called an *arc representation* of G . We
 677 denote by A_v the arc corresponding to v in \mathcal{A} . Let G be a graph and let
 678 $l : V(G) \rightarrow \mathbb{N}$ and $s : E(G) \rightarrow \mathbb{N}$ be two functions. An arc representation \mathcal{A}
 679 of G is *l-respecting* if A_v has length $l(v)$ for any $v \in V(G)$, *s-respecting* if
 680 $A_v \cap A_u$ has length $s(u, v)$ for all $uv \in E(G)$, and *(l, s)-respecting* if it is both
 681 *l-respecting* and *s-respecting*. One can easily adapt the algorithm given by
 682 Köbler et al. [?] for *(l, s)-respecting* interval representations to decide in
 683 linear time whether a graph admits an *(l, s)-respecting* arc representation in
 684 \mathbb{C}_n for every integer n .

685 **Claim 9.** Let H be a connected hypergraph on n vertices. There is a cy-
 686 cle \mathcal{P} -overlying H if and only if $IG(H)$ admits an (l_H, s_H) -respecting arc
 687 representation into \mathbb{C}_n .

688 *Proof of claim:* Assume that H is \mathcal{P} -overlaid by a cycle $C = (v_0, v_1, \dots, v_{n-1}, v_0)$.
 689 There is a canonical embedding of C to \mathbb{C}_n in which every vertex v_i is mapped

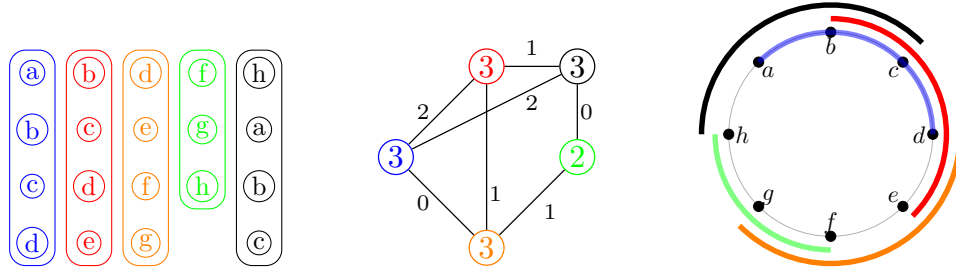


Figure 5: Given a hypergraph H (left), we determine its graph $IG(H)$ (middle) with each vertex represents the a hyperedge of the same color in which the value of any vertex $S \in V(H)$ is $l_H(S)$ and the weight of an edge between S, S' is $s_H(S, S')$. And a circular arc representation (each arc is associated to a hyperedge of the same color) is shown in the right which is (l, s) -respecting.

690 to i and every edge $v_i v_{i+1}$ to the circular arc $[i, i + 1]$. For every hyperedge
 691 $S \in E(H)$, $P[S]$ is a subpath, which is mapped to the circular arc A_S of \mathbb{C}_n
 692 that is the union of the circular arcs to which its edges are mapped. Clearly,
 693 $\mathcal{A} = \{A_S \mid S \in E(H)\}$ is an (l_H, s_H) -respecting interval representation of
 694 $IG(H)$.

695 Conversely, assume that $IG(H)$ admits an (l_H, s_H) -respecting interval
 696 representation $\mathcal{A} = \{A_S \mid S \in E(H)\}$ into \mathbb{C}_n . Let S_0 be a hyperedge
 697 of minimum size. Free to rotate all intervals, we may assume that A_{S_0} is
 698 $[1, |S_0|]$. Now since \mathcal{A} is (l_H, s_H) -respecting and H is connected, we deduce
 699 that the extremities of A_S are integers for all $S \in E(H)$. Let v_1 be a vertex
 700 of H that belongs to the hyperedges whose corresponding arcs of \mathcal{A} contain
 701 1. Then for all $i = 2$ to $n = |V(H)|$, denote by v_i an arbitrary vertex not in
 702 $\{v_1, \dots, v_{i-1}\}$ that belongs to the hyperedges whose corresponding arcs of \mathcal{A}
 703 contain i . Such a vertex exists because \mathcal{A} is (l_H, s_H) -respecting. Observe that
 704 such a construction yields $S = \{v_i \mid i \in A_S\}$ for all $S \in E(H)$. Furthermore,
 705 the cycle $C = (v_1, \dots, v_n, v_1)$ \mathcal{P} -overlays H . Indeed, for each $S \in E(H)$,
 706 $C[S]$ is the subpath corresponding to A_S , that is $V(C[S]) = \{v_i \mid i \in A_S\}$
 707 and $E(C[S]) = \{v_i v_{i+1} \mid [i, i + 1] \subseteq A_S\}$. \triangleleft

708 The algorithm to solve $(\Delta \leq 2)$ - \mathcal{P} -OVERLAY for a connected hypergraph
 709 H in linear time is thus the following:

- 710 1. Construct the intersection graph $IG(H)$ and compute the associated
 711 functions l_H and s_H .

712 2. Check whether graph $IG(H)$ has an (l_H, s_H) -respecting interval repre-
 713 sentation. If it is the case, return ‘Yes’. If not return ‘No’.

714 □

715 **Remark 1.** We can also detect in polynomial time whether a connected
 716 hypergraph H is \mathcal{P} -overlaid by a path. Indeed, similarly to Claim ??, one
 717 can show that there is a path \mathcal{P} -overlying H if and only if $IG(H)$ admits
 718 an (l_H, s_H) -respecting interval representation.

719 *4.3. On neat hypergraphs*

720 Let $F = (V, E)$ be a graph. We denote by V_i the set of vertices having
 721 degree i in F . For every increasing sequence of nonnegative integers $i_1 <$
 722 $i_2 < \dots < i_p$, an (i_1, i_2, \dots, i_p) -graph is a graph F such that $V_i = \emptyset$ if
 723 $i \notin \{i_1, i_2, \dots, i_p\}$. In particular, an (r) -graph is a r -regular graph.

724 Given a hypergraph H , we denote by $d_H(v)$ the number of hyperedges
 725 containing v for any $v \in V(H)$.

726 **Theorem 8.** *Let i_1, i_2, k be three non-negative integers with $i_1 < i_2$. For*
 727 *every (i_1, i_2) -graph F , one can solve $(\Delta \leq k)$ - F -OVERLAY in polynomial*
 728 *time on neat hypergraphs.*

729 *Proof.* Let H be a neat $|F|$ -uniform hypergraph.

730 If there is a vertex v such that $i_1 \cdot d_H(v) > k$, then we answer ‘No’. Indeed
 731 in an (H, F, k) -graph G , the vertex v has degree at least i_1 in each subgraph
 732 of G induced by a hyperedge to which it belongs, and so degree at least
 733 $i_1 \cdot d_H(v)$ in G because H is neat. Hence, we assume that $i_1 \cdot d_H(v) \leq k$ for
 734 every vertex.

735 Let $B(H)$ be the incidence graph of H : its vertex set is $E(H) \cup V(H)$
 736 and there is an edge between a hyperedge $S \in E(H)$ and a vertex $v \in V(H)$
 737 if and only if $v \in S$.

738 **Claim 10.** *There is an (H, F, k) -graph if and only if $B(H)$ has a subgraph*
 739 *in which every hyperedge vertex has degree $|V_{i_2}|$ and every vertex has degree*
 740 *at most $\lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$.*

741 *Proof of claim:* Assume that there is an (H, F, k) -graph G . Observe that
 742 since H is a neat hypergraph, free to remove some edges, we may assume
 743 that $G[S]$ is isomorphic to F for every hyperedge S of H . Let B' be the
 744 subgraph of $B(H)$ in which a vertex v is adjacent to a hyperedge S if and

745 only if v has degree i_2 in $G[S]$. By definition every hyperedge has degree $|V_{i_2}|$
746 in B' . Consider now a vertex v . Its degree in G is the sum of its degree in
747 the subgraphs induced by the hyperedges S to which it belongs because H is
748 neat. Hence $d_G(v) = i_2 \cdot d_{B'}(v) + i_1(d_H(v) - d_{B'}(v)) = i_1 \cdot d_H(v) + (i_2 - i_1)d_{B'}(v)$.
749 But G is an (H, F, k) -graph, so $d_G(v) \leq k$. With the previous equality, this
750 yields $d_{B'}(v) \leq \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1}$, and so $d_{B'}(v) \leq \lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$.

751 Conversely, assume $B(H)$ has a subgraph B' in which every $S \in E(H)$
752 has degree $|V_{i_2}|$ and any $v \in V(H)$ has degree at most $\lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$. We
753 construct G as follows: for each hyperedge S , the induced subgraph $G[S]$
754 is a copy of F such that any vertex v where $(S, v) \in E(B')$ is of degree
755 i_2 (there are $|V_{i_2}|$ such vertices) and the remaining vertices have degree i_1 .
756 Clearly, G F -overlays H . Moreover every vertex $v \in V(H)$ has degree i_2
757 in at most $\lfloor \frac{k - i_1 \cdot d_H(v)}{i_2 - i_1} \rfloor$ subgraphs and degree i_1 in the remaining subgraphs
758 induced by hyperedges to which it belongs. Hence v has degree at most k in
759 G . Therefore G is an (H, F, k) -graph. \square

760 Now the existence of a subgraph of $B(H)$ as shown in Claim ?? can be
761 done in polynomial time (see [?] Chap. 10). \square

762 **Theorem 9.** *Let i_1, i_2, i_3, k be three non-negative integers with $i_1 < i_2 < i_3$.
763 If F is a (i_1, i_2, i_3) -graph F such that all vertices of $V_{i_1} \cup V_{i_2}$ have the same
764 number of neighbours in V_{i_3} and $k < i_3 + i_1$, then one can solve $(\Delta \leq k)$ - \mathcal{F} -
765 OVERLAY in polynomial time on neat hypergraphs.*

766 *Proof.* Let j be the number of neighbours in V_{i_3} that each vertex of $V_{i_1} \cup V_{i_2}$
767 has.

768 Let H be an $|F|$ -uniform hypergraph. Observe that if a vertex has degree
769 i_3 in the subgraph induced by a hyperedge, then it cannot be in any other
770 hyperedge, for otherwise it would have degree at least $i_3 + i_1 > k$. Therefore,
771 we first check whether every hyperedge S contains a set $I(S)$ of $|V_{i_3}|$ vertices
772 that are uniquely in S . If not, then we return ‘No’. If it is the case, let $F' =$
773 $F - V_{i_3}$ and H' be the hypergraph whose vertex set is $V(H) \setminus \bigcup_{S \in E(H)} I(S)$
774 and whose hyperedge set is $\{S \setminus I(S) \mid S \in E(H)\}$, and let $k'(v) = k - j \cdot d_H(v)$
775 for every $v \in V(H')$. Since all vertices of $V_{i_1} \cup V_{i_2}$ have exactly j neighbours
776 in V_{i_3} , it is then easy to prove the following claim:

777 **Claim 11.** *There is an (H, F, k) -graph G if and only if there is a graph G'
778 F' -overlying H' such that $d_G(v) \leq k'(v)$ for all $v \in V(H')$.*

779 Deciding whether such a graph G' exists can be done similarly as the
780 proof of Theorem ?? replacing k by $k'(v)$. \square

781 **5. Further research**

782 Theorem ?? characterizes the complexity of MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY
 783 when \mathcal{F} contains a unique graph. It would be nice to extend this character-
 784 ization to families \mathcal{F} of arbitrary size.

785 **Problem 1.** *Characterize the pairs (\mathcal{F}, k) for which MAX $(\Delta \leq k)$ - \mathcal{F} -*
 786 *OVERLAY is polynomial-time solvable and those for which it is \mathcal{NP} -complete.*

787 Theorem ?? and the results obtained in Section ?? give a first view of
 788 the complexity of $(\Delta \leq k)$ - F -OVERLAY. A natural problem is to close the
 789 dichotomy:

790 **Problem 2.** *Characterize the pairs (F, k) for which $(\Delta \leq k)$ - F -OVERLAY*
 791 *is polynomial-time solvable and those for which it is \mathcal{NP} -complete.*

792 It would be interesting to consider the complexity of this problem when
 793 F is k -regular but non-connected, and when F is a cycle. In order to attack
 794 Problem ??, it would be helpful to prove the following conjecture.

795 **Conjecture 1.** *If $(\Delta \leq k)$ - F -OVERLAY is \mathcal{NP} -complete, then $(\Delta \leq k+1)$ -*
 796 *F -OVERLAY is also \mathcal{NP} -complete.*

797 Furthermore, for each pair (\mathcal{F}, k) such that MAX $(\Delta \leq k)$ - \mathcal{F} -OVERLAY
 798 is \mathcal{NP} -complete and $(\Delta \leq k)$ - \mathcal{F} -OVERLAY is polynomial-time solvable, it
 799 is natural to consider the parameterized complexity of MAX $(\Delta \leq k)$ - \mathcal{F} -
 800 OVERLAY when parameterized by $|E(H)| - s$, because $(\Delta \leq k)$ - \mathcal{F} -OVERLAY
 801 is the case $s = 0$.

802 It would also be interesting to obtain approximation algorithms for MAX
 803 $(\Delta \leq k)$ - \mathcal{F} -OVERLAY when this problem is \mathcal{NP} -complete.

804 As shown in Subsection ??, we proved that $(\Delta \leq k)$ - F -OVERLAY is some-
 805 times easier when restricted to neat hypergraphs. It would then be interesting
 806 to determine the complexity of $(\Delta \leq k)$ - F -OVERLAY on neat hypergraphs.
 807 So far, we only proved some of these problems to be polynomial-time solvable,
 808 but we believe some are \mathcal{NP} -complete.

809 **Problem 3.** *Characterize the pairs (F, k) for which $(\Delta \leq k)$ - F -OVERLAY*
 810 *on neat hypergraphs is polynomial-time solvable and those for which it is*
 811 *\mathcal{NP} -complete.*