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Hierarchical Symbolic Factorization for Sparse Matrices

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Hierarchical Symbolic Factorization for Sparse Matrices

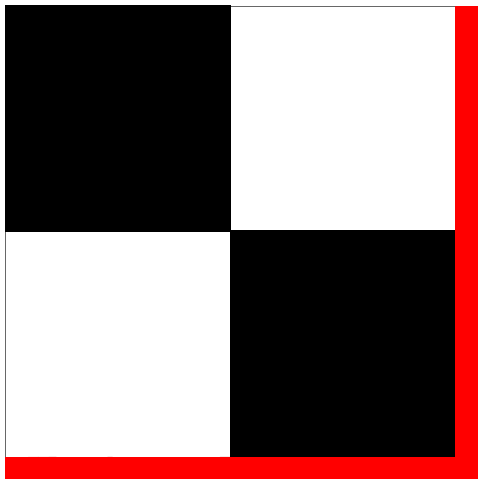
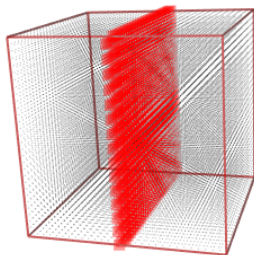
Aurélien FALCO

joint work with:

Emmanuel AGULLO, Luc GIRAUD, Guillaume SYLVAND
(Airbus & Inria HiePACS team)

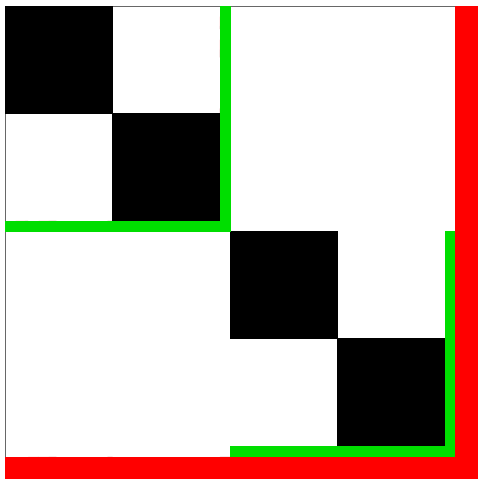
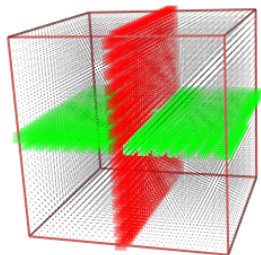
Sparse direct methods using low-rank compression

Nested dissection (ND) [George, 1973]



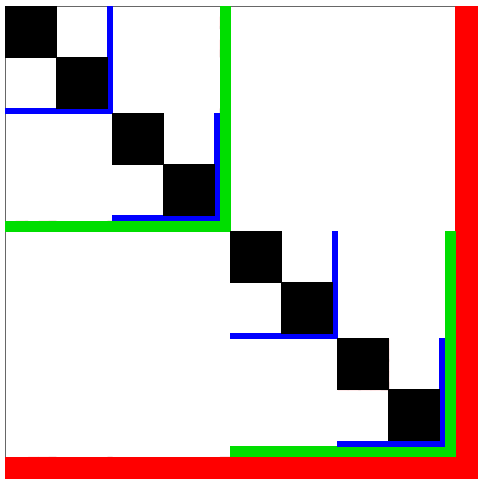
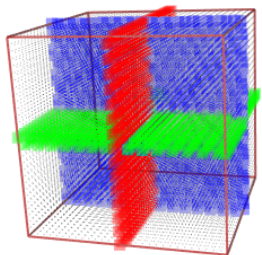
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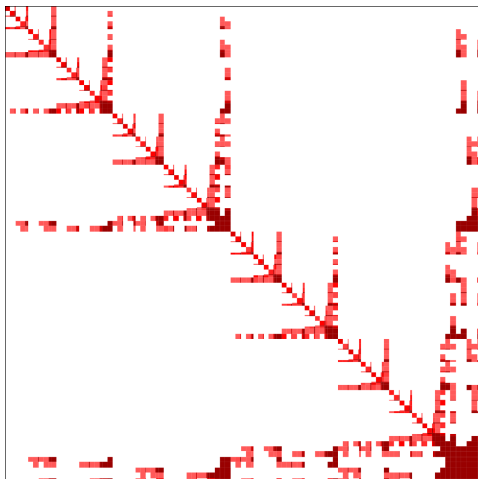
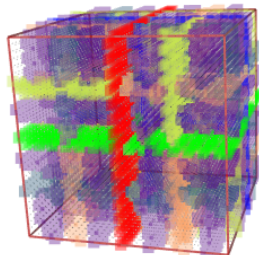
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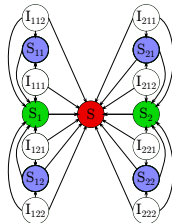
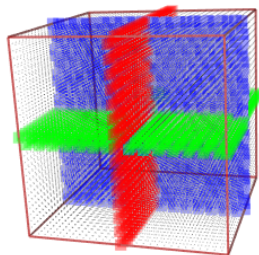
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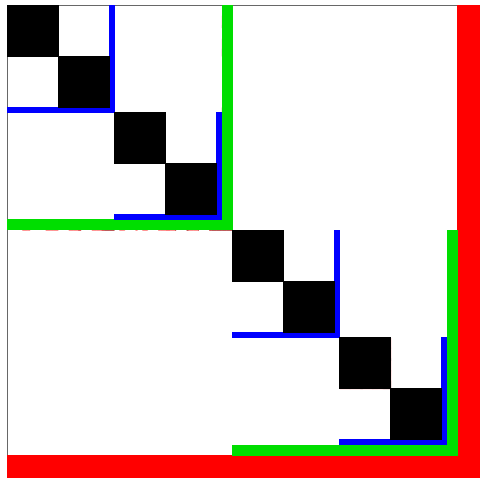


Sparse direct methods using low-rank compression

Symbolic Factorization



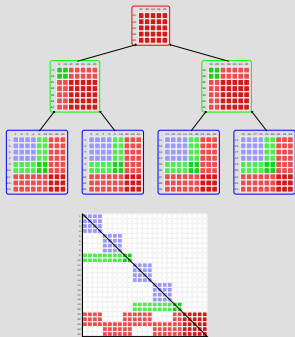
Quotient Graph



Matrix

Sparse direct methods using low-rank compression

- Multifrontal methods
 - Dense frontal matrices
[Ghysels et al., 2016, Xia, 2013, Xia et al., 2009, Amestoy et al., 2015, Aminfar et al., 2016, Mary, 2017]
- Supernodal methods
 - Dense (supernodal) sub-matrices
[Pichon et al., 2017a]



Sparse methods towards \mathcal{H}

- Advantages
 - Orderings
 - Symbolic factorization
- Hierarchical compression used locally on sub-matrices

Sparse direct methods using low-rank compression

- Multifrontal methods

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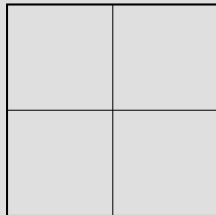
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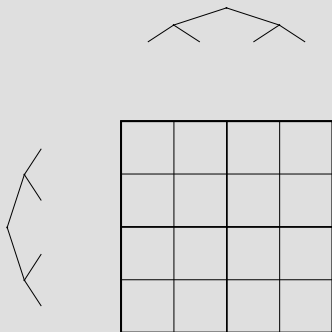
Hierarchical methods for dense linear systems

- Many Formats:
 - \mathcal{H} -Matrix, \mathcal{H}^2 -matrix, HSS, HODLR, BLR
- Clustering method
 - Bisection
[Hackbusch, 1999, Hackbusch, 2015]



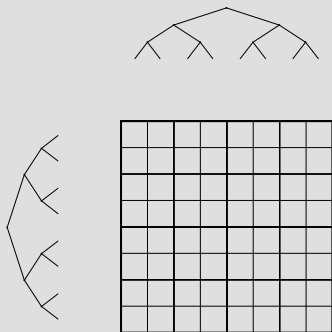
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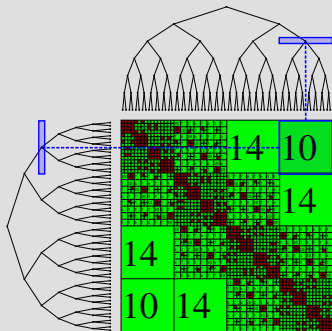
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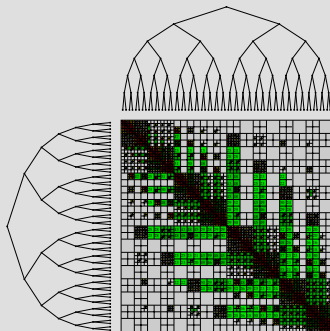
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Hierarchical methods exploiting sparsity

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Hierarchical methods exploiting sparsity

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- Clustering method
 - Nested dissection [Le Borne et al., 2007, Grasedyck et al., 2008, Grasedyck et al., 2009, Kriemann, 2013]



\mathcal{H} -Matrices towards sparse

- Advantages
 - Global hierarchical structure
 - Compression
- How to manage and locate fill-in?

Problem formulation

Sparse methods towards \mathcal{H}

- Advantages
 - Orderings lead to less fill-in
 - Symbolic factorization
- Current limits?
 - Hierarchical compression used locally on sub-matrices

\mathcal{H} -Matrices towards sparse

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- Current limits?
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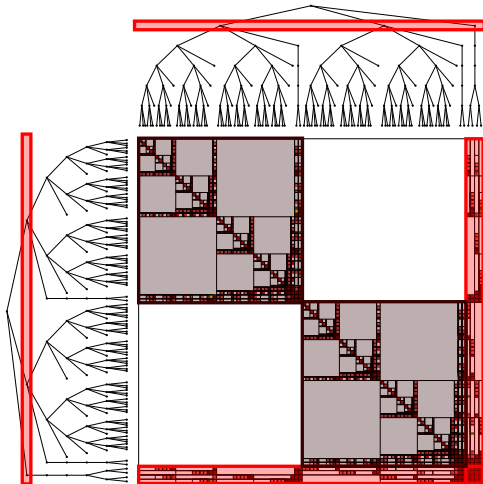
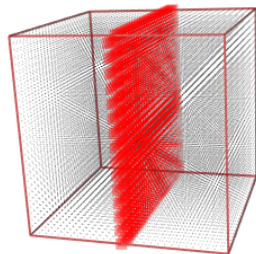
Objectives

Combine hierarchical and sparse techniques

- Number of non-zeros as low as sparse direct methods
- A global hierarchical structure

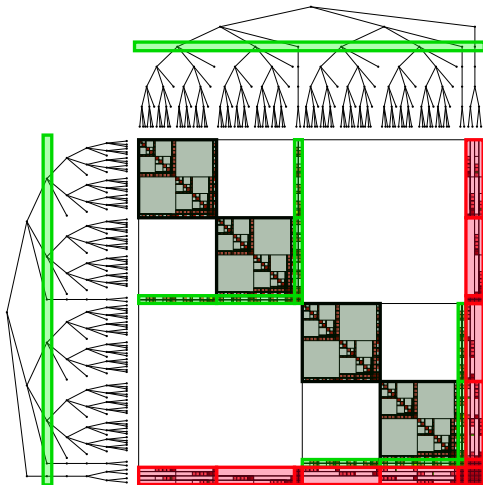
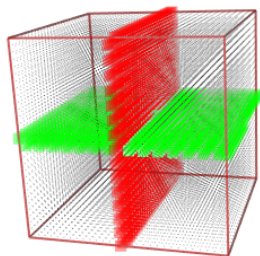
→ Hierarchical Symbolic Factorization

Nested dissection clustering [George, 1973]



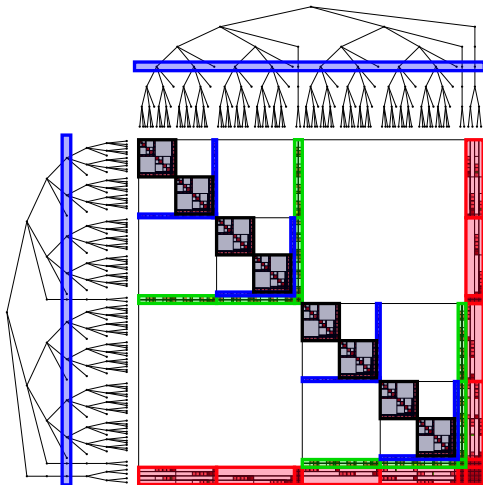
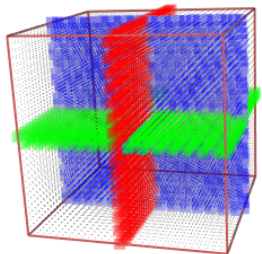
[Le Borne et al., 2007, Grasedyck et al., 2008,
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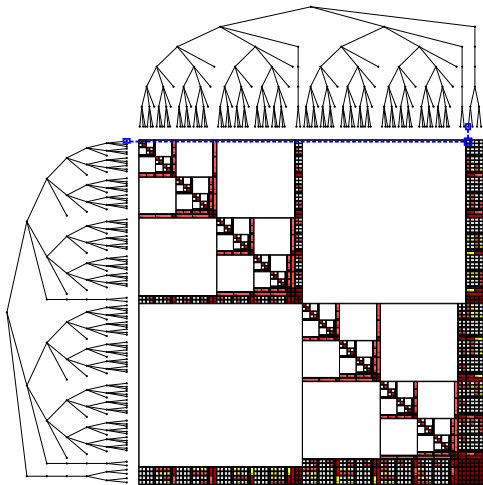
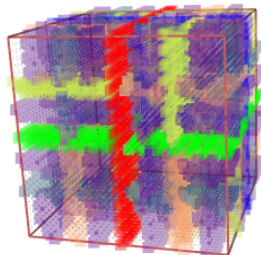
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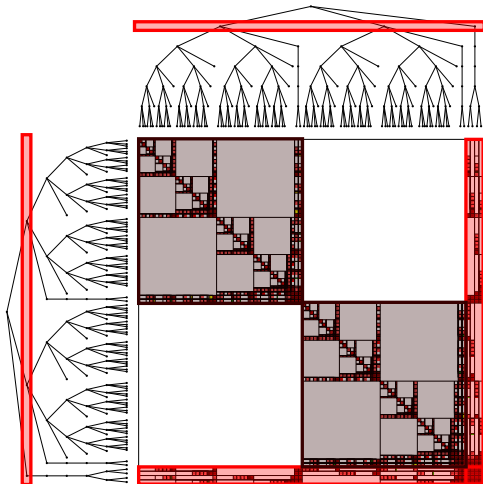
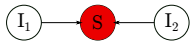
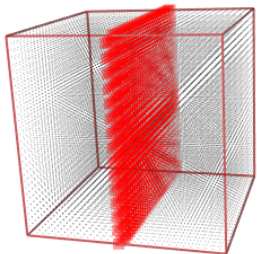
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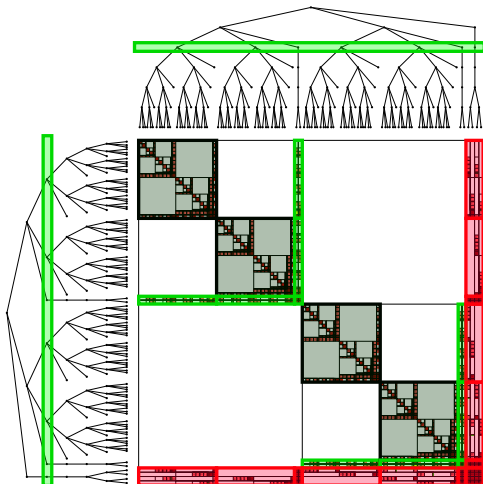
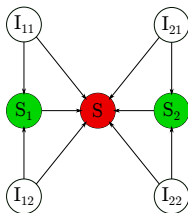
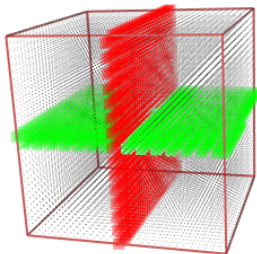
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Hierarchical Block Symbolic Factorization (TD-HBSF)



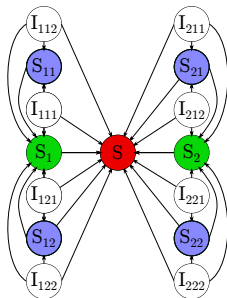
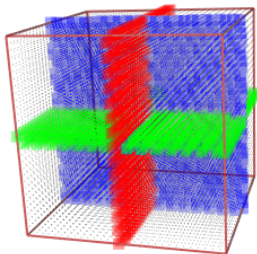
Quotient Graph

Hierarchical Block Symbolic Factorization (TD-HBSF)

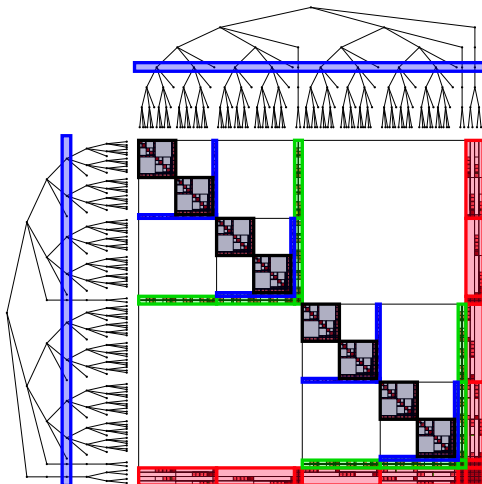


Quotient Graph

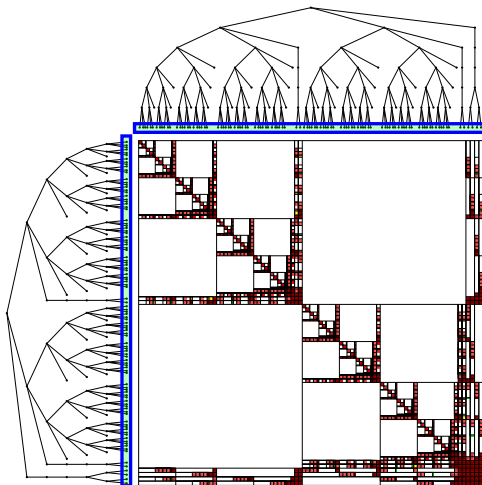
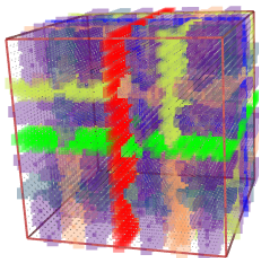
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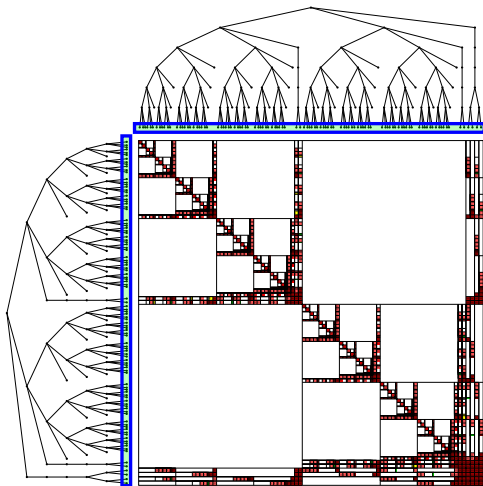
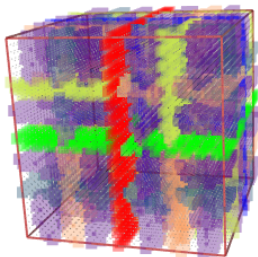
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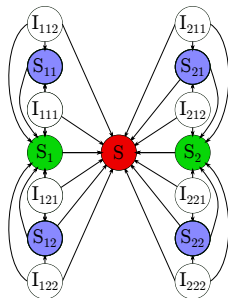
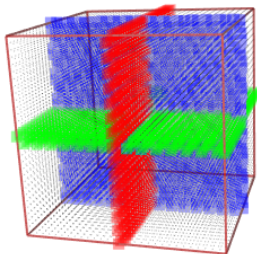
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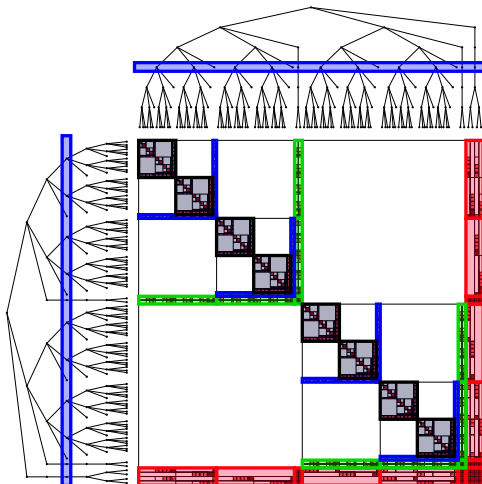
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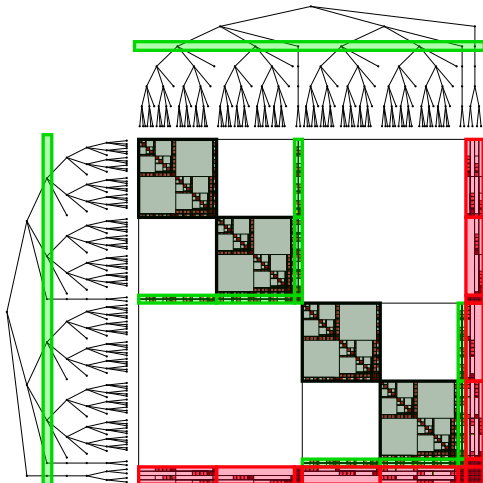
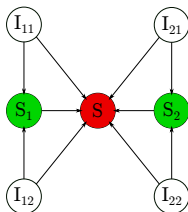
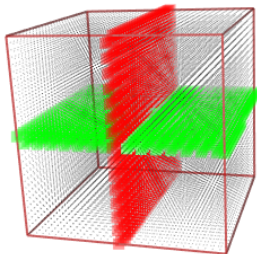
Hierarchical Block Symbolic Factorization (BU-HBSF)



Quotient Graph

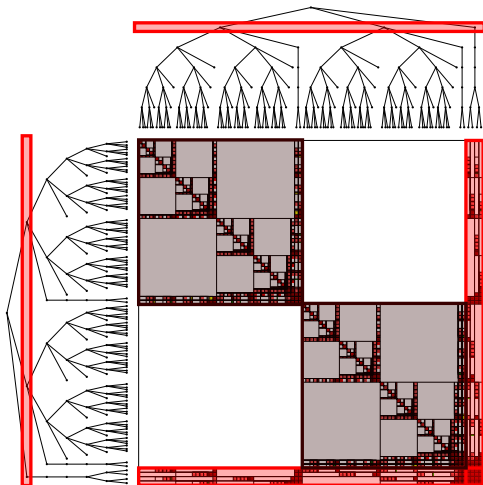
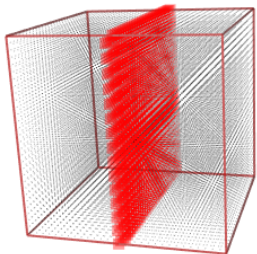


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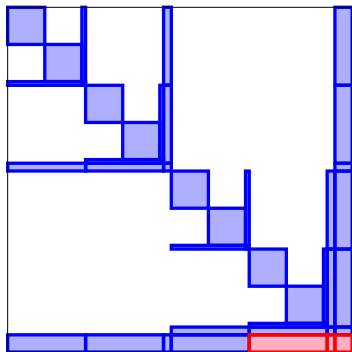
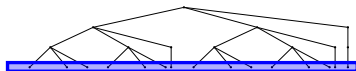
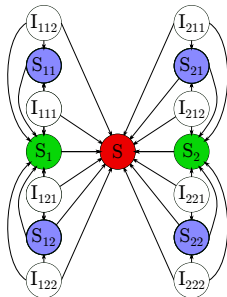
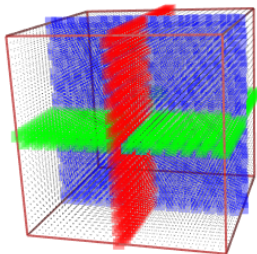
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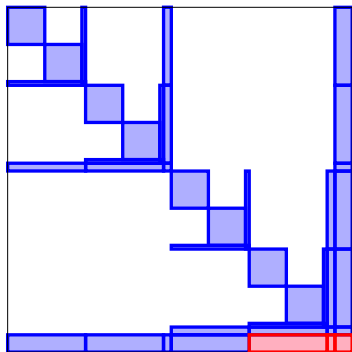
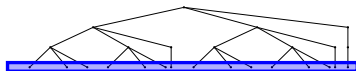
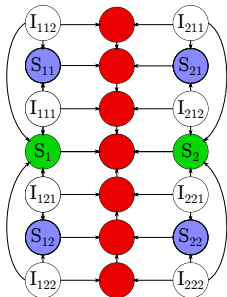
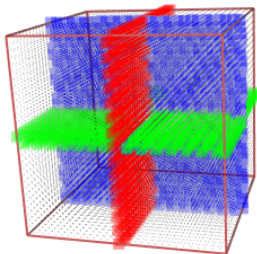


Quotient Graph

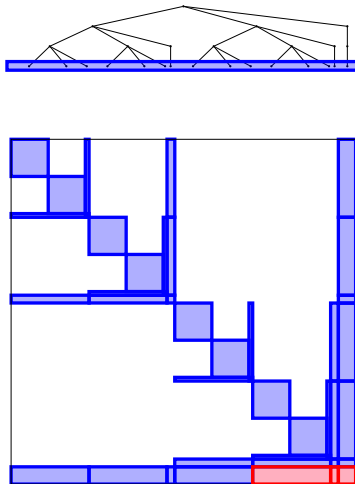
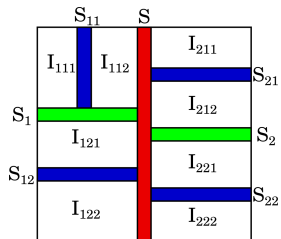
Separator Division



Separator Division

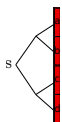
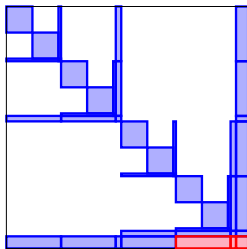
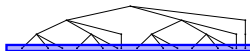
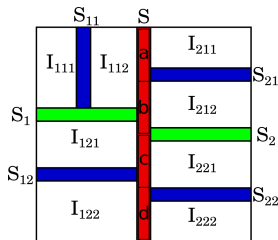


Separator Division



Minimizing extra-diagonal blocks is NP-hard
[Pichon et al., 2017b]

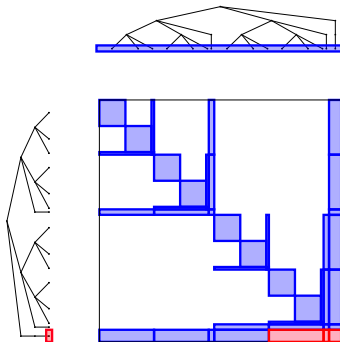
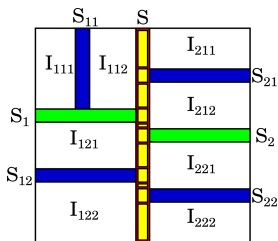
Interactions-Oblivious Separator Bisection



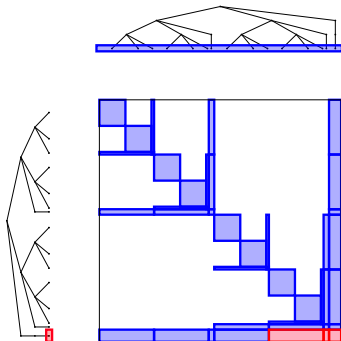
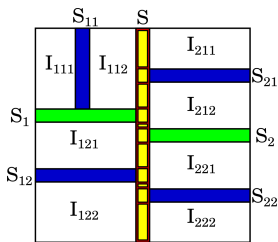
| | | | | | | | | | | | | | | | | |
|----|---|---|---|--|--|---|----|---|---|----|-----|-----|-----|-----|-----|-----|
| 3 | 2 | | | | | 9 | | | | 10 | 100 | 100 | 100 | 100 | 100 | 100 |
| | | | | | | | 2 | 1 | | | 10 | 10 | 100 | 100 | 100 | 100 |
| 11 | 9 | 9 | 8 | | | | | | | 97 | | | 98 | 100 | 100 | 100 |
| | | | | | | | 11 | 9 | 9 | 10 | | | 97 | 100 | 100 | 98 |

[Le Borne et al., 2007, Grasedyck et al., 2008,
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Interactions-Aware Flat Separator Division (IA-FSD)

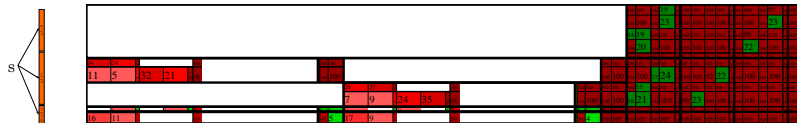
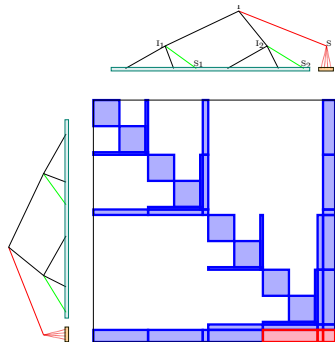
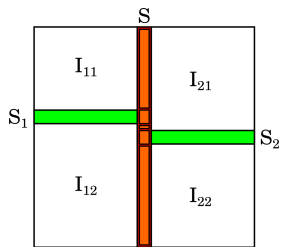


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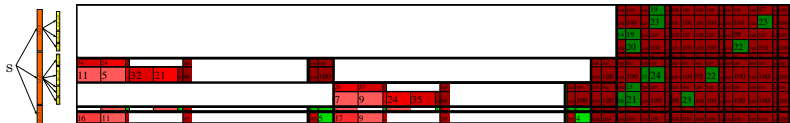
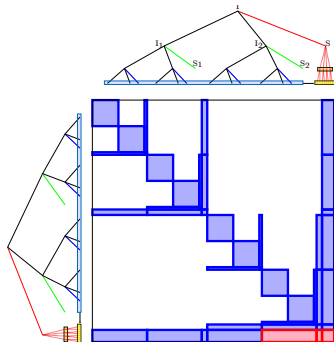
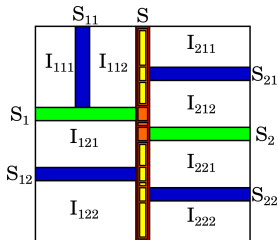


Related to T4 supernodes [Demmel et al., 1999] but too rare and therefore usually rejected

Interactions-Aware Hierarchical Separator Division (IA-HSD)

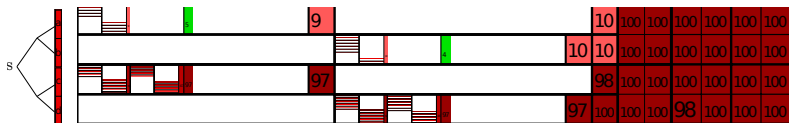
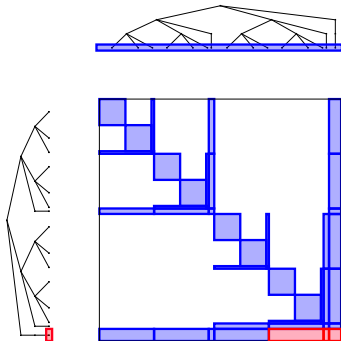
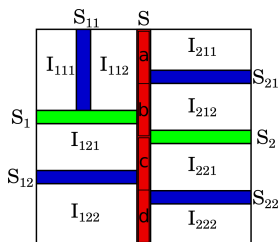


Interactions-Aware Hierarchical Separator Division (IA-HSD)



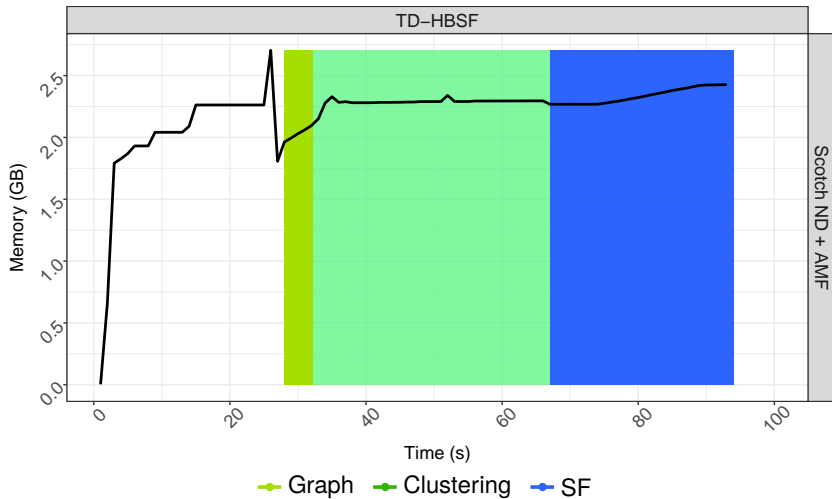
Interactions-Aware Leaf Separator Division (IA-LSD)

Using Block Column Symbolic Factorization (BCSF) Information



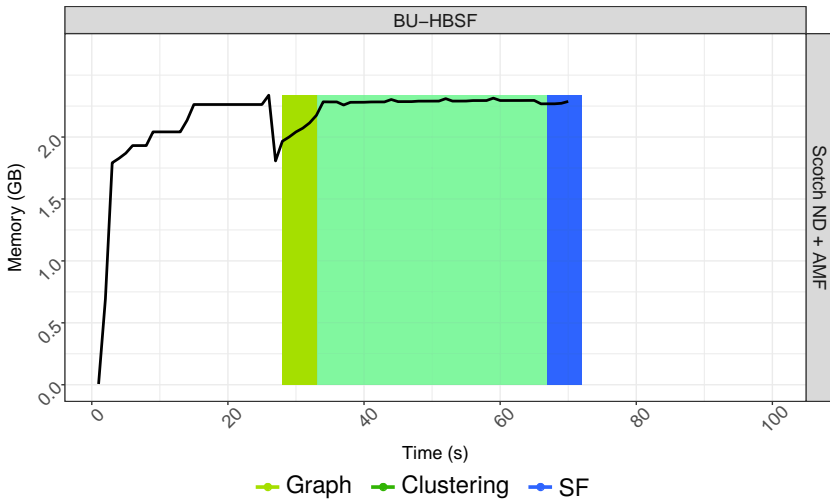
TD-HBSF Memory Profile

$N = 1e^6$



BU-HBSF Memory Profile

$N = 1e^6$



Conclusions




- Performing a (bottom-up) hierarchical symbolic factorization is nearly costless
- Possible divisions of separators
 - Bisection
 - Interactions-Aware Flat Separator Division (IA-FSD)
 - Interactions-Aware Hierarchical Separator Division (IA-HSD)
 - Interactions-Aware Leaf Separator Division (IA-LSD)

Perspectives & Work in Progress



- Extensive comparison between separator division methods
- Study the impact of each algorithm on numerical factorization
- Parallel implementation
- Extensive comparison with reference sparse solvers

Thank you for your attention!




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